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# Scanning Superfluid-Turbulence Cascade by Its Low-Temperature Cutoff

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On the basis of recently proposed scenario of the transformation of the Kolmogorov cascade into the Kelvin-wave cascade, we develop a theory of low-temperature cutoff. The theory predicts a specific behavior of the quantized vortex line density,  $L$ , controlled by the frictional coefficient,  $\alpha(T) \ll 1$ , responsible for the cutoff. The curve  $\ln L(\ln \alpha)$  is found to directly reflect the structure of the cascade, revealing four qualitatively distinct wavenumber regions. Excellent agreement with recent experiment by Walmsley *et al.* [Phys. Rev. Lett. 99, 265302 (2007)]—in which  $L(T)$  has been measured down to  $T \sim 0.08$  K—implies that the scenario of low-temperature superfluid turbulence is now experimentally validated, and allows to quantify the Kelvin-wave cascade spectrum.

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In the studies of superfluid turbulence (ST) (for introduction, see, e.g., [1, 2]), the case of very low temperatures remains intriguing and challenging [3]. A wealth of theoretical work has been devoted to ST in this regime in the past decade and especially in the past few years [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], and a detailed theoretical picture seems to emerge. In this picture, the most interesting physics is associated with the short length scales, where quantized nature of vortices manifests itself and the key role is played by Kelvin waves (KWs)—the distortion waves on quantized vortex lines. However, the feasibility of experimental confirmation of these predictions with the current technique seemed vague, which to many made the theory appear purely academic. Until very recently, the only experimentally confirmed fact had been the existence of cascades in the  $T \rightarrow 0$  limit [15, 16].

In their recent remarkable work [17], Walmsley *et al.* measured the vortex line density  $L$  (the length of vortex lines per unit volume), of the Kolmogorov cascade in superfluid  $^4\text{He}$  down to temperatures  $T \sim 0.08$  K. They found a striking temperature dependence of  $L$  and argued that it should be due to the Kelvin-wave cascade [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] being extended to higher wavenumbers with decreasing temperature. Better understanding of this process was formulated as a challenge for theorists.

In this Letter, we meet the above-mentioned challenge. Recently, the two of us proposed a detailed scenario [14] of how the zero-temperature Kolmogorov cascade transforms into the pure KW cascade. We argued that the transformation involves a chain of three intermediate regimes and thus occupies a significant interval in the wavenumber space. On the basis of this zero-temperature scenario, we now develop a theory of the low-temperature dissipative cutoff of the cascade. We show that the temperature dependence of  $L$  comes from the dependence of the wavelength scale  $\lambda_{\text{cutoff}}$ , at which the cascade ceases due to the mutual friction of vortex lines with the normal component, on the dimensionless friction coefficient  $\alpha \propto T^5$  ( $T \rightarrow 0$ ). A very characteristic shape of the function  $\ln L(\ln \alpha)$  following from our analysis is in an excel-

lent agreement with the experimental results of Walmsley *et al.* Remarkably, the shape of the curve  $\ln L(\ln \alpha)$  directly reflects four qualitatively distinct wavenumber regions of the cascade. The agreement between our theory and the data of Ref. [17] means that the scenario of low-temperature superfluid turbulence is now experimentally validated. Moreover, fitting the experimental data with theoretical curves allows us to quantify the KW cascade spectrum.

A cascade of energy—the decay mechanism of both classical and superfluid turbulence—implies the existence of an *inertial range*—a significant range of length scales,  $\lambda_{\text{en}} > \lambda > \lambda_{\text{cutoff}}$ , in which all the dissipative mechanisms are weak rendering the system essentially conservative. The long- and the short-wavelength ends of the inertial range play a special role: almost all the energy of the system is concentrated at  $\lambda_{\text{en}}$ , while the dissipation becomes appreciable only at  $\lambda \lesssim \lambda_{\text{cutoff}} \equiv \lambda_{\text{cutoff}}(T)$ . The relaxation process must lead to a transfer of the turbulent energy towards  $\lambda_{\text{cutoff}}$ , where it can be dissipated into heat. For the cascade regime to set in, it is also necessary that the dynamics within the inertial range is such that the “collisional” kinetic time  $\tau_{\text{coll}}(\lambda)$ , i.e. the time between elementary events of energy exchange at a certain scale  $\lambda$ , gets progressively shorter down the scales. In this case the decay is governed by the slowest kinetics at  $\lambda_{\text{en}}$ , where the energy flux (per unit mass)  $\varepsilon$  is formed, while the faster kinetic processes at shorter scales are able to instantly adjust to this flux supporting the transfer of energy between neighboring scales towards  $\lambda_{\text{cutoff}}$ . Thus, the cascade is a (quasi-)stationary regime in which the energy flux  $\varepsilon$  is constant through the length scales, while the variation of  $\varepsilon$  in time happens on the longest time scale of the system  $\tau_{\text{coll}}(\lambda_{\text{en}})$ .

