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Dmitry Pelinovsky

PG Kevrekidis
University of Massachusetts - Amherst, kevrekid@math.umass.edu

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Periodic oscillations of dark solitons
in parabolic potentials

Dmitry E. Pelinovsky[†] and Panayotis G. Kevrekidis[††]

[†] Department of Mathematics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

[††] Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003

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Abstract

We reformulate the Gross–Pitaevskii equation with an external parabolic potential as a discrete dynamical system, by using the basis of Hermite functions. We consider small amplitude stationary solutions with a single node, called dark solitons, and examine their existence and linear stability. Furthermore, we prove the persistence of a periodic motion in a neighborhood of such solutions. Our results are corroborated by numerical computations elucidating the existence, linear stability and dynamics of the relevant solutions.

1 Introduction

We address the Gross-Pitaevskii (GP) equation with an external parabolic potential

$$iU_T = -\frac{1}{2}U_{XX} + \epsilon^2 X^2 U + \sigma |U|^2 U, \tag{1.1}$$

where $U(X, T) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{C}$ is decaying to zero as $|x| \to \infty$, $\epsilon \in \mathbb{R}$ is the strength of the external potential and $\sigma = 1$ ($\sigma = -1$) is normalized for the defocusing (focusing) cubic nonlinearity. This equation is of particular interest in the context of Bose-Einstein condensates, i.e., dilute alkali vapors at near-zero temperatures, where dynamics of localized dips in the ground state trapped by the magnetically induced, confining potential $V(X) = \epsilon^2 X^2$ is studied in many recent papers, see review in [11]. A question of particular interest concerns whether the localized density dips oscillate periodically near the center point $X = 0$ of the potential $V(X)$. If the motion of a localized dip is truly periodic, the frequency of periodic oscillations is to be found [5], while if the periodic oscillations are destroyed due to emission of radiation, the gradual change in the amplitude of oscillations is to be followed for sufficiently small ϵ [16]. Numerical simulations show radiation and amplitude changes if the confining parabolic potential is perturbed by a periodic potential while no radiation and time-periodic oscillations in the case of purely parabolic confinement [18].

If $\sigma = 1$, a localized dip on the ground state of the GP equation (1.1) in the formal limit $\epsilon \to 0$ represents the so-called dark soliton of the defocusing nonlinear Schrödinger (NLS) equation, which is the reason why we use the term "dark soliton" for a localized solution. Persistence and stability of a dark soliton of the defocusing NLS equation in the presence of an exponentially decaying potential $V(X)$ was studied in our previous paper [17], where methods of Lyapunov–Schmidt reductions, Evans functions and the stability theory in Pontryagin space were employed. These methods can not be

applied to the potential $V(X) = \epsilon^2 X^2$ since the potential deforms drastically the spectrum of the linearized problem: the continuous spectral band at $\epsilon = 0$ becomes an infinite sequence of isolated eigenvalues for $\epsilon \neq 0$. Therefore, we do not use here the limit $\epsilon \to 0$. Moreover, we transform the GP equation (1.1) to the ϵ-independent form

$$iu_t = -\frac{1}{2}u_{xx} + \frac{1}{2}x^2 u + \sigma|u|^2 u, \tag{1.2}$$

by the scaling transformation $x = \lambda X$, $t = \lambda^2 T$, and $u(x,t) = \lambda^{-1}U(X,T)$ with $\lambda = 2^{1/4}\epsilon^{1/2}$.

Substitution $u(x,t) = e^{-\frac{i}{2}t - i\mu t}\phi(x)$ reduces equation (1.2) to the second-order non-autonomous ODE

$$-\frac{1}{2}\phi''(x) + \frac{1}{2}x^2\phi(x) + \sigma\phi^3(x) = \left(\mu + \frac{1}{2}\right)\phi(x), \tag{1.3}$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}$. A strong solution of the ODE (1.3) is *said* to be a dark soliton if $\phi(x)$ is odd on $x \in \mathbb{R}$, has no zeros on $x \in \mathbb{R}_+$, and decays to zero sufficiently fast as $|x| \to \infty$. A classification of all localized solutions of the second-order ODE (1.3) and their construction with a rigorous shooting method is suggested in recent work [3].

Substitution of $u(x,t) = e^{-\frac{i}{2}t - i\mu t}[\phi(x) + v(x,t) + iw(x,t)]$ reduces equation (1.2) to the PDE system

$$\begin{cases} v_t &= \mathcal{L}_- w + 2\sigma\phi(x)vw + \sigma(v^2 + w^2)w, \\ -w_t &= \mathcal{L}_+ v + \sigma\phi(x)(3v^2 + w^2) + \sigma(v^2 + w^2)v, \end{cases} \tag{1.4}$$

where $(v,w) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}^2$ and $\mathcal{L}_\pm$ are self-adjoint Schrödinger operators in $L^2(\mathbb{R})$

$$\begin{cases} \mathcal{L}_+ &= -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 - \frac{1}{2} - \mu + 3\sigma\phi^2(x), \\ \mathcal{L}_- &= -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 - \frac{1}{2} - \mu + \sigma\phi^2(x). \end{cases} \tag{1.5}$$

Solutions of the PDE (1.2) are considered in space

$$\mathcal{H}_1(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : \quad xu \in L^2(\mathbb{R})\} \tag{1.6}$$

equipped with the norm

$$\|u\|_{\mathcal{H}_1}^2 = \int_{\mathbb{R}} \left(|u'(x)|^2 + (x^2 + 1)|u(x)|^2\right) dx. \tag{1.7}$$

Similarly, the domain of operators $\mathcal{L}_\pm$ in (1.5) is defined in space

$$\mathcal{H}_2(\mathbb{R}) = \{u \in H^2(\mathbb{R}) : \quad x^2 u \in L^2(\mathbb{R})\}. \tag{1.8}$$

The PDE system (1.4) is a Hamiltonian system with the standard symplectic structure and the Hamiltonian function in the form

$$H = \frac{1}{2}(v, \mathcal{L}_+ v) + \frac{1}{2}(w, \mathcal{L}_- w) + \sigma(\phi v, v^2 + w^2) + \frac{\sigma}{4}(v^2 + w^2, v^2 + w^2), \tag{1.9}$$

where $(\cdot, \cdot)$ denotes a standard inner product in $L^2(\mathbb{R})$. The Hamiltonian function H is bounded if $v, w \in \mathcal{H}_1(\mathbb{R})$ and is constant in time t. Due to the gauge invariance of the PDE (1.2), there exists an additional quantity

$$Q = 2(\phi, v) + (v, v) + (w, w), \tag{1.10}$$

which is constant in time t. Global existence of solutions of the initial-value problem associated with the PDE (1.2) in space $u \in \mathcal{H}_1(\mathbb{R})$ for all $t \in \mathbb{R}_+$ has been proved (see Proposition 2.2 in [6]).

Considering the linear part of the PDE system (1.4), one can separate the variables in the form $v(x,t) = v(x)e^{\lambda t}$, $w(x,t) = w(x)e^{\lambda t}$ and obtain the linear problem

$$\mathcal{L}_+ v = -\lambda w, \quad \mathcal{L}_- w = \lambda v \qquad (1.11)$$

for the spectral parameter $\lambda \in \mathbb{C}$ and the eigenvector $(v,w) \in L^2(\mathbb{R}, \mathbb{C}^2)$. Fix $\mu \in \mathbb{R}$ such that a stationary solution of the ODE (1.3) exists and $\phi \in \mathcal{H}_1(\mathbb{R})$. Then, the linear problem (1.11) admits an exact solution

$$\lambda = \pm i : \qquad v = \phi'(x), \quad w = \mp i x \phi(x), \qquad (1.12)$$

and $(v,w) \in L^2(\mathbb{R}, \mathbb{C}^2)$. Additionally, for any $\mu \in \mathbb{R}$, for which the stationary solution $\phi(x)$ is smooth with respect to parameter μ, the linear problem (1.11) admits another exact solution for zero eigenvalue $\lambda = 0$ of geometric multiplicity one and algebraic multiplicity two:

$$\mathcal{L}_- \phi(x) = 0, \qquad \mathcal{L}_+ \partial_\mu \phi(x) = -\phi(x). \qquad (1.13)$$

The main part of this work is devoted to the study of periodic solutions of the PDE system (1.4) for values of μ near $\mu = 1$. This special value corresponds to the second eigenvalue of the linear Schrödinger operator

$$\mathcal{L} = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 - \frac{1}{2} \qquad (1.14)$$

with the eigenfunction $\phi(x) = \varepsilon x e^{-x^2/2}$. Here the parameter ε is arbitrary in the linear problem and it parameterizes the corresponding family of stationary solutions $(\mu, \phi(x))$ of the nonlinear ODE (1.3), which bifurcates from the small-amplitude eigenmode [3].

A periodic solution of the PDE system (1.4) bifurcates from the linear eigenmodes $v(x) = \delta e^{i\tau}\phi'(x)$ and $w(x) = \mp i \delta e^{i\tau} x \phi(x)$ corresponding to the eigenvalue pair $\lambda = \pm i$ of the linear problem (1.11). Here δ and τ are two real-valued parameters, which are arbitrary in the linear problem and parameterize the corresponding family of periodic solutions (v,w) of the PDE system (1.4). An additional parameter α comes from the projection of the solution (v,w) to the geometric kernel of the linear problem (1.11) with the eigenmode $v(x) = 0$ and $w(x) = \alpha \phi(x)$. Using this construction, the main result of our paper is described by the following theorem.

Theorem 1 *Let ε and δ be sufficiently small and let α, τ be arbitrary. There exists a unique family of solutions of the ODE (1.3) such that*

$$\|\phi - \varepsilon x e^{-x^2/2}\|_{\mathcal{H}_1} \leq C_1 \varepsilon^3, \quad \left| \mu - 1 - \frac{3\sigma \varepsilon^2}{\sqrt{32\pi}} \right| \leq C_2 \varepsilon^4, \qquad (1.15)$$

for some ε-independent constants $C_1, C_2 > 0$. There exists a family of time-periodic space-localized solutions of the PDE system (1.4) such that $(v,w) \in \mathcal{H}_1(\mathbb{R}, \mathbb{R}^2)$ for all $t \in \mathbb{R}$,

$$v\left(x, t + \frac{2\pi}{\Omega}\right) = v(x,t), \quad w\left(x, t + \frac{2\pi}{\Omega}\right) = w(x,t), \qquad \forall (x,t) \in \mathbb{R}^2, \qquad (1.16)$$

with the bounds

$$\|v(\cdot, t) - \delta \phi'(x) \cos(\Omega t + \tau)\|_{\mathcal{H}_1} \leq C_3 \varepsilon \delta^2, \qquad (1.17)$$
$$\|w(\cdot, t) - \delta [x\phi(x) \sin(\Omega t + \tau) + \alpha \phi(x)]\|_{\mathcal{H}_1} \leq C_4 \varepsilon \delta^2, \qquad (1.18)$$

and $|\Omega - 1| \leq C_5 \varepsilon^2 \delta^2$ for some (ε, δ)-independent constants $C_3, C_4, C_5 > 0$.

