Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein–Gordon equations

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ASYMPTOTIC BEHAVIOR OF SMALL SOLUTIONS FOR THE DISCRETE NONLINEAR SCHRÖDINGER AND KLEIN-GORDON EQUATIONS

A. STEFANOV AND P.G. KEVREKIDIS

Abstract. We show decay estimates for the propagator of the discrete Schrödinger and Klein-Gordon equations in the form $\|U(t)f\|_{l^\infty} \leq C(1+|t|)^{-d/3}\|f\|_{l^1}$. This implies a corresponding (restricted) set of Strichartz estimates. Applications of the latter include the existence of excitation thresholds for certain regimes of the parameters and the decay of small initial data for relevant l^p norms. The analytical decay estimates are corroborated with numerical results.

1. Introduction

A sequence of (time evolving) harmonic oscillators which interact only with their immediate neighbors is described by the discrete Schrödinger equation

$$\left| \begin{array}{l} iu_n'(t) + h^{-2}(u_{n+h}(t) + u_{n-h}(t) - 2u_n(t)) + F_n(t) = 0 \quad k \in h\mathcal{Z} \\ \{u_k(0)\} \in l^2 \end{array} \right.$$

where h is the distance between the oscillators and $h\mathcal{Z}$ is the lattice of points $\{hn : n - integer\}$. More generally,

$$\Delta_{discrete} = h^{-2}\sum_{j=1}^{d}(u_{n+he_j} + u_{n-he_j} - 2u_n)$$

and the equation describing the system is given by

$$iu_n'(t) + \Delta_{discrete}u + F_n(t) = 0, \qquad n \in h\mathcal{Z}^d,$$

As $h \to 0$, one obtains the continuous model.

For $F_n(t) = \pm|u|^{2\sigma}u$, the above equation becomes the discrete nonlinear Schrödinger equation. The latter is one of the prototypical differential-difference models that is both physically relevant and mathematically tractable. Perhaps, the most direct implementation of this equation can be identified in one-dimensional arrays of coupled optical waveguides [1, 2]. These may be multi-core structures created in a slab of a semiconductor material (such as AlGaAs), or virtual ones, induced by a set of laser

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beams illuminating a photorefractive crystal. In this experimental implementation, there are about forty lattice sites (guiding cores), and the localized, solitary wave structures that this model is well known to support [3, 4] may propagate over tens of diffraction lengths.

Photonic lattices [5, 6] have recently provided another application of the discrete nonlinear Schrödinger class of models. In this case, the refractive index of a nonlinear medium changes periodically due to a grid of strong beams, while a weaker probe beam is used to monitor the localized waves. This has created a large volume of recent activity in the direction of understanding discrete solitons in such photonic lattices.

Finally, besides its applications in nonlinear optics, the discrete nonlinear Schrödinger equation is a relevant model for Bose-Einstein condensates trapped in strong optical lattices (formed by the interference patterns of laser beams) [7, 8]. In this context, the model can be derived systematically by using the Wannier function expansions of [9].

These applications render the discrete nonlinear Schrödinger equation a particularly relevant dynamical lattice for a variety of physical applications.

Another example of a differential-difference equation that arises in many physical contexts consists of the nonlinear Klein-Gordon models. Perhaps the simplest possible implementation of such a lattice arises for an array of coupled torsion pendula under the effect of gravity [10]. However, such models are also relevant in condensed matter physics (e.g., describing the fluxon dynamics in arrays of superconducting Josephson-junctions) [11], as well as biophysics (e.g., describing the local denaturation of the DNA double strand [12]). It is also interesting to note that nonlinear Klein-Gordon models are intimately related to their Schrödinger siblings since the latter are the natural envelope wave reduction of the former [13].

The Klein-Gordon model is of the form

$$
\left| \begin{array}{l} \partial_t^2 u_n(t) - \Delta_d u + u_n + F_n(t) = 0 \\ u_n(0) = f_n \in l^2(\mathcal{Z}^d), \\ \partial_t u_n(0) = g_n \in l^2(\mathcal{Z}^d) \end{array} \right. \tag{1}
$$

where we take $h = 1$ and the nonlinearity will in general be assumed to be of the form $F_n = \pm |u|^{2\sigma} u$.

In this work, we examine such Schrödinger and Klein-Gordon models as follows: in section 2, we give a priori energy and decay estimates for the free propagation in these lattices, which we subsequently prove by means of Strichartz estimates in section 3. In sections 4 and 5, we use these estimates to examine solutions in the presence of nonlinearity for the Schrödinger and Klein-Gordon cases respectively. As an application, we show the existence of energy thresholds for the appearance of localized solutions (for appropriate regimes of the parameters) and illustrate the decay of various lattice norms, complementing the analysis with numerical simulations. A number of technical details are worked out in the appendix.

