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Two-loop analysis of axial vector current propagators in chiral perturbation theory

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(a) (b)

Fig. 1

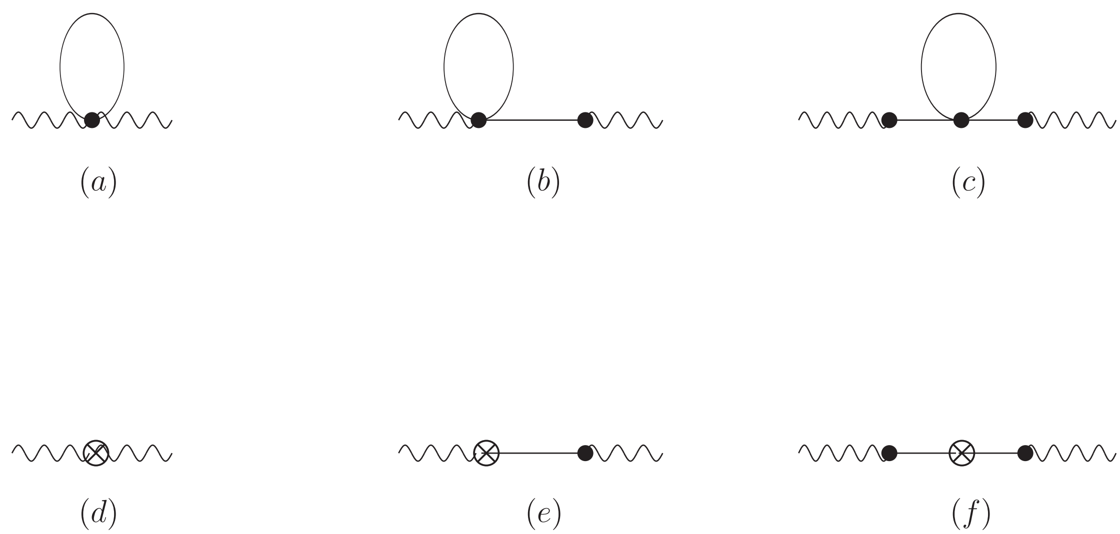
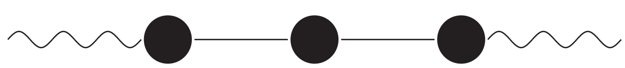


Fig. 2

(*a*)



(*b*)

Fig. 3

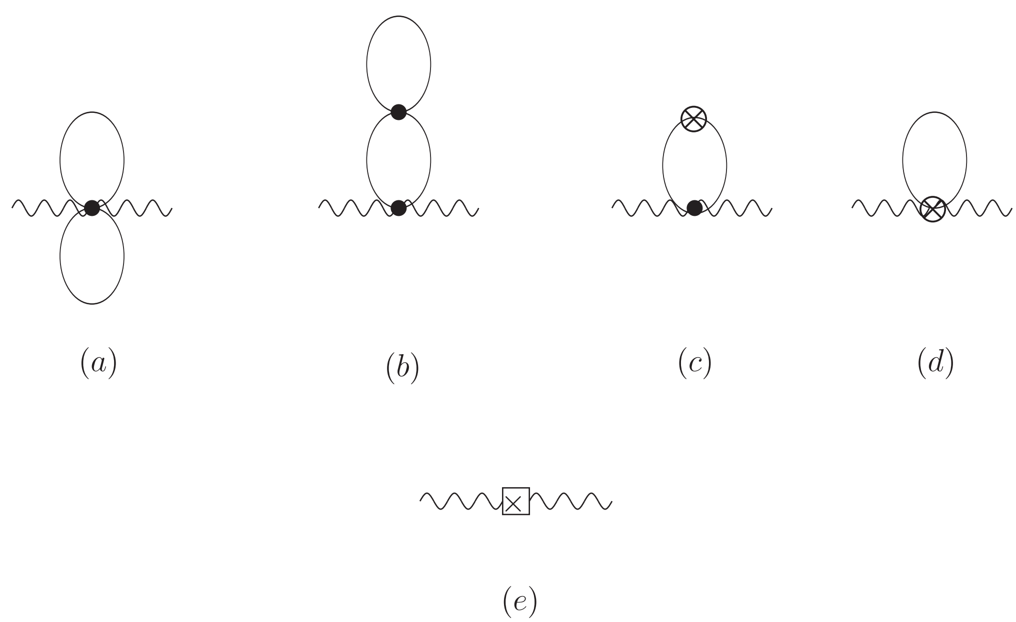


Fig. 4

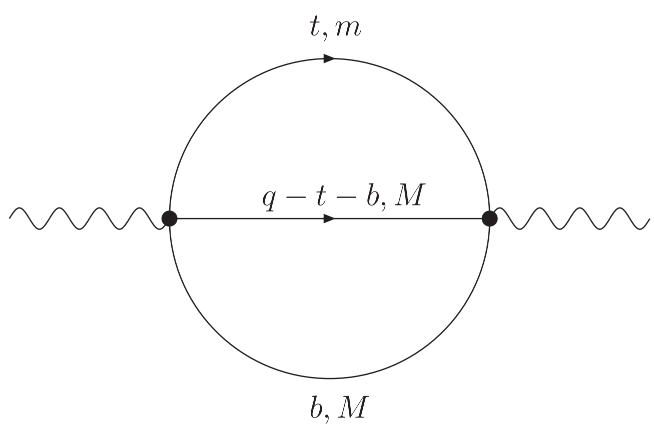


Fig. 5

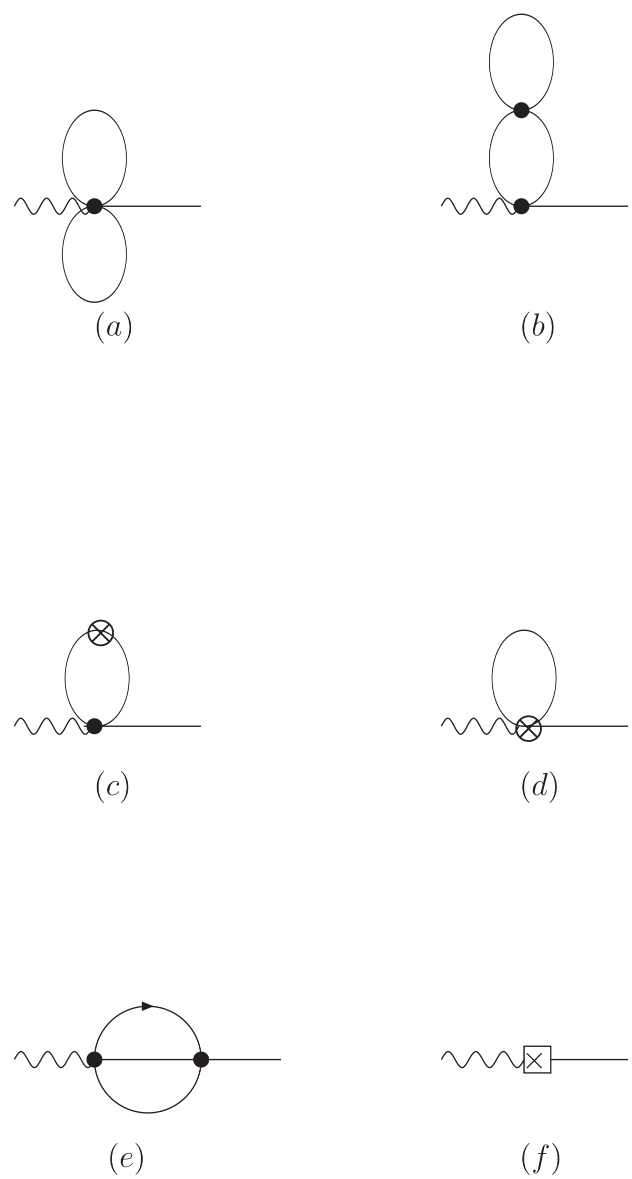


Fig. 6

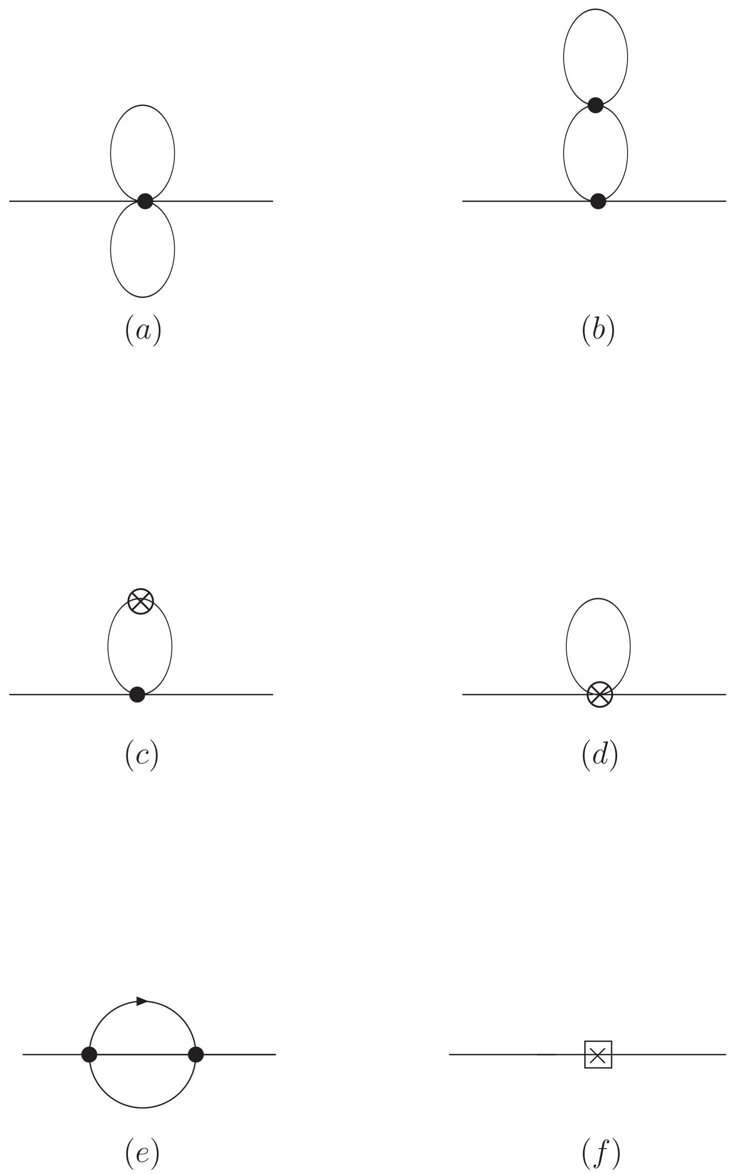


Fig. 7

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Two-loop Analysis of Axialvector Current Propagators in Chiral Perturbation Theory

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Abstract

We perform a calculation of the isospin and hypercharge axialvector current propagators ($\Delta^{\mu\nu}_{\rm A3}(q)$ and $\Delta^{\mu\nu}_{\rm A8}(q)$) to two loops in $SU(3)\times SU(3)$ chiral perturbation theory. A large number of $\mathcal{O}(p^6)$ divergent counterterms are fixed, and complete two-loop renormalized expressions for the pion and eta masses and decay constants are given. The calculated isospin and hypercharge axialvector polarization functions are used as input in new chiral sum rules, valid to second order in the light quark masses. Some phenomenological implications of these sum rules are considered.

I. INTRODUCTION

Although low energy quantum chromodynamics remains analytically intractable, the calculational scheme of chiral perturbation theory [1] (ChPT) has led to many valuable contributions. Following the seminal papers of Gasser and Leutwyler [2,3], numerous studies conducted in the following decade convincingly demonstrated the power of ChPT. The state of the art up to 1994 is summarized in several reviews *e.g.* [4,5] (see also [6,7]). The exploration of ChPT continues to this day, and two-loop studies represent an active frontier area of research. These include processes which have leading contributions in the chiral expansion at order p^4 [8–13] or even p^6 [14], as well as systems for which precision tests will soon be available, *e.g.* the low-energy behaviour of $\pi\pi$ scattering [15–17]. While the case of $SU(2)\times SU(2)$ ChPT to two-loop order has been relatively well explored (in particular see [17]), works in $SU(3)\times SU(3)$ ChPT are still few in number [9,11,13,18,19].

Recently, we performed a calculation of the isospin and hypercharge vector current propagators ($\Delta^{\mu\nu}_{\rm V3}(q)$ and $\Delta^{\mu\nu}_{\rm V8}(q)$) to two-loop order in $SU(3)\times SU(3)$ chiral perturbation theory [9]. A partial motivation for working in the three-flavour sector stems from its inherently richer phenomenology. In particular, it becomes possible to derive new chiral sum rules which explicitly probe the $SU(3)$-breaking sector. With the aid of improved experimental information on spectral functions with strangeness content, it should become possible to evaluate and test these sum rules.

We have completed this program of calculation by determining the corresponding isospin and hypercharge axialvector current propagators ($\Delta^{\mu\nu}_{\rm A3}(q)$ and $\Delta^{\mu\nu}_{\rm A8}(q)$). Determination of axialvector propagators is much more technically demanding than for the vector propagators, but at the same time yields an extended set of results, among which are:

1. a large number of constraints on the set of $\mathcal{O}(p^6)$ counterterms,
2. predictions for threshold behaviour of the 3π, $\bar{K}K\pi$, $\bar{K}K\pi$, $\eta\pi\pi$, *etc* axialvector spectral functions,
3. an extensive analysis of the so-called 'sunset' diagrams,
4. new axialvector spectral function sum rules,
5. a new contribution to the Das-Mathur-Okubo sum rule [20], and
6. a complete two-loop renormalization of the masses and decay constants of the pion and eta mesons. This final item places the axialvector problem at the heart of two-loop studies in $SU(3) \times SU(3)$ ChPT.

We begin the presentation in Sect. II by presenting basic definitions and describing the calculational approach. To illustrate the procedure, we summarize the results for the tree-level and one-loop sectors. Our calculation of the two-loop amplitudes and the corresponding enumeration of the $\mathcal{O}(p^6)$ counterterms is given in Sect. III. The construction of a proper renormalization procedure forms the subject of Sect. IV. It provides the framework for the removal of divergences, described in Sect. V, and leads to finite renormalized expressions for the meson masses, decay constants, and polarization functions. We give explicit expressions for the isospin polarization functions in Sect. VI and continue the presentation of finite results in Appendix B. Sect. VII deals with the determination of spectral functions. The subject of chiral sum rules is discussed in Sect. VIII and Sect. IX contains our conclusions. Various technical details regarding sunset integrals are presented in Appendix A and as mentioned, our final expressions for meson masses, decay constants and the hypercharge polarization amplitudes are collected in Appendix B. In addition, at several points in the paper we compare results as expressed in the '$\overline{\lambda}$-subtraction' renormalization used here with a variant of the $\overline{MS}$ scheme.

II. TREE-LEVEL AND ONE-LOOP ANALYSES

The one-loop chiral analysis of the isospin axialvector-current propagator was first carried out by Gasser and Leutwyler who used the background-field formalism and worked in an $SU(2)$ basis of fields [2]. We shall devote this section to a re-calculation of the isospin axialvector propagator through one-loop order, but now done within the context of a Feynman diagram calculation and using an $SU(3)$ basis of fields.

A. Basic Definitions and Calculational Procedure

Our normalization for the $SU(3)$ octet of axialvector currents is standard,

$$A_k^\mu = \bar{q}\frac{\lambda_k}{2}\gamma^\mu\gamma_5 q \qquad (k=1,\ldots,8) \quad . \tag{1}$$

In this paper, we shall deal with the axialvector current propagators

$$\Delta^{\mu\nu}_{\mathrm{A}a}(q) \equiv i\int d^4x\; e^{iq\cdot x}\;\langle 0|T\left(A_a^\mu(x)A_a^\nu(0)\right)|0\rangle \qquad (a=3,8 \text{ not summed}) \quad , \tag{2}$$

having the spectral content

$$\frac{1}{\pi}\mathcal{I}m\;\Delta^{\mu\nu}_{\mathrm{A}a}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu})\rho^{(1)}_{\mathrm{A}a}(q^2) + q^\mu q^\nu \rho^{(0)}_{\mathrm{A}a}(q^2) \;, \tag{3}$$

where $\rho^{(1)}_{\mathrm{A}a}$ and $\rho^{(0)}_{\mathrm{A}a}$ are the spin-one and spin-zero spectral functions. This motivates the following tensorial decomposition usually adopted in the literature,

$$\Delta^{\mu\nu}_{\mathrm{A}a}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{(1)}_{\mathrm{A}a}(q^2) + q^\mu q^\nu \Pi^{(0)}_{\mathrm{A}a}(q^2) \quad , \tag{4}$$

where $\Pi^{(1)}_{\mathrm{A}a}$ and $\Pi^{(0)}_{\mathrm{A}a}$ are the spin-one and spin-zero polarization functions. The low-energy behaviour of the spin-zero spectral function $\rho^{(0)}_{\mathrm{A}a}$ is dominated by the pole contribution associated with propagation of a Goldstone mode,

$$\rho^{(0)}_{\mathrm{A}a}(s) \equiv \; F_a^2\;\delta(s-M_a^2) + \bar{\rho}^{(0)}_{\mathrm{A}a}(s) \quad . \tag{5}$$

In the chiral limit, one has $M_a \to 0$ and $\bar{\rho}^{(0)}_{\mathrm{A}a}(s) \to 0$ as well.

The lowest-order chiral lagrangian $\mathcal{L}^{(2)}$ is given by

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4}\,\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{F_0^2}{4}\,\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right) \;, \tag{6}$$

where F_0 is the pseudoscalar meson decay constant to lowest order and $\chi = 2B_0\mathbf{m}$ is proportional to the quark mass matrix $\mathbf{m} = \mathrm{diagonal}\,(\hat{m},\hat{m},m_s)$ with

$$B_0 = -\frac{1}{F_0^2}\langle\bar{q}q\rangle \quad . \tag{7}$$

Note that we work in the isospin symmetric limit of $m_u = m_d \equiv \hat{m}$ and that to lowest order we may use the Gell Mann-Okubo relation,

$$m_\eta^2 = \frac{1}{3}\left(4m_K^2 - m_\pi^2\right) \quad . \tag{8}$$

The field variable U is defined in terms of the octet of pseudoscalar meson fields $\{\phi_k\}$,

$$U \equiv \exp(i\lambda_k\cdot\phi_k/F_0) \;, \tag{9}$$

and we construct the covariant derivative $D_\mu U$ via external axialvector sources a_μ,[1]

$$D_\mu U \equiv \partial_\mu U + i a_\mu U + i U a_\mu \quad . \tag{10}$$

The axialvector source a_μ has a component a_μ^k for each of the $SU(3)$ flavours,

$$a_\mu \equiv \frac{1}{2}\lambda_k \ a_\mu^k \quad . \tag{11}$$

Adopting the approach carried out in Ref. [9], we make use of external axialvector sources to determine the axialvector propagators. The procedure is simply to compute the $\mathcal{S}$-matrix element connecting initial and final states of an axialvector source. Analogous to the analysis of Ref. [9], the invariant amplitude is then guaranteed to be the axialvector-current propagator, say of flavour 'a',

$$\langle a_a(q', \lambda')|\mathcal{S} - 1|a_a(q, \lambda)\rangle = i(2\pi)^4 \delta^{(4)}(q' - q) \ \epsilon_\mu^\dagger(q', \lambda')\Delta_{\mathrm{A}a}^{\mu\nu}(q)\epsilon_\nu(q, \lambda) \ . \tag{12}$$

We shall typically use the invariant amplitude symbol $\mathcal{M}_{\mu\nu a}$ to denote various individual contributions (tree, tadpole, counterterm, 1PI, *etc*) to the full propagator.

B. Tree-level Analysis

For definiteness, the analysis in the remainder of this section will refer to the isospin flavour. Given the lagrangian of Eq. (6), it is straightforward to determine the lowest-order propagator contributions,

$$\mathcal{M}_{\mu\nu 3}^{(\mathrm{tree})} = F_0^2 g_{\mu\nu} - \frac{F_0^2}{q^2 - m_\pi^2 + i\epsilon} q_\mu q_\nu \quad . \tag{13}$$

The two terms represent respectively contributions from a contact interaction (Fig. 1(a)) and a pion-pole term (Fig. 1(b)). Although $\mathcal{M}_{\mu\nu 3}^{(\mathrm{tree})}$ has an exceedingly simple form, it is nonetheless worthwhile to briefly point out two of its features. First, as follows from unitarity there is an imaginary part corresponding to the pion single-particle intermediate state,

$$\mathcal{I}m \ \mathcal{M}_{\mu\nu 3}^{(\mathrm{tree})} = \pi F_0^2 \delta(q^2 - m_\pi^2) q_\mu q_\nu \quad . \tag{14}$$

However, there is also a non-pole contribution to $\mathcal{M}_{\mu\nu 3}^{(\mathrm{tree})}$. Its presence is needed to ensure the proper behaviour in the chiral limit ($\partial_\mu A_3^\mu = 0$), where $\mathcal{R}e \ \mathcal{M}_{\mu\nu 3}^{(\mathrm{tree})}$ is required to obtain a purely spin-one (or 'transverse') form. This is indeed the case, as we find by taking $m_\pi^2 \to 0$ in Eq. (13),

$$\mathcal{M}_{\mu\nu 3}^{(\mathrm{tree})}\Big|_{m_\pi = 0} = -\frac{F_0^2}{q^2} \ (q_\mu q_\nu - q^2 g_{\mu\nu}) \quad . \tag{15}$$

[1]Throughout this paper we adopt the phase employed in Ref. [21], which is opposite to that used in Ref. [3].

C. One-loop Analysis

At one-loop level, the axialvector-current propagators are determined from the lagrangian of Eq. (6) and correspond to the Feynman diagrams appearing in Fig. 2. The loop correction to the lowest-order contact amplitude appears in Fig. 2(a) and Figs. 2(b),(c) depict corrections (whose significance we shall consider shortly) to the pion-pole amplitude. We find for the one-loop ('tadpole') contributions to the isospin propagator,

$$\mathcal{M}_{\mu\nu 3}^{(\text{tadpole})} = -i\left(2A(m_\pi^2) + A(m_K^2)\right) g_{\mu\nu} + \frac{4i}{3}\frac{2A(m_\pi^2) + A(m_K^2)}{q^2 - m_\pi^2} q_\mu q_\nu$$
$$+ \frac{i}{6}\frac{A(m_\pi^2)(m_\pi^2 - 4q^2) + 2A(m_K^2)(m_K^2 - q^2) + A(m_\eta^2)m_\pi^2}{(q^2 - m_\pi^2)^2} q_\mu q_\nu \ , \qquad (16)$$

where all masses occurring in pole denominators are understood to have infinitesimal negative imaginary parts. In the above, A is the scalar integral

$$A(m^2) \equiv \int d\tilde{k}\ \frac{1}{k^2 - m^2} \quad , \qquad (17)$$

where $d\tilde{k} \equiv d^dk/(2\pi)^d$. Hereafter, any integration measure accompanied by a super-tilde will have a similar meaning. The evaluation of A is standard, and we have

$$A(m^2) = \frac{-i}{(4\pi)^{d/2}}\frac{\mu^{4-d}}{\mu^{4-d}}\Gamma\left(1 - \frac{d}{2}\right)(m^2)^{(d-2)/2} \qquad (18)$$

$$= \mu^{d-4}\left[-2im^2\overline{\lambda} - \frac{im^2}{16\pi^2}\log\left(\frac{m^2}{\mu^2}\right) + \ldots\right] \quad , \qquad (19)$$

with

$$\lambda = \mu^{d-4}\overline{\lambda} = \frac{\mu^{d-4}}{16\pi^2}\left[\frac{1}{d-4} - \frac{1}{2}\left(\log 4\pi - \gamma + 1\right)\right] \quad . \qquad (20)$$

The quantity μ introduced in Eq. (18) is the mass scale which enters the calculation via the use of dimensional regularization. The μ^{d-4} prefactor in Eq. (19) ensures that $A(m^2)$ has the proper units in d-dimensions.