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The periodic solution of Theorem 1 has four free parameters $(\varepsilon, \delta, \tau, \alpha)$ which are associated with projections to the four eigenmodes (1.12) and (1.13) of the linear problem (1.11). Parameters τ and α can be set to zero due to two obvious symmetries of the PDE (1.2): the gauge transformation $u(x,t) \mapsto u(x,t)e^{i\alpha}$, $\forall \alpha \in \mathbb{R}$ and the reversibility transformation $u(x,t) \mapsto \bar{u}(x,-t)$, $\forall t \in \mathbb{R}$.

Although the eigenmodes (1.12) and (1.13) persist for all $\mu \in \mathbb{R}$, existence of periodic orbits of the GP equation (1.2) is only proved near $\mu = 1$. This is due to the fact that the non-resonance conditions $n \neq \mathrm{Im}\lambda_m$, $\forall n, m \in \mathbb{N}$ are proved to be satisfied only in this domain, where λ_m denote other isolated eigenvalues of the linear problem (1.11) with $\mathrm{Im}\lambda_m > 0$ which are different from $\lambda = i$. Thus, resonances do not occur near the value $\mu = 1$. The construction of the periodic orbit is complicated due to the existence of translational eigenmodes associated with the double zero eigenvalue $\lambda = 0$ of the linear problem (1.11).

Our main result is in agreement with Theorem 2.1 in [9], where the Newton's law of particle dynamics is obtained in a more general context of multi-dimensional confining potentials and general nonlinear functions of the GP equation $i\dot{\psi} = -\nabla^2\psi + V(x)\psi - f(\psi)$. The Newton's law is derived for parameters (a, p) of the solitary wave solution of the unperturbed equation with $V(x) \equiv 0$ and it takes the form

$$\dot{a} = 2p, \qquad \dot{p} = -\nabla V(a). \tag{1.19}$$

Adopting our notations for the time variable and the potential function of the GP equation (1.2), we rewrite the Newton's law (1.19) in the explicit form $\ddot{a} + a = 0$, which recovers the frequency $\Omega = 1$ of the periodic solution of Theorem 1 in the linear approximation $\delta \to 0$.

There are several differences between results of Theorem 2.1 in [9] and our Theorem 1. First, the Newton's law (1.19) is valid on finite time intervals and in the limit when the localization length of the stationary solution $\phi(x)$ is much smaller than the confinement length of the potential $V(x)$. This situation corresponds to the original GP equation (1.1) in the limit $\epsilon \to 0$. Second, the exact periodicity is not guaranteed by the periodic solutions of the Newton's law (1.19) because of the remainder terms. Lastly, the frequency $\Omega = 1$ of the Newton's law is independent of the nonlinear function $f(\psi)$ and the nonlinear corrections in (a, p). In our case, the result of Theorem 1 is valid for all time intervals, the exact periodicity is guaranteed, and the frequency Ω changes with parameters δ. On the other hand, our results are valid in the limit $\mu \to 1$, which is far from the limit $\epsilon \to 0$ of the GP equation (1.1).

Note that the oscillations of the dark solitons in the GP equation (1.2) with the frequency $\Omega = 1$ were predicted from the Ehrenfest Theorem in much earlier works (see references in [5] and [9]). However, it was argued that this frequency is not observed in numerical simulations of the original GP equation (1.1) with $\sigma = 1$ for sufficiently small ϵ [5, 16, 18]. It was suggested in these works (see review in [11]) that dark solitons oscillate with a smaller frequency $\Omega = \frac{1}{\sqrt{2}}$. We will show that both frequencies occur in the spectrum of the linear problem (1.11) in the corresponding limit but the non-resonance conditions are not satisfied for either frequency in this limit.

Our strategy for the proof of Theorem 1 is to use a complete set of Hermite functions and to reformulate the evolution problem for the PDE (1.2) as an infinite-dimensional discrete dynamical system for coefficients of the decomposition (Section 2). Existence of stationary solutions $\phi(x)$ of the ODE (1.3) and spectral stability of stationary solutions in the linear problem (1.11) are studied in the framework of the discrete dynamical system (Section 3). The proof of existence of periodic solutions of the PDE system (1.4) relies on construction of periodic orbits in the discrete dynamical system (Section 4). The analytical results are verified with numerical approximations of solutions of the ODE (1.3), eigenvalues of the linear problem (1.11) and solutions of the GP equation (1.2) (Section 5). Distribution of eigenvalues of the linear problem (1.11) in the limit $\mu \to \infty$ for $\sigma = 1$ is also analyzed with formal asymptotic methods (Appendix A).

2   Formalism of the discrete dynamical system

The set of Hermite functions is defined by the standard expressions [1]:

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2}, \qquad \forall n = 0, 1, 2, 3, ..., \tag{2.1}$$

where $H_n(x)$ denote the Hermite polynomials, e.g. $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$, $H_3 = 8x^3 - 12x$, etc. Since the Hermite functions are eigenfunctions of the linear Schrödinger equation

$$-\frac{1}{2}\phi_n''(x) + \frac{1}{2}x^2\phi_n(x) = \left(n + \frac{1}{2}\right)\phi_n(x), \qquad \forall n = 0, 1, 2, 3, ..., \tag{2.2}$$

the Sturm–Liouville theory implies that the set of Hermite functions $\{\phi_n(x)\}_{n=0}^{\infty}$ forms an orthogonal basis in $L^2(\mathbb{R})$. The normalization coefficients in the expressions (2.1) ensure that the Hermite functions have unit L^2-norm, such that

$$(\phi_n, \phi_m) = \delta_{n,m}, \qquad \forall n, m = 0, 1, 2, 3, ... \tag{2.3}$$

We represent a solution $u(x, t)$ of the GP equation (1.2) by the series of eigenfunctions

$$u(x, t) = e^{-\frac{i}{2}t} \sum_{n=0}^{\infty} a_n(t)\phi_n(x) \tag{2.4}$$

where the components $(a_0, a_1, a_2, ...)$ form a vector $\mathbf{a}$ on $\mathbb{N}$. When the series representation (2.4) is substituted to the GP equation (1.2), the PDE problem is converted to the discrete dynamical system

$$i\dot{a}_n = n a_n + \sigma \sum_{(n_1, n_2, n_3)} K_{n, n_1, n_2, n_3} a_{n_1} \bar{a}_{n_2} a_{n_3}, \qquad \forall n = 0, 1, 2, 3, ..., \tag{2.5}$$

where $K_{n, n_1, n_2, n_3} = (\phi_n, \phi_{n_1}\phi_{n_2}\phi_{n_3})$. We shall use a convention to avoid specifying the range of non-negative integers (n_1, n_2, n_3) and n in the summation signs of the dynamical system (2.5). Let $l_s^2(\mathbb{N})$ be a weighted discrete l^2-space equipped with the standard norm

$$\|\mathbf{a}\|_{l_s^2}^2 = \sum_{n=0}^{\infty} (1 + n)^{2s}|a_n|^2 < \infty, \quad \forall s \in \mathbb{R}. \tag{2.6}$$

Since the set $\{\phi_n(x)\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(\mathbb{R})$, we note the isometry $\|u\|_{L^2}^2 = \|\mathbf{a}\|_{l^2}^2$, so that $u \in L^2(\mathbb{R})$ if and only if $\mathbf{a} \in l^2(\mathbb{N})$. On the other hand, we need an equivalence between the space $\mathcal{H}_1(\mathbb{R})$ for the function $u(x)$ and the space $l_s^2(\mathbb{N})$ for the vector $\mathbf{a}$. In addition, we need to determine the domain and range of the vector field of the discrete dynamical system (2.5). These results are described in Lemmas 1 and 2.

Lemma 1 *Let $u(x) = \sum_{m=0}^{\infty} a_n\phi_n(x)$. Then $u \in \mathcal{H}_1(\mathbb{R})$ if and only if $\mathbf{a} \in l_{1/2}^2(\mathbb{N})$.*

Proof. It follows directly that

$$\|u\|_{\mathcal{H}_1}^2 = \int_{\mathbb{R}} \left(|u'(x)|^2 + (x^2 + 1)|u(x)|^2\right) dx$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1} \bar{a}_{n_2} \int_{\mathbb{R}} \left[\phi'_{n_1}(x) \phi'_{n_2}(x) + (x^2 + 1) \phi_{n_1}(x) \phi_{n_2}(x) \right] dx$$

$$= 2 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1} \bar{a}_{n_2} (1 + n_2)(\phi_{n_1}, \phi_{n_2})$$

$$= 2 \sum_{n=0}^{\infty} (1 + n)|a_n|^2 = 2\|\mathbf{a}\|_{l_{1/2}^2}^2,$$

where the orthogonality relations (2.3) have been used. $\qquad \square$

Remark 1 By the same method, one can prove that $u \in \mathcal{H}_2(\mathbb{R})$ if and only if $\mathbf{a} \in l_1^2(\mathbb{N})$.

