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Discrete Klein-Gordon models with static kinks free of the Peierls-Nabarro potential

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For the nonlinear Klein-Gordon type models, we describe a general method of discretization in which the static kink can be placed anywhere with respect to the lattice. These discrete models are therefore free of the *static* Peierls-Nabarro potential. Previously reported models of this type are shown to belong to a wider class of models derived by means of the proposed method. A relevant physical consequence of our findings is the existence of a wide class of discrete Klein-Gordon models where slow kinks *practically* do not experience the action of the Peierls-Nabarro potential. Such kinks are not trapped by the lattice and they can be accelerated by even weak external fields.

I. INTRODUCTION

Discrete solitons and more specifically kink-like topological excitations are ubiquitous structures that arise in numerous physical applications ranging from dislocations or ferroelectric domain walls in solids, to bubbles in DNA, or magnetic chains and Josephson junctions, among others (see, e.g., [1] for a recent exposition of relevant applications). The mobility of such lattice kinks is one of the key issues in many of these applications, especially since the pioneering works of [2, 3] which illustrated that the kinematics on the lattice is dramatically different from the continuum analog of such equations where constant speed propagation is typical. Instead, on the discrete substrate, kinks need to overcome the, so-called, Peierls-Nabarro potential (PNp), constantly radiating their energy and being eventually trapped by the lattice. The *static* PNp refers to the energy difference between a stable inter-site centered discrete kink and an unstable, on-site centered discrete kink. Clearly, as a kink is travelling from one site to the next, it "wobbles" over this potential energy landscape [4]. However, even though clearly travelling is intimately connected with overcoming the static PNp without "radiating" energy [5], this connection is relatively subtle and the inter-dependence of these two features (static PNp and travelling) still remains elusive [6]. Typically, discrete kinks traveling with finite velocity have only been obtained for a discrete set of velocities [7] which makes the motion unstable with respect to perturbations. There exists a class of more exotic exact solutions (the so-called "nanoptera") where the kink propagates together with a plane wave having the same velocity [7].

While the travelling problem is extremely interesting in its own right, in the present work, we will start by examining the construction of discrete models with PNp-free kinks, using a simplified (quasi)static approach. Two classes of discrete models where static kink can be placed anywhere with respect to the lattice have been previously derived: one conserving energy [8] and another one conserving momentum [9]. In both cases the static kink solution can be obtained from a two-body nonlinear map.

In the present paper we demonstrate that, in general, a discrete version of the first integral of the static continuum Klein-Gordon field plays the role of this nonlinear map. Thus we derive a wide class of such models including the two above-mentioned classes as special cases. The advantage of this approach is that the kinks are no longer (typically) trapped by the lattice. Instead they can be accelerated by even weak external fields. However, a note of caution should be added here. While one might naively expect that such solutions would be intimately connected with slow travelling, it has been demonstrated numerically that travelling solutions (when they can be found as e.g. in [7, 10] for Klein-Gordon lattices, using the methods of [11]) have a sharp lower bound in their wave speed [12]. The existence of such a threshold illustrates the fact that one should be particularly careful in trying to infer features of the travelling problem from such "static" considerations. On the other hand, as the recent work of Barashenkov, Oxtoby and Pelinovsky demonstrates [13], discretizations without PNp are much more natural candidates for possessing travelling solutions for a isolated wave speeds (not close to zero).

The presentation of our results will be structured as follows. Section II will contain the setup and notations used for the Klein-Gordon models. Section III will present the general methodology for obtaining static PNp-free discretizations. Section IV will illustrate the connection to previously reported models. Section V will focus on the special case example of the ϕ^4 model, for which our numerical observations will be presented in section VI. Finally in section VII, we will summarize our findings and present our conclusions.

