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E Kozik

Boris Svistunov

University of Massachusetts - Amherst, svistunov@physics.umass.edu

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Scale Separation Scheme for Simulating Superfluid Turbulence: Kelvin wave Cascade

Evgeny Kozik¹ and Boris Svistunov^{1,2,3}

¹ *Department of Physics, University of Massachusetts, Amherst, MA 01003*

² *Russian Research Center “Kurchatov Institute”, 123182 Moscow, Russia*

³ *Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106*

A Kolmogorov-type cascade of Kelvin waves—the distortion waves on vortex lines—plays a key part in the relaxation of superfluid turbulence at low temperatures. We propose an efficient numeric scheme for simulating the Kelvin wave cascade on a single vortex line. The idea is likely to be generalizable for a full-scale simulation of different regimes of superfluid turbulence. With the new scheme, we are able to unambiguously resolve the cascade spectrum exponent, and thus to settle the controversy between recent simulations [1] and recently developed analytic theory [2].

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The superfluid turbulence (for an introduction see [3, 4]) can be viewed as a tangle of quantized vortex lines. In the case of an essentially finite temperature, the superfluid turbulence is perfectly described by the local induction approximation (LIA), as it has been shown in pioneering simulations by Schwarz [5, 6]. LIA reduces the problem of finding the velocity of an element of the vortex line to its local differential characteristics:

$$\dot{\mathbf{s}} = \beta \mathbf{s}' \times \mathbf{s}'' + \text{friction} , \quad (1)$$

$$\beta = \frac{\kappa}{4\pi} \ln(R/a_0) , \quad (2)$$

where $\mathbf{s} \equiv \mathbf{s}(\xi, t)$ is the evolving in time radius-vector of the vortex line element, the parameter ξ being the arc length, the dot and prime denote differentiation with respect to time and ξ , respectively; κ is the quantum of velocity circulation; R is the typical curvature radius (which is of the same order that the typical interline spacing), and a_0 is the vortex core radius. A necessary condition of applicability of LIA is

$$\ln(R/a_0) \gg 1 . \quad (3)$$

At zero temperature, a crucial role in the dynamics of superfluid turbulence is played by the Kelvin wave cascade [1, 2, 7, 8, 9, 10]. LIA fails to properly describe a considerable part of the cascade inertial range (see, e.g., [2] and references therein). In this case, one has to use the description in terms of full Biot-Savart equation (BSE):

$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \int (\mathbf{s}_0 - \mathbf{s}) \times d\mathbf{s}_0 / |\mathbf{s}_0 - \mathbf{s}|^3 . \quad (4)$$

Here the evolving in time radius-vector of the vortex line element, $\mathbf{s} \equiv \mathbf{s}(\zeta, t)$, is labelled by some parameter ζ (not necessarily the arc length), the vector \mathbf{s}_0 is of the same physical meaning as \mathbf{s} , understood as an integration variable; the integration is over all the vortex lines; $d\mathbf{s}_0 = (\partial\mathbf{s}_0/\partial\zeta) d\zeta$. The two conditions of applicability of Eq. (4) are: (i) the absence of the normal component implying the absence of the friction term, and (ii)

$$R \gg a_0 . \quad (5)$$

The need to use the full Biot-Savart law in dealing with the Kelvin wave cascade had been recognized in Refs. [1, 10]. It is worth noting that the Kelvin wave cascade is not the only problem where non-local vortex line interactions play a crucial part. The other examples include the conditions of stability of vortex knots revealed by Ricca *et al.* [11]: the vortex knots that are unstable under LIA acquire a greatly increased lifetime when BSE is used. In the simulation by Samuels [12], the non-local interaction of vortex rings in laminar flow of He II results in overall parabolic superfluid velocity profile matching the normal-fluid velocity profile. In the problem of superfluid turbulence decay at length scales greater than the interline spacing, the non-local interactions of vortex lines result in Kolmogorov cascade of eddies mimicking the behavior of the classical turbulence even without the normal component [13, 14, 15].