Lemma 2 *The vector field of the dynamical system (2.5) maps $l_{1/2}^2(\mathbb{N})$ to $l_{-1/2}^2(\mathbb{N})$.*

Proof. The vector field of the dynamical system (2.5) is decomposed into the linear $\mathbf{f}(\mathbf{a})$ and nonlinear $\sigma \mathbf{g}(\mathbf{a})$ parts, where

$$f_n = n a_n, \quad g_n = \sum_{(n_1, n_2, n_3)} K_{n, n_1, n_2, n_3} a_{n_1} \bar{a}_{n_2} a_{n_3}, \quad \forall n = 0, 1, 2, 3, \dots$$

The linear unbounded part satisfies the estimate

$$\|\mathbf{f}(\mathbf{a})\|_{l_s^2}^2 = \sum_{n=0}^{\infty} (1 + n)^{2s} n^2 |a_n|^2 \leq \|\mathbf{a}\|_{l_{s+1}^2}^2, \tag{2.7}$$

such that $\mathbf{f} : l_{s+1}^2(\mathbb{N}) \mapsto l_s^2(\mathbb{N})$ for all $s \in \mathbb{R}$. If $\mathbf{a} \in l_{1/2}^2(\mathbb{N})$, then $s = -\frac{1}{2}$. The nonlinear vector part satisfies the estimate

$$\begin{aligned}
\|\mathbf{g}(\mathbf{a})\|_{l_s^2}^2 &= \sum_{n=0}^{\infty} (1 + n)^{2s} \sum_{(n_1, n_2, n_3)} \sum_{(m_1, m_2, m_3)} K_{n, n_1, n_2, n_3} K_{n, m_1, m_2, m_3} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{m_1} a_{m_2} \bar{a}_{m_3} \\
&= \sum_{n=0}^{\infty} (1 + n)^{2s} \left| (\phi_n u, |u|^2) \right|^2 \leq \left(\sum_{n=0}^{\infty} (1 + n)^{2s} \|u \phi_n\|_{L^2}^2 \right) \|u\|_{L^4}^4 \\
&\leq \left(\sum_{n=0}^{\infty} (1 + n)^{2s} \|\phi_n\|_{L^4}^2 \right) \|u\|_{L^4}^6,
\end{aligned}$$

where $u(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$ and all $\phi_n(x)$ are real-valued. By the main theorem of [7], there exists a constant $C > 0$ such that

$$\|\phi_n\|_{L^4}^4 \leq C \frac{\log(1 + n)}{\sqrt{1 + n}}, \qquad \forall n = 0, 1, 2, \dots \tag{2.8}$$

Therefore, the series $\sum_{n=0}^{\infty} (1 + n)^{2s} \|\phi_n\|_{L^4}^2$ converges for all $s < -\frac{3}{8}$. The value $s = -\frac{1}{2}$ belongs to this interval. Finally, by the Sobolev embedding and Poincare inequality [2], there are constants $C, \tilde{C} > 0$ such that

$$\|u\|_{L^4}^4 \leq C \|(u^2)'\|_{L^2}^2 \leq 4C \|u\|_{L^\infty}^2 \|u'\|_{L^2}^2 \leq \tilde{C} \|u\|_{H^1}^4 \leq \tilde{C} \|u\|_{\mathcal{H}_1}^4.$$

Since the norm in $\mathcal{H}_1(\mathbb{R})$ for the function $u(x)$ is equivalent to the norm in $l^2_{1/2}(\mathbb{Z})$ for the vector $\mathbf{a}$ by Lemma 1, the estimate for the nonlinear vector field is completed by

$$\|\mathbf{g}(\mathbf{a})\|^2_{l^2_{-1/2}} \leq C_0 \|u\|^6_{L^4} \leq \tilde{C}_0 \|\mathbf{a}\|^6_{l^2_{1/2}}, \tag{2.9}$$

for some $C_0, \tilde{C}_0 > 0$. The interpolation argument for the bounds (2.7) and (2.9) concludes the proof that the nonlinear vector field $\mathbf{f}(\mathbf{a}) + \sigma\mathbf{g}(\mathbf{a})$ maps $l^2_{1/2}(\mathbb{N})$ to $l^2_{-1/2}(\mathbb{N})$. $\qquad\square$

Theorem 2 *The discrete dynamical system (2.5) is globally well-posed in the phase space $\mathbf{a} \in l^2_{1/2}(\mathbb{N})$.*

Proof. By Proposition 2.2 in [6], the GP equation (1.2) is globally well-posed in the phase space $u \in \mathcal{H}_1(\mathbb{R})$. By Lemma 1, the trajectory $u(t) \in \mathcal{H}_1(\mathbb{R})$ is equivalent to the trajectory $\mathbf{a}(t) \in l^2_{1/2}(\mathbb{N})$ on $t \in \mathbb{R}$. By Lemma 2, the vector field of the discrete dynamical system (2.5) is well-defined on $l^2_{1/2}(\mathbb{N}) \subset l^2(\mathbb{N})$, where it is equivalent to the vector field of the GP equation (1.2) by virtue of standard orthogonal projections. $\qquad\square$

3   Existence and stability of stationary solutions

Stationary solutions of the dynamical system (2.5) take the form $\mathbf{a}(t) = \mathbf{A}e^{-i\mu t}$, where $\mathbf{A}$ is a time-independent vector and μ is a parameter of the solution. If $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ and $\phi(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$, then $\phi \in \mathcal{H}_1(\mathbb{R})$ is a stationary solution of the GP equation (1.2), that is $\phi(x)$ satisfies the ODE (1.3). The vector $\mathbf{A}$ is found as a root of the infinite-dimensional cubic vector field $\mathbf{F} : l^2_{1/2}(\mathbb{N}) \times \mathbb{R} \mapsto l^2_{-1/2}(\mathbb{N})$, where the n-th component of $\mathbf{F}(\mathbf{A}, \mu)$ is given by

$$F_n = (\mu - n)A_n - \sigma \sum_{(n_1, n_2, n_3)} K_{n;n_1,n_2,n_3} A_{n_1} \bar{A}_{n_2} A_{n_3} = 0, \qquad \forall n = 0, 1, 2, \ldots \tag{3.1}$$

The Jacobian operator $D_{\mathbf{A}}\mathbf{F}(\mathbf{0}, \mu)$ is a diagonal matrix with entries $\mu - n$ and it admits a one-dimensional kernel if $\mu = n_0$ for any non-negative integer n_0. The corresponding eigenvector is $\mathbf{e}_{n_0}$, the unit vector in $l^2(\mathbb{N})$. By the local bifurcation theory [8], each eigenvector of $D_{\mathbf{A}}\mathbf{F}(\mathbf{0}, n_0)$ can be uniquely continued in a local neighborhood of the point $\mathbf{A} = \mathbf{0} \in l^2_{1/2}(\mathbb{N})$ and $\mu = n_0 \in \mathbb{R}$. We are particularly interested in the second eigenvalue $n_0 = 1$, which corresponds to the *dark soliton* $\phi(x)$ with a single zero (node) at $x = 0$. (Other bifurcations of stationary localized solutions $\phi(x)$ are considered in [3].) Details of this bifurcation are given in the following proposition.

Proposition 1 *Consider real-valued roots $(\mathbf{A}, \mu)$ of the vector field $\mathbf{F}(\mathbf{A}, \mu)$ such that $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$. There exists a unique family of solutions near $\mu = 1$ parameterized by ε such that*

$$\|\mathbf{A} - \varepsilon\mathbf{e}_1\|_{l^2_{1/2}} \leq C_1\varepsilon^3, \qquad \left|\mu - 1 - \frac{3\sigma\varepsilon^2}{\sqrt{32\pi}}\right| \leq C_2\varepsilon^4, \tag{3.2}$$

for some ε-independent constants $C_1, C_2 > 0$ and sufficiently small ε. Moreover, if $\sigma \neq 0$, the solution $(\mathbf{A}, \mu)$ is smooth with respect to ε for sufficiently small ε and $\frac{d}{d\mu}Q(\mathbf{A}) \neq 0$, where $Q(\mathbf{A}) = \|\mathbf{A}\|^2_{l^2}$.

Proof. Both $\mathbf{F}(\mathbf{A}, \mu)$ and $D_{\mathbf{A}}\mathbf{F}(\mathbf{A}, \mu)$ are continuous in a local neighborhood of $\mathbf{A} = \mathbf{0} \in l^2_{1/2}(\mathbb{N})$ and $\mu = 1 \in \mathbb{R}$. At the point $\mathbf{A} = \mathbf{0}$ and $\mu = 1$, the operator has a one-dimensional kernel with

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the eigenvector $\mathbf{e}_1 \in l^2(\mathbb{N})$. By using the method of Lyapunov–Schmidt reductions [8], we set $\mathbf{A} = \varepsilon \left[\mathbf{e}_1 + \tilde{\mathbf{A}} \right]$ and $\mu = 1 + \tilde{\mu}$, where $\tilde{\mathbf{A}}$ is an orthogonal complement of $\mathbf{e}_1$ in $l^2(\mathbb{N})$ such that $\tilde{A}_1 = 0$. The orthogonal projection of equation (3.1) to $\mathbf{e}_1$ gives a bifurcation equation for $\tilde{\mu}$

$$\tilde{\mu} = \sigma \varepsilon^2 \left[K_{1;1,1,1} + 3 \sum_{n_1} K_{1;1,1,n} \tilde{A}_{n_1} + 3 \sum_{(n_1,n_2)} K_{1;1,n_1,n_2} \tilde{A}_{n_1} \tilde{A}_{n_2} + \sum_{(n_1,n_2,n_3)} K_{1;n_1,n_2,n_3} \tilde{A}_{n_1} \tilde{A}_{n_2} \tilde{A}_{n_3} \right],$$

where the index for (n_1, n_2, n_3) in the summation signs runs on the set $\{0, 2, 3, ...\}$. Let P be an orthogonal projection from $l^2(\mathbb{N})$ to the orthogonal complement of $\mathbf{e}_1$. Then the inverse of $PD_{\mathbf{A}}\mathbf{F}(\mathbf{0}, 1)P$ exists and is a bounded operator from $l^2_{1/2}(\mathbb{N})$ to $l^2_{1/2}(\mathbb{N})$. By the Implicit Function Theorem, there exists a unique smooth solution $\tilde{\mathbf{A}}$ in the neighborhood of $\tilde{\mathbf{A}} = \mathbf{0} \in l^2_s(\mathbb{N})$ such that $\|\tilde{\mathbf{A}}\|_{l^2_{1/2}} \leq C_1 \varepsilon^2$ for some $C_1 > 0$. By the Implicit Function Theorem, there exists a unique smooth solution $\tilde{\mu}$ of the bifurcation equation in the neighborhood of $\tilde{\mu} = 0$ such that $|\tilde{\mu} - \varepsilon^2 \sigma K_{1,1,1,1}| \leq C_2 \varepsilon^4$ for some $C_2 > 0$. The value $K_{1,1,1,1} = \|\phi_1\|_{L^4}^4 = \frac{3}{\sqrt{32\pi}}$ is computed in Table I. Since $Q(\mathbf{A}) = \|\mathbf{A}\|_{l^2}^2 = \varepsilon^2 + \mathrm{O}(\varepsilon^4)$ and $\mu - 1 = \frac{3\sigma\varepsilon^2}{\sqrt{32\pi}} + \mathrm{O}(\varepsilon^4)$, then $\frac{d}{d\mu}Q(\mathbf{A}) \neq 0$ near $\mu = 1$ for $\sigma \neq 0$. $\qquad \square$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$K_{n,n,n,n}$	$\frac{1}{\sqrt{2\pi}}$	$\frac{3}{4\sqrt{2\pi}}$	$\frac{41}{64\sqrt{2\pi}}$	$\frac{147}{256\sqrt{2\pi}}$	$\frac{8649}{16384\sqrt{2\pi}}$	$\frac{32307}{65536\sqrt{2\pi}}$
$K_{1,n,n,1}$	$\frac{1}{2\sqrt{2\pi}}$	$\frac{3}{4\sqrt{2\pi}}$	$\frac{7}{16\sqrt{2\pi}}$	$\frac{11}{32\sqrt{2\pi}}$	$\frac{75}{256\sqrt{2\pi}}$	$\frac{133}{512\sqrt{2\pi}}$
$K_{0,1,1,n}$	$\frac{1}{2\sqrt{2\pi}}$	0	$\frac{1}{8\sqrt{\pi}}$	0	$-\frac{3\sqrt{3}}{32\sqrt{\pi}}$	0

Table I: Numerical values for $K_{n,n,n,n} = \|\phi_n\|_{L^4}^4$, $K_{1,n,n,1} = (\phi_1^2, \phi_n^2)$, and $K_{0,1,1,n} = (\phi_0\phi_n, \phi_1^2)$.