To deal with divergences arising from the loop corrections, one must include counterterm amplitudes. It suffices to employ the well-known list of counterterms $\{L_i\}$ $(i = 1, \ldots, 10)$ and $\{H_j\}$ $(j = 1, 2)$ appearing in the $\mathcal{O}(p^4)$ chiral lagrangian of Gasser and Leutwyler [3].[2] Analogous to Eq. (19) for the $A(m^2)$ integral, each $\mathcal{O}(p^4)$ counterterm is expressible as an expansion in $\overline{\lambda}$,

$$L_\ell = \mu^{(d-4)}\sum_{n=1}^{-\infty} L_\ell^{(n)}(\mu)\ \overline{\lambda}^n = \mu^{(d-4)}\left[\ L_\ell^{(1)}(\mu)\ \overline{\lambda}\ +\ L_\ell^{(0)}(\mu)\ + L_\ell^{(-1)}(\mu)\ \overline{\lambda}^{-1} + \ldots\ \right] \ , \quad (21)$$

[2]See also the discussion surrounding Eq. (26) of Ref. [9].

where the leading degree of singularity is seen to be linear. The counterterm diagrams involve a contact term (Fig. 2(d)) as well as contributions to the pion pole term (Figs. 2(e),(f)) and yield the following isospin counterterm amplitude,

$$\begin{aligned}
\mathcal{M}^{(\mathrm{CT})}_{\mu\nu 3} &= 2(L_{10}-2H_1)(q_\mu q_\nu - q^2 g_{\mu\nu}) + 8\left[(m_\pi^2+2m_K^2)L_4 + m_\pi^2 L_5\right] g_{\mu\nu} \\
&-16\frac{(m_\pi^2+2m_K^2)L_4 + m_\pi^2 L_5}{q^2-m_\pi^2} q_\mu q_\nu \\
&+8\frac{q^2\left((m_\pi^2+2m_K^2)L_4 + m_\pi^2 L_5\right) - 2m_\pi^2\left((m_\pi^2+2m_K^2)L_6 + m_\pi^2 L_8\right)}{(q^2-m_\pi^2)^2} q_\mu q_\nu \ .
\end{aligned} \tag{22}$$

Results through One-loop Order

Our complete expression for the isospin axialvector current propagator through one-loop order is given by the sum of Eqs. (13),(16),(22),

$$\Delta_{\mathrm{A}\mu\nu 3} = \mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3} + \mathcal{M}^{(\mathrm{tadpole})}_{\mu\nu 3} + \mathcal{M}^{(\mathrm{CT})}_{\mu\nu 3} \ . \tag{23}$$

The resulting expression is complicated and seems to lack immediate physical interpretation because we have not yet accounted for renormalizations of the pion's mass and decay constant. A detailed account of the renormalization procedure is deferred to Sect. IV. However, we note here that the renormalized masses and decay constants have the expansions

$$\begin{aligned}
F^2 &= F^{2(0)} + F^{2(2)} + F^{2(4)} + \ldots \\
M^2 &= M^{2(2)} + M^{2(4)} + M^{2(6)} + \ldots
\end{aligned} \tag{24}$$

where we have temporarily suppressed flavour notation and the superscript indices $\{(i)\}$ denote quantities evaluated at chiral order $\{p^i\}$. To one loop [3], the explicit expressions are given by[3]

$$\begin{aligned}
F_\pi^2 &= F_0^2 + 8\left[(m_\pi^2+2m_K^2)L_4^{(0)} + m_\pi^2 L_5^{(0)}\right] - \frac{2m_\pi^2}{16\pi^2}\ln\frac{m_\pi^2}{\mu^2} - \frac{m_K^2}{16\pi^2}\ln\frac{m_K^2}{\mu^2} \\
M_\pi^2 &= m_\pi^2 + \frac{1}{F_0^2}\left[\frac{m_\pi^4}{32\pi^2}\ln\frac{m_\pi^2}{\mu^2} - \frac{m_\pi^2 m_\eta^2}{96\pi^2}\ln\frac{m_\eta^2}{\mu^2}\right. \\
&\qquad \left. -8m_\pi^2\left((m_\pi^2+2m_K^2)(L_4^{(0)}-2L_6^{(0)}) + m_\pi^2(L_5^{(0)}-2L_8^{(0)})\right)\right] \ .
\end{aligned} \tag{25}$$

Upon combining the information gathered in Eqs. (23),(25) as well as the divergent part of the one-loop functional given in Ref. [3], one finds through one-loop for the renormalized isospin propagator,

$$\Delta^{\mu\nu}_{\mathrm{A}3} = F_\pi^2 g^{\mu\nu} + 2(L_{10}^{(0)} - 2H_1^{(0)})(q^\mu q^\nu - q^2 g^{\mu\nu}) - \frac{F_\pi^2}{q^2-M_\pi^2} q^\mu q^\nu \ . \tag{26}$$

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[3] At one-loop order, our counterterms $\{L_i^{(0)}\}$ and $\{H_i^{(0)}\}$ are equivalent to the $\{L_i^r\}$ and $\{H_i^r\}$ of Ref. [3].

The hypercharge propagator $\Delta^{\mu\nu}_{\rm A8}$ can be obtained analogously. We comment that both the isospin and hypercharge amplitudes contain the regularization dependent constant $H_1^{(0)}$, and are therefore unphysical. A physically observable quantity is obtained from the difference,

$$\Delta^{\mu\nu}_{\rm A3} - \Delta^{\mu\nu}_{\rm A8} = \left(F_\pi^2 - F_\eta^2\right) g^{\mu\nu} - q^\mu q^\nu \left[\frac{F_\pi^2}{q^2 - M_\pi^2} - \frac{F_\eta^2}{q^2 - M_\eta^2}\right] \quad . \tag{27}$$

III. TWO-LOOP ANALYSIS

The general structure of propagator corrections is displayed in Fig. 3, which includes the one-particle irreducible (1PI) diagrams of Fig. 3(a) and the one-particle reducible (1PR) diagrams of Fig. 3(b). The latter consists of both vertex and self-energy corrections. Throughout this section, we continue to focus on the isospin sector when giving explicit expressions for two-loop amplitudes.

A. Two-loop Analysis: 1PI Graphs

First, we consider the 1PI diagrams of Fig. 4 and Fig. 5. The graphs of Figs. 4(a)–(c) are simple in the sense that they are either equal to or the negative of identical graphs in which external vector sources occur. As such, they can be read off from the work of Ref. [9]. This is only partly true of Fig. 4(d), and the graph of Fig. 5 (the so-called 'sunset graph') has no counterpart in the vector system. For convenience, we compile definitions and explicit representations for the sunset-related functions in App. A, leaving detailed derivation of these results for another setting [22].

The amplitudes of Figs. 4(a)–(b) are of the general form

$$\mathcal{M}_{\mu\nu 3}[4a, 4b] = \frac{g_{\mu\nu}}{F_0^2} \sum_{k,\ell} a_{k\ell}^{(a,b)} A(k)\ A(\ell) \quad , \tag{28}$$

where the $\{a_{k\ell}^{(a,b)}\}$ are numerical coefficients (in some cases dependent on dimension d) and the $\{A(k)\}$ are the integrals of Eq. (17). The sum over the indices k, ℓ simply reflects the need to perform flavour sums independently for each of the two loops. The amplitude of Fig. 4(c) can be expressed similarly,

$$\mathcal{M}_{\mu\nu 3}[4c] = \frac{g_{\mu\nu}}{F_0^2} \sum_{k,\ell} a_{k\ell}^{(c)} A(k)\ L_\ell \quad , \tag{29}$$

except now the coefficients$\{a_{k\ell}^{(c)}\}$ are proportional to squared meson masses and there are $\{L_\ell\}$ factors arising from the $\mathcal{O}(p^4)$ lagrangian.

Each of the above amplitudes will diverge for $d \to 4$, and we write

$$\mathcal{M}_{\mu\nu 3}[4a, 4b, 4c] = \frac{g_{\mu\nu}}{F_0^2} \left[\overline{a}_2^{(a,b,c)} \bar{\lambda}^2 + \overline{a}_1^{(a,b,c)} \bar{\lambda}^1 + \ldots\right] \quad , \tag{30}$$

where the singular quantity $\overline{\lambda}$ has been previously defined in Eq. (20), $\overline{a}_2^{(a,b,c)}$ and $\overline{a}_1^{(a,b,c)}$ are numerical quantities, and the ellipses refer to terms which are nonsingular at $d = 4$. As expected for two-loop amplitudes, the leading singularity goes as $\overline{\lambda}^2$.

For Fig. 4(d), the contribution from L_{10} is the negative of the vector case, but there are also many new terms,

$$\begin{aligned}
\mathcal{M}_{\mu\nu 3}[4d] &= -(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{4i}{F_0^2}(2A(\pi) + A(K))L_{10} \\
&+\frac{ig_{\mu\nu}}{F_0^2}\Big[A(\pi)\Big(m_\pi^2[(48 + \frac{32}{d})L_1 + (16 + \frac{64}{d})L_2 + (24 + \frac{16}{d})L_3] \\
&\qquad - 8(5m_\pi^2 + 4m_K^2)L_4 - 28m_\pi^2 L_5\Big) \\
&+A(K)\Big(m_K^2[64L_1 + \frac{64}{d}L_2 + (16 + \frac{16}{d})L_3] \\
&\qquad - 8(m_\pi^2 + 6m_K^2)L_4 - 8(m_\pi^2 + m_K^2)L_5\Big) \\
&+A(\eta)\Big(m_\eta^2[16L_1 + \frac{16}{d}L_2 + (\frac{8}{3} + \frac{16}{3d})L_3] - 8m_\eta^2 L_4 - \frac{4}{3}m_\pi^2 L_5\Big)\Big] \ .
\end{aligned} \tag{31}$$

Aside from the presence of a transverse component proportional to $(q_\mu q_\nu - g_{\mu\nu}q^2)$, it resembles the form of Fig. 4(c) and shares the degree of singularity shown in Eq. (30).

Finally, there is the sunset contribution of Fig. 5, which we write as

$$\begin{aligned}
\mathcal{M}_{\mu\nu 3}[5] &= \frac{4}{9}\mathcal{H}_{\mu\nu}(q^2, m_\pi^2, m_\pi^2) + \frac{1}{6}\mathcal{H}_{\mu\nu}(q^2, m_\eta^2, m_K^2) \\
&\quad + \frac{1}{18}\mathcal{H}_{\mu\nu}(q^2, m_\pi^2, m_K^2) + \mathcal{L}_{\mu\nu}(q^2, m_\pi^2, m_K^2) \ .
\end{aligned} \tag{32}$$

The quantities $\mathcal{H}_{\mu\nu}$ and $\mathcal{L}_{\mu\nu}$ are defined in App. A by Eqs. (A1),(A4). The flavour dependence of the individual sunset contributions can be read off from the arguments of the above functions. The defining representations for $\mathcal{H}_{\mu\nu}$ and $\mathcal{L}_{\mu\nu}$ (*cf* Eqs. (A1),(A4)) are given as multidimensional Feynman integrals. It takes considerable analysis to reduce these integrals to forms which can be compared directly with the other two-loop amplitudes, and one finds [22] that the sunset amplitudes share the singular behaviour of Eq. (30).

B. Vertex Corrections

The 1PR graphs are of two types, the vertex modifications of Figs. 6(a)–(e) and the self-energy effects of Figs. 7(a)–(e).

The two-loop vertex amplitude $\Gamma_{\mu a}^{(4)}$ is defined by

$$\langle P_a(q')|\mathcal{S} - 1|a_a(q,\lambda)\rangle_{\text{Fig. 6}} = i(2\pi)^4\delta^{(4)}(q' - q)\ \epsilon^\mu(q,\lambda)\Gamma_{\mu a}^{(4)} \qquad (a = 3, 8) \ . \tag{33}$$

A typical example is the vertex amplitude of Fig. 6(b),

$$\begin{aligned}
\Gamma_{\mu 3}^{(4)}[b] &= \frac{iq_\mu}{18F_0^3}\Big[A(\pi)\Big(16A(\pi) + 12m_\pi^2 B(0,\pi)\Big) + 11A(K)A(\pi) \\
&+6A^2(K) + A(\eta)\Big(3A(K) - 4m_\pi^2 B(0,\pi) + 4m_K^2 B(0,K)\Big)\Big] \ ,
\end{aligned} \tag{34}$$

where $B(0,m^2)$ is defined and evaluated as

$$B(0,m^2) \equiv \int d\tilde{k}\ \frac{1}{(k^2-m^2)^2} = \frac{1}{m^2}\left[A(m^2) - \frac{4-d}{2}A(m^2)\right] \quad . \tag{35}$$

The amplitudes of Figs. 6(a),(b) have the generic form

$$\Gamma^{(4)}_{\mu 3}[a,b] = \frac{q_\mu}{F_0^2}\sum_{k,\ell} v^{(a,b)}_{k\ell} A(k)\ A(\ell) \quad , \tag{36}$$

with numerical coefficients $v^{(a,b)}_{k\ell}$ whereas those of Figs. 6(c),6(d) are

$$\Gamma^{(4)}_{\mu 3}[c,d] = \frac{q_\mu}{F_0^2}\sum_{k,\ell} v^{(c,d)}_{k\ell} A(k)\ L_\ell \quad , \tag{37}$$

with coefficients $v^{(c,d)}_{k\ell}$ proportional to the squared meson masses. As in Eq. (34)), all the invariant parts of non-sunset vertex functions are found to be independent of the propagator momentum q^2.

For the sunset contribution of Fig. 6(e), we write

$$\begin{aligned}\Gamma^{(4)}_{\mu 3}[e] = \frac{i}{F_0^3}\Big[&\frac{2}{9}I_{1\mu}\left(q^2;m_\pi^2;m_\pi^2;m_\pi^2\right) + \frac{1}{36}I_{1\mu}\left(q^2;m_\pi^2;m_K^2;2(m_\pi^2+m_K^2)\right)\\ &+\frac{1}{12}I_{1\mu}\left(q^2;m_\eta^2;m_K^2;\frac{2}{3}(m_\pi^2-m_K^2)\right) + \frac{1}{2}I_{2\mu}\left(q^2;m_\pi^2;m_K^2\right)\Big] \quad .\end{aligned} \tag{38}$$

The vector-valued quantities $I_{1\mu}$ and $I_{2\mu}$ are defined in Eqs. (A6),(A7) of App. A, and are found to share the singular behaviour of Eq. (30).

C. Self-energy Corrections

The two-loop self-energy $\Sigma^{(6)}_a$ arises from the meson-to-meson matrix element,

$$\langle P_a(q')|\mathcal{S}-1|P_a(q)\rangle_{\text{Fig. 7}} = i(2\pi)^4\delta^{(4)}(q'-q)\ \Sigma^{(6)}_a \qquad (a=3,8) \quad . \tag{39}$$

The non-sunset contributions of Figs. 7(a)–7(b) are proportional to $\{A(k)A(\ell)\}$ and those of Figs. 7(c)–7(d) are proportional to $\{A(k)L_\ell\}$. As an example, the isospin self-energy contribution of Fig. 7(a) is

$$\begin{aligned}F_0^4\Sigma^{(6)}_3[a] = &\left[-\frac{4q^2}{9} + \left(\frac{7}{24}-\frac{8}{9}\right)m_\pi^2\right]A^2(\pi) + \frac{5m_\pi^2}{36}A(\pi)A(\eta)\\ &+\left[-\frac{7q^2}{18} - \frac{7m_\pi^2+2m_K^2}{18} + \frac{2m_\pi^2+m_K^2}{9}\right]A(\pi)A(K)\\ &+\left[-\frac{2q^2}{15} - \frac{17m_K^2}{45} + \frac{7(m_\pi^2+2m_K^2)}{90}\right]A^2(K)\\ &+\left[-\frac{q^2}{30} - \frac{6m_K^2+m_\eta^2}{30} + \frac{m_\pi^2+m_K^2}{45}\right]A(K)A(\eta) + \frac{m_\pi^2}{72}A^2(\eta) \quad .\end{aligned} \tag{40}$$

As in Eq. (40), the remaining non-sunset self-energies are at most linear in the propagator momentum q^2.

Finally, the sunset contribution of Fig. 7(e) is given by

$$
\begin{aligned}
F_0^4 \Sigma_3^{(6)}[e] = & \frac{m_\pi^4}{6} S(q^2; m_\pi^2; m_\pi^2) + \frac{m_\pi^4}{18} S(q^2; m_\pi^2; m_\eta^2) \\
& + \frac{1}{9} R(q^2; m_\pi^2; m_\pi^2; m_\pi^2) + \frac{1}{72} R(q^2; m_\pi^2; m_K^2; 2(m_\pi^2 + m_K^2)) \\
& + \frac{1}{24} R(q^2; m_\eta^2; m_K^2; \frac{2}{3}(m_\pi^2 - m_K^2)) + \frac{1}{4} U(q^2; m_\pi^2; m_K^2) \quad ,
\end{aligned} \tag{41}
$$

where the quantities S, R and U are yet new sunset functions defined by Eqs. (A2),(A8),(A9) of App. A. Both they and the non-sunset self-energy contributions share the degree of singularity of the 1PI and vertex amplitudes (*cf* Eq. (30)).

D. $\mathcal{O}(p^6)$ Counterterms

To construct the counterterm amplitudes needed to subtract off divergences and scale dependence contained in the two-loop graphs, we refer to the lagrangian of $\mathcal{O}(p^6)$ counterterms of Fearing and Scherer [23]. From their compilation, we extract 23 counterterm operators which contribute to the axialvector propagators.

The $\mathcal{O}(p^6)$ counterterm amplitudes are computed in like manner to the two-loop contributions considered thus far in this section. For example, the isospin 1PI two-loop amplitudes will require a corresponding counterterm contribution $\mathcal{M}_{\mu\nu 3}^{(\text{ct})}$ as computed from

$$
\langle a_3(q', \lambda') | \mathcal{S}^{(\text{ct})} - 1 | a_3(q, \lambda) \rangle = i(2\pi)^4 \delta^{(4)}(q' - q) \; \epsilon^{\mu^\dagger}(q', \lambda') \; \mathcal{M}_{\mu\nu 3}^{(\text{ct})} \; \epsilon^\nu(q, \lambda) \; , \tag{42}
$$

and one obtains

$$
\begin{aligned}
\mathcal{M}_{\mu\nu 3}^{(\text{ct})} = & (q_\mu q_\nu - q^2 g_{\mu\nu}) \Big[4(m_\pi^2 + 2m_K^2)(B_{29} - 2B_{49}) \\
& - 4q^2(B_{33} - 2B_{32}) + 8m_\pi^2(B_{28} - B_{46}) \Big] \\
& + g_{\mu\nu} \Big[-4(m_\pi^2 + 2m_K^2)^2 B_{21} - 4(3m_\pi^4 - 4m_\pi^2 m_K^2 + 4m_K^4) B_{19} \\
& - 4m_\pi^2(m_\pi^2 + 2m_K^2)(B_{16} + B_{18}) + 4m_\pi^4(2B_{14} - B_{11} - B_{17}) \Big] \; .
\end{aligned} \tag{43}
$$

This has three independent $\mathcal{O}(p^6)$ counterterms for the 'transverse' amplitude proportional to $q_\mu q_\nu - q^2 g_{\mu\nu}$ and five for the 'longitudinal' amplitude proportional to $g_{\mu\nu}$.

Employing a relation analogous to Eq. (33) for the vertex counterterm amplitude, one finds for the isospin case,

$$
\begin{aligned}
\Gamma_{\mu 3}^{(\text{ct})} = i \; \frac{q_\mu}{F_0} \Big[& 4m_\pi^4(B_{14} - B_{17}) - 4\left(4m_K^4 - 4m_K^2 m_\pi^2 + 3m_\pi^4\right) B_{19} \\
& - (m_\pi^2 + 2m_K^2)\left(2m_\pi^2(B_{16} + 2B_{18}) + 4(m_\pi^2 + 2m_K^2) B_{21}\right) \Big] \quad ,
\end{aligned} \tag{44}
$$

and a relation analogous to Eq. (39) leads to the isospin counterterm self-energy

$$\Sigma_3^{(\mathrm{ct})} = -\frac{2}{F_0^2}\Big[m_\pi^6\,(3B_1+2B_2) + m_\pi^2(5m_\pi^4+4m_K^4)B_3 + (m_\pi^2+2m_K^2)\left(2m_\pi^4 B_4 + 3m_\pi^2(m_\pi^2+2m_K^2)B_6\right)\Big] - 4\frac{q^2}{F_0^2}\Big[m_\pi^4 B_{17} + (4m_K^4 - 4m_K^2 m_\pi^2 + 3m_\pi^4)B_{19} + (m_\pi^2+2m_K^2)\left(m_\pi^2 B_{18} + (m_\pi^2+2m_K^2)B_{21}\right)\Big] \ . \quad (45)$$

Finally, we express the $\{B_\ell\}$ in dimensional regularization as[4]

$$B_\ell = \mu^{2(d-4)} \sum_{n=2}^{-\infty} B_\ell^{(n)}(\mu)\ \overline{\lambda}^n = \mu^{2(d-4)}\left[\ B_\ell^{(2)}(\mu)\ \overline{\lambda}^2\ +\ B_\ell^{(1)}(\mu)\ \overline{\lambda}\ +\ B_\ell^{(0)}(\mu)\ +\dots\ \right] . \quad (46)$$

This representation, together with Eq. (19) for the A-integral and Eq. (21) for the $\mathcal{O}(p^4)$ counterterms $\{L_\ell\}$, expresses the axialvector propagator as an expansion in the singular quantity $\overline{\lambda}$.