2. Free Evolution: Decay and Energy Estimates

2.1. Schrödinger equation.

Theorem 1. *For the free discrete Schrödinger equation*

$$\left| \begin{array}{l} iu_n'(t) + \Delta_d u = 0 \\ u_n(0) \in l^2(\mathcal{Z}^d) \end{array} \right.$$

one has

$$\|\{u_n(t)\}\|_{l^2} = \|\{u_n(0)\}\|_{l^2} \qquad \text{energy identity} \tag{2}$$

$$\|\{u_n(t)\}\|_{l^\infty} \leq C\frac{h^{2d/3}}{|t|^{d/3}}\|\{u_n(0)\}\|_{l^1} \qquad \text{decay estimate} \tag{3}$$

Moreover, for the inhomogeneous equation

$$\left| \begin{array}{l} iu_n'(t) + \Delta_{discrete} u_n + F_n(t) = 0 \\ u_n(0) \in l^2 \end{array} \right.$$

one has the Strichartz estimates with $(q,r) \geq 2, 1/q + d/(3r) \leq d/6$ *and* $(q,r,d) \neq (2,\infty,3)$*. That is*

$$h^{2/q}\left(\int_0^\infty \left(\sum_{n\in h\mathcal{Z}^d} |u_n(t)|^r\right)^{q/r} dt\right)^{1/q} \leq C\left(\sum_{n\in h\mathcal{Z}^d} |u_n(0)|^2\right)^{1/2} +$$
$$+ \quad Ch^{2+2/\tilde{q}'}\left(\int_0^\infty \left(\sum_{n\in h\mathcal{Z}^d} |F_n(t)|^{\tilde{r}'}\right)^{\tilde{q}'/\tilde{r}'} dt\right)^{1/\tilde{q}'}$$

Remark Note that the decay rate (and consequently the Strichartz estimates) is smaller than the usual $t^{-d/2}$ that one has for the continuous analogue.

In the Klein-Gordon case, similarly to the Schrödinger models, we discuss various relevant estimates for the free evolution. We have the following analogue of Theorem 1:

Theorem 2. *For the solutions of the one-dimensional*[1] *homogeneous discrete Klein-Gordon equation* (1)*, one has the a priori decay and energy estimates*

$$\|\{u_n(t)\}\|_{l^2} \leq C(\|f_n\|_{l^2} + \|g_n\|_{l^2}), \quad \text{energy estimate} \tag{4}$$

$$\|\{u_n(t)\}\|_{l^\infty} \lesssim t^{-1/3}(\|f_n\|_{l^1} + \|g_n\|_{l^1}) \quad \text{decay estimate} \tag{5}$$

[1] At this stage, we have encountered some technical difficulties, when trying to extend the result to dimensions higher than one. In fact, the proof of the energy estimate (4) goes through in all dimensions, whereas the decay estimate (5) boils down to the exact rate of decay of an explicit d dimensional integral, see (10). While numerically, one can see that the relevant integral has the right rate of decay, its rigorous justification remains open.

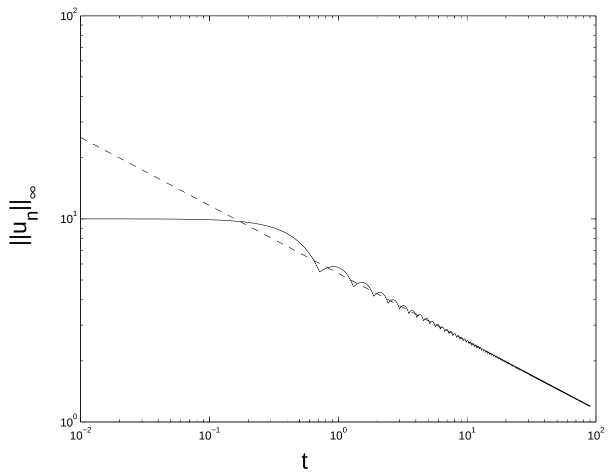

FIGURE 1. Log-log plot of the temporal evolution of the L^∞ norm in a linear one dimensional Schrödinger lattice. The intial condition contains one site excited with $u_0 = 10$. The dashed line shows the best fit (for times $50 \leq t \leq 90$) $\sim t^{-0.335}$.