Let $(\mathbf{A}, \mu)$ be a real-valued root of the nonlinear vector field (3.1) such that $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$. Spectral stability of the stationary solution is studied with the expansion

$$\mathbf{a}(t) = e^{-i\mu t} \left[\mathbf{A} + (\mathbf{B} - \mathbf{C}) e^{i\Omega t} + (\bar{\mathbf{B}} + \bar{\mathbf{C}}) e^{-i\bar{\Omega} t} + \mathrm{O}(\|\mathbf{B}\|^2 + \|\mathbf{C}\|^2) \right], \tag{3.3}$$

where the spectral parameter $\Omega \in \mathbb{C}$ and the eigenvector $(\mathbf{B}, \mathbf{C}) \in l^2(\mathbb{N}, \mathbb{C}^2)$ satisfy the linear problem

$$L_+ \mathbf{B} = \Omega \mathbf{C}, \qquad L_- \mathbf{C} = \Omega \mathbf{B}, \tag{3.4}$$

associated with matrix operators $L_\pm$. Their n-th components are defined in the form

$$\begin{cases} (L_+\mathbf{B})_n &= (n-\mu)B_n + 3\sigma \sum_{n_1} V_{n,n_1} B_{n_1}, \\ (L_-\mathbf{C})_n &= (n-\mu)C_n + \sigma \sum_{n_1} V_{n,n_1} C_{n_1}, \end{cases} \qquad \forall n = 0, 1, 2, 3, ..., \tag{3.5}$$

where $V_{n,n_1} = \sum_{(n_2,n_3)} K_{n,n_1,n_2,n_3} A_{n_2} A_{n_3}$. We have used here the symmetry of the coefficients K_{n,n_1,n_2,n_3} with respect to the interchange of (n_1, n_2, n_3).

Lemma 3 *Let $(\mathbf{A}, \mu)$ be a real-valued root of the vector field $\mathbf{F}(\mathbf{A}, \mu)$ such that $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$. Operators L_+ and L_- admit closed self-adjoint extensions in $l^2(\mathbb{N})$ with the domain in $l^2_1(\mathbb{N})$.*

Proof. The diagonal unbounded part of $L_\pm$ maps $l^2_1(\mathbb{N})$ to $l^2(\mathbb{N})$. We need to show that the non-diagonal part of $L_\pm$ represents a bounded perturbation from $l^2(\mathbb{N})$ to $l^2(\mathbb{N})$ if $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$. This is done by using

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the same ideas as in the proof of Lemma 2:

$$\sum_{n=0}^{\infty}\left|\sum_{n_1}V_{n,n_1}B_{n_1}\right|^2 = \sum_{n=0}^{\infty}\sum_{(n_1,n_2,n_3)}\sum_{(m_1,m_2,m_3)}K_{n,n_1,n_2,n_3}K_{n,m_1,m_2,m_3}A_{n_2}A_{n_3}A_{m_2}A_{m_3}B_{n_1}\bar{B}_{m_1}$$

$$= \sum_{n=0}^{\infty}|(\phi_n,u^2v)|^2 = \|u^2v\|_{L^2}^2 \le \|u\|_{L^\infty}^4\|v\|_{L^2}^2 \le C^4\|u\|_{\mathcal{H}_1}^4\sum_{n=0}^{\infty}|B_n|^2,$$

where $u(x) = \sum_{n=0}^{\infty}A_n\phi_n(x)$ and $v(x) = \sum_{n=0}^{\infty}B_n\phi_n(x)$. $\qquad\square$

Remark 2 The result of Lemma 3 is obvious from the equivalence between the space $\mathcal{H}_2(\mathbb{R})$ for the function $v(x)$ and the space $l_1^2(\mathbb{N})$ for the vector $\mathbf{B}$, see Remark 1. We recall that the differential operators $\mathcal{L}_\pm$ given by (1.5) are defined on the domain $\mathcal{H}_2(\mathbb{R})$ and the matrix operators $L_\pm$ given by (3.5) represent the action of differential operators on the basis of Hermite functions in $\mathcal{H}_2(\mathbb{R})$.

The linear problem (3.4) has eigenvalue $\Omega = 0$ of geometric multiplicity one and algebraic multiplicity two due to the exact solution

$$L_-\mathbf{A} = \mathbf{0}, \qquad L_+\partial_\mu\mathbf{A} = \mathbf{A}, \tag{3.6}$$

where the smoothness of $\mathbf{A}$ with respect to μ near $\mu = 1$ is guaranteed by Proposition 1.

When $\mathbf{A} = \mathbf{0}$ and $\mu = 1$, the spectrum of the eigenvalue problem (3.4) is known in the explicit form. It consists of eigenvalues $\Omega = 0$ and $\Omega = \pm 1$ of geometric and algebraic multiplicities two and simple eigenvalues $\Omega = \pm m$ for all $m = 2, 3, ...$. The double zero eigenvalue persists for any ε according to the exact solution (3.6), stemming from the underlying $U(1)$ invariance of the system. Splitting of all other eigenvalues in a local neighborhood of $\mathbf{A} = \mathbf{0}$ and $\mu = 1$ is described by the following proposition.

Proposition 2 *Let $(\mathbf{A}, \mu)$ be defined by Proposition 1 for sufficiently small ε. Non-zero eigenvalues of the linear problem (3.4) form a set $\{\pm\Omega_m\}_{m=0}^{\infty}$ of simple real symmetric eigenvalue pairs, such that*

$$|\Omega_0 - 1| \le C_0\varepsilon^4, \qquad \left|\Omega_1 - 1 + \frac{\varepsilon^2\sigma}{8\sqrt{2\pi}}\right| \le C_1\varepsilon^4 \tag{3.7}$$

and

$$\left|\Omega_m - m + \varepsilon^2\sigma\left(K_{1,1,1,1} - 2K_{m+1,1,1,m+1}\right)\right| \le C_m\varepsilon^4, \quad \forall m = 2, 3, ... \tag{3.8}$$

for some ε-independent constants $C_0, C_1, C_m > 0$.

Proof. Since the essential spectrum of the matrix operators $L_\pm$ is empty and the potential terms are bounded perturbations to the unbounded diagonal terms, isolated eigenvalues split according to the regular perturbation theory [10]. The formal power series expansion for a simple eigenvalue $\Omega = m = 2, 3, ...$ is defined by

$$\begin{cases} \mathbf{B} &= \mathbf{e}_{m+1} + \varepsilon^2\tilde{\mathbf{B}} + \mathrm{O}(\varepsilon^4), \\ \mathbf{C} &= \mathbf{e}_{m+1} + \varepsilon^2\tilde{\mathbf{C}} + \mathrm{O}(\varepsilon^4), \\ \Omega &= m + \varepsilon^2\tilde{\Omega} + \mathrm{O}(\varepsilon^4). \end{cases} \tag{3.9}$$

Projections to the component $n = m + 1$ lead to a linear system at the leading order $\mathrm{O}(\varepsilon^2)$

$$\begin{cases} m\left(\tilde{B}_{m+1} - \tilde{C}_{m+1}\right) &= \sigma\left[K_{1,1,1,1} - 3K_{m+1,1,1,m+1}\right] + \tilde{\Omega} \\ m\left(\tilde{C}_{m+1} - \tilde{B}_{m+1}\right) &= \sigma\left[K_{1,1,1,1} - K_{m+1,1,1,m+1}\right] + \tilde{\Omega}. \end{cases} \tag{3.10}$$

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The linear system has a solution if and only if $\tilde{\Omega} = \sigma\left(2K_{m+1,1,1,m+1} - K_{1,1,1,1}\right)$. Persistence of the eigenvalue by the perturbation theory results in the expansion (3.8). The power series expansion for the double eigenvalue $\Omega = 1$ is defined by

$$\begin{cases} \mathbf{B} & = & \alpha\mathbf{e}_0 + \beta\mathbf{e}_2 + \varepsilon^2\tilde{\mathbf{B}} + \mathrm{O}(\varepsilon^4), \\ \mathbf{C} & = & -\alpha\mathbf{e}_0 + \beta\mathbf{e}_2 + \varepsilon^2\tilde{\mathbf{C}} + \mathrm{O}(\varepsilon^4), \\ \Omega & = & 1 + \varepsilon^2\tilde{\Omega} + \mathrm{O}(\varepsilon^4), \end{cases} \qquad (3.11)$$

where (α, β) are arbitrary parameters. Projections to the components $n = 0$ and $n = 2$ leads to a linear system at the leading order $\mathrm{O}(\varepsilon^2)$

$$\begin{cases} \left(\tilde{B}_0 + \tilde{C}_0\right) & = & \sigma\left[3K_{0;1,1,0}\alpha + 3K_{0,1,1,2}\beta - K_{1,1,1,1}\alpha\right] + \tilde{\Omega}\alpha \\ -\left(\tilde{C}_0 + \tilde{B}_0\right) & = & \sigma\left[K_{0,1,1,0}\alpha - K_{0,1,1,2}\beta - K_{1,1,1,1}\alpha\right] + \tilde{\Omega}\alpha \\ \left(\tilde{B}_2 - \tilde{C}_2\right) & = & \sigma\left[K_{1,1,1,1}\beta - 3K_{2,1,1,0}\alpha - 3K_{2,1,1,2}\beta\right] + \tilde{\Omega}\beta \\ \left(\tilde{C}_2 - \tilde{B}_2\right) & = & \sigma\left[K_{1,1,1,1}\beta + K_{2,1,1,0}\alpha - K_{2,1,1,2}\beta\right] + \tilde{\Omega}\alpha \end{cases} \qquad (3.12)$$

The linear system has a solution if and only if (α, β) satisfies a homogeneous system

$$\begin{cases} \sigma\left(K_{1,1,1,1}\alpha - 2K_{0,1,1,0}\alpha - K_{0,1,1,2}\beta\right) & = & \tilde{\Omega}\alpha, \\ \sigma\left(-K_{1,1,1,1}\beta + K_{2,1,1,0}\alpha + 2K_{2,1,1,2}\beta\right) & = & \tilde{\Omega}\beta. \end{cases} \qquad (3.13)$$