IV. RENORMALIZATION PROCEDURE

The axialvector propagator $\Delta_{\mathrm{A}a}^{\mu\nu}$ will have contributions from both the $1PI$ part $\mathcal{M}_a^{\mu\nu}$ and the 1PR pole term,

$$\Delta_{\mathrm{A}a}^{\mu\nu}(q) \ = \ \mathcal{M}_a^{\mu\nu}(q) - q^\mu q^\nu \frac{\Gamma_a^2(q^2)}{q^2 - m_a^2 + \Sigma_a(q^2)} \qquad (a=3,8) \ . \quad (47)$$

In the above, the $\mathcal{O}(p^6)$ counterterms are understood to be already included in $\mathcal{M}_a^{\mu\nu}$, in the self energy Σ_a and also in $\Gamma_a(q^2)$. The latter is defined in terms of the vertex amplitude $\Gamma_a^\mu(q)$ as

$$\Gamma_a^\mu(q) \equiv i q^\mu \Gamma_a(q^2) \ . \quad (48)$$

The renormalized mass M_a and decay constant F_a are defined as parameters occurring in the meson pole term

$$q^\mu q^\nu \frac{\Gamma_a^2(q^2)}{q^2 - m_a^2 + \Sigma_a(q^2)} \equiv q^\mu q^\nu \left(\frac{F_a^2}{q^2 - M_a^2} + R_a(q^2)\right) \ , \quad (49)$$

where $R_a(q^2)$ is a remainder term having no poles.

---

[4]The dependence of $L_\ell^{(n)}(\mu)$ and $B_\ell^{(n)}(\mu)$ upon the scale μ is determined from the renormalization group equations and has been explicitly given in Ref. [9].

A. Identification of the Meson Mass

From Eqs. (47),(49), it follows that M_a^2 is a solution of the implicit relation

$$M_a^2 = m_a^2 - \Sigma_a(M_a^2) \quad . \tag{50}$$

Since we have already calculated $\Sigma(m^2)$ (in the following, we temporarily omit flavour indices), it makes sense to expand the self-energy $\Sigma(M^2)$ as

$$\Sigma(M^2) = \Sigma(m^2) + \Sigma'(m^2)(M^2 - m^2) + \dots \quad . \tag{51}$$

Then, expressing the squared-mass perturbatively,

$$M^2 = M^{(2)2} + M^{(4)2} + M^{(6)2} + \dots \quad , \tag{52}$$

and similarly for the self-energy, we obtain the perturbative chain

$$M^{(2)2} = m^2 \ , \tag{53}$$

$$M^{(4)2} = -\Sigma^{(4)}(m^2) \ , \tag{54}$$

$$M^{(6)2} = -\Sigma^{(6)}(m^2) + \Sigma^{(4)'}(m^2)\ \Sigma^{(4)}(m^2) \ , \tag{55}$$

where we have noted that

$$M^2 - m^2 = \mathcal{O}(q^4) = -\Sigma^{(4)}(m^2) + \dots \quad . \tag{56}$$

We briefly exhibit this procedure at one-loop level. Using the fourth-order isospin self-energy

$$\begin{aligned}\Sigma_3^{(4)}(q^2) = {} & \frac{i}{F_0^2}\left[\frac{m_\pi^2 - 4q^2}{6}A(\pi) + \frac{m_\pi^2 - q^2}{3}A(K) + \frac{m_\pi^2}{6}A(\eta)\right] \\ & + \frac{8}{F_0^2}\left[(m_\pi^2 + 2m_K^2)(q^2 L_4 - 2m_\pi^2 L_6) + m_\pi^2(q^2 L_5 - 2m_\pi^2 L_8)\right] \ ,\end{aligned} \tag{57}$$

we obtain from Eq. (54),

$$M_\pi^{(4)2} = \frac{i}{F_0^2}\left[\frac{3A(\pi) - A(\eta)}{6}\right] - \frac{8m_\pi^2}{F_0^2}\left[(m_\pi^2 + 2m_K^2)(L_4 - 2L_6) + m_\pi^2(L_5 - 2L_8)\right] \ . \tag{58}$$

Expanding the A-integrals in a Laurent series around $d = 4$, we readily obtain the result cited earlier in Eq. (25). In like manner, the one-loop hypercharge self-energy

$$\begin{aligned}\Sigma_8^{(4)}(q^2) = {} & \frac{i}{F_0^2}\left[\frac{m_\pi^2}{2}A(\pi) + \frac{16m_K^2 - 7m_\pi^2}{18}A(\eta) - (q^2 + \frac{m_\pi^2}{3})A(K)\right] \\ & + \frac{8}{F_0^2}\Big[(m_\pi^2 + 2m_K^2)(q^2 L_4 - 2m_\eta^2 L_6) + m_\eta^2 q^2 L_5 \\ & - \frac{16}{3}(m_K^2 - m_\pi^2)^2 L_7 - \frac{2}{3}(8m_K^4 - 8m_K^2 m_\pi^2 + 3m_\pi^4)L_8\Big] \ ,\end{aligned} \tag{59}$$

yields

$$M_\eta^{(4)2} = \frac{i}{F_0^2}\left[-\frac{m_\pi^2}{2}A(\pi) - \frac{16m_K^2 - 7m_\pi^2}{18}A(\eta) + \frac{4m_K^2}{3}A(K)\right]$$
$$-\frac{8}{F_0^2}\Big[m_\eta^2(m_\pi^2 + 2m_K^2)(L_4 - 2L_6) + m_\eta^4 L_5$$
$$-\frac{16}{3}(m_K^2 - m_\pi^2)^2 L_7 - \frac{2}{3}(8m_K^4 - 8m_K^2 m_\pi^2 + 3m_\pi^4)L_8\Big] \; . \qquad (60)$$

These agree, of course, in the $SU(3)$ limit of equal quark mass. The analysis at two-loop level proceeds analogously.

B. Identification of the Meson Decay Constant

With the aid of Eq. (50), it is straightforward to write the meson pole term in the form (again temporarily suspending flavour indices)

$$\frac{\Gamma^2(q^2)}{q^2 - m^2 - \Sigma(q^2)} = \frac{\Gamma^2(q^2)}{q^2 - M^2}\cdot\frac{1}{1+\tilde{\Sigma}(q^2)}$$
$$= \frac{\Gamma^2(q^2)\left(1 - \tilde{\Sigma}(q^2) + \tilde{\Sigma}^2(q^2) + \ldots\right)}{q^2 - M^2} \quad , \qquad (61)$$

where $\tilde{\Sigma}(q^2)$ is the divided difference

$$\tilde{\Sigma}(q^2) \equiv \frac{\Sigma(q^2) - \Sigma(M^2)}{q^2 - M^2} \quad . \qquad (62)$$

From the definition in Eq. (49) of the squared decay constant F^2, we have

$$F^2 = \lim_{q^2 = M^2}\left[\Gamma^2(q^2)\left(1 - \tilde{\Sigma}(q^2) + (\tilde{\Sigma}(q^2))^2 + \ldots\right)\right] . \qquad (63)$$

Let us analyze this relation perturbatively.

We begin by writng the vertex quantity $\Gamma(q^2)$ (evaluated at $q^2 = M^2$) as

$$\Gamma(M^2) = \Gamma^{(0)} + \Gamma^{(2)} + \Gamma^{(4)}(M^2) + \ldots \quad , \qquad (64)$$

where both $\Gamma^{(0)}$ and $\Gamma^{(2)}$ are independent of q^2. Expanding $\Gamma^{(4)}(M^2)$ as

$$\Gamma^{(4)}(M^2) = \Gamma^{(4)}(m^2) + \Gamma^{(4)\prime}(m^2)(M^2 - m^2) + \ldots \quad , \qquad (65)$$

we see from chiral counting that to the order at which we are working, one is justified in replacing $\Gamma^{(4)}(M^2)$ by $\Gamma^{(4)}(m^2)$. As for the self energy dependence in Eq. (63), we first observe that

$$\tilde{\Sigma}(M^2) = \lim_{q^2 = M^2}\tilde{\Sigma}(q^2) = \Sigma'(M^2) \quad . \qquad (66)$$

Recalling from Eqs. (57),(59) that $\Sigma^{(4)}(q^2)$ is linear in q^2, we have the perturbative expression

$$\Sigma'(M^2) = \Sigma^{(4)'} + \Sigma^{(6)'}(M^2) + \ldots = \Sigma^{(4)'} + \Sigma^{(6)'}(m^2) + \ldots \quad , \tag{67}$$

where the error in replacing $\Sigma^{(6)'}(M^2)$ by $\Sigma^{(6)'}(m^2)$ appears in higher order. Thus, the perturbative content of Eq. (63) reduces to

$$\begin{aligned} F^2 = &\left(\Gamma^{(0)} + \Gamma^{(2)} + \Gamma^{(4)}(m^2)\right)^2 \\ &\times \left(1 - \Sigma^{(4)'} - \Sigma^{(6)'}(m^2) + (\Sigma^{(4)'})^2\right) + \ldots \ . \end{aligned} \tag{68}$$

Upon organizing terms in ascending chiral powers, we obtain the following expression, valid through two-loop order, for the decay constant

$$\begin{aligned} F = \Gamma^{(0)} \ + &\left[\Gamma^{(2)} - \frac{1}{2}\Gamma^{(0)}\Sigma^{(4)'}\right] \ + \ \left[\Gamma^{(4)}(m^2) - \frac{1}{2}\Gamma^{(2)}\Sigma^{(4)'}\right. \\ &\left. + \ \Gamma^{(0)}\left(-\frac{1}{2}\Sigma^{(6)'}(m^2) + \frac{3}{8}(\Sigma^{(4)'})^2\right)\right] + \ldots \quad , \end{aligned} \tag{69}$$

where we have collected together terms of a given order.

As an example, Eq. (69) readily provides a determination of the isospin and hypercharge decay constants through one-loop order. The corresponding vertex quantities are

$$\begin{aligned} \Gamma_\pi &= F_0 - \frac{2i}{3F_0}\left[2A(\pi) + A(K)\right] + \frac{8}{F_0}\left[(m_\pi^2 + 2m_K^2)L_4 + m_\pi^2 L_5\right] + \ldots \ , \\ \Gamma_\eta &= F_0 - \frac{2i}{F_0}A(K) + \frac{8}{F_0}\left[(m_\pi^2 + 2m_K^2)L_4 + m_\eta^2 L_5\right] + \ldots \ . \end{aligned} \tag{70}$$

Upon inferring $\Sigma_k^{(4)'}$ $(k = 3, 8)$ from Eqs. (57),(59), we obtain

$$\begin{aligned} F_\pi^{(0)} + F_\pi^{(2)} &= F_0 - \frac{i}{2F_0}\left[2A(\pi) + A(K)\right] \\ &\quad + \frac{4}{F_0}\left[(m_\pi^2 + 2m_K^2)L_4 + m_\pi^2 L_5\right] \ , \\ F_\eta^{(0)} + F_\eta^{(2)} &= F_0 - \frac{3i}{2F_0}A(K) + \frac{4}{F_0}\left[(m_\pi^2 + 2m_K^2)L_4 + m_\eta^2 L_5\right] \ . \end{aligned} \tag{71}$$

The one-loop expressions for the pion decay constant (appearing in Eq. (25)) and the eta decay constant (not shown) follow from the above relation. The two-loop corrections are found analogously.

C. The Remainder Term

The preceding work allows extraction of the remainder term $R(q^2)$ defined earlier in Eq. (49). Making use of Eqs. (52)–(55) as well as Eq. (69), a straightforward calculation yields the expression

$$\begin{aligned} R(q^2) = &\, 2F_0\frac{\Gamma^{(4)}(q^2) - \Gamma^{(4)}(m^2)}{q^2 - m^2} \\ -&F_0^2\ \frac{\Sigma^{(6)}(q^2) - \Sigma^{(6)}(m^2) - (q^2 - m^2)\Sigma^{(6)'}(m^2)}{(q^2 - m^2)^2} \ . \end{aligned} \tag{72}$$

It is manifest that $R(q^2)$ has no pole at $q^2 = m^2$. Moreover, since the non-sunset vertex functions $\Gamma^{(4)}[a]$–$[d]$ of Sect. 3 are constant in q^2, they do not contribute to $R(q^2)$. Nor do the non-sunset self-energies $\Sigma^{(6)}[a]$–$[d]$ since they are at most linear in q^2. Thus, the only contributors to $R(q^2)$ are sunset amplitudes, and it is straightforward to obtain the isospin and hypercharge remainder functions directly from Eq. (72).

D. The Polarization Amplitudes $\hat{\Pi}^{(1)}$ and $\hat{\Pi}^{(0)}$

From the tree-level and one-loop results, we anticipate that the two-loop $1PI$ amplitude $\mathcal{M}^{(4)}_{\mu\nu}$ will contain an additive term involving the meson decay constant, and can thus be written

$$\mathcal{M}^{(4)}_{\mu\nu} \equiv g_{\mu\nu}(F^2)^{(4)} + \hat{\mathcal{M}}_{\mu\nu} \quad , \tag{73}$$

where $\hat{\mathcal{M}}_{\mu\nu}$ denotes the residual part of the $1PI$ amplitude. It turns out that much of the rather complicated content in the $1PI$ amplitudes of Sect. 3 is attributable to the two-loop squared decay constant $(F^2)^{(4)}$, as can be verified from decay constant results derived earlier in this section. First, we re-express the expansion of Eq. (68) as

$$F^2 = (F^2)^{(0)} + (F^2)^{(2)} + (F^2)^{(4)} + \dots \quad , \tag{74}$$

with $(F^2)^{(0)} = F_0^2$, $(F^2)^{(2)} = 2F_0F^{(2)}$ and

$$(F^2)^{(4)} = (F^{(2)})^2 + 2F_0F^{(4)} \quad . \tag{75}$$

One then compares the $g_{\mu\nu}$ part of $\mathcal{M}^{(4)}_{\mu\nu}$ with $(F^2)^{(4)}$. The residual amplitude $\hat{\mathcal{M}}_{\mu\nu}$ is simply the difference of these.

It is convenient to replace the residual amplitude $\hat{\mathcal{M}}^{\mu\nu}$ and the remainder term $-q^\mu q^\nu R(q^2)$ (which arises from the meson pole term but itself contains no poles) by equivalent quantities $\hat{\Pi}^{(1)}$ and $\hat{\Pi}^{(0)}$,

$$\hat{\mathcal{M}}^{\mu\nu}(q) - q^\mu q^\nu R(q^2) \equiv (q^\mu q^\nu - q^2 g^{\mu\nu})\hat{\Pi}^{(1)}(q^2) + g^{\mu\nu}\hat{\Pi}^{(0)}(q^2) \quad . \tag{76}$$

We shall employ $\hat{\Pi}^{(1)}$ and $\hat{\Pi}^{(0)}$ throughout the rest of the paper.

V. REMOVAL OF DIVERGENCES

Thus far, we derived lengthy expressions for the various two-loop components of the axialvector propagators and then determined the renormalization structure of the masses and decay constants. At this stage of the calculation, there are many terms which diverge as $d \to 4$ and which must therefore be removed from the description. Below, we carry out the subtraction procedure by expanding the relevant quantities in powers of the parameter $\bar{\lambda}$ and then using $\mathcal{O}(p^6)$ and $\mathcal{O}(p^4)$ counterterms to cancel the singular contributions. In particular, this process will determine a subset of the so-called β-functions of the complete $\mathcal{O}(p^6)$ lagrangian. This is of special interest because the divergence structure of the generating functional to two-loop level can be obtained in closed form [24], and our results derived in the following can be used as checks of such future calculations.

A. Removal of $\bar{\lambda}^2$ Dependence

It will simplify the following discussion to define the counterterm combinations

$$\begin{aligned} A &\equiv 2B_{14} - B_{17} \ , \qquad B \ \equiv \ B_{16} + B_{18} \ , \qquad C \ \equiv \ B_{15} - B_{20} \ , \\ D &= B_{19} + B_{21} \ , \qquad E \ \equiv \ B_{19} - B_{21} \ , \qquad F \ \equiv \ 3B_1 + 2B_2 \ . \end{aligned} \tag{77}$$

First we consider the decay constant sector. Upon demanding that all $\bar{\lambda}^2$ dependence vanish, we obtain six equations containing five variables. There are *six* equations because the pion and eta constants each have explicit dependence on m_π^4, $m_\pi^2 m_K^2$ and m_K^4 factors and the singular behaviour must be subtracted for each of these.

We find the equation set to be degenerate, and one obtains just the following four conditions,

$$\begin{aligned} A^{(2)} - 3E^{(2)} &= -\frac{31}{24F_0^2} \ , \qquad B^{(2)} - 2E^{(2)} \ = \ -\frac{53}{72F_0^2} \ , \\ C^{(2)} + E^{(2)} &= \frac{13}{18F_0^2} \ , \qquad D^{(2)} \ = \ \frac{73}{144F_0^2} \ . \end{aligned} \tag{78}$$

The information contained in the above set is unique and any other way of expressing the solution must be equivalent.

In the mass sector, we find subtraction at the $\bar{\lambda}^2$ level to yield *seven* equations in eleven variables. The number of equations follows from the dependence of each mass on m_π^6, $m_\pi^4 m_K^2$ and $m_\pi^2 m_K^4$ factors (this implies six equations), along with the fact that m_K^6 dependence is absent from $M_\pi^{(6)2}$. This latter fact arises becauses there is no m_K^6 counterterm contribution in $M_\pi^{(6)2}$, and thus there must be a cancellation between sunset and nonsunset numerical terms. Such a cancellation occurs and constitutes an important check on our determination of the sunset contribution. The equation set for the $M^{(6)2}$ masses is found to be degenerate and just five constraints can be obtained, *e.g.*

$$\begin{aligned} B_3^{(2)} &= \frac{4}{27F_0^2} - \frac{1}{6}F^{(2)} - B_4^{(2)} - B_5^{(2)} - 3B_7^{(2)} \ , \\ B_6^{(2)} &= -\frac{16}{81F_0^2} + \frac{1}{18}F^{(2)} + \frac{1}{3}B_4^{(2)} + \frac{1}{3}B_5^{(2)} + B_7^{(2)} \ , \\ B_{14}^{(2)} &= \frac{1}{48}\left[-\frac{37}{F_0^2} + 72B_4^{(2)} + 72B_5^{(2)} + 216B_7^{(2)}\right] \ , \\ B_{15}^{(2)} &= \frac{307}{216F_0^2} - \frac{1}{3}F^{(2)} - B_4^{(2)} - 2B_5^{(2)} - 3B_7^{(2)} \ , \\ B_{16}^{(2)} &= -\frac{91}{72F_0^2} + \frac{1}{6}F^{(2)} + 2B_4^{(2)} + B_5^{(2)} + 3B_7^{(2)} \ . \end{aligned} \tag{79}$$

Finally, removal of $\bar{\lambda}^2$-dependence for the polarization functions $\hat{\Pi}^{(0)}$ and $\hat{\Pi}^{(1)}$ yields a final set of constraints at this order,

$$\begin{aligned} B_{11}^{(2)} = B_{13}^{(2)} &= 0 \ , \qquad B_{33}^{(2)} \ = \ 2B_{32}^{(2)} \ , \\ B_{28}^{(2)} - B_{46}^{(2)} &= -\frac{3}{16F_0^2} \ , \qquad B_{29}^{(2)} - 2B_{49}^{(2)} = -\frac{1}{8F_0^2} \ . \end{aligned} \tag{80}$$

B. Removal of $\bar{\lambda}$ Dependence

In a similar manner, we obtain constraints for the $\{B^{(1)}\}$ counterterms. We list these below without further comment, beginning with those following from decay constants,

$$
\begin{aligned}
A^{(1)} - 3E^{(1)} &= \frac{1}{F_0^2}\left[-\frac{175}{9216\pi^2} - \frac{28}{3}L_1^{(0)} - \frac{34}{3}L_2^{(0)} - \frac{25}{3}L_3^{(0)}\right.\\
&\qquad \left. + \frac{26}{3}L_4^{(0)} - \frac{8}{3}L_5^{(0)} - 12L_6^{(0)} + 12L_8^{(0)}\right] \ ,\\
B^{(1)} - 2E^{(1)} &= \frac{1}{F_0^2}\left[\frac{19}{1536\pi^2} - \frac{32}{9}L_1^{(0)} - \frac{8}{9}L_2^{(0)} - \frac{8}{9}L_3^{(0)}\right.\\
&\qquad \left. + \frac{106}{9}L_4^{(0)} - \frac{22}{9}L_5^{(0)} - 20L_6^{(0)}\right] \ ,\\
C^{(1)} + E^{(1)} &= \frac{1}{F_0^2}\left[\frac{691}{82944\pi^2} + \frac{28}{9}L_1^{(0)} + \frac{34}{9}L_2^{(0)} + \frac{59}{18}L_3^{(0)}\right.\\
&\qquad \left. - \frac{26}{9}L_4^{(0)} + 3L_5^{(0)} + 4L_6^{(0)} - 6L_8^{(0)}\right] \ ,\\
D^{(1)} &= \frac{1}{F_0^2}\left[\frac{43}{3072\pi^2} + \frac{104}{9}L_1^{(0)} + \frac{26}{9}L_2^{(0)} + \frac{61}{18}L_3^{(0)}\right.\\
&\qquad \left. - \frac{34}{9}L_4^{(0)} + L_5^{(0)} - 4L_6^{(0)} - 2L_8^{(0)}\right] \ ,
\end{aligned} \tag{81}
$$

then masses,

$$
\begin{aligned}
B_3^{(1)} &= \frac{1}{F_0^2}\left[\frac{5}{216\pi^2} - \frac{20}{3}L_4^{(0)} - \frac{23}{3}L_5^{(0)} + \frac{40}{3}L_6^{(0)}\right.\\
&\qquad \left. + 40L_7^{(0)} + \frac{86}{3}L_8^{(0)}\right] - \frac{1}{6}F^{(1)} - B_4^{(1)} - B_5^{(1)} - 3B_7^{(1)} \ ,\\
B_6^{(1)} &= \frac{1}{648}\left[\frac{1}{F_0^2}\left(-\frac{5}{\pi^2} - 3168L_4^{(0)} - 24L_5^{(0)} + 6336L_6^{(0)} - 9792L_7^{(0)}\right.\right.\\
&\qquad \left.\left. - 3216L_8^{(0)}\right) + 36F^{(1)} + 216B_4^{(1)} + 216B_5^{(1)} + 648B_7^{(1)}\right] \ ,\\
B_{14}^{(1)} &= \frac{1}{F_0^2}\left[-\frac{167}{4608\pi^2} + 8L_6^{(0)} - 64L_7^{(0)} - \frac{62}{3}L_8^{(0)}\right]\\
&\qquad + \frac{3}{2}B_4^{(1)} + \frac{3}{2}B_5^{(1)} + \frac{9}{2}B_7^{(1)} \ ,\\
B_{15}^{(1)} &= \frac{1}{F_0^2}\left[\frac{371}{20736\pi^2} + 2L_5^{(0)} - \frac{16}{3}L_6^{(0)} + 72L_7^{(0)} + 24L_8^{(0)}\right]\\
&\qquad - \frac{1}{3}F^{(1)} - B_4^{(1)} - 2B_5^{(1)} - 3B_7^{(1)} \ ,\\
B_{16}^{(1)} &= \frac{1}{F_0^2}\left[-\frac{9}{512\pi^2} - L_5^{(0)} - \frac{152}{3}L_7^{(0)} - \frac{62}{3}L_8^{(0)}\right]\\
&\qquad + \frac{1}{6}F^{(1)} + 2B_4^{(1)} + B_5^{(1)} + 3B_7^{(1)} \ ,
\end{aligned} \tag{82}
$$

and finally from the polarizations $\hat{\Pi}^{(0)}$ and $\hat{\Pi}^{(1)}$,

$$B_{11}^{(1)} = -\frac{49}{576}\frac{1}{16\pi^2 F_0^2} \ , \qquad B_{13}^{(1)} = \frac{173}{5184}\frac{1}{16\pi^2 F_0^2} \ , \qquad 2B_{32}^{(1)} - B_{33}^{(1)} = \frac{3}{64}\frac{1}{16\pi^2 F_0^2} \ ,$$
$$B_{28}^{(1)} - B_{46}^{(1)} = \frac{1}{F_0^2}\left[-\frac{5}{64}\frac{1}{16\pi^2} + \frac{3}{2}L_{10}^{(0)}\right] \ , \qquad B_{29}^{(1)} - 2B_{49}^{(1)} = \frac{1}{F_0^2}\left[-\frac{17}{96}\frac{1}{16\pi^2} + L_{10}^{(0)}\right] \ . \quad (83)$$

This completes the subtraction part of the calculation.