For the solutions of the inhomogeneous equation, one has the Strichartz estimates with $(q,r) \geq 2, 1/q + 1/(3r) \leq 1/6$. *That is*

$$\left(\int_0^\infty \left(\sum_{n\in\mathcal{Z}} |u_n(t)|^r\right)^{q/r} dt\right)^{1/q} \leq C\left(\sum_{n\in\mathcal{Z}} |f_n|^2 + |g_n|^2\right)^{1/2} +$$
$$+ \quad C\left(\int_0^\infty \left(\sum_{n\in\mathcal{Z}} |F_n(t)|^{\tilde{r}'}\right)^{\tilde{q}'/\tilde{r}'} dt\right)^{1/\tilde{q}'}$$

3. STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER AND KLEIN-GORDON EQUATIONS

In this section, we present the proof of Theorem 1.

Proof. (Theorem 1) By rescaling in time, it suffices to consider the case $h = 1$. Next, associate $\{u_n\} \longleftrightarrow f(k) = \sum_{n\in\mathcal{Z}^d} u_n e^{2\pi i n\cdot k}$. Then,

$$\Delta_{discrete} = \sum_{j=1}^{d} (S_j + S_j^* - 2Id)$$

where S_j is the shift operator in the direction of e_j. On the spaces of trigonometric polynomials,

$$\Delta_d f = \sum_{j=1}^{d} (e^{2\pi i k_j} + e^{-2\pi i k_j} - 2) f(x) = \left[-4\sum_{j=1}^{d} \sin^2(\pi k_j)\right] f(k).$$

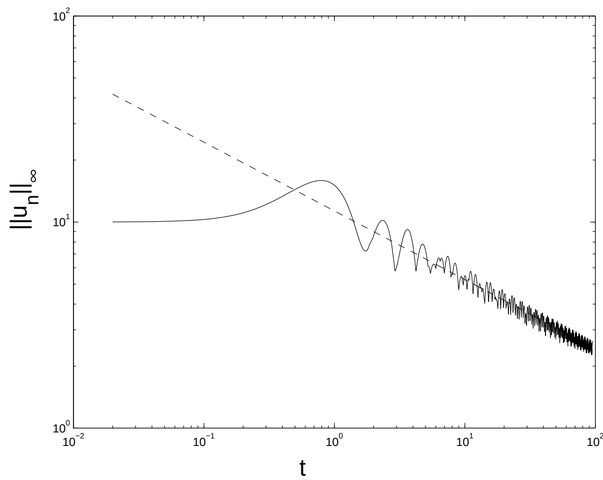

FIGURE 2. Log-log plot of the temporal evolution of the L^∞ norm in a linear one dimensional Klein-Gordon lattice. The intial condition contains one site excited with $u_0 = 10$. The dashed line shows the best fit (for times $20 \leq t \leq 90$) $\sim t^{-0.3335}$.

Since $\|\{u_n\}\|_{l^2} = \|f\|_{L^2([0,1]^d)}$, it is easy to see that $U(t)u_0 = e^{-4it\sum_{j=1}^d \sin^2(\pi k_j)} f(k)$ is an isometry in l^2.

For the decay estimate (3), write $U(t)u_0 = e^{-4it\sum_{j=1}^d \sin^2(\pi k_j)} \left(\sum u_n(0) e^{2\pi i k\cdot n}\right)$, whence

$$u_n(t) = \sum_{m\in\mathcal{Z}^d} u_m(0) \int\limits_{[0,1]^d} e^{-4it\sum_{j=1}^d \sin^2(\pi k_j)} e^{2\pi i(m-n)\cdot k} dk. \tag{6}$$

Thus, (3), reduces to showing

$$\sup_{m,n\in\mathcal{Z}^d} \left| \int\limits_{[0,1]^d} e^{-4it\sum_{j=1}^d \sin^2(\pi k_j)} e^{2\pi i(m-n)\cdot k} dk \right| \leq C \min(1, |t|^{-d/3}).$$

Since the integrals split into 1 D integrals, it suffices to show (after elementary change of variables)

$$\sup_{s\in\mathbf{R}^1} \left| \int_0^1 e^{-it(\sin^2(x)-sx)} dx \right| \leq C \min(1, |t|^{-1/3}).$$

The estimate by 1 is trivial by taking absolute values, while the estimate by $|t|^{-1/3}$ follows from the Van der Corput lemma, see [14], p. 332. Indeed, we have to verify that the phase function $\varphi(x) = \sin^2(x) - sx$ satisfies
$\max(|\varphi'(x)|, |\varphi''(x)|, |\varphi'''(x)|) \geq 1$, which is verified since $(\varphi'')^2(x) + (\varphi''')^2(x) = 4(\cos^2(2x) + \sin^2(2x)) = 4$.