In Ref. [14] we argued that at the effective zero temperature the inertial range is highly nontrivial, since the energy transport mechanisms change several times within it. We first briefly outline the  $T = 0$  case (see [14] for details), and then proceed to include the effect of mutual friction due to the finite  $T$ . In this scenario, the key

dynamical parameter is

$$\Lambda = \ln(l_0/a_0) \gg 1, \quad (1)$$

where  $a_0$  is the vortex core radius and  $l_0$  is the typical separation between the vortex lines determined by the Kolmogorov energy flux  $\varepsilon$  [14]. The parameter  $\Lambda$  controls the competition between the local self-induced motion of the vortex lines and the coupling between them. At the largest length scales in the inertial range,  $\lambda \gg r_0 \sim \Lambda^{1/2}l_0$ , the vortex lines are coupled in bundles that move coherently mimicking the velocity profile of the classical-fluid Kolmogorov turbulence. (The indistinguishability of ST from its classical counterpart at large scales has been attracting much attention since it was first predicted in Ref. [5].) At the scale  $r_0$ , the self-induced motion of vortex lines becomes appreciable and the cascade enters the *quantized* regime, where each vortex line evolves independently (apart from reconnections). Before the purely non-linear KW cascade [10] can develop on separate vortex lines at  $k \gg k_* \sim \Lambda^{1/2}/l_0$  the cascade experiences a complex transformation stage necessary to crossover from spatially organized bundles at  $k \ll r_0^{-1}$  and to generate the KWs. In the crossover, which (given  $\Lambda \sim 10$ ) extends for about a decade in the wavenumber space, the vortex-line reconnections [4] become the main driving mechanism of the energy transport in the cascade. Remarkably, the crossover range is further split into three (sub-)regimes distinguished by their specific types of reconnections: (1) reconnections of vortex-line bundles ( $r_0^{-1} \ll k \ll k_b \sim 1/\Lambda^{1/4}l_0$ ); (2) reconnections between nearest-neighbor lines ( $k_b \ll k \ll k_c \sim \Lambda^{1/4}/l_0$ ); (3) self-reconnections on single lines ( $k_c \ll k \ll k_*$ ). It is important here that each regime features a distinct spectrum of KW amplitude  $b_k$  summarized in Fig. 1.

If the vortex lines were smooth, the vortex-line density  $L$  would be simply related to the interline separation as  $L = l_0^{-2}$ . The increase of  $L$  due to the creation of KWs on the vortex lines is related to their spectrum by [4]

$$\ln [L(\alpha)/L_0] = \int_{\tilde{k}}^{k_{\text{cutoff}}(\alpha)} (b_k k)^2 dk/k. \quad (2)$$

Here  $k_{\text{cutoff}} \sim 1/\lambda_{\text{cutoff}}$ ,  $\tilde{k}$  is the smallest wavenumber of the KW cascade (not to be confused with the smallest wavenumber of the Kolmogorov cascade) at which the concept of a definite cutoff scale is meaningful, and  $L_0$  is the “background” line density corresponding to  $k_{\text{cutoff}} \sim \tilde{k}$ . There is an ambiguity in the definition of  $b_k$  associated with the choice of the spectral width of the scale  $k$ , which is fixed in Eq. (2) by setting the proportionality constant between the l.h.s. and the r.h.s. to unity.

At  $T = 0$ , the cascade is cut off by the radiation of sound (at least in  $^4\text{He}$ ) [6] at the length scale  $\lambda_{\text{cutoff}} = \lambda_{\text{ph}}$  [11], [14]. As we show below, changing the temperature one controls  $\lambda_{\text{cutoff}}(T) > \lambda_{\text{ph}}$  in Eq. (2), which allows one to *scan* the KW cascade observing qualitative changes in  $L(T)$  as  $\lambda_{\text{cutoff}}$  traverses different cascade regimes. The existence of a well-defined cutoff is due to the fact that