C. $\overline{\lambda}$-Subtraction and $\overline{MS}$ Renormalization Schemes

The renormalization procedure employed originally in Ref. [9] and adopted in this paper amounts to $\overline{\lambda}$-subtraction, *cf* Eqs. (21),(46). Alternatively, one could employ minimal subtraction (MS),

$$\begin{aligned} L_\ell(d) &= \frac{\mu^{2\omega}}{(4\pi)^2}\left[\frac{\Gamma_\ell}{2\omega} + L_{\ell,r}^{\rm MS}(\mu,\omega) + \ldots\right] \\ B_\ell(d) &= \frac{\mu^{4\omega}}{(4\pi)^4}\left[\frac{B_\ell^{(2){\rm MS}}}{(2\omega)^2} + \frac{B_\ell^{(1){\rm MS}}}{2\omega} + B_{\ell,r}^{(0){\rm MS}}(\mu,\omega) + \ldots\right] \ , \end{aligned} \quad (84)$$

where $\omega \equiv d/2 - 2$, or modified minimal subtraction ($\overline{\rm MS}$),

$$\begin{aligned} L_\ell(d) &= \frac{(\mu c)^{2\omega}}{(4\pi)^2}\left[\frac{\Gamma_\ell}{2\omega} + L_{\ell,r}^{\overline{\rm MS}}(\mu,\omega) + \ldots\right] \\ B_\ell(d) &= \frac{(\mu c)^{4\omega}}{(4\pi)^4}\left[\frac{B_\ell^{(2){\rm MS}}}{(2\omega)^2} + \frac{B_\ell^{(1){\rm MS}}}{2\omega} + (4\pi)^4 B_{\ell,r}^{(0)\overline{\rm MS}}(\mu) + \ldots\right] \ , \end{aligned} \quad (85)$$

where we make the standard ChPT choice

$$\ln c = -\frac{1}{2}\left[1 - \gamma_{\rm E} + \ln(4\pi)\right] \equiv -C \ . \quad (86)$$

Of course, there must be only finite differences between these three procedures, amounting to additional finite renormalizations.

As an illustration, we relate the $\{B_\ell^{(n)}\}$ and $\{L_\ell^{(n)}\}$ renormalization constants employed here to those defined in the $\overline{\rm MS}$ approach of Ref. [17]. In the latter scheme, one writes further that

$$L_{\ell,r}^{\overline{\rm MS}}(\mu,\omega) = L_{\ell,r}^{\overline{\rm MS}}(\mu,0) + L_{\ell,r}^{\overline{\rm MS}\prime}(\mu,0)\ \omega + \ldots \quad (87)$$

and sets

$$L_{\ell,r}^{\overline{\rm MS}\prime}(\mu,0) = 0 \ . \quad (88)$$

The ‘convention’ established by Eq. (88) is allowed because it can be shown [17] that the effect of the quantity $L_{\ell,r}^{\overline{\rm MS}\prime}(\mu,0)$ is to add a local contribution at order p^6, which can always be abosrbed into the couplings of the $\mathcal{O}(p^6)$ lagrangian. Comparison of the two approaches yields

$$
\begin{aligned}
L_\ell^{(1)} &= \Gamma_\ell \ , \\
L_\ell^{(0)} &= \frac{1}{(4\pi)^2} L_{\ell,r}^{\overline{\rm MS}}(\mu,0) \equiv L_\ell^r(\mu) \ , \\
L_\ell^{(-1)} &= \frac{C}{(4\pi)^4}\left[-L_{\ell,r}^{\overline{\rm MS}}(\mu,0) + \frac{C\Gamma_\ell}{2}\right] \ ,
\end{aligned} \tag{89}
$$

and

$$
\begin{aligned}
B_\ell^{(2)} &= B_\ell^{(2){\rm MS}} \ , \\
B_\ell^{(1)} &= \frac{1}{(4\pi)^2} B_\ell^{(1){\rm MS}} \ , \\
B_\ell^{(0)}(\mu) &= B_{\ell,r}^{\overline{\rm MS}}(\mu) - \frac{C}{(4\pi)^4} B_\ell^{(1)MS} + \frac{C^2}{(4\pi)^4} B_\ell^{(2)MS} \ .
\end{aligned} \tag{90}
$$

We stress that the content of Eqs. (89),(90) is partly a reflection of the convention of Eq. (88). Finally, one can combine the relations of Eq. (90) to write

$$
B_{\ell,r}^{\overline{\rm MS}}(\mu) = B_\ell^{(0)}(\mu) + \frac{C}{(4\pi)^2} B_\ell^{(1)} - \frac{C^2}{(4\pi)^4} B_\ell^{(2)} \quad . \tag{91}
$$

We shall return to the comparison between the $\overline{\lambda}$-subtraction and $\overline{\rm MS}$ renormalizations at the ends of Sect. VI and of App. B.

VI. THE RENORMALIZED ISOSPIN POLARIZATION FUNCTIONS

Having performed the removal of $\bar{\lambda}^2$ and $\bar{\lambda}^1$ singular dependence from the theory, we are left with the $\bar{\lambda}^0$ sector in which all quantities are finite. We shall express such contributions entirely in terms of physical quantities by replacing the tree-level parameters $m_\pi^2, m_K^2, m_\eta^2, F_0$ with $M_\pi^2, M_K^2, M_\eta^2, F_\pi$. Any error thereby induced would appear in still higher orders.

For any observable $\mathcal{O}$ (*e.g.* $\mathcal{O} = \hat{\Pi}_3^{(1)}$, M_π, F_π, *etc*) evaluated at two-loop level, there will be generally three kinds of finite contributions,

$$
\mathcal{O} = \mathcal{O}_{\rm rem} + \mathcal{O}_{\rm CT} + \mathcal{O}_{\rm YZ} \quad . \tag{92}
$$

$\mathcal{O}_{\rm rem}$ refers to the finite $\bar{\lambda}^0$ 'remnants' in $\{A(k)A(\ell)\}$ or $\{A(k)L_\ell\}$ contributions and arise from either the product of two $\bar{\lambda}^0$ factors or the product of $\bar{\lambda}^1$ and $\bar{\lambda}^{-1}$ factors. $\mathcal{O}_{\rm CT}$ denotes any term containing the $\{B_\ell^{(0)}\}$ p^6-counterterms, whereas $\mathcal{O}_{\rm YZ}$ represents contributions from the finite Y, Z integrals (*cf* Eqs. (A20),(A21) of App. A) which occur solely in sunset amplitudes.

Before proceeding, we address a technical issue related to the presence of $L_\ell^{(-1)}$ terms appearing in the 'remnant' amplitudes. Such terms are always multiplied by polynomials in the quark masses and hence can be absorbed by the $\mathcal{O}(p^6)$ counterterms, as expected from general renormalization theorems [17]. In the vector current analysis of Ref. [9], we defined the following dimensionless quantities (now expressed in terms of the $\{B_\ell\}$),

$$
\begin{aligned}
P &\equiv 4F_\pi^2\left(-2B_{30}^{(0)}+B_{31}^{(0)}\right)+4L_9^{(-1)} \quad , \\
Q &\equiv 2F_\pi^2 B_{47}^{(0)}-3\left(L_9^{(-1)}+L_{10}^{(-1)}\right) \quad , \\
R &\equiv 2F_\pi^2 B_{50}^{(0)}-\left(L_9^{(-1)}+L_{10}^{(-1)}\right) \quad .
\end{aligned} \tag{93}
$$

and thus removed all explicit $L_\ell^{(-1)}$ dependence. We repeat that procedure here by defining axialvector quantities

$$
\begin{aligned}
P_{\rm A} &\equiv 4F_\pi^2\left(-2B_{32}^{(0)}+B_{33}^{(0)}\right) \quad , \\
Q_{\rm A} &\equiv 2F_\pi^2\left(-B_{28}^{(0)}+B_{46}^{(0)}\right)+3L_{10}^{(-1)} \quad , \\
R_{\rm A} &\equiv F_\pi^2\left(-B_{29}^{(0)}+2B_{49}^{(0)}\right)+L_{10}^{(-1)} \quad ,
\end{aligned} \tag{94}
$$

such that the counterterm dependence of $\Pi_{\rm A3,8}^{(1)}$ (*cf* Eqs. (97),(B2)) is identical to that established originally for $\Pi_{\rm V3,8}$ [9].

Since our finite results are quite lengthy, we restrict the discussion in this section to just the *isospin* polarization functions. Expressions for all the other observables (masses, decay constants and hypercharge polarization functions) are compiled in Appendix B.

A. The Spin-one Isospin Polarization Amplitude

Turning now to the isospin transverse polarization amplitude, we have for the remnant piece,

$$
\begin{aligned}
F_\pi^2\hat\Pi_{3,\rm rem}^{(1)}(q^2) &= \frac{M_\pi^2}{\pi^4}\left[\frac{49}{13824}-\frac{C}{192}\right. \\
&+\log\frac{M_\pi^2}{\mu^2}\left(\frac{1}{288}-\frac{\pi^2}{2}L_{10}^{(0)}-\frac{1}{768}\log\frac{M_\pi^2}{\mu^2}-\frac{1}{768}\log\frac{M_K^2}{\mu^2}\right) \\
&\left.+\log\frac{M_K^2}{\mu^2}\left(\frac{1}{576}+\frac{1}{1536}\log\frac{M_K^2}{\mu^2}\right)\right] \\
&+\frac{M_K^2}{\pi^4}\left[\frac{5}{36864}-\frac{17\ C}{3072}+\log\frac{M_K^2}{\mu^2}\left(\frac{17}{3072}-\frac{\pi^2}{4}L_{10}^{(0)}-\frac{1}{1024}\log\frac{M_K^2}{\mu^2}\right)\right] \\
&+\frac{q^2}{\pi^4}\left[-\frac{283}{294912}+\frac{3\ C}{4096}-\frac{1}{3072}\log\frac{M_\pi^2}{\mu^2}-\frac{5}{12288}\log\frac{M_K^2}{\mu^2}\right] \quad ,
\end{aligned} \tag{95}
$$

where the constant C has the same meaning as in Ref. [9],

$$
C \equiv \frac{1}{2}\left[1-\gamma_{\rm E}+\ln(4\pi)\right] \quad . \tag{96}
$$

Next is the counterterm contribution,

$$
\hat\Pi_{3,\rm CT}^{(1)}(q^2) = -\frac{q^2}{F_\pi^2}P_{\rm A}-\frac{8M_K^2}{F_\pi^2}R_{\rm A}-\frac{4M_\pi^2}{F_\pi^2}\left(Q_{\rm A}+R_{\rm A}\right) \ , \tag{97}
$$

and finally upon defining

$$\mathcal{H}^{qq} \equiv S - 6\overline{S} + 9S_1 \quad , \tag{98}$$

we have the so-called YZ piece,

$$F_\pi^2 \hat{\Pi}^{(1)}_{3,\mathrm{YZ}}(q^2) = \frac{4}{9}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2, M_\pi^2, M_\pi^2) + \frac{1}{6}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2, M_\eta^2, M_\pi^2)$$
$$+\frac{1}{18}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2, M_\pi^2, M_K^2) + \frac{1}{3}\mathcal{K}_{1,\mathrm{YZ}}(q^2, M_\pi^2, M_K^2) - R_{3,\mathrm{YZ}}(q^2) \quad . \tag{99}$$

The quantity $R_{3,\mathrm{YZ}}$ is rather complicated, so before writing it down explicitly we first develop some useful notation. For a quantity $f(q^2, \ldots)$, we define the auxiliary functions

$$\overline{f}(q^2, \ldots) \equiv f(q^2, \ldots) - f(M^2, \ldots)$$
$$\breve{f}(q^2, \ldots) \equiv \overline{f}(q^2, \ldots) - (q^2 - M^2) f'(M^2, \ldots) \quad , \tag{100}$$

where the 'M^2' quantities become M_π^2 for the case of isospin flavour and M_η^2 for hypercharge flavour. Then we have for $R_{3,\mathrm{YZ}}$ the expression

$$R_{3,\mathrm{YZ}}(q^2) \equiv \frac{2}{q^2 - M_\pi^2}\Big[\frac{2}{9}\overline{I}_{1,\mathrm{YZ}}(q^2; M_\pi^2; M_\pi^2; M_\pi^2)$$
$$+\frac{1}{36}\overline{I}_{1,\mathrm{YZ}}(q^2; M_\pi^2; M_K^2; 2(M_\pi^2 + M_K^2))$$
$$+\frac{1}{12}\overline{I}_{1,\mathrm{YZ}}(q^2; M_\eta^2; M_K^2; \frac{2(M_\pi^2 - M_K^2)}{3}) + \frac{1}{2}\overline{I}_{2,\mathrm{YZ}}(q^2; M_\pi^2; M_K^2)\Big]$$
$$-\frac{1}{(q^2 - M_\pi^2)^2}\Big[\frac{M_\pi^4}{6}\breve{S}_{\mathrm{YZ}}(q^2; M_\pi^2; M_\pi^2) + \frac{M_\pi^4}{18}\breve{S}_{\mathrm{YZ}}(q^2; M_\pi^2; M_\eta^2)$$
$$+\frac{1}{4}\breve{U}_{\mathrm{YZ}}(q^2; M_\pi^2; M_K^2) + \frac{1}{9}\breve{R}_{\mathrm{YZ}}(q^2; M_\pi^2; M_\pi^2; M_\pi^2)) \tag{101}$$
$$+\frac{1}{72}\breve{R}_{\mathrm{YZ}}(q^2; M_\pi^2; M_K^2; 2(M_\pi^2 + M_K^2))$$
$$+\frac{1}{24}\breve{R}_{\mathrm{YZ}}(q^2; M_\eta^2; M_K^2; \frac{2(M_\pi^2 - M_K^2)}{3})\Big] \ .$$

For example, we obtain at $q^2 = 0$ the numerical value

$$F_\pi^2 \hat{\Pi}^{(1)}_{3,\mathrm{YZ}}(0) = 1.927 \times 10^{-6}\ \mathrm{GeV}^2 \quad . \tag{102}$$

B. The Spin-zero Isospin Polarization Amplitude

For the isospin polarization function $\hat{\Pi}^{(0)}_3$, we find for the remnant contribution,

$$F_\pi^2 \hat{\Pi}^{(0)}_{3,\mathrm{rem}}(q^2) = \frac{M_\pi^4}{\pi^4}\left[-\frac{361}{294912} + \frac{49\ C}{36864} - \frac{1}{3072}\log\frac{M_\pi^2}{\mu^2}\right.$$
$$\left.-\frac{11}{12288}\log\frac{M_K^2}{\mu^2} - \frac{1}{9216}\log\frac{M_\eta^2}{\mu^2}\right] \ , \tag{103}$$

whereas the counterterm piece is given by

$$\hat{\Pi}^{(0)}_{3,\mathrm{CT}} = -4M_\pi^4 B_{11}^{(0)} \quad . \tag{104}$$

For the piece coming from the finite functions, we have

$$\begin{aligned}
F_\pi^2\hat{\Pi}^{(0)}_{3,\mathrm{YZ}}(q^2) &= 4S_{2,\mathrm{YZ}}(q^2,M_\pi^2,M_\pi^2) + \frac{3}{2}S_{2,\mathrm{YZ}}(q^2,M_\eta^2,M_\pi^2)\\
&+\frac{1}{2}S_{2,\mathrm{YZ}}(q^2,M_\pi^2,M_K^2) + \frac{1}{3}\mathcal{K}_{2,\mathrm{YZ}}(q^2,M_\pi^2,M_K^2)\\
&+q^2\Big(\frac{4}{9}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2,M_\pi^2,M_\pi^2) + \frac{1}{6}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2,M_\eta^2,M_\pi^2)\\
&+\frac{1}{18}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2,M_\pi^2,M_K^2) + \frac{1}{3}\mathcal{K}_{1,\mathrm{YZ}}(q^2,M_\pi^2,M_K^2) - R_{3,\mathrm{YZ}}(q^2)\Big)\\
&-2\Big[\frac{2}{9}I_{1,\mathrm{YZ}}\left(M_\pi^2;M_\pi^2;M_\pi^2;M_\pi^2\right) + \frac{1}{2}I_{2,\mathrm{YZ}}\left(M_\pi^2;M_\pi^2;M_K^2\right)\\
&+\frac{1}{36}I_{1,\mathrm{YZ}}\left(M_\pi^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2)\right) \qquad (105)\\
&+\frac{1}{12}I_{1,\mathrm{YZ}}\left(M_\pi^2;M_\eta^2;M_K^2;\frac{2}{3}(M_\pi^2-M_K^2)\right)\\
&-\frac{M_\pi^4}{12}S'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2) - \frac{M_\pi^4}{36}S'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_\eta^2)\\
&-\frac{1}{18}R'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_\pi^2;M_\pi^2) - \frac{1}{144}R'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2))\\
&-\frac{1}{48}R'_{\mathrm{YZ}}(M_\pi^2;M_\eta^2;M_K^2;\frac{2}{3}(M_\pi^2-M_K^2)) - \frac{1}{8}U'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2)\Big] \quad . \qquad (106)
\end{aligned}$$

The numerical value of $q^2 = 0$ is found to be

$$F_\pi^2\hat{\Pi}^{(0)}_{3,\mathrm{YZ}}(0) = -2.954\times 10^{-9}\ \mathrm{GeV}^4 \quad . \tag{107}$$

As mentioned earlier, the remaining finite results (hypercharge polarization amplitudes, *etc*) appear in Appendix B.

C. Isospin Polarization Functions and $\overline{MS}$ Renormalization

The discussion in Sect. VI-C allows one to re-express the isospin polarization functions $\hat{\Pi}_3^{(1,0)}$ in the $\overline{MS}$ renormalization. The result of this is, in essence, to replace the finite renormalization constants obtained in $\overline{\lambda}$-subtraction by the corresponding $\overline{\mathrm{MS}}$ quantities and at the same time to omit all terms from the remnant contributions containing the constant C.

As an example, let us determine the relation between the renormalization constants P_{A} and $P_{\mathrm{A}}^{\overline{\mathrm{MS}}}$. Starting from Eq. (94) and applying the relations in Eq. (90), we find

$$P_{\mathrm{A}}^{\overline{\mathrm{MS}}} = P_{\mathrm{A}} + \frac{F_\pi^2 C}{\pi^2}\left(-2B_{32}^{(1)} + B_{33}^{(1)}\right) - \frac{F_\pi^2 C^2}{(2\pi)^4}\left(-2B_{32}^{(2)} + B_{33}^{(2)}\right) \quad . \tag{108}$$

Then from Eqs. (80),(83), we obtain

$$P_{\rm A}^{\overline{\rm MS}} = P_{\rm A} - \frac{3}{16}\frac{C}{(4\pi)^4} \quad . \tag{109}$$

This result is entirely consistent with the form obtained earlier in this section for $\hat{\Pi}_3^{(1)}$. That is, from Eqs. (95),(97) we have

$$\hat{\Pi}_3^{(1)}(q^2) = \frac{q^2}{F_\pi^2}\left[-P_{\rm A} + \frac{3C}{4096\pi^4} + \ldots\right] \quad . \tag{110}$$

But this is just the combination of factors appearing in Eq. (109) and we conclude

$$\hat{\Pi}_3^{(1)\overline{\rm MS}}(q^2) = \frac{q^2}{F_\pi^2}\left[-P_{\rm A}^{\overline{\rm MS}} + \ldots\right] \quad . \tag{111}$$

Analogous steps lead to the further relations

$$\begin{aligned} Q_{\rm A}^{\overline{\rm MS}} &= Q_{\rm A} + \frac{5}{32}\frac{C}{(4\pi)^4} \\ R_{\rm A}^{\overline{\rm MS}} &= R_{\rm A} + \frac{17}{96}\frac{C}{(4\pi)^4} \\ B_{11,r}^{\overline{\rm MS}} &= B_{11}^{(0)} - \frac{49}{576}\frac{C}{(4\pi)^4 F_\pi^2} \\ B_{13,r}^{\overline{\rm MS}} &= B_{13}^{(0)} + \frac{173}{5184}\frac{C}{(4\pi)^4 F_\pi^2} \quad . \end{aligned} \tag{112}$$

Again, the forms for $\hat{\Pi}_3^{(1,0)}$ obtained by us in $\overline{\lambda}$-subtraction are found to convert to $\overline{MS}$ renormalization by simply removing all C dependence from the polarization functions and employing the $\overline{MS}$ finite counterterms.

Finally, we point out that for the renormalization constants P, Q, and R (*cf* Eq. (93)) which appeared in our two-loop analysis of vector-current propagators [9], there is no difference between the $\overline{\lambda}$-subtraction and $\overline{MS}$ schemes. This can be traced to the fact that the quantities $P^{(1)}$, $Q^{(1)}$, and $R^{(1)}$ have only contributions from $L_9^{(0)}$ and $L_{10}^{(0)}$.

VII. SPECTRAL FUNCTIONS

As noted in Sect. II, there will generally exist *two* spectral functions, $\rho_{{\rm A}a}^{(1)}(q^2)$ and $\rho_{{\rm A}a}^{(0)}(q^2)$ for the system of axialvector propagators. In the following, we determine the 3π contribution to the spectral functions for the isospin case $a = 3$. The $K\bar{K}\pi$ and $K\bar{K}\eta$ components will have thresholds at higher energies.