The homogeneous system for (α, β) has a non-zero solution if and only if $\tilde{\Omega}$ satisfies a quadratic equation, roots of which are given by

$$\tilde{\Omega} = \sigma\left(K_{2,1,1,2} - K_{0,1,1,0} \pm \sqrt{(K_{1,1,1,1} - K_{0,1,1,0} - K_{2,1,1,2})^2 - K_{0,1,1,2}^2}\right). \qquad (3.14)$$

It follows from Table I that $\sqrt{(K_{1,1,1,1} - K_{0,1,1,0} - K_{2,1,1,2})^2 - K_{0,1,1,2}^2} = \frac{1}{16\sqrt{2\pi}}$ and $K_{2,1,1,2} - K_{0,1,1,0} = -\frac{1}{16\sqrt{2\pi}}$. Persistence of the eigenvalues by the perturbation theory results in the expansion (3.7). $\quad\square$

Corollary 1 *Let $[\mathbf{B}_m, \mathbf{C}_m]^T$ be an eigenvector of the linear problem (3.4) for the eigenvalue $\Omega_m \in \mathbb{R}_+$ for any $m = 0, 1, 2, 3...$ in Proposition 2. For sufficiently small ε, the eigenvalue Ω_0 has positive signature of $\langle\mathbf{B}_0, L_+\mathbf{B}_0\rangle$, the eigenvalue Ω_1 has negative signature of $\langle\mathbf{B}_1, L_+\mathbf{B}_1\rangle$, while all other eigenvalues Ω_m with $m = 2, 3, ...$ have positive signature of $\langle\mathbf{B}_m, L_+\mathbf{B}_m\rangle$, where $\langle\cdot, \cdot\rangle$ denotes a standard inner product in $l^2(\mathbb{N})$.*

Proof. In the case $\tilde{\Omega} = 0$, the homogeneous system (3.13) for (α, β) has a one-parameter family of solutions with $\beta = -\sqrt{2}\alpha$, such that $\langle\mathbf{B}_0, L_+\mathbf{B}_0\rangle = -|\alpha|^2 + |\beta|^2 + \mathrm{O}(\varepsilon^2) > 0$ for sufficiently small ε. In the case $\tilde{\Omega} \neq 0$, the homogeneous system (3.13) for (α, β) has a one-parameter family of solutions with $\alpha = -\sqrt{2}\beta$, such that $\langle\mathbf{B}_1, L_+\mathbf{B}_1\rangle = -|\alpha|^2 + |\beta|^2 + \mathrm{O}(\varepsilon^2) < 0$ for sufficiently small ε. In the case of other eigenvalues, it is obvious from the proof of Proposition 2 that $\langle\mathbf{B}_m, L_+\mathbf{B}_m\rangle = m + \mathrm{O}(\varepsilon^2)$ for $m = 2, 3, ...$ $\quad\square$

Remark 3 The double zero eigenvalue is associated with the expansions

$$\mathbf{A} = \varepsilon\mathbf{e}_1 + \mathrm{O}(\varepsilon^3), \qquad \partial_\mu\mathbf{A} = \frac{\sqrt{32\pi}}{3\sigma\varepsilon}\mathbf{e}_1 + \mathrm{O}(\varepsilon), \qquad (3.15)$$

where ε is sufficiently small. As a result, $\langle\mathbf{A}, \partial_\mu\mathbf{A}\rangle = \frac{\sqrt{32\pi}}{3\sigma} + \mathrm{O}(\varepsilon^2)$.

Lemma 4 *Let ε be sufficiently small. Let $\{[\mathbf{B}_m, \mathbf{C}_m]^T\}_{m=0}^\infty$ be a set of real-valued eigenvectors of the linear problem (3.4) for the set of positive eigenvalues $\{\Omega_m\}_{m=0}^\infty$. The set of eigenvectors is symplectically orthogonal such that*

$$\langle \mathbf{B}_{m'}, \mathbf{C}_m \rangle = 0, \quad \forall m' \neq m \qquad \langle \mathbf{B}_m, \mathbf{C}_m \rangle \neq 0, \quad \forall m = 0, 1, 2, 3... \tag{3.16}$$

In addition, two eigenvectors $\{[\mathbf{0}, \mathbf{A}]^T, [\partial_\mu \mathbf{A}, \mathbf{0}]^T\}$ for the double zero eigenvalue $\Omega = 0$ are symplectically orthogonal to other eigenvectors and $\langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle \neq 0$. The set of eigenvectors

$$\{[\mathbf{B}_m, \mathbf{C}_m]^T\}_{m=0}^\infty \oplus \{[\mathbf{B}_m, -\mathbf{C}_m]^T\}_{m=0}^\infty \oplus \{[\mathbf{0}, \mathbf{A}]^T, [\partial_\mu \mathbf{A}, \mathbf{0}]^T\} \tag{3.17}$$

is a basis in $l^2(\mathbb{N}, \mathbb{R}^2)$ which is orthogonal with respect to the symplectic projections (3.16).

Proof. All eigenvalues $\{\Omega_m\}_{m=0}^\infty$ are positive and simple for sufficiently small ε by Proposition 2. Since $L_\pm$ are self-adjoint in $l^2(\mathbb{N})$ and Ω_m is a real eigenvalue, then the eigenvector $[\mathbf{B}_m, \mathbf{C}_m]^T$ of the linear problem (3.4) can be chosen to be real-valued. The orthogonality relations (3.16) follow by direct computations from the linear problem (3.4) for distinct eigenvalues $\Omega_{m'} \neq \Omega_m$ for all $m' \neq m$. Values of $\langle \mathbf{B}_m, \mathbf{C}_m \rangle$ are proportional to the values of $\langle \mathbf{B}_m, L_+ \mathbf{B}_m \rangle$ for $\Omega_m \neq 0$ and they are non-zero for sufficiently small ε by Corollary 1. The value of $\langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle = \frac{1}{2} \frac{d}{d\mu} Q(\mathbf{A})$ is non-zero for sufficiently small ε by Proposition 1. By the proof of Proposition 2 and Remark 3, the eigenvectors of the set (3.17) are represented for sufficiently small ε by the standard basis $\{\mathbf{e}_m\}_{m=0}^\infty \oplus \{\mathbf{e}_m\}_{m=0}^\infty$ perturbed by a bounded perturbation in $l^2(\mathbb{N})$ of the order $O(\varepsilon^2)$. Also

$$\Omega_m = m + O(\varepsilon^2), \quad \langle \mathbf{B}_m, \mathbf{C}_m \rangle = \frac{\langle \mathbf{B}_m, L_+ \mathbf{B}_m \rangle}{\Omega_m} = 1 + O(m^{-1}\varepsilon^2), \quad \forall m = 2, 3, ..., \tag{3.18}$$

for sufficiently small ε, uniformly in m. Since no other eigenvalues exist, the set of linearly independent eigenvectors (3.17) is complete in $l^2(\mathbb{N}, \mathbb{R}^2)$. According to the Banach Theorem for non-self-adjoint operators, the set is a basis if and only if the spectral projections are bounded from below by a non-zero constant in the limit $m \to \infty$, which follows from the uniform asymptotic distribution (3.18). Therefore, the set (3.17) is a basis in $l^2(\mathbb{N}, \mathbb{R}^2)$. $\qquad \square$

Lemma 5 *Fix $\varepsilon \neq 0$ sufficiently small. Simple positive eigenvalues of the set $\{\Omega_m\}_{m=0}^\infty$ lie in the intervals*

$$\sigma > 0: \quad \Omega_0 = 1 \ \text{ and } \ m - C_m^- \varepsilon^2 < \Omega_m < m, \quad \forall m \in \mathbb{N} \tag{3.19}$$

and

$$\sigma < 0: \quad \Omega_0 = 1 \ \text{ and } \ m < \Omega_m < m + C_m^+ \varepsilon^2, \quad \forall m \in \mathbb{N} \tag{3.20}$$

for some ε-independent constants $C_m^\pm > 0$.

Proof. The eigenvalue $\Omega_0 = 1$ persists for any $\varepsilon \in \mathbb{R}$ due to equivalence of the linear eigenvalue problems (1.11) and (3.4) for $\phi \in \mathcal{H}_1(\mathbb{R})$ and $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ and the existence of the exact solution (1.12) of the linear eigenvalue problem (1.11). The corresponding eigenvector $(\mathbf{B}, \mathbf{C})$ of the linear eigenvalue problem (3.4) is found from the series representation

$$\phi'(x) = \sum_{n=0}^\infty B_n \phi_n(x), \qquad -x\phi(x) = \sum_{n=0}^\infty C_n \phi_n(x).$$

The eigenvalue Ω_1 satisfies the bounds (3.19)–(3.20) due to the explicit bound (3.7). We use the bound (3.8) to prove the bounds (3.19)–(3.20) for eigenvalues Ω_m for all $m = 2, 3, \ldots$. The values of $K_{1,1,1,1} - 2K_{m+1,1,1,m+1}$ are positive for the first values of $m = 3, 4, \ldots$ as follows from Table I, e.g.

$$K_{1,1,1,1} - 2K_{3,1,1,3} = \frac{1}{16\sqrt{2\pi}}, \quad K_{1,1,1,1} - 2K_{4,1,1,4} = \frac{21}{128\sqrt{2\pi}}, \quad K_{1,1,1,1} - 2K_{5,1,1,5} = \frac{59}{256\sqrt{2\pi}}.$$

Note that the positive numerical values are monotonically increasing. According to the main theorem in [7], the sequence $\{\|\phi_n\|_{L^4}\}_{n\in\mathbb{N}}$ is monotonically decreasing to zero with the bound (2.8). Since $K_{m+1,1,1,m+1} \leq \|\phi_1\|_{L^4}^2 \|\phi_{m+1}\|_{L^4}^2$, then

$$K_{1,1,1,1} - 2K_{m+1,1,1,m+1} \geq \|\phi_1\|_{L^4}^2 \left(\|\phi_1\|_{L^4}^2 - 2\|\phi_{m+1}\|_{L^4}^2 \right), \qquad \forall m = 2, 3, \ldots$$

Since $\|\phi_{m+1}\|_{L^4}^2$ decays monotonically to zero as $m \to \infty$, there exists M sufficiently large, such that the lower bound above is strictly positive for $m \geq M$. $\qquad\square$