the cascade is supported by rare kinetic events in the sense that  $\tau_{\text{coll}} \equiv \tau_{\text{coll}}(\varepsilon, k)$  is much larger than the KW oscillation period,  $\tau_{\text{per}} \equiv \tau_{\text{per}}(k)$ . The dissipative time  $\tau_{\text{dis}} \equiv \tau_{\text{dis}}(\alpha, k) \sim \tau_{\text{per}}/\alpha$ , as we show below, is the typical time of the frictional decay of a KW at the scale  $k$ . Thus, the cutoff condition is

$$\tau_{\text{dis}}(\alpha, k) \sim \tau_{\text{coll}}(\varepsilon, k), \quad (3)$$

which implies that the energy dissipation rate at a given wavenumber scale becomes comparable to the energy being transferred to higher wavenumber scales per unit time by the cascade. It is this condition that defines the cutoff wavenumber  $k_{\text{cutoff}} \equiv k_{\text{cutoff}}(\varepsilon, \alpha)$ . Decreasing  $T$  and thus  $\alpha(T)$ , one gradually increases  $k_{\text{cutoff}}$  thereby scanning the cascade. In view of Eq. (2), this in principle allows one to extract the KW spectrum.

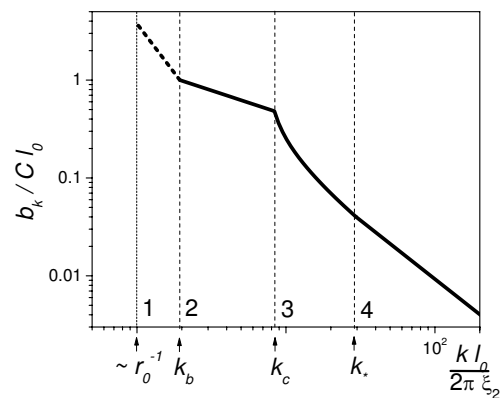


FIG. 1: Spectrum of Kelvin waves in the quantized regime as predicted in Ref. [14] and quantified here, apart from the regime (1), by the fit to experimental data, Fig. 2. In view of Eq. (2), the constants  $C$  and  $\xi_2$  can be found only in the combination  $\xi_2 C \approx 0.049$ . The inertial range consists of a chain of cascades driven by different mechanisms: (1) reconnections of vortex-line bundles, (2) reconnections between nearest-neighbor vortex lines in a bundle, (3) self-reconnections on single vortex lines, (4) non-linear dynamics of single vortex lines without reconnections. The regimes (3) and (4) are familiar in the context of non-structured vortex tangle decay [4, 10].

At finite  $T$ , dissipative dynamics of a vortex line element is described by the equation (omitting the third term in the r.h.s., which is irrelevant for dissipation) [1, 2]

$$\dot{\mathbf{s}} = \mathbf{v}(\mathbf{s}) + \alpha \mathbf{s}' \times [\mathbf{v}_n(\mathbf{s}) - \mathbf{v}(\mathbf{s})]. \quad (4)$$

Here  $\mathbf{v}(\mathbf{r})$  is the superfluid velocity field,  $\mathbf{v}_n(\mathbf{r})$  is the normal velocity field,  $\mathbf{s} = \mathbf{s}(\xi, t)$  is the time-evolving radius-vector of the vortex line element parameterized by the arc length, the dot and the prime denote differentiation with respect to time and the arc length, respectively.

At  $\alpha \sim 1$  the superfluid and normal components are strongly coupled and the KWs are suppressed. In this case, the cascade must cease *before* it enters the quantized regime, i.e.  $\lambda_{\text{cutoff}} \gtrsim r_0$ . In this Letter, we do not

discuss the strongly coupled case, since it has no control parameter for a theory. Consideration regarding energy spectra in this regime can be found in Ref. [18].

If, however, the mutual friction is small, then  $\alpha^{-1} \gg 1$  gives the characteristic number of KW oscillations required for the wave to decay. Indeed, to the first approximation in  $1/\Lambda$ ,  $\mathbf{v}(\mathbf{s})$  in Eq. (4) is given by the local induction approximation (LIA) [1],

$$\mathbf{v}(\mathbf{s}) \approx \Lambda_R \frac{\kappa}{4\pi} \mathbf{s}' \times \mathbf{s}'', \quad \Lambda_R = \ln(R/a_0), \quad (5)$$