II. SETUP

We consider the Lagrangian of the Klein-Gordon field,

$$L = \int_{-\infty}^{\infty} \left[\frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - V(\phi) \right] dx, \qquad (1)$$

and the corresponding equation of motion,

$$\phi_{tt} = \phi_{xx} - V'(\phi) \equiv D(x). \tag{2}$$

Topological solitons (kinks) are possible only if $V(\phi)$ has at least two minima ϕ_{01} and ϕ_{02}, where $V'(\phi_{0i}) = 0$ and $V''(\phi_{0i}) > 0$. Obviously, $\phi = \phi_{01}$ and $\phi = \phi_{02}$ are the stationary solutions to Eq. (2). We will study the properties of kinks that interpolate between these two stationary solutions. Our considerations allow one to treat the cases when other minima appear in between the two minima, ϕ_{01} and ϕ_{02}, connected by the kink.

Equation (2) will be discretized on the lattice $x = nh$, where $n = 0, \pm 1, \pm 2...$, and h is the lattice spacing.

For brevity, when possible, we will use the notations

$$\phi_{n-1} \equiv l, \quad \phi_n \equiv m, \quad \phi_{n+1} \equiv r. \tag{3}$$

We would like to construct a nearest-neighbor discrete analog to Eq. (2) of the form

$$\ddot{m} = D(C, l, m, r), \tag{4}$$

where $C > 0$ is a parameter related to the lattice spacing h as $C = 1/h^2$, such that in the continuum limit ($C \to \infty$), $D(C, l, m, r) \to D(x) = \phi_{xx} - V'(\phi)$.

Note that in this context, the "standard" discretization emerges in the form: $D(C, l, m, r) = C(l - 2m + r) - V'(m)$. Generalizations of this model will be discussed in the form

$$\ddot{m} = C(l - 2m + r) - B(l, m, r), \tag{5}$$

where $B(l, m, r)$ has $V'(\phi)$ as the continuum limit.

We will characterize a model as PNp-free if a *static* kink can be placed anywhere with respect to the lattice (continuum, rather than discrete, set of equilibrium solutions). This is equivalent to demanding that the kink must have an neutral direction, or (from Noether's theory [14]) a Goldstone translational mode. It is natural to categorize this definition of PNp-free model as "static" or "quasi-static", in the sense that it does not involve the kinematic or dynamical properties of the model.

On the other hand, one can demand the absence of PNp at finite kink velocities. This can be transformed to the demand that the discrete model supports the exact traveling wave solutions and this demand can be called "dynamic" definition; see e.g. [6] for such travelling wave examples, where the "static" definition of the PNp clearly fails.

In this paper we aim to construct the models PNp-free in the static sense as a first (yet nontrivial) step towards understanding the nature of the discrete travelling problem (see also the comments above).

We will also focus on the existence of physically motivated conserved quantities for the derived models. Hamiltonian models are energy-conserving models and the models with $dM/dt = 0$, where

$$M = \sum_{n=-\infty}^{\infty} \dot{\phi}_n (\phi_{n+1} - \phi_{n-1}), \tag{6}$$

will be called momentum-conserving models. As was shown in [9], the discrete model of Eq. (4) conserves the momentum of Eq. (6), if it can be presented in the form

$$\ddot{m} = \frac{\mathcal{H}(m, r) - \mathcal{H}(l, m)}{r - l}. \tag{7}$$

This can be verified by calculating

$$\frac{dM}{dt} = \sum_n \ddot{\phi}_n (\phi_{n+1} - \phi_{n-1})$$
$$= \sum_n [\mathcal{H}(\phi_n, \phi_{n+1}) - \mathcal{H}(\phi_{n-1}, \phi_n)] = 0, \tag{8}$$

where we have used the fact that the terms $\dot{\phi}_n(\dot{\phi}_{n+1} - \dot{\phi}_{n-1})$ cancel out due to telescopic summation.

III. STATIC PNP-FREE DISCRETIZATION

Our aim here will be to discretize Eq. (2) in a symmetric way, so that the static kink solution can be found from a reduced first-order difference equation. According to [8], if we achieve that, then we are going to have a one-parameter family of solutions with the possibility to place equilibrium kinks anywhere with respect to the lattice (and hence, PNp-free in the static sense).

The first integral of the steady state problem in Eq. (2), $\phi_x - \sqrt{2V(\phi)} = 0$ (with zero integration constant), can be written in the form

$$w(x) \equiv g(\phi_x) - g\left(\sqrt{2V(\phi)}\right) = 0, \tag{9}$$

where g is a continuous function.