A. Three-pion Contribution to Isospin Spectral Functions

Spectral functions can be determined from the imaginary parts of polarization functions (*cf* Eqs. (3),(4)),

$$\rho_{{\rm A}a}^{(1)}(q^2) = \frac{1}{\pi}\,\mathcal{I}m\ \Pi_a^{(1)}(q^2) \qquad \text{and} \qquad \rho_{{\rm A}a}^{(0)}(q^2) = \frac{1}{\pi}\,\mathcal{I}m\ \Pi_a^{(0)}(q^2) \quad . \tag{113}$$

In two-loop ChPT, the imaginary parts arise solely from sunset graphs, and in terms of the notation established in App. A we find the 3π component of $\rho_{\rm A3}^{(1)}$ to be

$$\rho_{\rm A3}^{(1)}[3\pi] = -\frac{2}{\pi F_\pi^2}\frac{1}{(16\pi^2)^2}\mathcal{I}m\,\left[q^2\left(2\bar{Y}_0^{(3)} - 3\bar{Y}_0^{(2)} + \bar{Y}_0^{(1)}\right)\right.$$
$$+M_\pi^2\left(4\bar{Y}_0^{(1)} - 3\bar{Y}_0^{(2)} - \bar{Y}_0^{(0)}\right) + 4M_\pi^2\left(2\bar{Z}_0^{(2)} - \bar{Z}_0^{(1)}\right)$$
$$\left.+\frac{M_\pi^4}{q^2}\left(\bar{Y}_0^{(1)} - \bar{Y}_0^{(0)} - 4\bar{Z}_0^{(1)}\right)\right]\ , \tag{114}$$

where the above finite functions are evaluated with 3π mass values. There is a comparable, but rather more complicated, expression for $\rho_{\rm A3}^{(0)}[3\pi]$ which we do not display here.

B. Unitarity Determination of $\rho_{\rm A3}^{(0,1)}[3\pi]$

Unitarity provides an alternative determination of the three-pion component of the isospin spectral functions $\rho_{\rm A3}^{(1)}[3\pi]$ and $\rho_{\rm A3}^{(0)}[3\pi]$. The first step is to relate the spectral functions to the fourier transform of a non-time-ordered product,

$$\rho_{\rm A3}^{(1)}(q^2)\ (q^\mu q^\nu - q^2 g^{\mu\nu}) + \rho_{\rm A3}^{(0)}(q^2)\ q^\mu q^\nu = \frac{1}{2\pi}\int d^4x\ e^{iq\cdot x}\langle 0|A_3^\mu(x)A_3^\nu(0)|0\rangle\ . \tag{115}$$

One obtains the three-pion contribution by simply inserting the $3\pi^0$ and $\pi^+\pi^0\pi^-$ intermediate states in the above integral. To determine the relevant $\mathcal{S}$-matrix element, we employ the lowest order chiral langrian of Eq. (6) to find

$$\begin{aligned}&\langle a_3(q,\lambda)|\mathcal{S}|\pi^0(p_1)\pi^0(p_2)\pi^0(p_3)\rangle = \\ &\qquad i(2\pi)^4\delta^{(4)}(q-p_1-p_2-p_3)\epsilon_\mu^*(q,\lambda)\mathcal{M}_{000}^\mu\ , \\ &\langle a_3(q,\lambda)|\mathcal{S}|\pi^+(p_1)\pi^-(p_2)\pi^0(p_3)\rangle = \\ &\qquad i(2\pi)^4\delta^{(4)}(q-p_1-p_2-p_3)\epsilon_\mu^*(q,\lambda)\mathcal{M}_{+-0}^\mu\ ,\end{aligned} \tag{116}$$

where the invariant amplitudes are given by

$$\begin{aligned}\mathcal{M}_{000}^\mu &= \frac{i}{\sqrt{6}F_\pi}\ q^\mu\frac{M_\pi^2}{q^2-M_\pi^2}\ , \\ \mathcal{M}_{+-0}^\mu &= \frac{i}{F_\pi}\ \left[2\ p_0^\mu + q^\mu\ \frac{M_\pi^2 - 2q\cdot p_0}{q^2-M_\pi^2}\right]\ .\end{aligned} \tag{117}$$

Observe that $\mathcal{M}_{+-0}^\mu$ has two distinct contributions, a direct coupling and a pion pole term, whereas $\mathcal{M}_{000}^\mu$ has only a pion pole term. In the chiral limit of massless pions, the above amplitudes are conserved ($q_\mu\mathcal{M}_{3\pi}^\mu = 0$) as required by chiral symmetry.

To determine the spectral functions from Eq. (115), we take $\mu = \nu = 3$ for $\rho_{\rm A3}^{(1)}[3\pi]$ and $\mu = \nu = 0$ for $\rho_{\rm A3}^{(0)}[3\pi]$. Throughout, we work in the Lorentz frame where $q_\mu = (q^0, \mathbf{0})$. With the tacit understanding that we consider only the three-pion component and take $q^2 > 9M_\pi^2$ in the following, this leads to

$$q^2\rho^{(1)}_{\rm A3}(q^2) = \frac{1}{128\pi^6 F_\pi^2} I^{(1)} \quad ,$$
$$I^{(1)} \equiv \int \frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2}\frac{d^3p_3}{E_3}(p_3^3)^2 \delta^{(4)}(q-p_1-p_2-p_3) \quad , \tag{118}$$

and to

$$q^2\rho^{(0)}_{\rm A3}(q^2) = \frac{1}{512\pi^6 F_\pi^2}\cdot\frac{M_\pi^4}{(q^2-M_\pi^2)^2} I^{(0)} \quad , \tag{119}$$
$$I^{(0)} \equiv \int \frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2}\frac{d^3p_3}{E_3}\left(\frac{7}{6}q^2 - 4p_3^0\sqrt{q^2} + 4(p_3^0)^2\right)\delta^{(4)}(q-p_1-p_2-p_3) \ .$$

The former experiences only the $\pi^+\pi^-\pi^0$ contribution whereas the latter has contributions from both the $\pi^+\pi^-\pi^0$ and $3\pi^0$ intermediate states.

Integration of the above integrals is staightforward and one obtains a form involving integration over the invariant squared-mass of the $\pi_1\pi_2$ subsystem. For example, for the spectral function $\rho^{(1)}_{\rm A3}(q^2)$ of Eq. (118) we obtain

$$q^2\rho^{(1)}_{\rm A3}(q^2) = \frac{1}{768\pi^4 F_\pi^2}\ \frac{1}{(q^2)^3}\int_{4M_\pi^2}^{(\sqrt{q^2}-M_\pi)^2} da\ \sqrt{\frac{a-4M_\pi^2}{a}}\lambda^{3/2}(q^2,a,M_\pi^2) \ , \tag{120}$$

where

$$\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2xz-2yz \quad . \tag{121}$$

With the change of variable

$$a = \frac{1}{2}\left(\sqrt{q^2}-M_\pi\right)^2 + 2M_\pi^2 + \left(\frac{1}{2}\left(\sqrt{q^2}-M_\pi\right)^2 - 2M_\pi^2\right)\cos\phi \ , \tag{122}$$

we obtain the final form,

$$\rho^{(1)}_{\rm A3}(q^2) = \frac{1}{768\pi^4}\cdot\frac{q^2}{F_\pi^2}\theta(q^2-9M_\pi^2)\ \bar{I}^{(1)}(x) \ , \tag{123}$$

where $x \equiv M_\pi/\sqrt{q^2}$ and $\bar{I}^{(1)}(x)$ is the dimensionless integral

$$\bar{I}^{(1)}(x) = \left[\frac{1}{2}(1-x)^2 - 2x^2\right]^{3/2}\int_0^\pi d\phi\ \sin\phi\ (1+\cos\phi)^{1/2}$$
$$\times\frac{\left[-4x^2 + \left(\frac{1}{2}(1-x)^2 + x^2 - 1 + \left[\frac{1}{2}(1-x)^2 - 2x^2\right]\cos\phi\right)^2\right]^{3/2}}{\left[\frac{1}{2}(1-x)^2 + 2x^2 + \left[\frac{1}{2}(1-x)^2 - 2x^2\right]\cos\phi\right]^{1/2}} \ . \tag{124}$$

In like manner, we find

$$\rho^{(0)}_{\rm A3}(q^2) = \frac{1}{512\pi^4}\cdot\frac{q^2}{F_\pi^2}\cdot\frac{M_\pi^4}{(q^2-M_\pi^2)^2}$$
$$\times\left[\frac{7}{3}\ \bar{I}_0^{(0)}(x) - 4\ \bar{I}_1^{(0)}(x) + 2\ \bar{I}_2^{(0)}(x)\right]\theta(q^2-9M_\pi^2) \ , \tag{125}$$

where

$$\bar{I}_n^{(0)}(x) = \left[\frac{1}{2}(1-x)^2 - 2x^2\right]^{3/2} \int_0^\pi d\phi\ \sin\phi\ (1+\cos\phi)^{1/2}$$
$$\times \left[-4x^2 + \left(\frac{1}{2}(1-x)^2 + x^2 - 1 + \left[\frac{1}{2}(1-x)^2 - 2x^2\right]\cos\phi\right)^2\right]^{1/2}$$
$$\times \frac{\left(1 - x^2 - \frac{1}{2}(1-x)^2 - \left[\frac{1}{2}(1-x)^2 - 2x^2\right]\cos\phi\right)^n}{\left[\frac{1}{2}(1-x)^2 + 2x^2 + \left[\frac{1}{2}(1-x)^2 - 2x^2\right]\cos\phi\right]^{1/2}} . \tag{126}$$

The spectral functions obtained in the unitarity approach described here agree precisely with those obtained from the imaginary parts of the sunset amplitudes.

VIII. CHIRAL SUM RULES

In previous sections, we have determined the isospin and hypercharge axialvector propagators to two-loop order in ChPT. Essential to the success of this program is the renormalization procedure by which the results are rendered finite. As a consequence of the renormalization paradigm, however, the physical results contain a number of undetermined finite counterterms. In particular, the real parts of the polarization functions $\Pi_{\mathrm{A}a}^{(1)}$ and $\Pi_{\mathrm{A}a}^{(0)}$ ($a = 3, 8$) contain two such constants from one-loop order ($L_{10}^{(0)}$ and $H_1^{(0)}$) and five independent combinations of the $\{B_\ell^{(0)}\}$ from two-loop order. Of these, the constants $H_1^{(0)}$ and $B_{11}^{(0)}$ are related to contact terms which are regularization dependent and thus physically unobservable. However, the remaining counterterms (called 'low energy constants' or LEC) must be extracted from data. In this section, we describe how some of the $\mathcal{O}(p^6)$ counterterm coupling constants are obtainable from chiral sum rules, and as an example we study a specific case involving broken $SU(3)$.

A derivation of the chiral sum rules together with an application of one of them appears in Ref. [25]. We refer the reader to that article for a general orientation. For the purpose of writing dispersion relations it suffices to note that at low energies the polarization functions are already determined from the results obtained in previous sections, although some care must be taken with the kinematic poles at $q^2 = 0$ in the individual functions $\Pi_{\mathrm{A}a}^{(1)}$ and $\Pi_{\mathrm{A}a}^{(0)}$. As regards high energy behaviour, the large-s limit of spectral functions relevant to our analysis can be read off from the work in Refs. [26,27]. However, since we are calculating up to order p^6, terms up to and including quadratic dependence in the light quark masses must be included. Thus, for example, the asymptotic expansion for the isospin axialvector spectral function summed over spin-one and spin-zero reads

$$\rho_{\mathrm{A}3}^{(1+0)}(s) = \frac{1}{8\pi^2}\left(1 + \frac{\alpha_s}{\pi}\left[1 + 12\frac{\hat{m}^2}{q^2}\right] + \mathcal{O}(\alpha_s^2, 1/s^2)\right) . \tag{127}$$

Similar expressions hold for the other components, and we summarize their collective leading asymptotic behaviour by

$$\rho_{\mathrm{A}a}^{(1)}(s) \sim \mathcal{O}(1) , \quad \rho_{\mathrm{A}a}^{(0)}(s) \sim \mathcal{O}(s^{-1}) , \quad (\rho_{\mathrm{A}3}^{(1)} - \rho_{\mathrm{A}8}^{(1)})(s) \sim \mathcal{O}(s^{-1}) . \tag{128}$$

The real parts of the polarization functions show exactly the same asymptotic behaviour as the imaginary parts, *i.e.* expressions analogous to Eq. (128) hold. The Källén-Lehmann spectral representation of two-point functions then implies the following dispersion relations for the axialvector polarization functions of a given flavour $a = 3, 8$,

$$q^2 \Pi^{(0)}_{\mathrm{A}a}(q^2) - \lim_{q^2=0} \left(q^2 \Pi^{(0)}_{\mathrm{A}a}(q^2) \right) = q^2 \int_0^\infty ds \, \frac{\rho^{(0)}_{\mathrm{A}a}(s)}{s - q^2 - i\epsilon} \quad , \tag{129}$$

$$q^2 \Pi^{(1)}_{\mathrm{A}a}(q^2) - \lim_{q^2=0} \left(q^2 \Pi^{(1)}_{\mathrm{A}a}(q^2) \right) - \lim_{q^2=0} \frac{d}{dq^2} \left(q^2 \Pi^{(1)}_{\mathrm{A}a}(q^2) \right) = q^4 \int_0^\infty ds \, \frac{\rho^{(1)}_{\mathrm{A}a}(s)}{s\,(s - q^2 - i\epsilon)} \quad . \tag{130}$$

We work with $q^2 \Pi^{(0),(1)}_{\mathrm{A}a}(q^2)$ due to the presence of $q^2 = 0$ kinematic poles. Moreover, the subtraction constants have been placed on the left hand side in Eqs. (129),(130) in order to equate only physically observable quantities. Dispersion relations involving $SU(3)$-breaking combinations have an improved asymptotic behaviour, such as

$$(\Pi^{(1)}_{\mathrm{A}3} + \Pi^{(0)}_{\mathrm{A}3} - \Pi^{(1)}_{\mathrm{A}8} - \Pi^{(0)}_{\mathrm{A}8})(q^2) = \int_0^\infty ds \, \frac{(\rho^{(1)}_{\mathrm{A}3} + \rho^{(0)}_{\mathrm{A}3} - \rho^{(1)}_{\mathrm{A}8} - \rho^{(0)}_{\mathrm{A}8})(s)}{s - q^2 - i\epsilon} \tag{131}$$

and

$$q^2 (\Pi^{(1)}_{\mathrm{A}3} - \Pi^{(1)}_{\mathrm{A}8})(q^2) - \lim_{q^2=0} \left(q^2 \left(\Pi^{(1)}_{\mathrm{A}3} - \Pi^{(1)}_{\mathrm{A}8} \right)(q^2) \right) = q^2 \int_0^\infty ds \, \frac{\left(\rho^{(1)}_{\mathrm{A}3} - \rho^{(1)}_{\mathrm{A}8} \right)(s)}{s - q^2 - i\epsilon} \quad . \tag{132}$$

Sum rules are obtained by evaluating arbitrary derivatives of such relations at $q^2 = 0$. For sum rules inferred from Eqs. (129),(130) it is preferable to express the left hand side in terms of $\hat{\Pi}^{(0,1)}_{\mathrm{A}a}$,

$$\frac{1}{n!} \left[\frac{d}{dq^2} \right]^n \hat{\Pi}^{(0)}_{\mathrm{A}a}(0) = \int_0^\infty ds \, \frac{\bar{\rho}^{(0)}_{\mathrm{A}a}(s)}{s^n} \qquad (n \geq 1) \; , \tag{133}$$

$$\frac{1}{(n-1)!} \left[\frac{d}{dq^2} \right]^{n-1} \hat{\Pi}^{(1)}_{\mathrm{A}a}(0) - \frac{1}{n!} \left[\frac{d}{dq^2} \right]^n \hat{\Pi}^{(0)}_{\mathrm{A}a}(0) = \int_0^\infty ds \, \frac{\rho^{(1)}_{\mathrm{A}a}(s)}{s^n} \qquad (n \geq 2) \quad , \tag{134}$$

where $\bar{\rho}^{(0)}_{\mathrm{A}a}(s)$ is defined in Eq. (5). Finally, Eq. (132) leads directly to the following sequence of sum rules explicitly involving broken $SU(3)$,

$$\frac{1}{n!} \left[\frac{d}{dq^2} \right]^n \left(\Pi^{(1)}_{\mathrm{A}3} + \Pi^{(0)}_{\mathrm{A}3} - \Pi^{(1)}_{\mathrm{A}8} - \Pi^{(0)}_{\mathrm{A}8} \right)(0) = \int_0^\infty ds \, \frac{(\rho^{(1)}_{\mathrm{A}3} + \rho^{(0)}_{\mathrm{A}3} - \rho^{(1)}_{\mathrm{A}8} - \rho^{(0)}_{\mathrm{A}8})(s)}{s^{n+1}} \; , \tag{135}$$

where $n \geq 0$.

For this last sum rule, let us consider in some detail the case $n = 0$. An equivalent form, better suited for phenomenological analysis, is given by

$$\left(\hat{\Pi}^{(1)}_{\mathrm{A}3} - \hat{\Pi}^{(1)}_{\mathrm{A}8} \right)(0) - \frac{d}{dq^2} \left(\hat{\Pi}^{(0)}_{\mathrm{A}3} - \hat{\Pi}^{(0)}_{\mathrm{A}8} \right)(0) = \int_0^\infty ds \, \frac{\left(\rho^{(1)}_{\mathrm{A}3} - \rho^{(1)}_{\mathrm{A}8} \right)(s)}{s} \quad . \tag{136}$$

Evaluation of the left hand side (LHS) of this sum rule yields

$$\begin{aligned}
\mathrm{LHS} = & \frac{16}{3}\frac{M_K^2 - M_\pi^2}{F_\pi^2} Q_{\mathrm{A}}(\mu) \ + \ 0.001053 \\
& + \frac{1}{F_\pi^2 (16\pi^2)^2} \Big[M_\pi^2 \log\frac{M_\pi^2}{\mu^2}\left(\frac{8}{9} - \frac{1}{3}\log\frac{M_\pi^2}{\mu^2} - \frac{1}{3}\log\frac{M_K^2}{\mu^2}\right) \\
& \qquad + M_\pi^2 \log\frac{M_K^2}{\mu^2}\left(-\frac{1}{18} + \frac{1}{6}\log\frac{M_K^2}{\mu^2}\right) \\
& \qquad + M_K^2 \log\frac{M_K^2}{\mu^2}\left(-\frac{5}{6} + \frac{1}{2}\log\frac{M_K^2}{\mu^2}\right) \\
& \qquad + 128\pi^2 L_{10}^{(0)} \left(M_K^2 \log\frac{M_K^2}{\mu^2} - M_\pi^2 \log\frac{M_\pi^2}{\mu^2}\right) \Big] \ . \qquad (137)
\end{aligned}$$

Recall from the discussion at the beginning of Sect. VI that there are three distinct sources for the finite low energy terms: (i) $\mathcal{O}(p^6)$ CTs $\{B_\ell^{(0)}\}$, (ii) the 'remnant' contributions, and (iii) the finite Y, Z integrals (*cf* Eq. (92)). It is a combination of the latter two which give rise to the numerical term (which is scale-independent and vanishes in the $SU(3)$ limit of equal masses) in the first line. We have displayed all chiral logs explicitly, and $Q_{\mathrm{A}}(\mu)$ is the $\mathcal{O}(p^6)$ counterterm defined earlier in Eq. (94). Since the full expression is scale-independent, this allows one to directly read off the variation of the contributing counterterm combination at renormalization scale μ.

We can use the sum rule of Eq. (137) to numerically estimate $Q_{\mathrm{A}}(\mu)$. One needs to evaluate the spectral integral on the right hand side (RHS) of Eq. (136). For our purposes, it is sufficient to approximate the contribution of the isospin spectral function in terms of the a_1 resonance taken in narrow width approximation, $\rho_{\mathrm{A3}}^{(1)\mathrm{res}}(s) \simeq g_{a_1}\delta(s - M_{a_1}^2)$. Employing resonance parameters as obtained from the fit in Ref. [29], we obtain

$$\int_0^\infty ds\ \frac{\rho_{\mathrm{A3}}^{(1)\mathrm{res}}}{s} \simeq 0.0189 \ . \qquad (138)$$

Although consistency with QCD dictates that we also include the large-s continuum [30], the leading-order contributions would cancel in Eq. (136) and the remaining mass corrections are small. As regards the hypercharge spectral function $\rho_{\mathrm{A8}}^{(1)}$, little is presently known. The lowest lying resonances which contribute are f_1 (1285) and f_1 (1510) but the couplings of these resonances to the axialvector current have not been determined. Since the corresponding sum rule for vector current spectral functions [14] exhibits large cancellations between the contributions from ρ (770), ω (782) and Φ (1020), we expect a similar cancellation to be at work in the axialvector sector. To obtain a rough estimate, we assume the two resonances f_1 (1285) and f_1 (1510) can be approximated by a single effective resonance with spectral function $\rho_{\mathrm{A8}}^{(1)\mathrm{eff}}(s) \simeq g_{a_8}\delta(s - M_{a_8}^2)$. Assuming further $g_{a_8} \simeq g_{a_1}$ and $M_{a_8} \simeq 1.4$ GeV we estimate the hypercharge contribution to the RHS of Eq. (136) as 0.012. Allowing for a 50 % error in this estimate places the RHS in the range $0.001 \le RHS \le 0.013$ and leads finally to

$$0.000043 \le Q_{\mathrm{A}}(M_{a_1}) \le 0.000130 \ \ , \qquad (139)$$

where the renormalization scale $\mu = M_{a_1}$ has been adopted. This is clearly to be taken as just a rough estimate. Only experimental determination of the missing coupling constants can provide a more reliable estimate. In addition, a more thorough phenomenological analysis will involve use of the entire spectrum. However, this example serves to illustrate the general procedure.

We have not touched on chiral sum rules involving *both* vector and axialvector spectral functions. The most prominent example of this type is the Das-Mathur-Okubo (DMO) sum rule [20] which, in modern terminology, has been employed to determine the LEC $L_{10}^{(0)}$ [3,28]. In Ref. [25] we have shown how the DMO sum rule must be modified to be valid to second order in the light quark masses. Recently, τ-decay data has renewed interest in this sum rule from the experimental side [31]. We have begun a phenomenological study of the DMO sum rule using the two-loop results of polarization functions obtained both here and in Ref. [9]. Results will be reported elsewhere.

Finally, there are also those sum rules involving *no* $\mathcal{O}(p^6)$ counterterm coupling constants, *i.e.* those obtained by taking appropriately many derivatives of the dispersion relations Eqs. (129)-(132). From experience with the corresponding inverse moment sum rules of vector current spectral functions [14], we expect these sum rules in general not to be verified. This is because the relevant physics (which involves the low-lying resonances) enters the relations only in higher order of the chiral expansion. A quantitative study of these sum rules is deferred to a forthcoming publication.

IX. CONCLUSIONS

Our analysis of axialvector current propagators in two-loop ChPT has led to a complete two-loop renormalization of the pion and eta masses and decay constants as well as the real-valued parts of the isospin and hypercharge polarization functions. It has yielded predictions for axialvector spectral functions and has allowed the derivation of spectral function sum rules.