4 Existence of periodic solutions

Let $(\mathbf{A}, \mu)$ be a real-valued root of the nonlinear vector field $\mathbf{F}(\mathbf{A}, \mu)$ such that $\mathbf{A} \in l_{1/2}^2(\mathbb{N})$. We use a decomposition $\mathbf{a}(t) = e^{-i\mu t} [\mathbf{A} + \mathbf{B}(t) + i\mathbf{C}(t)]$ with real-valued vectors $\mathbf{B}$ and $\mathbf{C}$ to rewrite the discrete dynamical system (2.5) in the form

$$\dot{\mathbf{B}} = L_-\mathbf{C} + \sigma\mathbf{N}_-(\mathbf{B}, \mathbf{C}), \qquad -\dot{\mathbf{C}} = L_+\mathbf{B} + \sigma\mathbf{N}_+(\mathbf{B}, \mathbf{C}), \tag{4.1}$$

where the operators $L_\pm$ are defined by (3.5) and the vector fields $\mathbf{N}_\pm(\mathbf{B}, \mathbf{C})$ contains quadratic and cubic terms with respect to $(\mathbf{B}, \mathbf{C})$. By Theorem 2, the initial-value problem for system (4.1) is globally well-posed and the solution set $(\mathbf{B}, \mathbf{C}) \in l_{1/2}^1(\mathbb{N}, \mathbb{R}^2)$ is equivalent to the solution set $(v, w) \in \mathcal{H}_1(\mathbb{R}, \mathbb{R}^2)$ of the PDE system (1.4). The discrete dynamical system (4.1) inherits the Hamiltonian function (1.9) in the form

$$\begin{aligned}
H &= \frac{1}{2}\langle \mathbf{B}, L_+\mathbf{B}\rangle + \frac{1}{2}\langle \mathbf{C}, L_-\mathbf{C}\rangle + \sigma \sum_{(n,n_1,n_2,n_3)} K_{n,n_1,n_2,n_3} A_{n_1}(B_{n_2}B_{n_3} + C_{n_2}C_{n_3})B_n \\
&\quad + \frac{\sigma}{4} \sum_{(n,n_1,n_2,n_3)} K_{n,n_1,n_2,n_3} \left(B_{n_1}B_{n_2}B_{n_3}B_n + 2B_{n_1}B_{n_2}C_{n_3}C_n + C_{n_1}C_{n_2}C_{n_3}C_n \right)
\end{aligned} \tag{4.2}$$

and the conserved quantity (1.10) in the form

$$Q = 2\langle \mathbf{A}, \mathbf{B}\rangle + \langle \mathbf{B}, \mathbf{B}\rangle + \langle \mathbf{C}, \mathbf{C}\rangle. \tag{4.3}$$

Using Lemma 4, we represent a solution $(\mathbf{B}, \mathbf{C})$ of the discrete system (4.1) by the series of eigenvectors (3.17) associated with the linear problem (3.4):

$$\begin{cases} \mathbf{B}(t) &= \sum_{m=0}^{\infty} b_m(t)\mathbf{B}_m + \sum_{m=0}^{\infty} \bar{b}_m(t)\mathbf{B}_m + \beta(t)\partial_\mu\mathbf{A}, \\ \mathbf{C}(t) &= i\sum_{m=0}^{\infty} b_m(t)\mathbf{C}_m - i\sum_{m=0}^{\infty} \bar{b}_m(t)\mathbf{C}_m + \gamma(t)\mathbf{A}, \end{cases} \tag{4.4}$$

where $b_0(t)$, $\mathbf{b}(t) = (b_1, b_2, \ldots)$ are complex-valued and $\beta(t)$, $\gamma(t)$ are real-valued. The linear part of system (4.1) becomes block-diagonal in the representation (4.4), yielding the evolution equations

$$\dot{b}_m - i\Omega_m b_m = \sigma N_m(b_0, \mathbf{b}, \beta, \gamma), \quad \forall m = 0, 1, 2, 3\ldots \tag{4.5}$$

and

$$\dot{\beta} = \sigma S_0(b_0, \mathbf{b}, \beta, \gamma), \quad \dot{\gamma} + \beta = \sigma S_1(b_0, \mathbf{b}, \beta, \gamma), \tag{4.6}$$

where

$$N_m(b_0, \mathbf{b}, \beta, \gamma) = \frac{\langle \mathbf{C}_m, \mathbf{N}_-(\mathbf{B}, \mathbf{C}) \rangle + i \langle \mathbf{B}_m, \mathbf{N}_+(\mathbf{B}, \mathbf{C}) \rangle}{2 \langle \mathbf{C}_m, \mathbf{B}_m \rangle}, \quad \forall m = 0, 1, 2, 3, \dots$$

and

$$S_0(b_0, \mathbf{b}, \beta, \gamma) = \frac{\langle \mathbf{A}, \mathbf{N}_-(\mathbf{B}, \mathbf{C}) \rangle}{\langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle}, \qquad S_1(b_0, \mathbf{b}, \beta, \gamma) = -\frac{\langle \partial_\mu \mathbf{A}, \mathbf{N}_+(\mathbf{B}, \mathbf{C}) \rangle}{\langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle}.$$

Using conservation of Q given by (4.3) and the decomposition (4.4), one can integrate the first equation of system (4.6) in the form

$$\beta = \frac{Q - \|\mathbf{B}\|_{l^2}^2 - \|\mathbf{C}\|_{l^2}^2}{2 \langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle}, \tag{4.7}$$

where Q is constant in time $t \in \mathbb{R}$. As a result, the second equation of system (4.6) is rewritten explicitly in the form

$$\dot{\gamma} = \frac{\|\mathbf{B}\|_{l^2}^2 + \|\mathbf{C}\|_{l^2}^2 - 2\sigma \langle \partial_\mu \mathbf{A}, \mathbf{N}_+(\mathbf{B}, \mathbf{C}) \rangle - Q}{2 \langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle}. \tag{4.8}$$

We are now ready to apply the method of Lyapunov–Schmidt reductions to the proof of Theorem 1.

Proof of Theorem 1: The vector space $(\mathbf{B}, \mathbf{C}) \in l_{1/2}^2(\mathbb{N}, \mathbb{R}^2)$ is equivalent to the vector space $\mathbf{b} \in l_{1/2}^2(\mathbb{N})$ because of the asymptotic distribution (3.18). For instance, one obtains that

$$\sum_{n=0}^{\infty} (1 + n)|B_n| \sim \langle \mathbf{B}, L_+ \mathbf{B} \rangle = 2 \sum_{m=0}^{\infty} \Omega_m \langle \mathbf{C}_m, \mathbf{B}_m \rangle |b_m|^2 + |\beta|^2 \langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle \sim \sum_{n \in \mathbb{N}} (1 + n)|b_n|^2.$$

We should work in the space of T-periodic functions $b_0(t)$, $\mathbf{b}(t) \in l_{1/2}^2(\mathbb{N})$, $\beta(t)$ and $\gamma(t)$ on $t \in \mathbb{R}$, where T is close to 2π. This period corresponds to the eigenvalue $\Omega_0 = 1$ which persists for any $\varepsilon \in \mathbb{R}$. By Lemma 5, all other eigenvalues of the linear problem (3.4) satisfy the non-resonance conditions $n \neq \Omega_m, \forall n, m \in \mathbb{N}$ for any fixed $\varepsilon \neq 0$ sufficiently small. As a result, we define periodic functions $\mathbf{b}(t)$, $\beta(t)$ and $\gamma(t)$ in terms of the periodic function $b_0(t)$, which solves a reduced evolution problem. Let δ be sufficiently small. We shall prove that there exist solutions of system (4.5), (4.7) and (4.8) which are T-periodic on $t \in \mathbb{R}$ satisfying the apriori bounds

$$|b_0(t)| \leq \varepsilon \delta C_0, \quad \|\mathbf{b}(t)\|_{l_{1/2}^2} \leq \varepsilon \delta^2 C_b, \quad |\beta(t)| \leq \varepsilon^2 \delta^2 C_\beta, \quad |\gamma(t) - \delta\alpha| \leq \varepsilon^2 \delta^2 C_\gamma, \quad \forall t \in \mathbb{R}, \ \forall \alpha \in \mathbb{R}, \tag{4.9}$$

for some (ε, δ)-independent constants $C_0, C_b, C_\beta, C_\gamma > 0$. If $b_0(t)$, $\mathbf{b}(t) \in l_{1/2}^2(\mathbb{N})$, $\beta(t)$ and $\gamma(t)$ are T-periodic functions on $t \in \mathbb{R}$ satisfying the bounds (4.9), then $(\mathbf{B}(t), \mathbf{C}(t)) \in l_{1/2}^2(\mathbb{N}, \mathbb{R}^2)$ is a T-periodic function on $t \in \mathbb{R}$ satisfying the bound

$$\|\mathbf{B}(t)\|_{l_{1/2}^2} + \|\mathbf{C}(t)\|_{l_{1/2}^2} \leq C\varepsilon\delta, \qquad \forall t \in \mathbb{R}, \ \forall \alpha \in \mathbb{R}, \tag{4.10}$$

for some (ε, δ)-independent constant $C > 0$. Here we recall the expansion (3.15) for $\mathbf{A}$, $\partial_\mu \mathbf{A}$ and the fact that $(\mathbf{B}_m, \mathbf{C}_m)^T$ are close to the unit vectors $\mathbf{e}_m$ for sufficiently small ε. Since $\mathbf{N}_\pm(\mathbf{B}, \mathbf{C})$ is cubic with respect $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, contains quadratic terms in $(\mathbf{B}, \mathbf{C})$, and maps $l_{1/2}^2(\mathbb{N}, \mathbb{R}^2)$ to $l_{-1/2}^2(\mathbb{N}, \mathbb{R}^2)$, we obtain the bound

$$\|\mathbf{N}_\pm(\mathbf{B}(t), \mathbf{C}(t))\|_{l_{-1/2}^2} \leq C_\pm \varepsilon^3 \delta^2, \qquad \forall t \in \mathbb{R}, \ \forall \alpha \in \mathbb{R}, \tag{4.11}$$

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for some (ε, δ)-independent constants $C_\pm > 0$. By the Implicit Function Theorem to the right-hand-side of equation (4.8), there exists a unique constant Q in the interval $|Q| \leq C_Q \varepsilon^2 \delta^2$ for some $C_Q > 0$, such that the periodic function in the right-hand-side of equation (4.8) has zero mean on $t \in \mathbb{R}$. In this case, there exists a periodic solution $\gamma(t) = \delta\alpha + \tilde{\gamma}(t)$ of the differential equation (4.8), where $\tilde{\gamma}(t)$ is a uniquely defined varying part and $\delta\alpha$ is an arbitrary mean part. The varying part $\tilde{\gamma}(t)$ satisfies the last bound in the list (4.9). The function $\beta(t)$ is uniquely defined by the explicit representation (4.7) and it hence satisfies the third bound in the list (4.9).

Consider now system (4.5) for $m \in \mathbb{N}$. Recall that $\Omega_m - m = O(\epsilon^2)$ for $m = 1, 2, \dots$ uniformly in $m \in \mathbb{N}$ for sufficiently small ε. By the Implicit Function Theorem, there exists a unique solution $\mathbf{b}(t) \in l^2_{1/2}(\mathbb{N})$ defined by the periodic function $b_0(t)$ and parameter $\alpha \in \mathbb{R}$ for sufficiently small δ provided that the distance $|\Omega_m - m| \neq 0$ and the frequency Ω of the periodic function $b_0(t)$ is such that $\Omega \to 1$ as $\delta \to 0$. By the bound (4.11) and the distribution $\Omega_m - m = O(\epsilon^2)$ for all $m \in \mathbb{N}$, the function $\mathbf{b}(t)$ satisfies the second bound in the list (4.9).