Our plan will then be the following:

- discretize the first-order differential equation of Eq. (9) using a first order difference scheme $w(l, m) = 0$.

- Then express the right-hand side of Eq. (2) as a sum of terms containing derivatives, e.g., dw/dx, $dw/d\phi$, etc.

- As a result, discretizations of such terms, e.g., $dw/dx \sim \sqrt{C}[w(m, r) - w(l, m)]$, vanish for $w(l, m) = 0$ (or otherwise stated: the construction of the equilibrium solution is converted to a first order difference problem). Then, the static kink (PNp-free, by construction) solutions for the obtained discrete model can be found from this two-site problem.

In the following, we will consider a particular case of Eq. (9) with $g(\xi) = \xi^2$, for which we introduce the notation

$$u(x) \equiv \phi_x^2 - 2V(\phi) = 0, \tag{10}$$

and the following two-site discrete analog

$$u(l, m) \equiv C(m - l)^2 - 2V(l, m) = 0 \,. \qquad (11)$$

We will also use the shorthand notations,

$$u_l = u(l, m) \,, \qquad u_m = u(m, r) \,. \qquad (12)$$

We have assumed that the Klein-Gordon field supports kink solutions. Then, at least for the case of weak discreteness, Eq. (11) also supports static kinks because it is nothing but a discretization of the first integral of static version of Eq. (2) (see also [8]).

The next step is then to find a discretization of the right-hand side of Eq. (2), $D(x)$, which vanishes when Eq. (11) is fulfilled.

One simple possibility comes from the following finite difference

$$D_1(l, m, r) \equiv \frac{u_m - u_l}{r - l} \to \frac{1}{2}\frac{du}{d\phi} = D(x) \,. \qquad (13)$$

One can also consider, more generally, continuous functions $q(u_l, h)$ such that $q(0, h) = 0$ and, in the continuum limit, $q(u, 0) = u$ and $\frac{dq}{du}(u, 0) = 1$. For example, one can take $q = (e^{hu} - 1)/h$ or $q = u + \sum_{n>1} A_n h^{n-1} u^n$ with constant A_n, etc. Then,

$$\frac{1}{2}\frac{dq}{d\phi}\left(\frac{dq}{du}\right)^{-1} = D(x) \,. \qquad (14)$$

Discretizing the left-hand side of Eq. (14) we obtain

$$D_2 = \frac{1}{2}\frac{q(u_m, h) - q(u_l, h)}{r - l}\left[\frac{1}{q'(u_l)} + \frac{1}{q'(u_m)}\right] \,. \qquad (15)$$

Inspired by [8], we note that, in the continuum limit,

$$\frac{v(m, r)}{r - m} - \frac{v(l, m)}{m - l} \to \frac{dv}{d\phi} - v\frac{\phi_{xx}}{\phi_x^2} \,, \qquad (16)$$

and find

$$D_3 \equiv \frac{u_m}{r - m} - \frac{u_l}{m - l} + \sqrt{2V(l, m, r)} \times$$
$$\left(\frac{\sqrt{C(r - m)^2 - u_m}}{r - m} - \frac{\sqrt{C(m - l)^2 - u_l}}{m - l}\right)$$
$$\to \frac{du}{d\phi} - u\frac{\phi_{xx}}{\phi_x^2} + \sqrt{2V}\left(\frac{d\sqrt{2V}}{d\phi} - \sqrt{2V}\frac{\phi_{xx}}{\phi_x^2}\right)$$
$$= D(x) \,. \qquad (17)$$

Since the expressions for $D_i(l, m, r)$ given by Eqs. (13),(15) and (17) tend to $D(x)$ in the continuum limit, one can write the following discrete analog to the Klein-Gordon equation Eq. (2)

$$\ddot{m} = \sum_i b_i D_i(l, m, r), \quad \text{where} \quad \sum_i b_i = 1 \,. \qquad (18)$$

Then, by construction, any structure derived from the two-site problem of Eq. (11) is a static solution of Eq. (18) and hence, the latter is the static PNp-free discrete model.