Note that if $s = 1$, we may have $\max(|\varphi'(x)|, |\varphi''(x)|) << 1$ for $x \sim \pi/4$, indicating that this is the sharp rate of decay, at least as far as the Van der Corput lemma is

concerned.
Corroborating this estimate in a direct numerical simulation, we have considered a linear one-dimensional lattice with $h = 1$. The result is shown in Fig. 1 for the temporal evolution of the L^∞ norm. The best fit of the numerical result in the log-log plot of the figure very closely follows the theoretical prediction since the corresponding decay exponent is found to be ≈ 0.335.

The Strichartz estimates follow from the abstract Strichartz estimate by Keel and Tao, [15], which states that energy and decay estimates imply Strichartz estimates. The last statement is also implicit in the earlier work of Ginibre and Velo, [16]. □

The proof of the Strichartz estimate for the discrete Klein-Gordon equation (DKG) follows closely the proof for the Schrödinger case. There are however some distinctive differences, which we try to highlight in the argument.

Proof. (Theorem 2) Let us consider the homogeneous equation first.
The energy estimate is derived in all dimensions. Let $\{u_n\}$ be a solution and multiply both sides of the (DKG) by $\partial_t u_n(t)$ and sum in n. We have

$$\partial_t \sum_n (\partial_t u_n)^2 - \sum_n (\Delta_d u_n)\partial_t u_n + \partial_t \sum_n u_n^2 = 0$$

By the summation by parts formula, (e.g. (2.4) in [17], it is not hard to see that

$$-\sum_n (\Delta_d u_n)\partial_t u_n = \partial_t \sum_{r=1}^d \sum_n |u_n - u_{n+e_r}|^2/2.$$

Thus,

$$\partial_t(\sum_n (\partial_t u_n)^2 + \sum_{r=1}^d \sum_n |u_n - u_{n+e_r}|^2 + \sum_n u_n^2) = 0,$$

implying that

$$\begin{aligned}\sum_n u_n^2(t) &\leq (\sum_n (\partial_t u_n)^2 + \sum_{r=1}^d \sum_n |u_n - u_{n+e_r}|^2 + \sum_n u_n^2) = \\ &= (\sum_n g_n^2 + \sum_{r=1}^d \sum_n |f_n - f_{n+e_r}|^2 + \sum_n f_n^2) \leq C(\sum_n g_n^2 + \sum_n f_n^2),\end{aligned}$$

which is (4).
We have already introduced and studied the action of Δ_{disc} (where for simplicity $\Delta_{disc} = \Delta_1$ is the discrete Laplacian in 1 D), see (6). Following the same idea, consider $\{u_n\} \longleftrightarrow f = \sum_n u_n e^{2\pi n\cdot k}$. Then, from Duhamel's formula for the wave equation (and also by straightforward verification), the solution to the inhomogeneous equation is given by

$$u_n(t) = \cos(t\sqrt{1-\Delta_{disc}})f_n + \frac{\sin(t\sqrt{1-\Delta_{disc}})}{\sqrt{1-\Delta_{disc}}}g_n + \int_0^t \frac{\sin((t-s)\sqrt{1-\Delta_{disc}})}{\sqrt{1-\Delta_{disc}}}F_n(s)ds.$$

where $\sin(x), \cos(x)$ are expressed via the Euler's formula in terms of e^{itx} and a (continuous) function of $\sqrt{1-\Delta_{disc}}$ is the operator given by

$$(h(\sqrt{1-\Delta_{disc}})u)_n(t) = \sum_{m\in\mathcal{Z}} u_m(0)\int_0^1 h(1+4\sin^2(\pi k))e^{2\pi i(m-n)k}dk.$$

We have the following technical lemma, whose proof is postponed for the appendix.

Lemma 1. *The operator $h(\sqrt{1-\Delta_{disc}}) : l^p \to l^p$ for all $1 \le p \le \infty$ for all sufficiently smooth functions h.*

By the abtract result of Keel and Tao, the energy estimate (4) and Lemma 1, matters reduce to establishing the decay estimate

$$\left\| e^{it\sqrt{1-\Delta_{disc}}}u\right\|_{l^\infty} \le Ct^{-1/3}\|u_n\|_{l^1}. \tag{7}$$

Indeed, for Duhamel's term, we have by the above mentioned result of Keel and Tao

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{1-\Delta_{disc}})}{\sqrt{1-\Delta_{disc}}}F_n(s)ds\right\|_{L^q l^r} \lesssim \left\|(1-\Delta_{disc})^{-1/2}F\right\|_{L^{q'}l^{r'}} \lesssim \|F\|_{L^{q'}l^{r'}},$$

where the last inequality follows from Lemma 1.
As in the proof of (6), one needs to show

$$\sup_{m\in\mathcal{Z}} \int_0^1 e^{it\sqrt{1+4\sin^2(\pi k)}}e^{-2\pi imk}dk \le Ct^{-1/3}, \tag{8}$$

which by an elementary change of variables would follow from

$$\sup_{s\in\mathbf{R}^1} \int_0^1 e^{it(\sqrt{1+\sin^2(x)}-sx)}dx \le Ct^{-1/3}, \tag{9}$$