In theoretical studies of turbulent vortex dynamics, an analytic solution is normally not available, and numeric simulations are of crucial importance. Clearly, the computational price of a direct simulation of BSE is macroscopically larger than that of LIA, since to perform one step in time for a given element of a line one has to address all the line elements. In this Letter, we observe that this extensive addressing is actually extremely excessive and can be easily eliminated at the expense of a *controllable* systematic error. The physical reason for that is as follows. The interactions at a distance r are significant only for the Kelvin waves of the wavelength $\lambda \gtrsim r$. The characteristic time of evolution of a harmonic of the wavelength λ scales like λ^2 , which means that during the time interval much shorter than λ^2 the configuration of these harmonics remains essentially unchanged. Correspondingly, the maximal reasonable frequency of addressing the distances $\sim r$ from a given element of the vortex line scales like $1/r^2$. This leads to a numeric scheme where the long-range interactions of Kelvin waves do not reduce the simulation efficiency. We employ this scheme to simulate the Kelvin wave cascade on a single vortex line at zero temperature. Our data for the cascade spectrum are in an excellent agreement with the predictions of analytic theory [2].

Scale separation scheme.—To be specific, we consider

the zero-temperature dynamics of Kelvin waves on a straight vortex line. We separate different scales of distance in the integral in Eq. (4) by writing BSE in the form

$$\dot{\mathbf{s}} = Q_1 + Q_2 + Q_3 + \dots + Q_n + \dots, \quad (6)$$

where

$$Q_n(\mathbf{s}) = \frac{\kappa}{4\pi} \int_{2^{n-1}a_0 < |\mathbf{s}_0 - \mathbf{s}| < 2^n a_0} \frac{(\mathbf{s}_0 - \mathbf{s}) \times d\mathbf{s}_0}{|\mathbf{s}_0 - \mathbf{s}|^3}. \quad (7)$$

Strictly speaking, Eqs. (6)-(7) are meaningful only for $n \gg 1$, since at distances $\lesssim a_0$ from the vortex core the velocity field is model-dependent. With the same accuracy, one may replace in Eq. (6) the $n \sim 1$ non-local terms with a local term. (As we do it in our simulation scheme.—See below.)

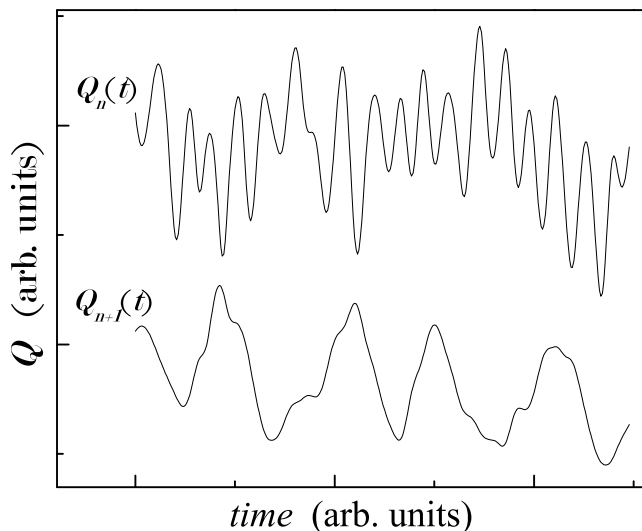


FIG. 1: Comparing functions $Q_n(t)$ and $Q_{n+1}(t)$ for a fixed vortex line element. The data are obtained by simulating Kelvin wave cascade with full BSE.

The idea behind the separation of scales of distance in Eq. (6) is clearly seen in Fig. 1, where we compare two functions, $Q_n(t)$ and $Q_{n+1}(t)$, in the regime of Kelvin wave cascade. The characteristic period of variation of the function $Q_{n+1}(t)$ is approximately 4 times larger than that of Q_n . This is precisely what one would expect on the basis of qualitative analysis, since the typical wavelength of harmonics contributing to Q_n is $\lambda_n = 2^n a_0$, which leads to the estimate

$$\tau_n \propto \lambda_n^2 \quad (8)$$

for the typical time of variation of Q_n . The contributions to Q_n from the harmonics with $\lambda \ll \lambda_n$ are suppressed by the denominator in Eq. (7) in combination with a random interference in the numerator.