where  $\kappa$  is the circulation quantum and  $R$  is the typical curvature radius. Taking into account that  $\Lambda_R$  is a very weak function of  $R$ , we shall treat it as a constant of the typical value  $\Lambda_R \sim \Lambda$ . As long as  $\alpha \ll 1$ , the disturbance of the normal component caused by vortex line motion can be neglected in Eq. (4), so that  $\mathbf{v}_n(\mathbf{s})$  is irrelevant for KW dissipation. In this case, Eqs. (4), (5) give the rate at which the amplitude  $b_k$  of a KW with wavenumber  $k$  decays due to the mutual friction:

$$\dot{b}_k \sim -\alpha \omega_k b_k, \quad (6)$$

while the KW dispersion is  $\omega_k = (\kappa/4\pi)\Lambda k^2$ . Here and below we omit factors of order unity, which are subject to the definition of the spectral width of the wave. These factors can not be found within our theory, but can be extracted from experimental data (as we do it below) and from numerical simulations. Since the energy per unit line length associated with the wave is  $E_k \sim \kappa \rho \omega_k b_k^2$ , the power dissipated (per unit line length) at the scale  $\sim k^{-1}$  is given by

$$\Pi(k) \sim \alpha \kappa \rho \omega_k^2 b_k^2, \quad (7)$$

where  $\rho$  is the fluid density. In the following, we analyze ST as  $\alpha(T)$  scans through the regimes shown in Fig. 1.

*Regime (1).* As it was already mentioned, at this stage the vortex lines are organized in bundles; the amplitudes of waves on these lines are given by [14]  $b_k \sim r_0^{-1} k^{-2}$ . One can estimate the total power lost due to the friction of these bundles at the wavenumber scale  $r_0^{-1} \ll k \ll k_b$  per unit mass of the fluid as

$$\varepsilon_{\text{dis}}(k) = \frac{1}{\rho b_k^2} \frac{b_k^2}{l_0^2} \Pi(k) \sim \alpha \kappa^3 \Lambda / l_0^4. \quad (8)$$

Here, the first factor in the r.h.s. is associated with the correlation volume at this scale  $\sim b_k^2 k^{-1}$  [14] and the second one stands for the number of vortex lines in the volume. Note that the dissipated power is constant at all the length scales within this regime and, since according to Ref. [14] the interline separation is related to the Kolmogorov flux by  $\varepsilon \sim \Lambda \kappa^3 / l_0^4$ , it is simply given by  $\alpha \varepsilon$ . Thus, when  $\alpha \ll 1$  the kinetic channel in the whole regime (1) becomes efficient and the cascade reaches the scale  $\lambda_b$ , where the notion of bundles becomes meaningless. That is the regime (1), as opposed to the regimes (2)-(4), is *not* actually scanned by  $\alpha$ . Basically, the inertial range of the regime (1) develops as a whole while  $\alpha$

evolves from the values of order unity to the values much smaller than unity.

*Regime (2).* In the range  $k_b \ll k \ll k_c$ , the spectrum of KWs is given by [14]  $b_k \sim l_0 (k_b/k)^{1/2}$ , which for the dissipated power yields

$$\varepsilon_{\text{dis}}(k) \sim \frac{1}{\rho l_0^2} \Pi(k) \sim \alpha \kappa^3 \Lambda^{7/4} k^3 / l_0. \quad (9)$$

The condition  $\varepsilon_{\text{dis}}(k) \sim \varepsilon$  gives the cutoff wavenumber

$$k_{\text{cutoff}} = \xi_2 \Lambda^{-1/4} \alpha^{-1/3} (2\pi/l_0), \quad k_b \ll k_{\text{cutoff}} \ll k_c, \quad (10)$$

where  $\xi_2$  is some constant of order unity. Then, from Eq. (2) we obtain

$$\ln \frac{L(k_{\text{cutoff}})}{L_0} = C^2 (k_b l_0)^2 [k_{\text{cutoff}}/k_b - 1]. \quad (11)$$

Here, we set  $b_k = C l_0 (k_b/k)^{1/2}$ , where  $C$  is a constant of order unity. The overall magnitude of  $b_k$  in the other regimes follows then from the continuity.

*Regime (3).* In this regime, supported by self-reconnections, the spectrum is given by  $b_k \sim k^{-1}$  (up to a logarithmic pre-factor). The corresponding energy balance condition yields the cutoff scale:

$$k_{\text{cutoff}} = \xi_3 (\Lambda \alpha)^{-1/2} (2\pi/l_0), \quad k_c \ll k_{\text{cutoff}} \ll k_*. \quad (12)$$

With the logarithmic pre-factor taken into account [4], the spectrum in this regime reads  $b_k = C [1 + c_3^2 \ln(k/k_c)]^{-1/2} (\sqrt{k_c k_b}/k) l_0$ , where  $c_3$  is a constant of order unity. Then, the relative increase of vortex line density through this regime is given by

$$\frac{L(k_{\text{cutoff}})}{L(k_c)} = \left[ 1 + c_3^2 \ln \frac{k_{\text{cutoff}}}{k_c} \right]^\nu, \quad (13)$$

where  $\nu = C^2 k_c k_b l_0^2 / c_3^2$ .