Thus, the phase function is $\psi(x) = \sqrt{1+\sin^2(x)} - sx$ and let us denote the function $\varphi(x) = \sqrt{1+\sin^2(x)}$. Denote $\alpha = \arccos(2-\sqrt{2})$.
We have that the solutions to $\psi''(x) = 0$ in $[0, 2\pi]$ are $x = \alpha, \pi-\alpha, \pi+\alpha, 2\pi-\alpha$. As we know by the Van der Corput lemma, the worst rate of decay occurs, when some consecutive derivatives (starting from ψ') vanish at a point. In our case, we have that only for $s = \varphi'(\alpha), \varphi'(\pi-\alpha), \varphi'(\pi+\alpha), \varphi'(2\pi-\alpha)$, one has $\psi'(x) = \psi''(x) = 0$ (at the points $x = \alpha, \pi-\alpha, \pi+\alpha, 2\pi-\alpha$). It is easy to check that for these values of s, one has $\max(\psi'(x), \psi''(x), \psi'''(x)) \ge 1/2$, which guarantees by the Van der Corput lemma a decay of at least $t^{-1/3}$.

□

Remark The proof of the higher dimensional analogue of such result would involve showing that

$$\sup_{s\in\mathbf{R}^d} \int_{[0,1]^d} e^{it(\sqrt{1+\sum_{j=1}^d \sin^2(x_j)}-\langle s,x\rangle)}dx \le Ct^{-d/3}, \tag{10}$$

Clearly, this integral does not reduce to the one dimensional case as in the Schrödinger case, which makes it harder object to study. Numerical simulations seem to confirm the validity of (10) since in a two dimensional numerical experiment for a DKG lattice, the best fit to the decay was found to be $\sim t^{-0.675}$.

4. Applications to the discrete Schrödinger equation

Consider

$$\left| \begin{array}{l} i\partial_t u_n + \Delta_d u \pm |u_n|^{2\sigma} u_n = 0 \\ u_k(0) \in l^2 \end{array} \right. \tag{11}$$

4.1. **Global solutions for small data in the regime $\sigma \geq 3/d$.** For (11), we establish the global existence of small solutions, provided $\|\{u_n(0)\}\|_{l^2} << 1$ and $\sigma \geq 3/d$.

Theorem 3. *Let $\sigma \geq 3/d$. There exists an $\varepsilon = \varepsilon(d) > 0$ and a constant $C = C(d)$, so that whenever $\|\{u_n(0)\}\|_{l^2} \leq \varepsilon$, a unique global solution to (11) exists, which satisfies*

$$\|\{u_n(t)\}\|_{L^q l^r} \leq C\varepsilon,$$

for all Strichartz admissible pairs (q, r). In particular, $\|u_n(t)\|_{l^r} \to 0$ as $t \to \infty$ for every $r > 2$.

Proof. We perform an iteration procedure for the equation

$$u_n(t) = \Lambda u_n = e^{it\Delta_d} u_n(0) \pm i \int_0^t e^{i(t-s)\Delta_d} |u_n|^{2\sigma} u_n(s) ds$$

in the (metric) space $X = \{u : \sup_{(q,r)-admissible} \|u\|_{L^q l^r} < 2C\|u_n(0)\|_{l^2}\}$, where C is the constant in the Strichartz inequality, that is, we are seeking a fixed point of the map Λ. Clearly, by the Strichartz estimates with (q, r) and $\tilde{q}' = 1, \tilde{r}' = 2$, we have

$$\begin{aligned} \sup_{(q,r)-admissible} \|\Lambda u_n(t)\|_{L^q l^r} &\leq C\|u_n(0)\|_{l^2} + C_1 \left\| |u_n|^{2\sigma+1} \right\|_{L^1 l^2} \leq \\ \leq \ & C\|u_n(0)\|_{l^2} + C_1 \|u_n\|^{2\sigma+1}_{L^{2\sigma+1} l^{2(2\sigma+1)}} \leq C\|u_n(0)\|_{l^2} + C_1 \|u_n\|^{2\sigma+1}_X \leq \\ \leq \ & C\|u_n(0)\|_{l^2} + C_1 (2C\varepsilon)^{2\sigma+1}. \end{aligned}$$