The behavior of Q_n 's suggests the following numeric procedure. Each $Q_n(t)$ is approximated by a piecewise

polynomial function $\tilde{Q}_n(t)$ based on an appropriate mesh $\{t_j^{(n)}\}$, $j = 1, 2, 3, \dots$, and corresponding approximate values of $Q_n(t = t_j^{(n)})$ obtained by integrating BSE with $Q_n(\mathbf{s}, t) \rightarrow \tilde{Q}_n(\mathbf{s}, t)$. In the present study, we take the values of Q_n calculated at the time moments $t_{j-1}^{(n)}$ and $t_j^{(n)}$, and use the linear extrapolation to get the approximation $\tilde{Q}_n(t)$ for $Q_n(t)$ at any t in the interval $[t_j^{(n)}, t_{j+1}^{(n)}]$.

The key point then is to properly choose the spacing of the mesh in accordance with the estimate (8):

$$t_{j+1}^{(n)} - t_j^{(n)} \propto \gamma \lambda_n^2, \quad (9)$$

where γ is a global (n -independent) small parameter controlling the accuracy of the scheme. (The scheme is asymptotically exact in the limit of $\gamma \rightarrow 0$.) With such a scaling of $t_{j+1}^{(n)} - t_j^{(n)}$ the long-range interactions do not form a bottleneck for the efficiency of the numeric process. Indeed, to numerically evaluate the integral (7) one has to perform $\propto \lambda_n$ operations, but the time interval between two integrations scales like $\propto \lambda_n^2$, which means that on average this integration costs only $\propto 1/\lambda_n$ operations per unit time. To control the accuracy, we perform a test simulation (at a smaller system size) and compare the results to the simulation of full BSE. At a reasonably small γ , we see no difference (up to a statistical noise, like the one noticeable in Fig. 2) in the evolution of the distribution of harmonics.

Kelvin wave cascade.—The decay of superfluid turbulence at $T \rightarrow 0$ leads to a cascade of Kelvin waves generated by vortex line reconnections [7, 8]. As it was argued by one of us [7], in the limit of $R/a_0 \rightarrow \infty$, when the dynamics at length scales $\sim R$ is well captured by LIA, in the low-energy part of the inertial range there takes place a specific regime when the energy flux in the wavenumber space is assisted by local self-crossings of fractionalized vortex lines. Corresponding spectrum of Kelvin waves (Kelvons) has the form

$$n_k \propto k^{-3}/(\ln k)^\nu, \quad (10)$$

where n_k is the Kelvon occupation number (see below) and ν is some dimensionless constant of order unity. The special role of local self-crossings follows from the integrability of Eq. (1) (the friction is absent at $T = 0$).—The existence of extra constants of motion excludes the possibility of the purely non-linear cascade. Vortex line self-crossings lift these constraints and push the cascade down to smaller wavelengths.

Vinen conjectured [8] that—apart from the logarithmic denominator—the spectrum (10) can be derived just on the basis of dimensional considerations and is thus insensitive to the particular cascade scenario. If true, this statement would be of crucial importance, since at large enough wavenumbers the cascade should radically change its microscopic character. From dimensional estimates it is readily seen that with increasing wavenumber non-local interactions eventually become dominant, and

the cascade is driven by purely non-linear dynamics of Kelvons. According to Vinen, the change of the regime should not affect the Kelvin spectrum. Recent numeric simulations [1, 10] seem to support this idea. However, an inconsistency can be seen in a dimensional analysis. Indeed, the spectrum $n_k \propto k^{-3}$ implies that the typical amplitude b_λ of the Kelvin turbulence with the wavelength λ is of the order of λ . This leads to a strong non-linearity of BSE, which allows one to estimate the *kinetic* time τ_{kin} as the *dynamic* time corresponding to $b_\lambda \sim \lambda$. It follows then that $\tau_{\text{kin}}(\lambda) \propto \lambda^2$. With this estimate for the kinetic time, the spectrum $n_k \propto k^{-3}$ is inconsistent with the λ -independent energy flux (see, e.g., Ref. [2] for the relation between spectrum, kinetic time, and energy flux). In our recent paper [2], we argue that the regime of the pure Kelvin wave cascade is characterized by the inequality