The model of Eq. (18) can be generalized in a number of ways. For example, function D_3, Eq. (17), can be modified choosing different functions $V(l, m, r)$ to discretize $V(\phi)$. Then, the modified $\tilde{D}_3$ can be added to the linear combination in the right-hand side of Eq. (18).

The model of Eq. (18) can be also generalized by appending terms which disappear in the continuum limit and ones that vanish upon substituting $u_l = 0$ and $u_m = 0$. For example, the derivative $df(u)/d\phi$ can be discretized as $2[f(u_m) - f(u_l)]/(r - l)$ or as $2f'(u_l/2 + u_m/2)(u_m - u_l)/(r - l)$. If then we have difference of such terms in the equation of motion, then in the continuum limit they will cancel out.

Any term in the right-hand side of Eq. (18) can be further modified by multiplying by a continuous function $e(C, l, m, r)$, whose continuum limit is unity (see e.g. [15] for such an example, also discussed in more detail below).

Generally speaking, the discrete PNp-free Klein-Gordon models derived here do not conserve either an energy, or a momentum-like quantity. However, as it will be demonstrated below, they contain energy-conserving and momentum-conserving subclasses.

IV. CONNECTION WITH PREVIOUSLY REPORTED MODELS

One energy-conserving PNp-free Klein-Gordon model has been derived by Speight with co-workers [8] with the use of the Bogomol'nyi argument [16]. Their model, can be written in the form of Eq. (5), with the Lagrangian

$$L = \frac{1}{2}\sum_n \dot{\phi}_n^2 - \frac{C}{2}\sum_n (\phi_n - \phi_{n-1})^2$$
$$- \sum_n \left(\frac{G(\phi_n) - G(\phi_{n-1})}{\phi_n - \phi_{n-1}}\right)^2 \,,$$
$$\text{where} \quad G'(\phi) = \sqrt{V(\phi)} \,. \qquad (19)$$

The static kink solution can then be derived from the lattice Bogomol'nyi equation [8], which can be taken in the form

$$U(l, m) = C(m - l)^2 - 2\left(\frac{G(m) - G(l)}{m - l}\right)^2 = 0, \quad (20)$$

which is a particular case of Eq. (11). The equation of motion derived from Eq. (19), written in terms of Eq. (20), is

$$\ddot{m} = \frac{U_m}{r - m} - \frac{U_l}{m - l} + \sqrt{2V(m)} \times$$
$$\left(\frac{\sqrt{C(r - m)^2 - U_m}}{r - m} - \frac{\sqrt{C(m - l)^2 - U_l}}{m - l}\right) \,. \qquad (21)$$

The right-hand side of Eq. (21) is a particular case of $D_3(l, m, r)$ given by Eq. (17) with $V(l, m, r) = V(m)$.

Momentum-conserving PNp-free models were proposed in [9] and further studied in [17]. They are the non-Hamiltonian models of the form

$$\ddot{m} = D_1(l, m, r), \qquad (22)$$

where D_1 is given by Eq. (13). Notice that Eq. (22) can be mapped into the formulation of Eq. (7). Static kink solutions in this model can be found from Eq. (11).

If Eq. (11) is taken in the particular form of Eq. (20), then the momentum-conserving PNp-free model Eq. (22) and the energy-conserving PNp-free model Eq. (21) have exactly the same static kink solutions. It has been proved that a standard nearest-neighbor discrete Klein-Gordon model conserving both energy and momentum does not exist [17].

V. APPLICATION TO THE ϕ^4 MODEL

As an example, we will discretize the well-known ϕ^4 field theory with the potential

$$V(\phi) = \frac{1}{4} \left(1 - \phi^2\right)^2 . \qquad (23)$$

By construction, the PNp-free models derived above are written in singular form. In this form the equations are inconvenient in practical simulations and one may wish to find such particular cases when singularities disappear.