Despite the complexity of many of the individual steps and results, the sum of tree, one-loop and two-loop contributions to the axialvector propagator yields a simple overall structure,

$$\Delta_{\mathrm{A},\mu\nu}(q^2) = (F^2 + \hat{\Pi}_{\mathrm{A}}^{(0)}(q^2))g_{\mu\nu} - \frac{F^2}{q^2 - M^2}q_\mu q_\nu + (2L_{10}^{(0)} - 4H_1^{(0)} + \hat{\Pi}_{\mathrm{A}}^{(1)}(q^2))(q_\mu q_\nu - q^2 g_{\mu\nu}) \ , \tag{140}$$

where flavor labelling is suppressed. Comparing this to the general decomposition of Eq. (4) yields

$$\Pi_{\mathrm{A}}^{(1)}(q^2) = 2L_{10}^{(0)} - 4H_1^{(0)} + \hat{\Pi}_{\mathrm{A}}^{(1)}(q^2) - \frac{F^2 + \hat{\Pi}_{\mathrm{A}}^{(0)}(q^2)}{q^2} \ ,$$
$$\Pi_{\mathrm{A}}^{(0)}(q^2) = \frac{\hat{\Pi}_{\mathrm{A}}^{(0)}(q^2)}{q^2} - \frac{F^2 M^2}{q^2(q^2 - M^2)} \ . \tag{141}$$

As noted earlier, there are kinematic poles at $q^2 = 0$ in both the spin-one and spin-zero polarization functions, but the sum $\Pi_{\mathrm{A}}^{(1)} + \Pi_{\mathrm{A}}^{(0)}$ is free of such singularities.

A large number of $\mathcal{O}(p^6)$ counterterms entered the axialvector calculation, and many constraints among them were obtained from the subtraction procedure. Thus given the total of 23 $\mathcal{O}(p^6)$ counterterms which appeared by employing the basis of Ref. [23], each of the $\bar{\lambda}^2$ and $\bar{\lambda}$ subtractions were found to yield 14 constraints. The analysis of vector propagators in Ref. [9] yielded another 3 conditions for each of the $\bar{\lambda}^2$ and $\bar{\lambda}$ subtractions. This total of 17 conditions constraining the $\{B_\ell^{(2)}\}$ and $\{B_\ell^{(1)}\}$ counterterms is of course universal and can be used together with results of other two-loop studies. We can summarize the remaining nine finite $\mathcal{O}(p^6)$ counterterms as

$$\begin{aligned} \text{Polarization Amplitudes}: &\ P_{\rm A}, Q_{\rm A}, R_{\rm A}, B_{11}^{(0)}, B_{13}^{(0)} \\ \text{Decay Constants}: &\ \{\tilde{B}_\ell\} \qquad (\ell = 1, \dots 4) \\ \text{Masses}: &\ \{\tilde{B}_\ell\} \qquad (\ell = 1, \dots 9)\ , \end{aligned} \tag{142}$$

where $P_{\rm A}, Q_{\rm A}, R_{\rm A}$ are defined in Eq. (94) and the $\{\tilde{B}_\ell\}$ in Eqs. (B10),(B20). We have made preliminary numerical estimates for $Q_{\rm A}$ in this paper (*cf* Eq. (139)) and for $P_{\rm A}$ in Ref. [25]. The counterterm $B_{11}^{(0)}$ is related to a contact term and is regularization dependent, much the same as the constant $H_1^{(0)}$ appearing in Eq. (140). However, these terms always drop out when physical observable quantities are considered. The constant $R_{\rm A}$ is seen to contribute equally to all flavour components of the axialvector polarization functions. It cannot therefore be accessed by the chiral sum rules involving broken $SU(3)$ considered in the previous section. However, by combining the results obtained here with the two-loop analysis of the vector current two-point functions, the combination $R - R_{\rm A}$ is seen to constitute a mass correction to the Das-Mathur-Okubo sum rule. The details of this analysis will be presented elsewhere. Finally, little is known about the nine constants $\{\tilde{B}_\ell\}$ $(\ell = 1, \dots, 9)$ which determine (together with the calculated loop contributions) the p^6 corrections to masses and decay constants. We have not attempted to estimate these constants (*e.g.* by the resonance saturation hypothesis) since as far as the axialvector two-point function is concerned, their contribution is implicit. However, the explicit expressions given here can be used in further ChPT studies when expressing bare masses and decay constants in terms of fully renormalized physical quantities.

In view of the length and difficulty of the calculation, it is reassuring that a broad range of independent checks was available to gauge the correctness of our results. We list the most important of them here:

1. Because the calculation involved *independent* determinations of isospin and hypercharge channels at each stage, the $SU(3)$ limit of equal masses provided numerous tests among the set of isospin and hypercharge decay constants, masses and polarization functions. As a by-product, it also revealed the presence of previously unnoticed identities among the sunset amplitudes.

2. As shown in Sect. VII, it was possible to determine spectral functions directly from the two-loop analysis or equivalently from a unitarity approach which employed one-loop amplitudes as input. In this way, both the specific sunset integrals as well as the structural relations of Eq. (141) were able to be tested successfully.

3. It turned out that although most of the divergent terms could be subtracted away with counterterms, there occurs no m_K^6 counterterm contribution in $M_\pi^{(6)2}$. To avoid

disaster, there must thus be a cancellation between sunset and nonsunset numerical terms. Such a cancellation indeed occurs and constitutes a nontrivial check on our determination of the sunset contribution.

4. Given that $\hat{\Pi}_{\rm A}^{(0)}(q^2)$ can be shown to vanish in the limit of zero quark mass, it follows that our final result in Eq. (140) has the correct chiral limit.

5. Lastly, we have explicitly verified in $\overline{MS}$ renormalization that the constant C is absent, as must be the case.

Yet more on this subject remains to be done. This is especially true of the material composing Sect. VIII, where an application of $SU(3)$-breaking sum rules to determine finite $\mathcal{O}(p^6)$ counterterms was discussed and additional points were raised. Future work will be needed to carefully analyze the axialvector sum rules, particularly the role of existing data to provide as precise a determination of the counterterms as experimental uncertainties allow. We can, of course, combine the results of the present axialvector study with the vector results of Ref. [9] to study an even wider range of sum rules (*e.g.* as with the proposed determination of $R_{\rm A}$ discussed above). On an even more ambitious level, our experience with such relations makes us optimistic about the possibility of describing a possible framework for interpreting chiral sum rules to *arbitrary* order in the chiral expansion. Finally, it will be of interest to reconsider the phenomenological extraction of spectral functions such as $\rho_{\rm A33}^{(0)}[3\pi]$ and to stimulate experimental efforts to extract spectral function information involving non-pionic particles such as kaons and etas.

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APPENDIX A: SUNSET INTEGRALS

In the following, we compile mathematical details related to the sunset amplitudes which appear in the two-loop analysis. First, we give integral expressions for the quantities occurring in the 1PI, vertex and self-energy amplitudes. Then we write down the finite sunset amplitudes which remain after the singular parts have been identified. Finally we discuss certain identities which relate various sunset integrals. Additional work on sunset integrals can be found in Refs. [32,10,12,17] for the equal mass case and in Refs. [33–35,11] for the general mass case.

1. Definitions of Sunset Integrals

For the sunset amplitudes containing unequal masses, we shall denote the mass occurring twice as 'M' and the third mass as 'm' (*e.g.* for $\bar{K}K\pi$ amplitudes, we have $M \to m_K$ and $m \to m_\pi$). The quantity $\mathcal{H}_{\mu\nu}$ appearing in the 1PI sunset amplitude of Eq. (32) is defined by the integral expression

$$\begin{aligned} F_0^2 \mathcal{H}_{\mu\nu}(q^2, m^2, M^2) &\equiv \int d\tilde{t}\; \frac{(q-3t)_\mu (q-3t)_\nu}{t^2 - m^2} \int d\tilde{b}\; \frac{1}{(b^2 - M^2)\cdot((Q-b)^2 - M^2)} \\ &= q_\mu q_\nu S - 3 q_\mu S_\nu - 3 q_\nu S_\mu + 9 S_{\mu\nu} \quad , \end{aligned} \tag{A1}$$

where $Q \equiv q - t$ and

$$\{S; S_\mu; S_{\mu\nu}\} \equiv \int d\tilde{t}\; \frac{\{1; t_\mu; t_\mu t_\nu\}}{t^2 - m^2} \int d\tilde{b}\; \frac{1}{(b^2 - M^2)\cdot((Q-b)^2 - M^2)} \quad . \tag{A2}$$

From covariance, we have

$$\begin{aligned} S_\mu(q^2, m^2, M^2) &\equiv q_\mu \bar{S}(q^2, m^2, M^2) \ , \\ S_{\mu\nu}(q^2, m^2, M^2) &\equiv q_\mu q_\nu S_1(q^2, m^2, M^2) + g_{\mu\nu} S_2(q^2, m^2, M^2) \quad , \end{aligned} \tag{A3}$$

Also appearing in Eq. (32) is $\mathcal{L}_{\mu\nu}$, defined by

$$F_0^2 \mathcal{L}_{\mu\nu}(q^2, m^2, M^2) = \int d\tilde{t}\; \frac{1}{t^2 - m^2} \int d\tilde{b}\; \frac{(Q-2b)_\mu (Q-2b)_\nu}{(b^2 - M^2)\cdot((Q-b)^2 - M^2)} \quad , \tag{A4}$$

which can be expressed in the equivalent form

$$\begin{aligned} F_0^2 \mathcal{L}_{\mu\nu}(q^2, m^2, M^2) \equiv \frac{1}{d-1} \Big[& q_\mu q_\nu \mathcal{K}_1(q^2, m^2, M^2) \\ & + g_{\mu\nu} \mathcal{K}_2(q^2, m^2, M^2) + 4 g_{\mu\nu} \left(\frac{3}{2} - \frac{4-d}{2} \right) A(m^2) A(M^2) \Big] \quad , \end{aligned} \tag{A5}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are given respectively in Eqs. (A18),(A19) below.

The vector-valued integrals $I_{1\mu}$ and $I_{2\mu}$ which contribute to the axialvector vertex function in Eq. (38), are defined as

$$I_{1\mu}\left(q^2;m^2;M^2;\Lambda\right) \equiv q_\mu I_1(q^2;m^2;M^2;\Lambda)$$
$$= \int d\tilde{t}\ \frac{(q-3t)_\mu}{t^2-m^2}\ \int d\tilde{b}\ \frac{Q^2-2q\cdot t+2b\cdot(Q-b)+\Lambda}{(b^2-M^2)\cdot((Q-b)^2-M^2)}\ , \quad \text{(A6)}$$
$$I_{2\mu}\left(q^2;m^2;M^2\right) \equiv q_\mu I_2(q^2;m^2;M^2)$$
$$= \int d\tilde{t}\ \frac{(q+t)^\nu}{t^2-m^2}\ \int d\tilde{b}\ \frac{(Q-2b)_\mu(Q-2b)_\nu}{(b^2-M^2)\cdot((Q-b)^2-M^2)}\ . \quad \text{(A7)}$$

Finally, in the calculation of self-energies, there appear quantities S, R and U. The function S is already defined in Eq. (A2), and we have for R and U,

$$R(q^2;m^2;M^2;\Lambda) \equiv$$
$$\int \frac{d\tilde{t}}{t^2-m^2}\ \int d\tilde{b}\ \frac{\left[(q-t)^2-2q\cdot t+2b\cdot(Q-b)+\Lambda\right]^2}{(b^2-M^2)\cdot((Q-b)^2-M^2)}\ , \quad \text{(A8)}$$
$$U(q^2;m^2;M^2) \equiv$$
$$\int \frac{d\tilde{t}}{t^2-m^2}\ \int d\tilde{b}\ \frac{\left((q+t)\cdot(2b+t-q)\right)^2}{(b^2-M^2)\cdot((Q-b)^2-M^2)}\ . \quad \text{(A9)}$$

Analysis reveals that all the sunset contributions can be expressed in terms of the functions S, $\bar{S}$, S_1, S_2, $\mathcal{K}_1$ and $\mathcal{K}_2$. This is already evident from Eq. (A1) for $\mathcal{H}_{\mu\nu}$ and from Eq. (A5) for $\mathcal{L}_{\mu\nu}$. One can deduce the additional relations

$$\begin{aligned} I_{1\mu}\left(q^2;m^2;M^2;\Lambda\right) &= -2q_\mu\left[2A^2(M^2)+A(m^2)A(M^2)\right] \\ &+\left(q_\mu S(q^2;m^2;M^2)-3S_\mu(q^2;m^2;M^2)\right)\left[2(q^2+m^2-M^2)+\Lambda\right] \\ &-6\left(q_\mu q_\nu S^\nu(q^2;m^2;M^2)-3q^\nu S_{\mu\nu}(q^2;m^2;M^2)\right) \end{aligned} \quad \text{(A10)}$$
$$\begin{aligned} I_{2\mu}\left(q^2;m^2;M^2\right) &= q_\mu\frac{2(3\ +\ d-4)}{d-1}A(m^2)A(M^2) \\ &+\frac{2F_0^2}{d-1}q_\mu\left(q^2\mathcal{K}_1(q^2;m^2;M^2)+\mathcal{K}_2(q^2;m^2;M^2)\right)\ , \end{aligned} \quad \text{(A11)}$$

as well as

$$\begin{aligned} R(q^2;m^2;M^2;\Lambda) &= \\ &\Lambda^2 S(q^2;m^2;M^2)-4\Lambda\Big[-(q^2+m^2-M^2)S(q^2;m^2;M^2) \\ &+3q_\mu S^\mu(q^2;m^2;M^2)-A^2(M^2)+A(M^2)A(m^2)\Big] \\ &+36q_\mu q_\nu S^{\mu\nu}(q^2;m^2;M^2)+24(M^2-m^2-q^2)q_\mu S^\mu(q^2;m^2;M^2) \\ &+4(q^2+m^2-M^2)^2S(q^2;m^2;M^2)+A^2(M^2)(4m^2-12q^2) \\ &+A(m^2)A(M^2)(8M^2-6q^2-6m^2) \end{aligned} \quad \text{(A12)}$$
$$\begin{aligned} U(q^2;m^2;M^2) &= \\ &\frac{4}{d-1}\Big[+(q^2+m^2)\left(\frac{3}{2}+\frac{d-4}{2}\right)A(m^2)A(M^2) \\ &+q^2\left(q^2\mathcal{K}_1(q^2;m^2;M^2)+\mathcal{K}_2(q^2;m^2;M^2)\right)\Big]\ . \end{aligned} \quad \text{(A13)}$$

The functions S, $\bar{S}$, S_1, S_2, $\mathcal{K}_1$ and $\mathcal{K}_2$ can in turn each be written as the sum of terms (which diverge in the $d \to 4$ limit) proportional to gamma functions plus finite-valued functions $\{Y_c^{(n)}(q^2, m^2, M^2)\}$ and $\{Z_c^{(n)}(q^2, m^2, M^2)\}$. Thus, we have

$$\begin{aligned} S(q^2, m^2, M^2) &= \frac{\Gamma^2(2-d/2)}{(4\pi)^d}\,(M^2)^{d-4} \\ &\quad \times \frac{1}{d-2}\Big[-2\left(\left(\frac{m^2}{M^2}\right)^{d/2-1} + \frac{1}{d-3}\right)M^2 + \frac{1}{5-d}m^2 \\ &\quad + \frac{4-d}{d(5-d)}q^2\Big] - Y_0^{(0)}m^2 + (2Y_0^{(1)} - Y_0^{(0)})q^2 \ , \end{aligned} \tag{A14}$$

$$\begin{aligned} \bar{S}(q^2, m^2, M^2) &= \frac{\Gamma^2(2-d/2)}{(4\pi)^d}\,(M^2)^{d-4} \\ &\quad \times \frac{1}{d}\left(-\frac{2}{d-3}M^2 - \frac{4-d}{(5-d)(d-2)}m^2 + \frac{4-d}{(5-d)(d+2)}q^2\right) \\ &\quad + (Y_0^{(0)} - 2Y_0^{(1)})m^2 + (3Y_0^{(2)} - 2Y_0^{(1)})q^2 + Z_0^{(1)} \ , \end{aligned} \tag{A15}$$

$$\begin{aligned} S_1(q^2, m^2, M^2) &= \frac{\Gamma^2(2-d/2)}{(4\pi)^d}\,(M^2)^{d-4} \\ &\quad \times \frac{1}{d+2}\left(-\frac{2}{d-3}M^2 - \frac{4-d}{d(5-d)}m^2 + \frac{4-d}{(d+4)(5-d)}q^2\right) \\ &\quad + (2Y_0^{(1)} - 3Y_0^{(2)})m^2 + (4Y_0^{(3)} - 3Y_0^{(2)})q^2 + 2Z_0^{(2)} \end{aligned} \tag{A16}$$

$$\begin{aligned} S_2(q^2, m^2, M^2) &= \frac{\Gamma^2(2-d/2)}{(4\pi)^d}\,(M^2)^{d-4} \\ &\quad \times \Big[\Big(-\frac{2}{d(d-2)}\left(\left(\frac{m^2}{M^2}\right)^{d/2} + \frac{2}{d-2}\right)M^4 - \frac{2}{d(d-2)(d-3)}m^2M^2 \\ &\quad + \frac{2}{d(d-3)(d+2)}q^2M^2 + \frac{1}{d(5-d)(d-2)}m^4 \\ &\quad + \frac{2(4-d)}{d(5-d)(d^2-4)}q^2m^2 + \frac{d-4}{d(5-d)(d+2)(d+4)}q^4\Big)\Big] \\ &\quad + \frac{1}{2}\Big((Y_0^{(1)} - Y_0^{(0)})m^4 + (2Y_0^{(3)} - 3Y_0^{(2)} + Y_0^{(1)})q^4 \\ &\quad + (4Y_0^{(1)} - 3Y_0^{(2)} - Y_0^{(0)})m^2q^2 - Z_0^{(1)}m^2 + (2Z_0^{(2)} - Z_0^{(1)})q^2\Big) \ , \end{aligned} \tag{A17}$$

and

$$\begin{aligned} \mathcal{K}_1(q^2, m^2, M^2) &= \frac{\Gamma^2(2-d/2)}{(4\pi)^d}\,(M^2)^{d-4}\,\frac{d-1}{d-2}\Big[-\frac{16}{d(d-3)(5-d)(d+2)}M^2 \\ &\quad + \left(-\frac{2}{3}\left(\frac{m^2}{M^2}\right)^{d/2-2} + \frac{24}{d(5-d)(7-d)(d+2)}\right)m^2 \\ &\quad + \frac{24(4-d)}{d(7-d)(5-d)(d+2)(d+4)}q^2\Big] + (6Y_1^{(1)} - 3Y_1^{(2)} - 3Y_1^{(0)})m^2 \\ &\quad + (4Y_1^{(3)} - 9Y_1^{(2)} + 6Y_1^{(1)} - Y_1^{(0)})q^2 + 2Z_1^{(2)} - 2Z_1^{(1)} \end{aligned} \tag{A18}$$

$$\mathcal{K}_2(q^2,m^2,M^2) = \frac{\Gamma^2(2-d/2)}{(4\pi)^d}\ (M^2)^{d-4}\ \frac{d-1}{d(d-2)}\ \Big[\frac{2(d-1)}{(d-3)(5-d)}m^2M^2$$
$$-\frac{4(d-1)}{(d-2)(d-3)}M^4 + \left(\frac{2(d-1)}{3}\left(\frac{m^2}{M^2}\right)^{d/2-2} - \frac{d-1}{(5-d)(7-d)}\right)m^4$$
$$+\left(\frac{2d}{3}\left(\frac{m^2}{M^2}\right)^{d/2-2} + \frac{2(d^2-5d-8)}{(5-d)(7-d)(d+2)}\right)q^2m^2$$
$$+\frac{2(6+3d-d^2)}{(d-3)(5-d)(d+2)}q^2M^2 + \frac{(4-d)(d^2-3d-22)}{(5-d)(7-d)(d+2)(d+4)}q^4\Big]$$
$$+\left(-7Y_1^{(3)} + \frac{27}{2}Y_1^{(2)} - \frac{15}{2}Y_1^{(1)} + Y_1^{(0)}\right)q^4$$
$$+\left(\frac{15}{2}Y_1^{(2)} - 12Y_1^{(1)} + \frac{9}{2}Y_1^{(0)}\right)m^2q^2 + \frac{3}{2}\left(Y_1^{(0)} - Y_1^{(1)}\right)m^4$$
$$+\left(\frac{7}{2}Z_1^{(1)} - 5Z_1^{(2)}\right)q^2 + \frac{3}{2}Z_1^{(1)}m^2\ . \tag{A19}$$

For the sake of simplicity, we have omitted the arguments of the $\{Y_c^{(n)}\}$ and the $\{Z_c^{(n)}\}$. We have verified that Eq. (A14) agrees in the equal mass limit with the explicit expression appearing in Ref. [17].