$$\alpha_\lambda \equiv \frac{b_\lambda}{\lambda} \ll 1, \quad (11)$$

with the parameter α_λ becoming progressively smaller with decreasing λ . This small parameter allows one to develop a quantitative kinetic theory that predicts the spectrum

$$n_k \propto k^{-17/5}. \quad (12)$$

In terms of b_λ , Eq. (12) is equivalent to

$$b_\lambda \propto \lambda^{6/5}, \quad (13)$$

while the Vinen's spectrum means $b_\lambda \propto \lambda$, that is $\alpha_\lambda \sim 1$ for any λ .

Since the number $17/5 = 3.4$ is rather close to 3, the accuracy of the data of Refs. [1, 10] turns out to be insufficient to distinguish between the two exponents.

Numeric simulation.—The main goal of our simulation is to observe the exponent $17/5$, and thus to verify the theory of Ref. [2].

We consider a straight vortex line along the Cartesian z -direction with small ($\alpha_\lambda \ll 1$) distortions in the (x, y) -plane; with the periodic boundary conditions. We use the z -coordinate as a parameter of the curve, and introduce the complex-valued function $w(z, t) = x(z, t) + iy(z, t)$ to specify the configuration of the vortex line at a given time moment t . To proceed numerically, we employ the Hamiltonian approach in terms of the discrete complex canonical variable $w_j(t) = x_j(t) + iy_j(t)$, $j = 1, 2, 3, \dots, N$, with the Hamiltonian function [mesh spacing is set equal to unity]

$$H = C \sum_{j=1}^N \sqrt{1 + |w_{j+1} - w_j|^2} + \sum_{j,l=1}^N \frac{1 + [(w_{j+1} - w_{j-1})(w_{l+1}^* - w_{l-1}^*)/8 + \text{c.c.}]}{\sqrt{(j-l)^2 + |w_j - w_l|^2}}, \quad (14)$$

and corresponding equation of motion

$$i\dot{w}_j = \frac{\partial H}{\partial w_j^*}. \quad (15)$$

Here C is a constant of order unity. In the long-wave limit, the discreteness is irrelevant and the model (14)-(15) is equivalent to the Hamiltonian form of BSE in terms of the function $w(z, t)$, with $z \equiv j$ [7]. From the theoretical point of view, the particular value of C , and the very presence of the first term in (14), is not important. In practice, we introduce this local term to render the model well-behaved at large momenta, where the spectrum of Kelvons is very sensitive to the short-range structure of the Hamiltonian. In our simulations, we used $C = 23$.

The Kelvin occupation number is given by the square of the absolute value of the Fourier transform of w_j [2]:

$$n_k = |w_k|^2, \quad (16)$$

$$w_k = N^{-1/2} \sum_{j=1}^N w_j e^{ikj}, \quad (17)$$

where $k = 2\pi m/N$, $m \in [-(N-1)/2, (N-1)/2]$ is an integer (we used odd N).

There are different ways to generate a Kolmogorov-type cascade. Perhaps the most natural one is to create a steady-state regime with a source and drain, as it was done in the simulation of Ref. [1]. However, an external source (random low-frequency force) is not necessary, since the cascade, by its definition, is a quasi-steady-state phenomenon, where the behavior of short-wavelength harmonics is adjusted to the energy flux coming from the long-wave harmonics. In a quasi-steady-state regime, the role of a source can be played by just the long-wavelength harmonics, as, e.g., in the simulation of Ref. [10]. In both cases—with or without an external force—the cascade regime requires some time to develop. We employ the following trick to circumvent the problem of transient period. We start with occupation numbers that progressively exceed the quasi-steady-state ones in the direction of shorter wavelengths. With such an initial condition, the cascade develops in the form of a wave in the momentum space propagating from the region of higher wavenumbers towards the region of lower wavenumbers, the distribution *behind* the wave corresponding to the quasi-steady-state cascade [16]. For our purpose of resolving the spectrum (12) from the spectrum $n_k \propto k^{-3}$, it is convenient to use the latter distribution of occupation numbers as an initial condition.