For example, for the energy-conserving PNp-free model expressed by Eqs. (19)-(21), singularity always disappears when $G(\phi)$ is polynomial [8]. Particularly, for the ϕ^4 model with the potential Eq. (23), one obtains from Eq. (21) the following energy-conserving PNp-free discretization derived in [8]

$$\ddot{m} = \left(C + \frac{1}{6}\right)(l + r - 2m) + m$$
$$-\frac{1}{18}\left[2m^3 + (m + l)^3 + (m + r)^3\right], \qquad (24)$$

whose static kink solution can be found from Eq. (20), which, for the ϕ^4 potential, obtains the form

$$3\sqrt{2C}(m - l) + m^2 + lm + l^2 - 3 = 0. \qquad (25)$$

Now let us turn to the momentum-conserving model. Substituting Eq. (11) into Eq. (22) we obtain

$$\ddot{m} = C(r - 2m + l) - 2\frac{V(m, r) - V(l, m)}{r - l}. \qquad (26)$$

To remove the singularity, $V(l, m)$ should be taken in the symmetric form $V(l, m) = V(m, l)$, e.g., as

$$V(l, m) = (1/4) - (\alpha/2)(m^2 + l^2) + (\alpha - 1/2)ml$$
$$+ (\beta/2)\left(m^3 + l^3\right) - (\beta/2)ml(m + l)$$
$$+ (\gamma/2)\left(m^4 + l^4\right) + (\delta/2)ml\left(m^2 + l^2\right)$$
$$- (\gamma + \delta - 1/4)m^2l^2, \qquad (27)$$

with free parameters α, β, γ, and δ. In the continuum limit, when $l \to m$ and $r \to m$, Eq. (27) reduces to $V(\phi)$. Substituting Eq. (27) into Eq. (26) we obtain the following momentum-conserving PNp-free ϕ^4 model derived in [17]

$$\ddot{m} = (C + \alpha)(l - 2m + r) + m$$
$$- \beta(l^2 + lr + r^2) + \beta m(l + r + m)$$
$$- \gamma(l^3 + r^3 + l^2r + lr^2) - \delta m(l^2 + m^2 + r^2 + lr)$$
$$+ (2\gamma + 2\delta - 1/2)m^2(l + r). \qquad (28)$$

The momentum-conserving model Eq. (28) with $\alpha = \beta = \gamma = \delta = 0$ can be written in the form

$$\ddot{m} = \left(1 - \frac{m^2}{2C}\right)C(l - 2m + r) + m - m^3. \qquad (29)$$

The following energy-conserving model, studied in [15],

$$\ddot{m} = C(l - 2m + r) + \frac{m - m^3}{1 - m^2/(2C)}, \qquad (30)$$

has the same continuum limit as model Eq. (29). Furthermore, it can be derived from Eq. (29) by multiplication with a factor $e(C, l, m, r) = 1/(1 - m^2/(2C))$, which possesses a unit continuum limit. The model Eq. (29) is PNp-free and thus, model Eq. (30) is also PNp-free since they have the same static solutions derivable from $C(m - l)^2 - (1 - ml)^2/2 = 0$. Thus, we have another example when energy-conserving and momentum-conserving PNp-free models have exactly the same static kink solutions.

It is interesting to note that the energy-conserving model of Eq. (30) cannot be constructed by the method reported in [8] where discretization of the anharmonic term always involves ϕ_{n-1} and ϕ_{n+1}. More generally than it is done in [8], the problem of finding the energy-conserving PNp-free models can be formulated as follows. We need to discretize the potential energy of the Lagrangian Eq. (1) in a way that the corresponding equation of static equilibrium is satisfied when Eq. (9) is satisfied. Both energy-conserving models discussed above are the solutions of this problem.

As an example of model conserving neither energy, nor momentum we take Eq. (15) for the case of $q(u, h) = u + Ahu^2$ with constant A and obtain

$$\ddot{m} = \frac{u_m - u_l}{r - l}\frac{(1 + Ahu_l + Ahu_m)^2}{(1 + 2Ahu_l)(1 + 2Ahu_m)}. \qquad (31)$$

This model can be obtained from the momentum-conserving model defined by Eq. (13) by multiplying by another function that reduces to unity in the continuum limit ($h \to 0$). Obviously, the original momentum-conserving model and model Eq. (31) have the same static kink solutions. It can be demonstrated that these two models also have the same spectra of small amplitude vibrations and the same frequencies of kink internal modes.