*Regime (4).* Since the spectrum of the purely nonlinear regime,  $b_k \sim k^{-6/5}$ , is steeper than the marginal  $b_k \sim k^{-1}$  meaning that the integral in Eq. (2) builds up at the lower limit, as soon as  $k_{\text{cutoff}} \gtrsim k_*$  the line density  $L(k_{\text{cutoff}})$  starts to saturate and becomes independent of  $k_{\text{cutoff}}$  at  $k_{\text{cutoff}} \gg k_*$ . The energy balance gives the dependence  $k_{\text{cutoff}}$  on  $\alpha$  in the form

$$k_{\text{cutoff}} = \xi_4 \Lambda^{-3/4} \alpha^{-5/8} (2\pi/l_0), \quad k_{\text{cutoff}} \gg k_*, \quad (14)$$

where  $\xi_4$  is an unknown constant. The coefficient in the KW spectrum is fixed by continuity with the previous regime, yielding  $b_k = C [1 + c_3^2 \ln(k_*/k_c)]^{-1/2} \sqrt{k_c k_b} k_*^{1/5} k^{-6/5} l_0$ . Eq. (2) thus yields

$$\frac{L(k_{\text{cutoff}})}{L(k_*)} \approx 1 + \frac{(5C^2/2) k_c k_b l_0^2}{1 + c_3^2 \ln(k_*/k_c)} \left[ 1 - \left( \frac{k_*}{k_{\text{cutoff}}} \right)^{2/5} \right]. \quad (15)$$

The continuity of  $k_{\text{cutoff}}$  leads to the following constraints on the free coefficients in Eqs. (10), (12), (14),

$\xi_3 = \xi_2 \Lambda^{1/4} \alpha_c^{1/6}$ ,  $\xi_4 = \xi_2 \Lambda^{1/2} \alpha_c^{1/6} \alpha_*^{1/8}$ , where  $\alpha_c \sim \Lambda^{-3/2}$  is the value of the friction coefficient at which the cascade is cut off at the crossover between the regimes (2) and (3) and  $\alpha_* \sim 1/\Lambda^2$  corresponds to the one between (3) and (4). Introducing  $\alpha_b \lesssim 1$ , corresponding to the crossover from the regime (1) to (2), we rewrite Eqs. (11), (13), (15) in terms of  $\alpha$ :

$$\begin{aligned} \ln \frac{L(\alpha)}{L_0} &= A_2 \left[ (\alpha_b/\alpha)^{1/3} - 1 \right], & \alpha_c \ll \alpha \ll \alpha_b, \\ \frac{L(\alpha)}{L(\alpha_c)} &= \left[ 1 + (c_3^2/2) \ln \frac{\alpha_c}{\alpha} \right]^\nu, & \alpha_* \ll \alpha \ll \alpha_c, \\ \frac{L(\alpha)}{L(\alpha_*)} &\approx 1 + A_4 \left[ 1 - \left( \frac{\alpha}{\alpha_*} \right)^{1/4} \right], & \alpha \ll \alpha_*, \end{aligned}$$

where  $A_2 = \frac{(2\pi)^2 \xi_2^2 C^2 / \Lambda^{1/2} \alpha_b^{2/3}}{(2\pi)^2 \xi_2^2 C^2 / \Lambda^{1/2} (\alpha_c \alpha_b)^{1/3} c_3^2}$ ,  $\nu = \frac{5(2\pi)^2 \xi_2^2 C^2 / 2\Lambda^{1/2} [1 + (c_3^2/2) \ln(\alpha_c/\alpha_*)] (\alpha_c \alpha_b)^{1/3}}{(2\pi)^2 \xi_2^2 C^2 / \Lambda^{1/2} \alpha_b^{2/3}}$ , and  $A_4 = \frac{5(2\pi)^2 \xi_2^2 C^2 / 2\Lambda^{1/2} [1 + (c_3^2/2) \ln(\alpha_c/\alpha_*)] (\alpha_c \alpha_b)^{1/3}}{(2\pi)^2 \xi_2^2 C^2 / \Lambda^{1/2} \alpha_b^{2/3}}$ .