Introducing a drain—a decay mechanism for high-frequency harmonics—significantly improves the quality of simulation, because the large-amplitude high-frequency modes distort the short-wavelength part of the cascade spectrum. We arrange the drain by periodically eliminating all the harmonics with $|k| > k_{\text{cutoff}}$, with $k_{\text{cutoff}} = \pi/5$. Note that the momentum k_{cutoff} is 5 times smaller than the largest momentum in our system. We find it important to severely cut off the high-frequency harmonics because their dispersion strongly differs from the parabolic dispersion of Kelvin waves in the continuous space.

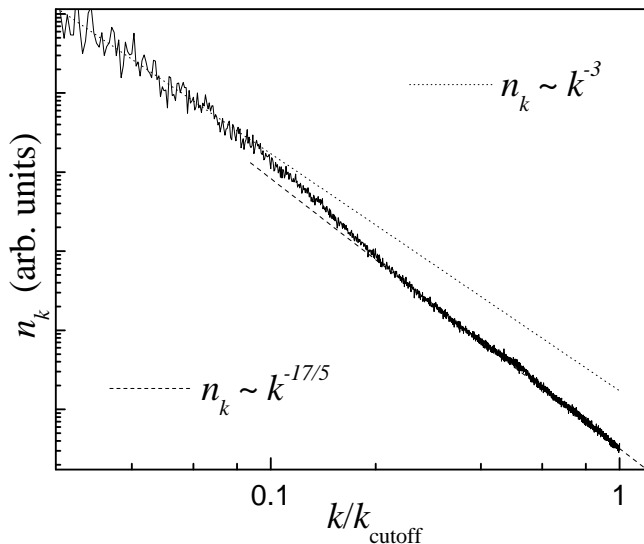


FIG. 2: A snapshot of time-averaged distribution of Kelvin occupation numbers. For time averaging we use time interval which is considerably shorter (by a factor of ~ 3) than the typical time of evolution.

We simulated a system of the size $N = 16385$. Our main results are presented in Fig. 2. We do not show a large low-frequency part of the distribution with the spectrum $n_k \propto k^{-3}$. Having these ballast long-wave harmonics is important to guarantee that the inertial range is far away from the region where the wavelength is com-

parable to the system size and the kinetics is strongly affected by the wavenumber discreteness.

Fig. 2 reveals the wave converting the initial distribution $n_k \propto k^{-3}$ into a new power-law distribution $n_k \propto k^{-\beta}$. Quantitative analysis shows that the exponent β coincides with $17/5$ with an absolute error less than 0.1, in an excellent agreement with the prediction of Ref. [2].

Unfortunately, we were unable to check the prediction of Ref. [2] for the relation between the energy flux and the amplitude A of the distribution $n_k = Ak^{-17/5}$. The problem is in calculating the mean value of the energy flux. With our quasi-steady-state setup, we can use only time averaging—with an essentially limited averaging period. This turns out to be insufficient to suppress huge fluctuations of the instant values of the energy flux. For these purposes, the steady-state setup of Ref. [1] seems to be more adequate.

In conclusion, we have proposed an efficient numeric approach for simulating the dynamics of superfluid turbulence in the framework of Biot-Savart equation. The general principle is to use a less detailed addressing for longer-range interactions. The idea is implemented in a scheme for simulating Kelvin wave cascade on a single vortex line. With the new scheme we were able to accurately resolve the spectrum of the pure Kelvin wave cascade, observing an excellent agreement with the prediction of the kinetic theory of Ref. [2].

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