VI. NUMERICS

In our recent work [17], some properties of kinks were compared for the "standard" energy-conserving ϕ^4 discretization having PNp,

$$\ddot{m} = C(l + r - 2m) + m - m^3, \qquad (32)$$

with the PNp-free models conserving energy Eq. (24) and momentum Eq. (29).

It was found that the mobility of kinks in the PNp-free models is higher and also that in the momentum-conserving, PNp-free models a kink self-acceleration effect may be observed. The origin of the effect is the non-conservative (non self-adjoint) nature of the model which, however, can be noticed only for asymmetric trajectories of particles when kink passes by [17]. If the trajectories are symmetric, there is no energy exchange with the surroundings and kink dynamics is the same as in energy-conserving models, e.g., the kink self-acceleration effect disappears. Kinks in some of the momentum-conserving models was found to have internal modes with frequencies above the phonon spectrum. Such modes do not radiate and they can have large amplitudes storing a considerable amount of energy.

Here we present/compare results for the energy-conserving PNp-free model Eq. (30) and the PNp-free model of Eq. (31), generally speaking, conserving neither energy nor momentum. For the latter model we take u_l in the form of Eq. (11) where the ϕ^4 potential is discretized according to Eq. (27) and, for the sake of simplicity, we set $\alpha = \beta = \gamma = \delta = 0$. We obtain

$$\ddot{m} = \left[\left(1 - \frac{m^2}{2C} \right) C(l - 2m + r) + m - m^3 \right] \times$$
$$\frac{(1 + Ahu_l + Ahu_m)^2}{(1 + 2Ahu_l)(1 + 2Ahu_m)},$$
$$\text{where} \quad u_l = C(m - l)^2 - (1 - ml)^2/2. \qquad (33)$$

For $A = 0$, Eq. (33) coincides with the momentum-conserving model Eq. (29).

In the model Eq. (33), the static kink solutions, phonon spectra, and frequencies of kink internal modes are A-independent. The energy-conserving model Eq. (30) has the same static kink solutions as model Eq. (33) but their spectra are different. The linear vibration spectrum of the vacuum for Eq. (33) is $\omega^2 = 2 + (4C - 2)\sin^2(\kappa/2)$ and that for Eq. (30) is $\omega^2 = 4C/(2C - 1) + 4C\sin^2(\kappa/2)$, while the one for the classical model Eq. (32) is $\omega^2 = 2 + 4C\sin^2(\kappa/2)$.

The top panels of Fig. 1 present the boundaries of the linear vibration spectrum of the vacuum (solid lines) and the kink internal modes (dots) as the functions of lattice spacing h for (a) the classical ϕ^4 model of Eq. (32), (b) the PNp-free model of Eq. (30) conserving energy, and (c) the PNp-free model of Eq. (33) at $A = 0$ conserving momentum. In PNp-free models kinks possess a zero frequency, Goldstone translational mode. Since all models
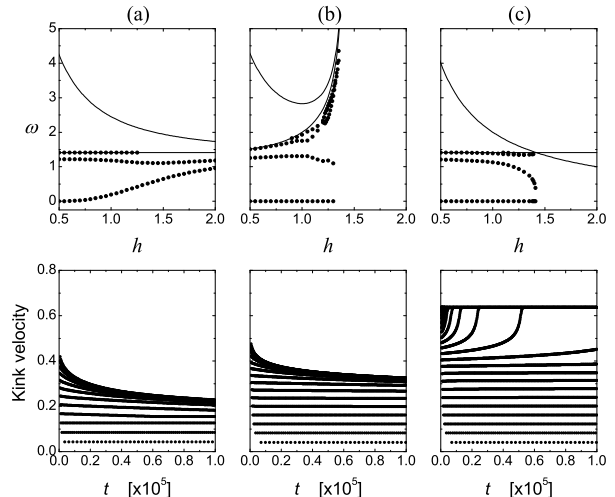