Eliminating the components $\mathbf{b}$, β and γ from equation (4.5) for $n = 0$, we obtain a reduced evolution problem for $b_0(t)$ in the form

$$\dot{b}_0 = ib_0 + R(b_0; \alpha), \tag{4.12}$$

where $R(b_0; \alpha)$ is a remainder term. Explicit computations of $N_0(b_0, \mathbf{b}, \beta, \gamma)$ show that

$$
\begin{aligned}
R(b_0; \alpha) &= \varepsilon \left[iK_1(\varepsilon)b_0^2 + iK_2(\varepsilon)\bar{b}_0^2 + iK_3(\varepsilon)|b_0|^2 + iK_4(\varepsilon)\delta^2\alpha^2 + K_5(\varepsilon)\delta\alpha\bar{b}_0 \right] \\
&\quad + O\left(|b_0|^3, \varepsilon^2\delta^2\alpha^2|b_0|, \varepsilon|b_0|\|\mathbf{b}\| \right),
\end{aligned}
\tag{4.13}
$$

where $K_{1,2,3,4,5}$ are real-valued constants which are bounded for sufficiently small ε. We are looking for T-periodic functions $b_0(t)$ which satisfy the evolution problem (4.12), have the leading order $b_0 \sim \varepsilon\delta e^{it+i\tau}$, where $\tau \in \mathbb{R}$ is arbitrary, and satisfy the first bound in the list (4.9). By the normal form analysis of the ODE (4.12) (see [14]), the quadratic terms in the remainder (4.13) do not change the frequency Ω of oscillations of the periodic function $b_0(t)$ at the leading order and therefore, $|\Omega - 1| \leq C_\Omega \varepsilon^2 \delta^2$ for some $C_\Omega > 0$. Since the Hamiltonian function (4.2) of system (4.1) is constant in time, it remains constant when the function $b_0(t)$ solves the reduced evolution problem (4.12) and the functions $\mathbf{b}(t)$, $\beta(t)$ and $\gamma(t)$ are constructed above. By the normal form analysis of reversible systems, there exists a two-dimensional invariant manifold of system (4.12) filled with periodic solutions of frequencies close to $\Omega = 1$ and parameterized by (δ, τ) in addition to parameter (ε, α). $\qquad\square$

Remark 4 Theorem 1 is reminiscent of an infinite-dimensional analogue of the Lyapunov Theorem for persistence of periodic orbits in Hamiltonian systems (see Chapter II, Section 45 on pp. 166–180 of [15]). However, due to the symmetries, a double zero eigenvalue occurs in the linear problem (3.4), and the proof of Theorem 1 is complicated by the analysis of the associated two-dimensional subspace. Similar theorems on persistence of k-dimensional tori in n-dimensional Hamiltonian system with $k-1$ additional conserved quantities were studied in the Nekhoroshev–Kuksin Theorems (see Theorem 2.3 on p. 4 of [4] and Theorem 1 on p. xiii of [13]).

Remark 5 The periodic solution of Theorem 1 has the smallest frequency in the focusing case $\sigma = -1$, since $\Omega_1 > 1$ in the bound (3.20) for sufficiently small ε. However, it is not the smallest frequency in the defocusing case $\sigma = 1$ since $\Omega_1 < 1$ in the bound (3.19). Persistence of the periodic solution for the smallest frequency Ω_1 can not be proved by a simple application of the Lyapunov Theorem since the bound (3.19) does not guarantee that the non-resonance conditions $n\Omega_1 \neq \Omega_m$ are satisfied for all $n \in \mathbb{N}$ and $m = 2, 3, \dots$. By the same reason, persistence of quasi-periodic oscillations on the tori with two and more frequencies $\{1, \Omega_1, \Omega_2, \dots\}$ can not be proved for small ε.

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Remark 6 Persistence of quasi-periodic oscillations on the tori along the Cantor set of parameter values was proved in Section 2.5 on p. 33 of [13] for the Hartree nonlinear functions and a perturbation of the parabolic potential $V(x) = \frac{1}{2}x^2$ by a localized potential $V_0(x)$. Our main result is stronger than this application of the main theorem in [13] since the periodic orbit is continuous with respect to parameters of the PDE problem rather than along the Cantor set of parameter values.

5   Numerical Results

We illustrate results of our manuscript with some numerical approximations. First, we identify the relevant branch of stationary solutions of the ODE (1.3). To do so, we use a fixed point method (Newton-Raphson iteration) to solve a discretized boundary-value problem. A centered-difference scheme is applied to the second-order derivatives with a typical spacing $\Delta x \in [0.025, 0.1]$. We are using a sufficiently large computational domain $x \in [-L, L]$ such that the boundary conditions do not affect the approximations within the considered numerical precision. The solutions $\phi(x)$ are obtained, using continuation, as a function of parameter μ. The continuation of the solution branches is performed from the linear limit $\mu = 1$, both for the cases $\sigma = 1$ and $\sigma = -1$. The results are shown in Figure 1, illustrating the quantity $Q = \|\phi\|_{L^2}^2$ as a function of μ. The numerical findings are also compared to the asymptotic result (3.2) of Proposition 1 indicating the good agreement of the latter prediction with our computational results for a fairly wide parametric window.

Once the corresponding numerical solution is identified (for a given σ and μ), the linear eigenvalue problem (1.11) is approximated numerically. We use again a discretization of differential operators on a finite grid, such that the spectral problem (1.11) becomes a matrix eigenvalue problem that is solved through standard numerical linear algebra routines. The relevant lowest eigenvalues are presented in Figure 2 and are also compared with the corresponding asymptotic results (3.7)–(3.8) of Proposition 2. The dashed lines show asymptotic results (A.7)–(A.8) of Appendix A derived in the limit $\mu \to \infty$ for $\sigma = 1$. Once again, the good agreement offers us a quantitative handle on the relevant eigenvalues.

Finally, we have also examined periodic oscillations of dark solitons in the numerical simulations of the GP equation (1.2). A typical example is shown in Figure 3 for $\sigma = 1$ and $\mu = 1.1$ for the initial condition $u(x, 0) = \phi(x) + \delta\phi'(x)$ with $\delta = 10^{-3}$. The top left panel shows the space-time contour plot of $|u(x, t)|^2$, clearly highlighting that this is a small (imperceptible, at the scale of this panel) perturbation of a stable stationary solution $\phi(x)$. The bottom left panel shows the space-time contour plot of $|u(x, t)|^2 - \phi^2(x)$, emphasizing the time-periodic oscillations of the perturbation to the stationary solution. The periodic oscillations are also visible on the top right panel where $|u(x_0, t)|^2$ is plotted versus t for $x_0 = 2$. Finally, the bottom right panel illustrates the Fourier transform of the time series of $|u(x_0, t)|^2$ (normalized to its maximum). It shows a high peak of the frequency spectrum near the value $\Omega = 1$, in agreement with the results of the main Theorem 1.

A   Asymptotic distribution of eigenvalues

Let us consider the case $\sigma = 1$, when the solution $\phi(x)$ of the ODE (1.3) bifurcates to the interval $\mu > 1$ (see Proposition 1 and Figure 1). We are interested in the distribution of eigenvalues of the
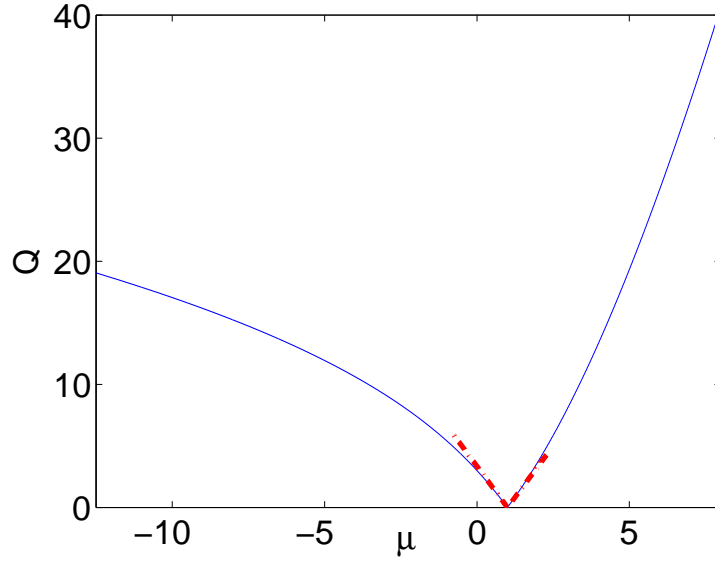
Figure 1: Branches of dark solitons versus μ both for the case of $\sigma = -1$ (when $\mu < 1$) and $\sigma = 1$ (when $\mu > 1$). The numerically obtained solution is shown by solid line and the asymptotic solution (3.2) is shown by dash-dotted line.
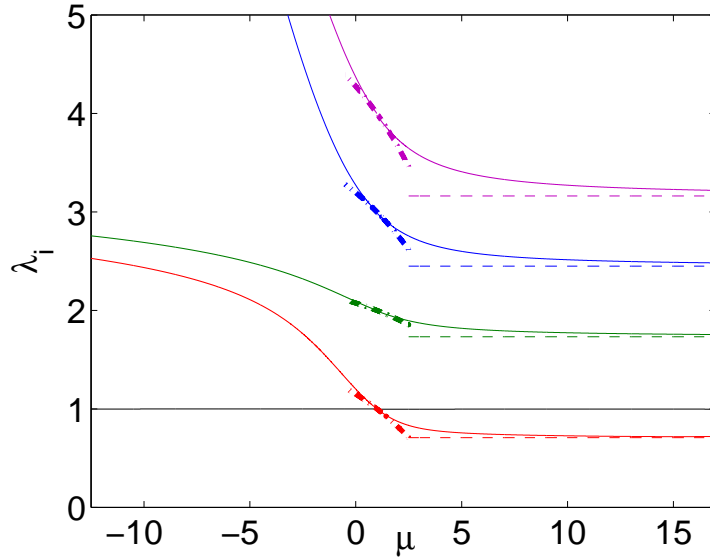