In the above sequence of inequalities, we have used that since $\sigma \geq 3/d$, $(2\sigma + 1, 2(2\sigma + 1))$ is a Strichartz admissible pair and therefore controllable by the norm in X. Thus, for an appropriate absolute $\varepsilon > 0$, one has

$$\sup_{(q,r)-admissible} \|\Lambda u_n(t)\|_{L^q l^r} \leq 2C\|u_n(0)\|_{l^2},$$

that is $\Lambda : X \to X$.
Similarly one verifies that

$$\begin{aligned} \|\Lambda(u_n(t) - v_n(t))\|_{L^q l^r} &\lesssim \|u_n - v_n\|_{L^{2\sigma+1} l^{2(2\sigma+1)}} (\|u_n\|^{2\sigma}_{L^{2\sigma+1} l^{2(2\sigma+1)}} + \|v_n\|^{2\sigma}_{L^{2\sigma+1} l^{2(2\sigma+1)}}) \lesssim \\ &\lesssim \|u_n - v_n\|_{L^{2\sigma+1} l^{2(2\sigma+1)}} \|u_n(0)\|^{2\sigma}_{l^2}. \end{aligned}$$

which by the smallness of $\|u_n(0)\|_{l^2}$ implies that the map $\Lambda : X \to X$ is a contraction. This shows the existence of a global solution of $u = \Lambda u$ and by construction $\|u\|_{L^q l^r} < 2C\|u_0\|_{l^2}$. □

4.2. **Decay of small solutions and the Weinstein conjecture.** M. Weinstein has proved that for $\sigma \geq 2/d$, one has an energy excitation treshold, [17], i.e., that there exists $\varepsilon = \varepsilon(d)$, so that every standing wave solution $\{e^{i\Lambda t}\phi_n\}$ must satisfy $\|\phi\|_{l^2} \geq \varepsilon$. In the same paper, he has also conjectured that for sufficiently small solutions, one has $\lim_{t\to\infty} \|u(t)\|_{l^p} = 0$ for all $p \leq \infty$.

In the next theorem, we give conditions under which small solutions *will actually decay like the free solution in the corresponding l^p norms.* This of course is a statement implying the Weinstein conjecture and is similar to a result that he has established for the continuous equation in an earlier paper[2] [18], see also the excellent expository paper [19].

Theorem 4. *Let $\sigma > 2/d$ and $d \leq 2$. There exists an ε, so that whenever $\|u_n(0)\|_{l^{(8+2d)/(d+7)}} \leq \varepsilon$, one has for all $p : 2 \leq p \leq (8+2d)/(d+1)$,*

$$\|\{u_n(t)\}\|_{l^p} \leq Ct^{-d(p-2)/(3p)}\|u_n(0)\|_{p'}. \tag{12}$$

which is the generic rate of decay for the free solutions (see (13) *below). In particular, no standing wave solutions are possible under the smallness assumptions outlined above.*

Proof. By interpolation between (2) and (3), we obtain

$$\left\|\{e^{it\Delta_d}u_n(0)\}\right\|_{l^p} \leq C < t >^{-d(p-2)/(3p)} \|\{u_n(0)\}\|_{l^{p'}}. \tag{13}$$

valid for all $2 \leq p \leq \infty$.
Note that from (13), we deduce that for every $p \geq 2$, $\left\|\{e^{it\Delta_d}u_n(0)\}\right\|_{l^p} \lesssim t^{-d(p-2)/(3p)}$.

We would like to establish an a priori bound (which simultaneously implies existence) for the solution $\{u_n\}$, that is we want to place it in

$$X = \{\{u_n\} : \|\{u_n(t)\}\|_{l^p} \leq 2Ct^{-d(p-2)/(3p)}\|\{u_n(0)\}\|_{l^{p'}}\}.$$

where C is the constant in the Strichartz inequality. We have by the decay estimate (13)

$$\begin{aligned}\|\{\Lambda u_n(t)\}\|_{l^p} &\leq C < t >^{-d(p-2)/(3p)} \|u_n(0)\|_{l^{p'}} + \\ &+ C_1 \int_0^t \frac{1}{< t-s >^{d(p-2)/(3p)}} \|u_n(s)\|_{l^{(2\sigma+1)p'}}^{2\sigma+1} ds.\end{aligned}$$

[2]Weinstein's result makes significant use of the pseudoconformal invariance of the continuous equation, which is unfortunately not present for the discrete problem. In particular, for the decaying result, he imposes the condition $\|xu_0\|_{L^2} < \infty$ on the initial data. In our results, it suffices to assume somewhat less - in the one dimensional case: $\left\||n|^{3/10}u_n\right\|_{l^{2,1}(\mathcal{Z})} << 1$ and in the two dimensional case, one has to impose $\left\||n|^{1/2}u_n\right\|_{l^{2,1}(\mathcal{Z}^2)} << 1$, see the precise statement of Theorem 4 for conditions in terms of the l^p spaces.