FIG. 1: Upper panels: boundaries of the linear spectrum of the vacuum (solid lines) and kink internal mode frequencies (dots) as functions of the lattice spacing $h = 1/\sqrt{C}$. Lower panels: time evolution of kink velocity for different initial velocities and $h = 0.7$. The results are shown for (a) classical ϕ^4 model, Eq. (32), (b) PNp-free model conserving energy, Eq. (30), and (c) PNp-free model conserving momentum, Eq. (33) at $A = 0$.
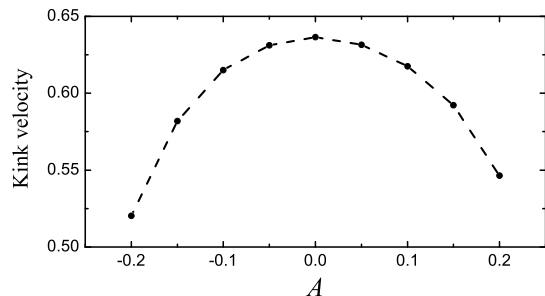

FIG. 2: The kink velocity in the regime of steady motion [see bottom panel in Fig. 1 (c)] for the PNp-free ϕ^4 model Eq. (33) is shown as a function of parameter A. For $|A| > 0.2$, the kink self-acceleration effect disappears.

presented in Fig. 1 share the same continuum ϕ^4 limit, their spectra are very close for small $h (< 0.5)$.

The bottom panels of Fig. 1 show the time evolution of kink velocity for the corresponding models at $h = 0.7$ for kinks launched with different initial velocities. To boost the kink we used the semi-analytical solution for the normalized Goldstone mode, whose amplitude serves as a measure of the initial kink velocity. One can see that the mobility of kinks in the PNp-free models shown in (b) and (c) is higher than in the classical model having

PNp and shown in (a). In the energy-conserving models shown in (a) and (b), the kink velocity decreases monotonically due to the energy radiation. Non-Hamiltonian momentum-conserving model in (c) shows the effect of kink self-acceleration discussed in [17].

It is interesting to study what happens when the parameter A in Eq. (33) deviates from zero and the conservation law of the model (momentum conservation) disappears. We found that the effect of kink self-acceleration, which can be seen in the bottom panel of Fig. 1 (c) for $A = 0$, remains for $|A| < 0.2$ but the value of the kink velocity in the steady motion regime decreases with increase in $|A|$ as it is presented in Fig. 2. For $|A| > 0.2$ kink self-acceleration effect disappears and kink velocity gradually decreases with time. From the above, we infer that properties such as the self-acceleration (for momentum-conserving models) or the Bogomol'nyi bounds (for energy-conserving discretizations) render such models rather special within the broader class of PNp free models. However, the critical ingredient for the more general feature of (*static*) PN absence exists in the form of a reduction of the second order problem into a first order.

VII. CONCLUSIONS

A general procedure for deriving discrete Klein-Gordon models whose static kinks can be placed anywhere with respect to the underlying lattice was described. Such models are called *static* PNp-free models. It was demon-strated that the models of this kind derived earlier [8, 9, 15, 17] are special cases of the wider family of models derived here.

Static kink solutions for the PNp-free models can be found from the nonlinear algebraic equation of the form $u(l, m) = 0$, which is a discrete analog of the first integral of the static continuum Klein-Gordon equation of motion. This ensures the existence of static kink solutions at least for the regime of sufficiently weak discreteness and smooth background potential. The range of the discreteness parameter supporting stable static kinks varies according to the specific properties of the model.

In this paper we have discussed only nearest-neighbor discretizations. However, one can easily write down a PNp-free model involving more distant neighbors by replacing Eq. (18) with higher-order finite difference operators approximating Eq. (2), keeping the two-point approximation, Eq. (11), for the first integral of Eq. (10).

Discrete kinks in the static PNp-free models possess the zero-frequency translational Goldstone mode and they can (almost) freely move with at least infinitesimally small velocity. Such kinks are not trapped by the lattice and they can be accelerated by even weak external fields.

As a topic for future studies, it would be interesting to find any possible relation between models constructed here and models that support traveling kink solutions for finite kink velocity. Such connections are apparently under intense investigation [13] and should provide a framework for understanding travelling in dispersive lattice systems.

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