Figure 2: Smallest purely imaginary eigenvalues of the linear eigenvalue problem (1.11) versus μ. The numerically obtained eigenvalues are shown by solid lines, the asymptotic results (3.7)-(3.8) are shown by dash-dotted lines, and the asymptotic results (A.7)–(A.8) are shown by dashed lines.
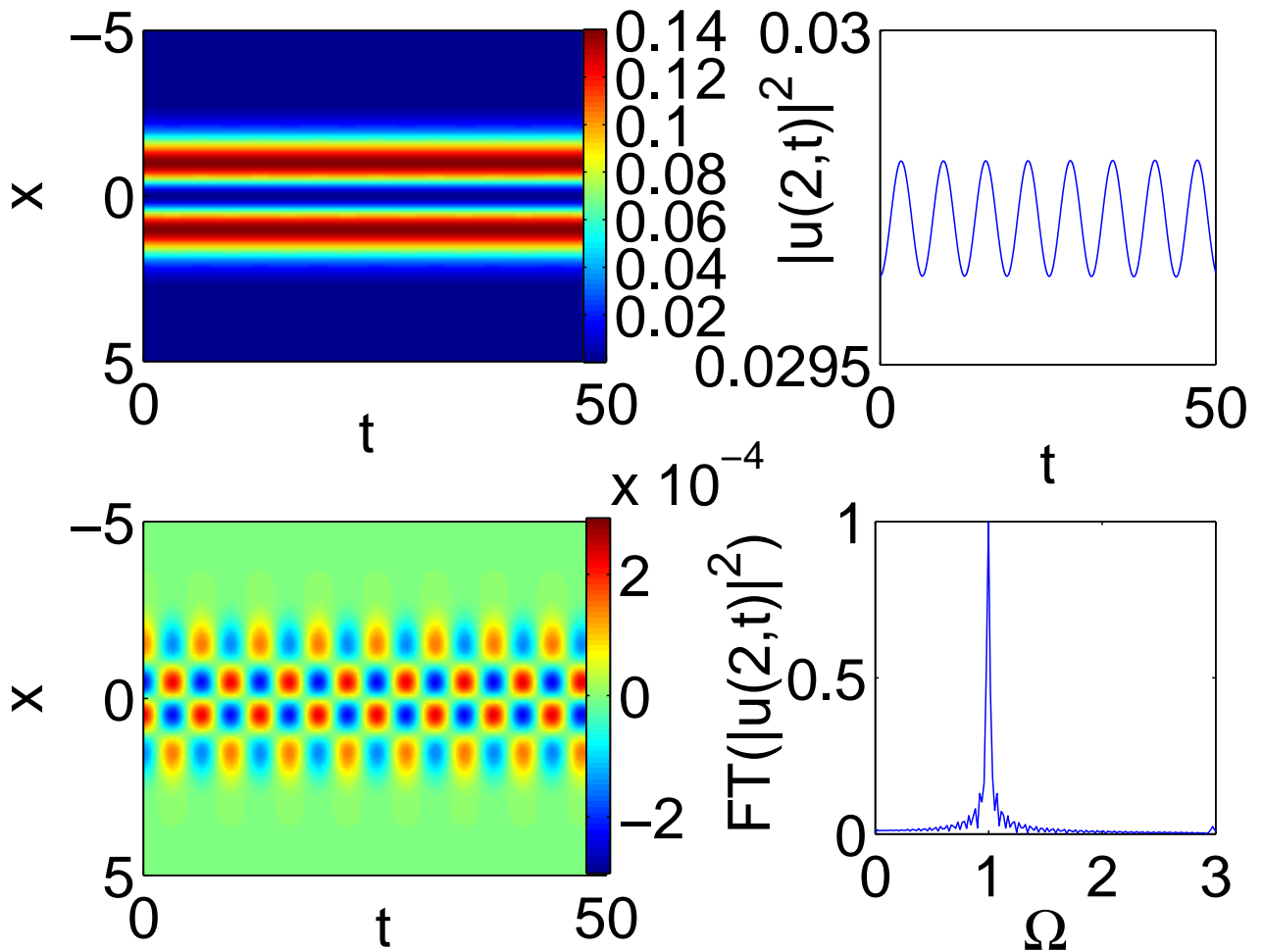
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Figure 3: A typical example of the robust time-periodic solution of the Gross-Pitaevskii equation (1.2) for $\sigma = 1$, $\mu = 1.1$ and $u(x,0) = \phi(x) + \delta\phi'(x)$ with $\delta = 10^{-3}$. The top left panel shows the space-time contour plot of $|u(x,t)|^2$, the bottom left panel shows the space-time contour of $|u(x,t)|^2 - \phi^2(x)$. The top right panel shows the time evolution of $|u(x_0,t)|^2$ with $x_0 = 2$, while the bottom right panel shows the Fourier transform of the time series of $|u(x_0,t)|^2$, featuring a peak at $\Omega \approx 1$.

linear problem (1.11) as $\mu \to \infty$, assuming that the solution $\phi(x)$ persists in this limit. It follows from the scaling transformation below equation (1.2) that the limit $\mu \to \infty$ of the normalized equation (1.2) corresponds to the limit $\epsilon \to 0$ in the original GP equation (1.1). We shall replace $\mu + \frac{1}{2} = \tilde{\mu}$ and drop tilde notations for the sake of simplicity. We report here formal results based on asymptotic methods. Rigorous justification of these results is beyond the scope of our work.

Denote the ground state of the ODE (1.3) by $\phi_0(x)$ such that $\phi_0(x)$ is even and positive on $x \in \mathbb{R}$ and it decays to zero as $|x| \to \infty$ sufficiently fast. Using the substitution $\phi_0(x) = \sqrt{\mu q(\xi)}$ and $\xi = \frac{x}{\sqrt{2\mu}}$, we obtain an equation for $q(\xi)$,

$$q = 1 - \xi^2 + \frac{1}{4\mu^2 \sqrt{q}} \frac{d^2}{d\xi^2} \sqrt{q}, \qquad \forall \xi \in \mathbb{R}, \tag{A.1}$$

which is solvable with the nonlinear WKB series [12]. The main result of the formal WKB theory is that there exists a classical solution $q_\mu(\xi)$ of the ODE (A.1) for sufficiently large $\mu > 1$ such that

$$\lim_{\mu \to \infty} q_\mu(x) = \begin{cases} 1 - \xi^2, & \forall |\xi| \leq 1 \\ 0, & \forall |\xi| > 1 \end{cases} \tag{A.2}$$

The linear problem (1.11) associated with the ground state $\phi_0(x)$ for $\sigma = 1$ and $\mu + \frac{1}{2} \to \mu$ can be written in variables $v = V(\xi)$, $w = W(\xi)$ and $\lambda = \mu\Lambda$ for sufficiently large $\mu > 1$. In new variables, it takes the form

$$L_+ V = -\Lambda W, \qquad L_- W = \Lambda V, \tag{A.3}$$

where

$$L_+ = 3q(\xi) - 1 + \xi^2 - \frac{1}{4\mu^2} \frac{d^2}{d\xi^2}, \quad L_- = q(\xi) - 1 + \xi^2 - \frac{1}{4\mu^2} \frac{d^2}{d\xi^2}. \tag{A.4}$$

Eliminating $V(x)$, we close the linear problem (A.3) at the fourth-order ODE

$$L_+ L_- W = \Gamma W, \qquad \Gamma = -\Lambda^2. \tag{A.5}$$

By using the WKB theory (A.2), we consider the auxiliary eigenvalue problem

$$\frac{1}{16\mu^4} W^{(\text{iv})} - \frac{(1 - \xi^2)}{2\mu^2} W'' = \Gamma W(\xi), \qquad \forall \xi \in [-1, 1], \tag{A.6}$$

for $W \in L^2([-1, 1])$. The entire spectrum of the problem (A.6) is defined by a set of polynomial solutions $W = P_m(\xi) = \xi^m + \alpha_{m,m-2}\xi^{m-2} + ... + \alpha_{m,k}\xi^k$, $\forall m \in \mathbb{N}$, where $k = 1$ if m is odd and $k = 0$ if m is even. The balance of the largest term in the ODE (A.6) shows that the eigenvalue $\Gamma = \Gamma_m$ is found explicitly as $\Gamma_m = \frac{m(m-1)}{2\mu^2}$, while all coefficients $\{\alpha_{m,m-2k}\}_{k=1}^{[m/2]}$ are uniquely defined. Converting the values of Γ to the values of λ, we have found that the linear problem (1.11) associated with the ground state $\phi_0(x)$ has a set of simple purely imaginary and symmetric eigenvalue pairs $\{\pm i\Omega_m\}_{m \in \mathbb{N}}$, such that

$$\lim_{\mu \to \infty} \Omega_m = \frac{\sqrt{m(m+1)}}{\sqrt{2}}, \qquad \forall m \in \mathbb{N}, \tag{A.7}$$

in addition to the double zero eigenvalue $\lambda = 0$.

Finally, the dark soliton $\phi(x)$ of the ODE (1.3) is obtained asymptotically from the ground state $\phi_0(x)$ by the factorization $\phi(x) = \phi_0(x)\psi(x)$, where $\psi(x)$ is odd on $x \in \mathbb{R}$, positive on $x \in \mathbb{R}_+$ and may approach to the constant values as $|x| \to \infty$ [16]. Using this factorization and the formal asymptotic

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analysis, it was shown in [16] that the spectrum of the linear problem (1.11) associated with the dark soliton $\phi(x)$ admits a pair of simple purely imaginary eigenvalues $\pm i\Omega_0$, such that

$$\lim_{\mu \to \infty} \Omega_0 = \frac{1}{\sqrt{2}}. \tag{A.8}$$

Although the analysis of [16] was directed to the original GP equation (1.1) in the limit of small ϵ and the eigenvalue pair was found to be $\tilde{\lambda} \to \pm i\epsilon$, the scaling transformation to the normalized GP equation (1.2) implies that $\lambda = \frac{\tilde{\lambda}}{2^{1/2}\epsilon} \to \pm \frac{i}{\sqrt{2}}$.

Numerical computations (see Figure 2) suggests that the entire spectrum of the linear problem (1.11) associated with the dark soliton $\phi(x)$ is a superposition between an infinite set of eigenvalues (A.7) of the linear problem (1.11) associated with the ground state $\phi_0(x)$ and the additional pair of eigenvalues (A.8).

Note that the linear eigenmode corresponding to the smallest eigenvalue $\Omega_0 = \frac{1}{\sqrt{2}}$ may not result in the periodic solution of the nonlinear PDE system (1.4) because the non-resonance condition $n \neq \sqrt{m(m+1)}$ for all $n, m \in \mathbb{N}$ is violated in the limit $n, m \to \infty$. Similarly, the linear eigenmode corresponding to the second eigenvalue $\Omega_1 = 1$ may not result in the periodic solution of the PDE system (1.4) because the non-resonance condition $n \neq \frac{\sqrt{m(m+1)}}{\sqrt{2}}$ for all $n, m = 2, 3, ...$ is violated at least for $n = 6$ and $m = 8$. In both cases, the Lyapunov Theorem for persistence of periodic orbit in Hamiltonian dynamical systems can not be applied [15].

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