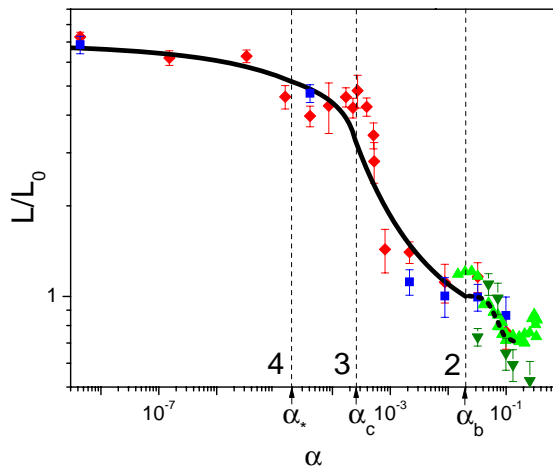


FIG. 2: (Color online.) Fit of the experimental data (squares and diamonds) adapted from Ref. [17] ( $l_0 \approx 5 \times 10^{-3} \text{ cm}$  giving  $\Lambda \approx 13$ ). The high temperature measurements of Refs. [21], [22] are represented by triangles and inverted triangles respectively. The form of  $\alpha(T)$  is taken according to Ref. [19] at  $T \gtrsim 0.5 \text{ K}$  (roton scattering) and according to Ref. [20] at  $T \lesssim 0.5 \text{ K}$  (phonon scattering). The fitting parameters are  $A_2 \approx 0.36$ ,  $A_4 \approx 0.33$ ,  $c_3 \approx 2.9$ ,  $\nu \approx 0.18$  with  $\alpha_b \approx 2 \times 10^{-2}$ ,  $\alpha_c \approx 2.5 \times 10^{-4}$ ,  $\alpha_* \approx 2 \times 10^{-5}$ .

In conclusion, we discuss the qualitative form of

$L(\alpha)$  in connection with the alternative scenario of the crossover from the Kolmogorov to the KW cascade proposed by L'vov *et al.* [13]. In Ref. [14] we argued that this scenario mistakenly leaves out the reconnection-driven regimes. In this scenario the classical regime extends down to the scale of interline separation  $l_0$ , where it is *immediately* followed by the purely non-linear KW cascade [regime (4) in our notation]. Since the kinetics of the regime (4) are too weak to support the flux  $\varepsilon$  at this scale, the vorticity was proposed in [13] to build up in the classical regime at the scales adjacent to  $l_0$ —the so-called bottleneck effect. Although this picture would also naturally predict a significant increase of  $L$ , the peculiar behavior of  $L(\alpha)$  observed in [17] essentially rules out the scenario of Ref. [13]. The reason is that the dramatic rise of  $L$  happens only at  $\alpha$  as low as  $10^{-3}$ , while the bottleneck accumulation of classical vorticity must manifest itself as soon as KWs become an unavoidable relaxational channel, i.e. already at  $\alpha \lesssim 1$ .

Instead,  $L(\alpha)$  exhibits the qualitative form peculiar to our scenario of Ref. [14]. As  $\alpha$  decreases from  $\alpha \sim 1$  to the values significantly smaller than unity, the line density  $L$  increases only by some factor close to unity ( $\sim 1.5$  in the experiment), which reflects the formation of the regime (1) driven by the reconnections of vortex bundles. During the crossover from the region (1) to (2), the increase of  $L$  is minimal—a *shoulder* in the curve  $L(\alpha)$  arises, in contrast to a large increase of  $L$  expected in the bottleneck picture. It is only well inside the region (2) that the increase of  $L$  becomes progressively pronounced, and, at the crossover to the region (3), the function  $L(\alpha)$  achieves its maximal slope, determined by the fractalization of the vortex lines necessary to support the cascade within the interval (3). As the cutoff moves along the interval (3) towards higher wavenumbers, the slope of  $L(\alpha)$  becomes less steep due to the decrease of the characteristic amplitude of KW turbulence. When the cutoff passes the crossover to the regime (4), the curve  $L(\alpha)$  gradually levels. Fitting the experimental data fixes the values of all the dimensionless parameters, thus revealing the *quantitative* form of the KW spectrum, Fig. 1. Note, however, that in view of the fact that the experimental  $\Lambda \sim 10$  is not so large we would expect certain systematic deviations, but a significant scatter of the experimental data does not allow us to assess them.

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