2. The Finite Sunset Integrals

Having identified the singular parts of the sunset functions by expanding these quantities in a Laurent series about $d = 4$, one can express the finite-valued functions $\{Y_c^{(n)}\}$ and $\{Z_c^{(n)}\}$ which remain by means of integral representations,

$$Y_c^{(n)} \equiv \frac{1}{(16\pi^2)^2}\int_{4M^2}^{\infty}\frac{d\sigma}{\sigma}\left(1-\frac{4M^2}{\sigma}\right)^{1/2+c}\int_0^1 dx\ x^n\ \ln(1+\Delta g)\quad , \tag{A20}$$

and

$$Z_c^{(n)} \equiv \frac{1}{(16\pi^2)^2}\int_{4M^2}^{\infty}d\sigma\ \left(1-\frac{4M^2}{\sigma}\right)^{1/2+c}\int_0^1 dx\ x^n\ \left(\ln(1+\Delta g) - \Delta g\right) \tag{A21}$$

where

$$\Delta g \equiv \left(\frac{m^2}{x} - q^2\right)\frac{1-x}{\sigma}\quad . \tag{A22}$$

For convenience, we shall introduce the dimensionless variables

$$\bar{q}^2 \equiv \frac{q^2}{4M^2} \qquad \text{and} \qquad r^2 \equiv \frac{m^2}{4M^2}\quad , \tag{A23}$$

and likewise work with the *reduced functions* $\bar{Y}_c^{(n)}$ and $\bar{Z}_c^{(n)}$,

$$Y_c^{(n)}(\bar{q}^2,r^2) \equiv \frac{1}{(16\pi^2)^2}\bar{Y}_c^{(n)}(\bar{q}^2,r^2)$$
$$Z_c^{(n)}(\bar{q}^2,r^2) \equiv \frac{4M^2}{(16\pi^2)^2}\bar{Z}_c^{(n)}(\bar{q}^2,r^2) \tag{A24}$$

One is allowed to express such finite quantities in terms of the physical meson masses, and it is unerstood we do so in the remainder of this section. For the six flavour configurations which can contribute to the sunset amplitude, the parameter r^2 takes on the numerical values

$$r^2 = \begin{cases} 0.016 & (\eta\eta\pi) \\ 0.020 & (\bar{K}K\pi) \\ 0.25 & (3\pi, 3\eta) \\ 0.31 & (\bar{K}K\eta) \\ 3.82 & (\pi\pi\eta) \ . \end{cases} \tag{A25}$$

a. Behaviour at $r^2 = 0$ and Near $q^2 = 0$

In the $r^2 = 0$ limit (*i.e.* $m^2 = 0$), analytic expressions can be obtained for $\bar{Y}_c^{(n)}$ and $\bar{Z}_c^{(n)}$

$$\bar{Y}_c^{(n)}(\bar{q}^2, 0) = -\sum_{k=1}^{\infty} \frac{B(k+1; n+1)\ B(k; a+3/2)}{k}\ \bar{q}^{2k} \quad , \tag{A26}$$

and

$$\bar{Z}_c^{(n)}(\bar{q}^2, 0) = -\sum_{k=2}^{\infty} \frac{B(k+1; n+1)\ B(k-1; a+3/2)}{k}\ \bar{q}^{2k} \quad , \tag{A27}$$

where $B(m; n)$ denotes the Euler beta function. Observe in the summations that the indices begin at $k = 1$ for $\bar{Y}_c^{(n)}$ and at $k = 2$ for $\bar{Z}_c^{(n)}$, *i.e.* that

$$\bar{Y}_c^{(n)}(0, 0) \ = \ \bar{Z}_c^{(n)}(0, 0) \ = \ \bar{Z}_c^{(n)\prime}(0, 0) \ = \ 0 \quad . \tag{A28}$$

For the more general case of nonzero r^2 but small q^2, it is useful to employ a power series

$$\begin{aligned} \bar{Y}_c^{(n)}(\bar{q}^2, r^2) &= \bar{Y}_c^{(n)}(0, r^2) + \bar{Y}_c^{(n)\prime}(0, r^2)\bar{q}^2 + \frac{1}{2}\bar{Y}_c^{(n)\prime\prime}(0, r^2)\bar{q}^4 + \ldots \\ \bar{Z}_c^{(n)}(\bar{q}^2, r^2) &= \bar{Z}_c^{(n)}(0, r^2) + \bar{Z}_c^{(n)\prime}(0, r^2)\bar{q}^2 + \frac{1}{2}\bar{Z}_c^{(n)\prime\prime}(0, r^2)\bar{q}^4 + \ldots \quad . \end{aligned} \tag{A29}$$

For nonzero r^2, one can obtain numerical values for the above $q^2 = 0$ derivatives of $\bar{Y}_c^{(n)}(\bar{q}^2, r^2)$ and $\bar{Z}_c^{(n)}(\bar{q}^2, r^2)$. Of course, the integral representations of Eqs. (A20),(A21) allow also for a straightforward numerical determination of the real part of the sunset amplitudes for arbitrary q^2. However, some care must be taken to obtain accurate values for q^2 close to or above three-particle thresholds.

b. Imaginary Parts

For $\bar{q}^2 < 1$, the finite sunset amplitudes are real-valued. However, $Y_c^{(n)}$ and $Z_c^{(n)}$ have a branch point singularity at $\bar{q}^2 = (1 + r)^2$ (corresponding to $q^2 = (2M + m)^2$) and become complex-valued for $\bar{q}^2 > (1+r)^2$. We shall be concerned here with determining the imaginary parts of these quantities.

Consider first the integral $X^{(n)}(\bar{q}^2, r^2)$ defined by

$$X^{(n)}(\bar{q}^2, r^2) \equiv \int_0^1 dx\ x^n\ \ln(1+\Delta g)\quad , \tag{A30}$$

which can be rewritten as

$$\begin{aligned} &X^{(n)}(\bar{q}^2, r^2) = \\ &\int_0^1 dx\ x^n \left[\ln\left(x^2 + x\left(\frac{1}{u\bar{q}^2} - 1 - \frac{r^2}{\bar{q}^2}\right) + \frac{r^2}{\bar{q}^2}\right) + \ln(u\bar{q}^2/x)\right] \\ &= \int_0^1 dx\ x^n \left[\ln\left((x-x_+)(x-x_-)\right) + \ln(u\bar{q}^2/x)\right]\quad , \end{aligned} \tag{A31}$$

where $x_\pm$ are given by

$$x_\pm = \frac{1}{2}\left[1 - \frac{1}{u\bar{q}^2} + \frac{r^2}{\bar{q}^2} \pm \sqrt{\left(1 - \frac{1}{u\bar{q}^2} + \frac{r^2}{\bar{q}^2}\right)^2 - \frac{4r^2}{\bar{q}^2}}\ \right]\quad . \tag{A32}$$

The imaginary part of $X^{(n)}(\bar{q}^2, r^2)$ will occur when the argument of the first logarithm in the above becomes negative,

$$\begin{aligned} \mathcal{I}m\ X^{(n)}(\bar{q}^2, r^2) &= \int_0^1 dx\ x^n\ \mathcal{I}m\ \ln\left((x-x_+)(x-x_-)\right) \\ &= -\pi \int_{x_-}^{x_+} dx\ x^n \\ &= -\frac{\pi}{n+1}\left(x_+^{n+1} - x_-^{n+1}\right)\quad , \end{aligned} \tag{A33}$$

so that

$$\mathcal{I}m\ \bar{Y}_c^{(n)}(\bar{q}^2, r^2) = -\frac{\pi}{n+1}\int_{u_0}^1 \frac{du}{u}(1-u)^{1/2+a}\left(x_+^{n+1} - x_-^{n+1}\right)\quad , \tag{A34}$$

where

$$u_0 = \frac{1}{\left(\sqrt{\bar{q}^2} - \sqrt{r^2}\right)^2}\quad . \tag{A35}$$

The lower limit u_0 on the u-integral is simply a reflection of the branch point occurring in the sunset amplitude at $q^2 = (2M+m)^2$. Proceeding in like manner leads to the following formula for $\mathcal{I}m\ Z_c^{(n)}$,

$$\mathcal{I}m\ \bar{Z}_c^{(n)}(\bar{q}^2, r^2) = -\frac{\pi}{n+1}\int_{u_0}^1 \frac{du}{u^2}(1-u)^{1/2+a}\left(x_+^{n+1} - x_-^{n+1}\right)\quad . \tag{A36}$$

3. Identities

Given the set of sunset integrals $\{S; S_\mu; S_{\mu\nu}\}$, it is not difficult to infer the following 'trace identity',

$$S^\mu_\mu(q^2, m^2, M^2) = m^2 S(q^2, m^2, M^2) + A^2(M^2) \quad , \tag{A37}$$

which is valid for arbitrary kinematics.

It turns out that several more identities become derivable in the equal mass limit of $SU(3)$ symmetry. This is a consequence of the symmetry constraint that the isospin and hypercharge results agree. Indeed, their direct comparison serves to check the correctness of the calculation. Interestingly, the identities discovered in the $SU(3)$ limit are typically not at all *a priori* obvious. Below, we list and indicate the source of relations:

1. Relating $\bar{S}$ to S:

$$\bar{S}(q^2, m^2, m^2) = \frac{1}{3} S(q^2, m^2, m^2) \quad . \tag{A38}$$

2. 1PI amplitudes:

$$\mathcal{H}_{\mu\nu}(q^2, m^2, m^2) = 3\mathcal{L}_{\mu\nu}(q^2, m^2, m^2) \quad . \tag{A39}$$

3. Vertex functions:

$$\begin{aligned} I_{1\mu}(q^2; m^2; m^2; \Lambda) &= I_{1\mu}(q^2; m^2; m^2; 0) \\ I_{1\mu}(q^2; m^2; m^2; 0) &= 3I_{2\mu}(q^2; m^2; m^2; 0) \end{aligned} \tag{A40}$$

4. Self-energies:

$$\begin{aligned} R(q^2; m^2; m^2; \Lambda) &= R(q^2; m^2; m^2; 0) + \Lambda^2 S(q^2; m^2; m^2) \\ R(q^2; m^2; m^2; 0) &= 3\, U(q^2; m^2; m^2) \quad . \end{aligned} \tag{A41}$$

APPENDIX B: COMPENDIUM OF FINITE RESULTS

In this Appendix we complete the compilation begun in Sect. VI of finite results in our two-loop calculation. We list in turn expressions for the hypercharge polarization functions, then the pion and eta decay constants and finally the pion and eta masses. The Appendix concludes with a brief summary relating our $\overline{\lambda}$-subtraction renormalization with the $\overline{MS}$ scheme of Ref. [17].

1. Hypercharge Polarization Functions

First we display the corresponding hypercharge results, beginning with the remnant piece of the polarization function $\hat{\Pi}_8^{(1)}$,

$$
\begin{aligned}
F_\pi^2 \hat{\Pi}^{(1)}_{8,\mathrm{rem}} = &\frac{M_\pi^2}{\pi^4}\left[\frac{13}{6144} - \frac{C}{512} + \frac{1}{512}\log\frac{M_K^2}{\mu^2}\right] \\
&+\frac{M_K^2}{\pi^4}\left[-\frac{1}{4096} - \frac{9\ C}{1024} + \log\frac{M_K^2}{\mu^2}\left(\frac{9}{1024} - \frac{3\pi^2}{4}L_{10}^{(0)} - \frac{3}{1024}\log\frac{M_K^2}{\mu^2}\right)\right] \\
&+\frac{q^2}{\pi^4}\left[-\frac{35}{32768} + \frac{3\ C}{4096} - \frac{3}{4096}\log\frac{M_K^2}{\mu^2}\right] \quad ,
\end{aligned} \tag{B1}
$$

and then the counterterm contribution,

$$
\hat{\Pi}^{(1)}_{8,\mathrm{CT}}(q^2) = -\frac{q^2}{F_\pi^2}P_\mathrm{A} - \frac{4M_\pi^2}{F_\pi^2}\left(R_\mathrm{A} - \frac{1}{3}Q_\mathrm{A}\right) - \frac{8M_K^2}{F_\pi^2}\left(R_\mathrm{A} + \frac{2}{3}Q_\mathrm{A}\right) \quad . \tag{B2}
$$

As was explained in Sect. VI, the $L_{10}^{(-1)}$ dependence from the polarizations $\hat{\Pi}^{(1)}_{3,8,\mathrm{rem}}$ has been removed via the definitions of $P_\mathrm{A}, Q_\mathrm{A}, R_\mathrm{A}$ given in Eq. (94).

Concluding with the part from the finite functions, we have

$$
\hat{\Pi}^{(1)}_{8,\mathrm{YZ}}(q^2) = \frac{1}{2}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2, M_\pi^2, M_K^2) + \frac{1}{2}\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2, M_\eta^2, M_K^2) - R_{8,\mathrm{YZ}} \quad , \tag{B3}
$$

where, making use of the notation established in Eq. (100),

$$
\begin{aligned}
R_{8,\mathrm{YZ}}(q^2) = &\frac{1}{2(q^2 - M_\eta^2)}\Big[\overline{I}_{1,\mathrm{YZ}}(q^2; M_\pi^2; M_K^2; \frac{2(M_\pi^2 - M_K^2)}{3}) \\
&+\overline{I}_{1,\mathrm{YZ}}(q^2; M_\eta^2; M_K^2; \frac{2(3M_K^2 - M_\pi^2)}{3})\Big] - \frac{1}{(q^2 - M_\eta^2)^2}\Big[\frac{M_\pi^4}{6}\check{S}_{\mathrm{YZ}}(q^2; M_\eta^2; M_\pi^2) \\
&+\frac{(16M_K^2 - 7M_\pi^2)^2}{486}\check{S}_{\mathrm{YZ}}(q^2; M_\eta^2; M_\eta^2) + \frac{1}{8}\check{R}(q^2; M_\pi^2; M_K^2; \frac{2(M_\pi^2 - M_K^2)}{3}) \\
&+\frac{1}{8}\check{R}(q^2; M_\eta^2; M_K^2; \frac{2(3M_K^2 - M_\pi^2)}{3})\Big] \quad ,
\end{aligned} \tag{B4}
$$

and at $q^2 = 0$, we find

$$
F_\pi^2 \hat{\Pi}^{(1)}_{8,\mathrm{YZ}}(0) = 5.494 \times 10^{-6}\ \mathrm{GeV}^2 \quad . \tag{B5}
$$

The last of the polarization functions is $\hat{\Pi}_8^{(0)}$, for which the remnant piece is

$$
\begin{aligned}
&F_\pi^2 \hat{\Pi}^{(0)}_{8,\mathrm{rem}}(q^2) = \\
&\frac{M_K^4}{\pi^4}\left(-\frac{2971}{497664} + \frac{307\ C}{62208} - \frac{1}{256}\log\frac{M_K^2}{\mu^2} - \frac{1}{972}\log\frac{M_\eta^2}{\mu^2}\right) \\
&+\frac{M_\pi^2 M_K^2}{\pi^4}\left(\frac{8347}{995328} - \frac{787\ C}{124416} + \frac{25}{4608}\log\frac{M_K^2}{\mu^2} + \frac{7}{7776}\log\frac{M_\eta^2}{\mu^2}\right) \\
&\frac{M_\pi^4}{\pi^4}\left(-\frac{28123}{7962624} + \frac{2707\ C}{995328} - \frac{1}{3072}\log\frac{M_\pi^2}{\mu^2} - \frac{49}{248832}\log\frac{M_\eta^2}{\mu^2}\right. \\
&\qquad\left. - \frac{9}{4096}\log\frac{M_K^2}{\mu^2}\right) \quad .
\end{aligned} \tag{B6}
$$

Next comes the counterterm contribution,

$$\hat{\Pi}^{(0)}_{8,\mathrm{CT}} = M_\pi^4 \left(-4B_{11}^{(0)} + \frac{32}{3} B_{13}^{(0)} \right) + M_\pi^2 M_K^2 \frac{32}{3} \left(B_{11}^{(0)} - 2B_{13}^{(0)} \right) + M_K^4 \frac{32}{3} \left(-B_{11}^{(0)} + B_{13}^{(0)} \right) \Big] \ , \tag{B7}$$

followed finally by the piece from the finite functions,

$$\begin{aligned} F_\pi^2 \hat{\Pi}^{(0)}_{8,\mathrm{YZ}}(q^2) &= \frac{9}{2} S_{2,\mathrm{YZ}}(q^2, M_\pi^2, M_K^2) + \frac{9}{2} S_{2,\mathrm{YZ}}(q^2, M_\eta^2, M_K^2) \\ &+ q^2 \Big(\frac{1}{2} \mathcal{H}^{qq}_{\mathrm{YZ}}(q^2, M_\pi^2, M_K^2) + \frac{1}{2} \mathcal{H}^{qq}_{\mathrm{YZ}}(q^2, M_\eta^2, M_K^2) - R_{8,\mathrm{YZ}}(q^2) \Big) \\ &- 2 \Big[\frac{1}{4} I_{1,\mathrm{YZ}} \left(M_\eta^2; M_\pi^2; M_K^2; \frac{2}{3}(M_\pi^2 - M_K^2) \right) \\ &+ \frac{1}{4} I_{1,\mathrm{YZ}} \left(M_\eta^2; M_\eta^2; M_K^2; \frac{2}{3}(3M_K^2 - M_\pi^2) \right) \\ &- \frac{M_\pi^4}{12} S'_{\mathrm{YZ}}(M_\eta^2; M_\eta^2; M_\pi^2) - \frac{(16M_K^2 - 7M_\pi^2)^2}{972} S'_{\mathrm{YZ}}(M_\eta^2; M_\eta^2; M_\eta^2) \\ &- \frac{1}{16} R'_{\mathrm{YZ}}(M_\eta^2; M_\pi^2; M_K^2; \frac{2}{3}(M_\pi^2 - M_K^2)) \\ &- \frac{1}{16} R'_{\mathrm{YZ}}(M_\eta^2; M_\eta^2; M_K^2; \frac{2}{3}(3M_K^2 - M_\pi^2)) \Big] \ . \end{aligned} \tag{B8}$$

The numerical value of the YZ-part at $q^2 = 0$ is

$$F_\pi^2 \hat{\Pi}^{(0)}_{8,\mathrm{YZ}}(0) = -1.753 \times 10^{-6}\ \mathrm{GeV}^4 \ . \tag{B9}$$

2. Meson Decay Constants

Before displaying explicit forms for $F^{(4)}_{\pi,\eta}$, we must (i) implement the procedure (*cf* Sect. VI) for removing all contributions from the $\{L_\ell^{(-1)}\}$ counterterms and (ii) re-express all one-loop mass and decay constant corrections in terms of one-loop renormalized quantities.

There are *a priori* seven of the $\mathcal{O}(p^4)$ counterterms $L_\ell^{(-1)}$ ($\ell = 1, \ldots, 6, 8$) which contribute to the decay constants $F^{(4)}_{(\pi,\eta),\mathrm{rem}}$. Analogous to the procedure followed for the polarization functions $\hat{\Pi}^{(1)}_{(3,8),\mathrm{rem}}$, we remove the $\{L_\ell^{(-1)}\}$ dependence by defining the following four effective $\mathcal{O}(p^4)$ counterterms,

$$\begin{aligned} \tilde{B}_1 &\equiv F_\pi^2 \left(A^{(0)} - 3E^{(0)} \right) + \frac{28}{3} L_1^{(-1)} + \frac{34}{3} L_2^{(-1)} + \frac{25}{3} L_3^{(-1)} \\ &\quad - \frac{26}{3} L_4^{(-1)} + \frac{8}{3} L_5^{(-1)} + 12 L_6^{(-1)} - 12 L_8^{(-1)} \ , \\ \tilde{B}_2 &\equiv F_\pi^2 \left(B^{(0)} - 2E^{(0)} \right) + \frac{32}{9} L_1^{(-1)} + \frac{8}{9} L_2^{(-1)} + \frac{8}{9} L_3^{(-1)} \\ &\quad - \frac{106}{9} L_4^{(-1)} + \frac{22}{9} L_5^{(-1)} + 20 L_6^{(-1)} \ , \end{aligned}$$

$$\tilde{B}_3 \equiv F_\pi^2 \left(C^{(0)} + E^{(0)}\right) - \frac{28}{9} L_1^{(-1)} - \frac{34}{9} L_2^{(-1)} - \frac{59}{18} L_3^{(-1)} + \frac{26}{9} L_4^{(-1)} - 3L_5^{(-1)} - 4L_6^{(-1)} + 6L_8^{(-1)} ,$$

$$\tilde{B}_4 \equiv F_\pi^2 D^{(0)} - \frac{104}{9} L_1^{(-1)} - \frac{26}{9} L_2^{(-1)} - \frac{61}{18} L_3^{(-1)} + \frac{34}{9} L_4^{(-1)} - L_5^{(-1)} + 4L_6^{(-1)} + 2L_8^{(-1)} \quad . \tag{B10}$$

It is natural to express the one-loop mass and decay constant corrections in terms of *one-loop renormalized quantities*. Thus, in the one-loop expressions for the decay constants and masses, we shall replace the tree-level parameters by their one-loop corrected counterparts,

$$m_i^2 \to M_i^2 + \Delta_i \ , \qquad F_0 \to F_i + \delta_i \ , \qquad (i = \pi, K, \eta) \ . \tag{B11}$$

The quantities Δ_i, δ_i $(i = \pi, K, \eta)$ are compiled in Ref. [9] (also see Eq. (25) of this paper). Thus, we write the decay constant relations for F_π and F_η through one-loop as

$$F_\pi = F_0 + \frac{1}{F_\pi}\Big[4L_4^{(0)}\left(M_\pi^2 + 2M_K^2\right) + 4L_5^{(0)} M_\pi^2 - \frac{M_\pi^2}{16\pi^2} \ln \frac{M_\pi^2}{\mu^2} - \frac{M_K^2}{32\pi^2} \ln \frac{M_K^2}{\mu^2}\Big] + \ldots$$

$$F_\eta = F_0 + \frac{1}{F_\pi}\Big[4L_4^{(0)}\left(M_\pi^2 + 2M_K^2\right) + 4L_5^{(0)} M_\eta^2 - \frac{3M_K^2}{32\pi^2} \ln \frac{M_K^2}{\mu^2}\Big] + \ldots \ , \tag{B12}$$

and the mass relations for M_π^2, M_K^2 and M_η^2 through one-loop as

$$M_\pi^2 = m_\pi^2 + \frac{M_\pi^2}{F_\pi^2}\Big[- 8\left(L_4^{(0)} - 2L_6^{(0)}\right)\left(M_\pi^2 + 2M_K^2\right) - 8\left(L_5^{(0)} - 2L_8^{(0)}\right) M_\pi^2 + \frac{M_\pi^2}{32\pi^2} \ln \frac{M_\pi^2}{\mu^2} - \frac{M_\eta^2}{96\pi^2} \ln \frac{M_\eta^2}{\mu^2}\Big] + \ldots \ ,$$

$$M_K^2 = m_K^2 + \frac{M_K^2}{F_\pi^2}\Big[- 8\left(L_4^{(0)} - 2L_6^{(0)}\right)\left(M_\pi^2 + 2M_K^2\right) - 8\left(L_5^{(0)} - 2L_8^{(0)}\right) M_K^2 + \frac{M_\eta^2}{48\pi^2} \ln \frac{M_\eta^2}{\mu^2}\Big] + \ldots \ ,$$

$$M_\eta^2 = m_\eta^2 + \frac{M_\eta^2}{F_\pi^2}\Big[- 8\left(L_4^{(0)} - 2L_6^{(0)}\right)\left(M_\pi^2 + 2M_K^2\right) - 8\left(L_5^{(0)} - 2L_8^{(0)}\right) M_\eta^2 + \frac{M_K^2}{16\pi^2} \ln \frac{M_K^2}{\mu^2} - \frac{M_\eta^2}{24\pi^2} \ln \frac{M_\eta^2}{\mu^2}\Big] + \frac{M_\pi^2}{F_\pi^2}\Big[- \frac{M_\pi^2}{32\pi^2} \ln \frac{M_\pi^2}{\mu^2} + \frac{M_K^2}{48\pi^2} \ln \frac{M_K^2}{\mu^2} + \frac{M_\eta^2}{96\pi^2} \ln \frac{M_\eta^2}{\mu^2}\Big] + \frac{128\ (M_K^2 - M_\pi^2)^2}{9\ F_\pi^2}\left(3L_7^{(0)} + L_8^{(0)}\right) \quad . \tag{B13}$$

The expressions in Eqs. (B12)-(B13) constitute our *conventions* for these quantities. In adopting our convention, we have made two kinds of choices:

1. The prefactors $1/F_0$ (for decay constants) and $1/F_0^2$ (for masses) have been replaced respectively by $1/F_\pi$ and $1/F_\pi^2$ since F_π is the most accurately determined decay constant.

2. We have employed M_η^2 explicitly throughout rather than use the GMO relation to replace it.

A consequence of the procedure just described is to introduce additional 'spill-over' corrections at two-loop order to the set of decay constants and masses which were calculated earlier in the paper. Such contributions depend on the choice of convention discussed above. It is understood that the two-loop decay constants and masses ($F_{\pi,\text{rem}}^{(4)}$, $F_{\eta,\text{rem}}^{(4)}$, $M_{\pi,\text{rem}}^{(6)2}$, $M_{\eta,\text{rem}}^{(6)2}$) listed in the remainder of this appendix contain these spill-over corrections. Finally, for convenience we use the GMO relation in two-loop contributions to eliminate all factors of the eta mass not occurring inside logarithms. Any error thereby made would appear in higher orders.