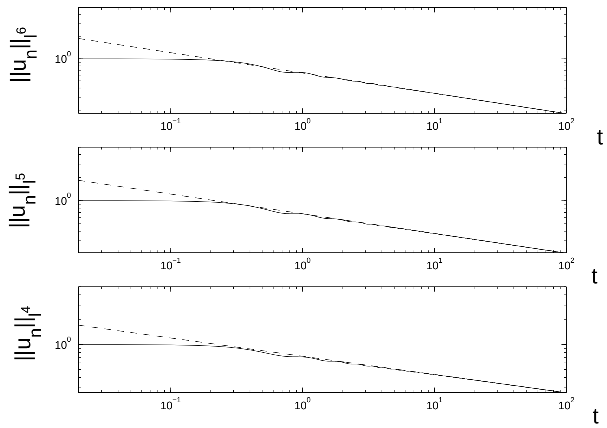

FIGURE 3. Log-log plot of the temporal evolution of the l^4 (bottom panel), l^5 (middle panel) and l^6 (top panel) norms in a nonlinear one dimensional Schrödinger lattice. The intial condition contains one site excited with $u_0 = 1$. The dashed line shows the best fits (for times $20 \leq t \leq 90$) that are given respectively by $||u_n||_{l^4} \sim t^{-0.221}$, $||u_n||_{l^5} \sim t^{-0.257}$ and $||u_n||_{l^6} \sim t^{-0.277}$.

Since $(2\sigma+1)p' > (4/d+1)(8+2d)/(d+7) \geq (8+2d)/(d+1) = p$ for $d \leq 2$, we have by the inclusion $l^p \hookrightarrow l^{(2\sigma+1)p'}$

$$\begin{aligned} \|\{\Lambda u_n(t)\}\|_{l^p} &\leq C < t >^{-d(p-2)/(3p)} \|u_n(0)\|_{l^{p'}} + \\ &+ C_1 \int_0^t \frac{1}{< t-s >^{d(p-2)/(3p)}} \|u_n(s)\|_{l^p}^{2\sigma+1} ds \leq \\ &\leq C < t >^{-d(p-2)/(3p)} \|u_n(0)\|_{l^{p'}} + \\ &+ C_2 \|u_n(0)\|_{l^{p'}}^{2\sigma+1} \int_0^t \frac{1}{< t-s >^{d(p-2)/(3p)}} \frac{1}{< s >^{d(p-2)(2\sigma+1)/(3p)}} ds. \end{aligned}$$

Note that $d(p-2)(2\sigma+1)/(3p) > d(p-2)(4/d+1)/(3p) > 1$, and therefore the integral term above is controlled by a constant times $t^{-d(p-2)/(3p)}$. As a consequence,

$$\|\{\Lambda u_n(t)\}\|_{l^p} \leq 2C < t >^{-d(p-2)/(3p)} \|u_n(0)\|_{l^{p'}},$$

provided $C_3 \|u_n(0)\|_{l^{p'}}^{2\sigma} \leq C_3 \varepsilon^{2\sigma} < C$.

The contractivity of the map $\Lambda : X \to X$ follows in the same manner, given that $\varepsilon << 1$. A fixed point argument shows that a solution exists together with the estimate $\|\{u_n(t)\}\|_{l^p} \lesssim t^{-d(p-2)/(3p)}$. □

The numerical simulations performed in a $d = 1$, $\sigma = 3$ lattice, with an initial condition of a single excited site with $u_0 = 1$ showed (see Fig. 3) that the actual decay rate is larger than that predicted by Theorem 4. In particular, for $p = 4$ and $p = 5$, the theorem predicts decay rates of $t^{-1/6}$ and $t^{-1/5}$ respectively, while the numerical simulations show decay rates of $t^{-0.221}$ and $t^{-0.257}$ respectively. For the

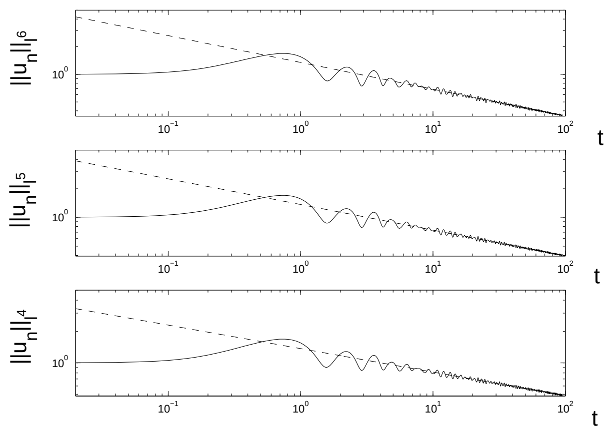

FIGURE 4. Same as the previous figure but for the nonlinear Klein-Gordon lattice. The dashed line shows the best fits (for times $20 \leq t \leq 90$) that are given respectively by $||u_n||_{l^4} \sim t^{-0.226}$, $||u_n||_{l^5} \sim t^{-0.267}$ and $||u_n||_{l^6} \sim t^{-0.292}$.

case of $p = 6$, the decay rate is faster and given by $t^{-0.277}$ (cf. with the theoretical prediction of $t^{-2/9}$).

5. Applications to the discrete Klein-Gordon equation

For the discrete Klein-Gordon equation, as mentioned above we consider nonlinearities of the form $|u|^{2\sigma}u$. We can deduce essentially the same results as in the case for the discrete nonlinear Schrödinger equation. We state them without proofs, since they follow by exactly the same proofs as in the Schrödinger case (by the exact same decay and Strichartrz estimates).

Theorem 5. *Let $\sigma \geq 3$. There exists an $\varepsilon > 0$ and a constant C, so that whenever $\|\{u_n(0)\}\|_{l^2}, \|\{\partial_t u_n(0)\}\|_{l^2} \leq \varepsilon$ there exists a unique global solution to the one-dimensional discrete Klein-Gordon equation with a nonlinearity $|u|^{2\sigma}u$, satisfying*

$$\|\{u_n(t)\}\|_{L^q l^r} \leq C\varepsilon,$$

for all Strichartz admissible pairs (q, r). In particular, $\|u_n(t)\|_{l^r} \to 0$ as $t \to \infty$ for every $r > 2$.

In the next theorem, we establish the Weinstein excitation threshold conjecture in the Klein-Gordon case.

Theorem 6. *Let $\sigma > 2$. There exists an ε, so that whenever $\|u_n(0)\|_{l^{5/4}} \leq \varepsilon$, $\|\partial_t u_n(0)\|_{l^{5/4}} \leq \varepsilon$, one has for all $p : 2 \leq p \leq 5$,*

$$\|\{u_n(t)\}\|_{l^p} \leq Ct^{-(p-2)/(3p)}\|u_n(0)\|_{p'}. \tag{14}$$

In particular, there are no small standing wave solutions to the discrete Klein-Gordon equation.

The numerical results, in this case as well, show a faster decay than theoretically predicted. Furthermore, the decay is slightly faster in this model for the same norms in comparison with the corresponding cases for the Schrödinger equation. More specifically, in the Klein-Gordon model, $||u_n||_{l^4} \sim t^{-0.226}$, $||u_n||_{l^5} \sim t^{-0.267}$ and $||u_n||_{l^6} \sim t^{-0.292}$. It is worth noting, however, that when we consider multiple sites excited by the initial condition, then we obtain decay rates which are much closer to the ones theoretically predicted above. More specifically, if we excite 2 nodes (rather than a single one), with 3 sites betwen them, then the decay rates are $t^{-0.207}$, t^{-245} and $t^{-0.270}$ which are considerably closer to the theoretically predicted exponents of 1/6, 1/5 and 2/9 respectively.

6. Appendix

Proof. (Lemma 1) we present the proof in the d dimensional case.
We use the representation of $h(\sqrt{1-\Delta_d})$ to conclude that it suffices to show that for every N, there exists C_N, so that

$$|b_m| := |\int\limits_{[0,1]^d} h(1+4\sum_{j=1}^{d} \sin^2(\pi k_j)) e^{2\pi i m\cdot k} dk| \leq C_N < m >^{-N} . \tag{15}$$

Indeed, if we had (15), then by the expression $(h(\sqrt{1-\Delta_d})u)_n = \sum_m b_{n-m} u_m$, we conclude

$$\left\| h(\sqrt{1-\Delta_d})u \right\|_{l^p} \lesssim \|b\|_{l^1} \|u\|_{l^p} \leq C_d \left\| < \cdot >^{-d+2} \right\|_{l^1} \|u\|_{l^p}.$$

For the proof of (15), use integration by parts and the fact that boundary terms disappear (due to the fact that $h(1+4\sum_{j=1}^{d} \sin^2(\pi k_j))$ is one-periodic in every variable k_j). We get

$$b_m = \frac{c}{m_j} \int_{[0,1]^d} h'(1+4\sum_{j=1}^{d} \sin^2(\pi k_j)) \sin(2\pi k_j) e^{2\pi m\cdot k} dk.$$

Clearly, one keeps integrating by parts to obtain arbitrary large rate of decay in all variables m_j. The lemma is proven. □

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