We have then for the pion decay constant,

$$
\begin{aligned}
F_\pi^3 F_{\pi,\text{rem}}^{(4)} = M_\pi^4 \Bigg(& -\frac{283}{589824\pi^4} - \frac{547\ C}{73728\pi^4} + \frac{1}{\pi^2}\left[-\frac{1}{8}L_1^{(0)} - \frac{37}{144}L_2^{(0)} - \frac{7}{108}L_3^{(0)}\right] \\
& + 8L_4^{(0)}\left[5L_4^{(0)} + 10L_5^{(0)} - 8L_6^{(0)} - 8L_8^{(0)}\right] + 8L_5^{(0)}\left[5L_5^{(0)} - 8L_6^{(0)} - 8L_8^{(0)}\right] \\
& + \frac{1}{\pi^4}\log\frac{M_K^2}{\mu^2}\left[-\frac{3}{8192} - \frac{1}{12288}\log\frac{M_K^2}{\mu^2}\right] \\
& + \frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[-\frac{1}{36864\pi^2} + \frac{1}{18}L_1^{(0)} + \frac{1}{72}L_2^{(0)} + \frac{1}{72}L_3^{(0)} - \frac{1}{24}L_4^{(0)}\right. \\
& \left. - \frac{1}{12288\pi^2}\log\frac{M_K^2}{\mu^2}\right] + \frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\left[\frac{119}{12288\pi^2} + \frac{7}{4}L_1^{(0)} + L_2^{(0)}\right. \\
& \left. + \frac{7}{8}L_3^{(0)} - \frac{9}{8}L_4^{(0)} - \frac{3}{4}L_5^{(0)} + \frac{1}{4096\pi^2}\log\frac{M_K^2}{\mu^2} + \frac{1}{1024\pi^2}\log\frac{M_\pi^2}{\mu^2}\right]\Bigg) \\
+ M_\pi^2 M_K^2 \Bigg(& \frac{101}{24576\pi^4} - \frac{19\ C}{6144\pi^4} + 32L_4^{(0)}\left[5L_4^{(0)} + 3L_5^{(0)}\right] - 128L_6^{(0)}\left[2L_4^{(0)} + L_5^{(0)}\right] \\
& + \frac{1}{\pi^2}\left[\frac{1}{18}L_2^{(0)} + \frac{1}{54}L_3^{(0)}\right] + \frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\left[\frac{13}{6144\pi^2} - \frac{1}{8}L_4^{(0)} - \frac{3}{8}L_5^{(0)}\right. \\
& \left. + \frac{7}{6144\pi^2}\log\frac{M_K^2}{\mu^2} + \frac{1}{512\pi^2}\log\frac{M_\pi^2}{\mu^2}\right] \\
& + \frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[\frac{1}{1536\pi^2} - \frac{4}{9}L_1^{(0)} - \frac{1}{9}L_2^{(0)} - \frac{1}{9}L_3^{(0)}\right. \\
& \left. + \frac{1}{3}L_4^{(0)} + \frac{1}{1536\pi^2}\log\frac{M_K^2}{\mu^2}\right] - \frac{1}{2\pi^2}L_4^{(0)}\log\frac{M_\pi^2}{\mu^2}\Bigg) \\
+ M_K^4 \Bigg(& -\frac{91}{24576\pi^4} - \frac{43\ C}{6144\pi^4} + 32L_4^{(0)}\left[5L_4^{(0)} + 2L_5^{(0)} - 8L_6^{(0)} - 4L_8^{(0)}\right]
\end{aligned}
\tag{B14}
$$

$$+\frac{1}{\pi^2}\left[-\frac{13}{36}L_2^{(0)}-\frac{43}{432}L_3^{(0)}\right]+\frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\left[\frac{1}{96\pi^2}+2L_1^{(0)}\right.$$
$$\left.+\frac{1}{2}L_2^{(0)}+\frac{5}{8}L_3^{(0)}-\frac{5}{4}L_4^{(0)}-\frac{1}{768\pi^2}\log\frac{M_K^2}{\mu^2}\right]$$
$$+\frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[-\frac{1}{768\pi^2}+\frac{8}{9}L_1^{(0)}+\frac{2}{9}L_2^{(0)}+\frac{2}{9}L_3^{(0)}\right.$$
$$\left.\left.-\frac{2}{3}L_4^{(0)}-\frac{1}{768\pi^2}\log\frac{M_K^2}{\mu^2}\right]\right)\ ,$$

and the corresponding counterterm amplitude is

$$F_\pi^3 F_{\pi,\mathrm{CT}}^{(4)}=M_\pi^4\left(2\tilde{B}_1-2\tilde{B}_2-4\tilde{B}_4\right)-4M_\pi^2M_K^2\tilde{B}_2-8M_K^4\tilde{B}_4\ . \tag{B15}$$

Finally, we find for the YZ contribution

$$\begin{aligned}F_\pi^3F_{\pi,\mathrm{YZ}}^{(4)}&=\frac{2}{9}I_{1,\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_\pi^2;M_\pi^2;)+\frac{1}{36}I_{1,\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2))\\&+\frac{1}{12}I_{1,\mathrm{YZ}}(M_\pi^2;M_\eta^2;M_K^2;\frac{2}{3}(M_\pi^2-M_K^2))+\frac{1}{2}I_{2,\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2)\\&-\frac{1}{2}\left[\frac{M_\pi^4}{6}S'_{\mathrm{YZ}}(M_\pi^2,M_\pi^2,M_\pi^2)+\frac{M_\pi^4}{18}S'_{\mathrm{YZ}}(M_\pi^2,M_\pi^2,M_\eta^2)\right.\\&+\frac{1}{9}R'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_\pi^2;M_\pi^2)+\frac{1}{72}R'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2))\\&\left.+\frac{1}{24}R'_{\mathrm{YZ}}(M_\pi^2;M_\eta^2;M_K^2;\frac{2}{3}(M_\pi^2-M_K^2))+\frac{1}{4}U'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2)\right]\end{aligned} \tag{B16}$$

The eta decay constant is treated analogously and one finds for the remnant contribution,

$$F_\pi^3F_{\eta,\mathrm{rem}}^{(4)}=M_\pi^4\left(\frac{139903}{15925248\pi^4}-\frac{5137\ C}{1990656\pi^4}+8L_4^{(0)}\left[5L_4^{(0)}-8L_6^{(0)}-8L_8^{(0)}\right]\right.$$
$$+\frac{8}{3}L_5^{(0)}\left[14L_4^{(0)}-L_5^{(0)}+8L_6^{(0)}-64L_7^{(0)}-24L_8^{(0)}\right]$$
$$+\frac{1}{\pi^2}\left[-\frac{1}{72}L_1^{(0)}-\frac{29}{144}L_2^{(0)}-\frac{5}{72}L_3^{(0)}\right]$$
$$+\frac{1}{\pi^4}\log\frac{M_K^2}{\mu^2}\left[\frac{49}{8192}+\frac{5}{4096}\log\frac{M_K^2}{\mu^2}\right]$$
$$+\frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[-\frac{145}{995328\pi^2}+\frac{1}{12}L_1^{(0)}+\frac{1}{12}L_2^{(0)}+\frac{1}{24}L_3^{(0)}\right.$$
$$\left.-\frac{1}{24}L_4^{(0)}-\frac{1}{4096\pi^2}\log\frac{M_K^2}{\mu^2}\right]$$
$$+\frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\left[-\frac{25}{12288\pi^2}+\frac{3}{2}L_1^{(0)}+\frac{3}{8}L_2^{(0)}+\frac{3}{8}L_3^{(0)}\right.$$
$$\left.\left.-\frac{9}{8}L_4^{(0)}+\frac{1}{12}L_5^{(0)}-\frac{9}{4096\pi^2}\log\frac{M_K^2}{\mu^2}\right]\right)$$
$$+M_\pi^2M_K^2\left(-\frac{3443}{248832\pi^4}-\frac{127\ C}{497664\pi^4}\right.$$

$$
\begin{aligned}
&+ 32L_4^{(0)}\left[5L_4^{(0)} - 8L_6^{(0)}\right] + \frac{32}{3}L_5^{(0)}\left[5L_4^{(0)} - 4L_6^{(0)} + 32L_7^{(0)} + 16L_8^{(0)}\right] \\
&+ \frac{1}{\pi^2}\left[\frac{1}{9}L_1^{(0)} + \frac{1}{9}L_2^{(0)} + \frac{1}{18}L_3^{(0)}\right] \\
&+ \frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\left[-\frac{1}{2}L_4^{(0)} - \frac{1}{3}L_5^{(0)} + \frac{3}{512\pi^2}\log\frac{M_K^2}{\mu^2}\right] \\
&+ \frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\left[-\frac{41}{18432\pi^2} - \frac{1}{8}L_4^{(0)} - \frac{5}{24}L_5^{(0)} - \frac{5}{2048\pi^2}\log\frac{M_K^2}{\mu^2}\right] \\
&+ \frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[\frac{187}{124416\pi^2} - \frac{2}{3}L_1^{(0)} - \frac{2}{3}L_2^{(0)} - \frac{1}{3}L_3^{(0)}\right. \\
&\left.\left. + \frac{1}{3}L_4^{(0)} + \frac{1}{512\pi^2}\log\frac{M_K^2}{\mu^2}\right]\right) \qquad \text{(B17)}\\
+M_K^4&\left(\frac{11531}{1990656\pi^4} - \frac{7303\ C}{497664\pi^4} + \frac{1}{\pi^2}\left[-\frac{2}{9}L_1^{(0)} - \frac{17}{36}L_2^{(0)} - \frac{19}{144}L_3^{(0)}\right]\right. \\
&+ 32L_4^{(0)}\left[5L_4^{(0)} - 8L_6^{(0)} - 4L_8^{(0)}\right] \\
&+ \frac{64}{3}L_5^{(0)}\left[7L_4^{(0)} + 2L_5^{(0)} - 8L_6^{(0)} - 8L_7^{(0)} - 8L_8^{(0)}\right] \\
&+ \frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\left[\frac{11}{512\pi^2} + 2L_1^{(0)} + \frac{1}{2}L_2^{(0)} + \frac{7}{8}L_3^{(0)} - \frac{5}{4}L_4^{(0)}\right. \\
&\left. - \frac{2}{3}L_5^{(0)} + \frac{1}{512\pi^2}\log\frac{M_K^2}{\mu^2}\right] + \frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[-\frac{211}{62208\pi^2} + \frac{4}{3}L_1^{(0)}\right. \\
&\left.\left. + \frac{4}{3}L_2^{(0)} + \frac{2}{3}L_3^{(0)} - \frac{2}{3}L_4^{(0)} - \frac{1}{256\pi^2}\log\frac{M_K^2}{\mu^2}\right]\right) \ ,
\end{aligned}
$$

whereas the counterterm and YZ amplitudes are respectively

$$
\begin{aligned}
F_\pi^3 F_{\eta,\mathrm{CT}}^{(4)} &= \frac{2}{3}M_\pi^4\left(3\tilde{B}_1 + \tilde{B}_2 + 8\tilde{B}_3 - 6\tilde{B}_4\right) \\
-\frac{4}{3}M_\pi^2 M_K^2&\left(4\tilde{B}_1 + \tilde{B}_2 + 8\tilde{B}_3\right) + \frac{8}{3}M_K^4\left(2\tilde{B}_1 - 2\tilde{B}_2 + 2\tilde{B}_3 - 3\tilde{B}_4\right) \ , \qquad \text{(B18)}
\end{aligned}
$$

and

$$
\begin{aligned}
&F_\pi^3 F_{\eta,\mathrm{YZ}}^{(4)} = \\
&\frac{1}{4}I_{1,\mathrm{YZ}}(M_\eta^2; M_\pi^2; M_K^2; \frac{2}{3}(M_\pi^2 - M_K^2)) \\
&+\frac{1}{4}I_{1,\mathrm{YZ}}(M_\eta^2; M_\eta^2; M_K^2; \frac{2}{3}(3M_K^2 - M_\pi^2)) \\
&-\frac{1}{2}\left[\frac{M_\pi^4}{6}S'_{\mathrm{YZ}}(M_\eta^2, M_\eta^2, M_\pi^2) + \frac{(16M_K^2 - 7M_\pi^2)^2}{486}S'_{\mathrm{YZ}}(M_\eta^2, M_\eta^2, M_\eta^2)\right. \\
&+\frac{1}{8}R'_{\mathrm{YZ}}(M_\eta^2; M_\pi^2; M_K^2; \frac{2}{3}(M_\pi^2 - M_K^2)) \\
&\left.+\frac{1}{8}R'_{\mathrm{YZ}}(M_\eta^2; M_\eta^2; M_K^2; \frac{2}{3}(3M_K^2 - M_\pi^2))\right] \ . \qquad \text{(B19)}
\end{aligned}
$$

3. Meson Masses

In order to remove the seven $L_\ell^{(-1)}$ $\mathcal{O}(p^4)$ counterterms ($\ell = 1, \ldots, 8$) from $M^{(6)2}_{(\pi,\eta),\mathrm{rem}}$, we define the following five effective $\mathcal{O}(p^6)$ counterterms,

$$\begin{aligned}
\tilde{B}_5 \equiv{}& F_\pi^2\left(B_3^{(0)} + \frac{1}{6}F^{(0)} + B_4^{(0)} + B_5^{(0)} + 3B_7^{(0)}\right) \\
&+\frac{20}{3}L_4^{(-1)} + \frac{23}{3}L_5^{(-1)} - \frac{40}{3}L_6^{(-1)} - 40L_7^{(-1)} - \frac{86}{3}L_8^{(-1)} \ , \\
\tilde{B}_6 \equiv{}& \frac{1}{648}\Big[F_\pi^2\left(648B_6^{(0)} - 36F^{(0)} - 216B_4^{(0)} - 216B_5^{(0)} - 648B_7^{(0)}\right) \\
&+3168L_4^{(-1)} + 24L_5^{(-1)} - 6336L_6^{(-1)} + 9792L_7^{(-1)} + 3216L_8^{(-1)}\Big] \ , \\
\tilde{B}_7 \equiv{}& F_\pi^2\left(B_{14}^{(0)} - \frac{3}{2}B_4^{(0)} - \frac{3}{2}B_5^{(0)} - \frac{9}{2}B_7^{(0)}\right) \\
&-8L_6^{(-1)} + 64L_7^{(-1)} + \frac{62}{3}L_8^{(-1)} \ , \\
\tilde{B}_8 \equiv{}& F_\pi^2\left(B_{15}^{(0)} + \frac{1}{3}F^{(0)} + B_4^{(0)} + 2B_5^{(0)} + 3B_7^{(0)}\right) \\
&-2L_5^{(-1)} + \frac{16}{3}L_6^{(-1)} - 72L_7^{(-1)} - 24L_8^{(-1)} \\
\tilde{B}_9 \equiv{}& F_\pi^2\left(B_{16}^{(0)} - \frac{1}{6}F^{(0)} - 2B_4^{(0)} - B_5^{(0)} - 3B_7^{(0)}\right) \\
&+L_5^{(-1)} + \frac{152}{3}L_7^{(-1)} + \frac{62}{3}L_8^{(-1)} \quad .
\end{aligned} \tag{B20}$$

Beginning with the pion squared-mass, we have

$$\begin{aligned}
F_\pi^4 M^{(6)2}_{\pi,\mathrm{rem}} ={}& M_\pi^6\Bigg(-\frac{3689}{884736\pi^4} + \frac{1403\ C}{110592\pi^4} \\
&+\frac{1}{\pi^2}\left[\frac{1}{4}L_1^{(0)} + \frac{37}{72}L_2^{(0)} + \frac{7}{54}L_3^{(0)}\right] \\
&+128L_4^{(0)}\left[-L_4^{(0)} - 2L_5^{(0)} + 4L_6^{(0)} + 4L_8^{(0)}\right] \\
&+128L_5^{(0)}\left[-L_5^{(0)} + 4L_6^{(0)} + 4L_8^{(0)}\right] \\
&-512L_6^{(0)}\left[L_6^{(0)} + 2L_8^{(0)}\right] - 512L_8^{(0)2} \\
&+\frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\Big[-\frac{13}{55296\pi^2} - \frac{1}{9}L_1^{(0)} - \frac{1}{36}L_2^{(0)} - \frac{1}{36}L_3^{(0)} + \frac{1}{6}L_4^{(0)} \\
&+\frac{2}{27}L_5^{(0)} - \frac{2}{9}L_6^{(0)} + \frac{4}{9}L_7^{(0)} + \frac{1}{18}L_8^{(0)} - \frac{31}{165888\pi^2}\log\frac{M_\eta^2}{\mu^2}\Big] \\
&+\frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\Big[-\frac{281}{18432\pi^2} - \frac{7}{2}L_1^{(0)} - 2L_2^{(0)} - \frac{7}{4}L_3^{(0)} + \frac{9}{2}L_4^{(0)} \\
&+3L_5^{(0)} - 8L_6^{(0)} - \frac{11}{2}L_8^{(0)} + \frac{5}{2048\pi^2}\log\frac{M_\pi^2}{\mu^2} \\
&-\frac{1}{3072\pi^2}\log\frac{M_\eta^2}{\mu^2}\Big] + \frac{1}{\pi^4}\log\frac{M_K^2}{\mu^2}\Big[-\frac{35}{36864} - \frac{13}{18432}\log\frac{M_K^2}{\mu^2}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{3}{2048}\log\frac{M_\pi^2}{\mu^2} - \frac{1}{18432}\log\frac{M_\eta^2}{\mu^2}\Bigg]\Bigg) \qquad \text{(B21)}\\
&+M_\pi^4 M_K^2 \Bigg(-\frac{49}{55296\pi^4} + \frac{47\ C}{13824\pi^4} + \frac{1}{\pi^2}\left[-\frac{1}{9}L_2^{(0)} - \frac{1}{27}L_3^{(0)}\right]\\
&\quad + 128 L_4^{(0)}\left[-4L_4^{(0)} - 3L_5^{(0)} + 16L_6^{(0)} + 6L_8^{(0)}\right]\\
&\quad + 256 L_6^{(0)}\left[3L_5^{(0)} - 8L_6^{(0)} - 6L_8^{(0)}\right]\\
&\quad + \frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\left[-\frac{1}{576\pi^2} + \frac{1}{2}L_4^{(0)} + \frac{1}{2}L_5^{(0)} - L_6^{(0)} - L_8^{(0)} - \frac{5}{4608\pi^2}\log\frac{M_K^2}{\mu^2}\right]\\
&\quad + \frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\Bigg[-\frac{7}{6912\pi^2} + \frac{8}{9}L_1^{(0)} + \frac{2}{9}L_2^{(0)} + \frac{2}{9}L_3^{(0)}\\
&\quad - \frac{7}{6}L_4^{(0)} - \frac{10}{27}L_5^{(0)} + \frac{13}{9}L_6^{(0)} - \frac{20}{9}L_7^{(0)} - \frac{2}{9}L_8^{(0)} + \frac{7}{20736\pi^2}\log\frac{M_\eta^2}{\mu^2}\Bigg]\\
&\quad + \frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\left[\frac{5}{2}L_4^{(0)} - 5L_6^{(0)} + \frac{5}{2304\pi^2}\log\frac{M_\eta^2}{\mu^2}\right]\Bigg)\\
&+M_\pi^2 M_K^4 \Bigg(\frac{91}{12288\pi^4} + \frac{43\ C}{3072\pi^4} + \frac{1}{\pi^2}\left[\frac{13}{18}L_2^{(0)} + \frac{43}{216}L_3^{(0)}\right]\\
&\quad + 128 L_4^{(0)}\left[-4L_4^{(0)} - L_5^{(0)} + 16L_6^{(0)} + 2L_8^{(0)}\right]\\
&\quad + 256 L_6^{(0)}\left[L_5^{(0)} - 8L_6^{(0)} - 2L_8^{(0)}\right]\\
&\quad + \frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\Bigg[-\frac{1}{48\pi^2} - 4L_1^{(0)} - L_2^{(0)} - \frac{5}{4}L_3^{(0)} + 5L_4^{(0)}\\
&\quad + L_5^{(0)} - 6L_6^{(0)} - 2L_8^{(0)} + \frac{11}{3072\pi^2}\log\frac{M_K^2}{\mu^2}\Bigg]\\
&\quad + \frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\Bigg[\frac{1}{384\pi^2} - \frac{16}{9}L_1^{(0)} - \frac{4}{9}L_2^{(0)} - \frac{4}{9}L_3^{(0)} + 2L_4^{(0)}\\
&\quad + \frac{8}{27}L_5^{(0)} - \frac{20}{9}L_6^{(0)} + \frac{16}{9}L_7^{(0)} + \frac{1}{1152\pi^2}\log\frac{M_K^2}{\mu^2} - \frac{1}{10368\pi^2}\log\frac{M_\eta^2}{\mu^2}\Bigg]\Bigg)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&F_\pi^4 M_{\pi,\mathrm{CT}}^{(6)2} =\\
&2M_\pi^6\left(-2\tilde{B}_1 + 2\tilde{B}_2 + 4\tilde{B}_4 + 5\tilde{B}_5 + 3\tilde{B}_6 + 4\tilde{B}_7 - 2\tilde{B}_9\right)\ ,\\
&+8M_\pi^4 M_K^2\left(\tilde{B}_2 + 3\tilde{B}_6 - \tilde{B}_9\right)\ ,\\
&+8M_\pi^2 M_K^4\left(2\tilde{B}_4 + \tilde{B}_5 + 3\tilde{B}_6\right)\ , \qquad \text{(B22)}
\end{aligned}
$$

and

$$
\begin{aligned}
&F_\pi^4 M_{\pi,\mathrm{YZ}}^{(6)2} =\\
&-\frac{M_\pi^4}{6}S_{\mathrm{YZ}}(M_\pi^2, M_\pi^2, M_\pi^2) - \frac{M_\pi^4}{18}S_{\mathrm{YZ}}(M_\pi^2, M_\pi^2, M_\eta^2) - \frac{1}{4}U_{\mathrm{YZ}}(M_\pi^2, M_\pi^2, M_K^2)\\
&-\frac{1}{9}R_{\mathrm{YZ}}(M_\pi^2; M_\pi^2; M_\pi^2; M_\pi^2) - \frac{1}{72}R_{\mathrm{YZ}}(M_\pi^2; M_\pi^2; M_K^2; 2(M_\pi^2 + M_K^2))
\end{aligned}
$$

$$-\frac{1}{24}R_{\rm YZ}(M_\pi^2;M_\eta^2;M_K^2;\frac{2}{3}(M_\pi^2-M_K^2)) \tag{B23}$$

Likewise, we find for the eta squared-mass,

$$\begin{aligned}
F_\pi^4 M^{(6)2}_{\eta,\rm rem} = M_\pi^6 \Big(&-\frac{13405}{7962624\pi^4}+\frac{14005\ C}{2985984\pi^4}\\
&+\frac{128}{3}L_4^{(0)}\left[L_4^{(0)}+\frac{2}{3}L_5^{(0)}-4L_6^{(0)}+24L_7^{(0)}+8L_8^{(0)}\right]\\
&+\frac{128}{9}L_5^{(0)}\left[-\frac{1}{3}L_5^{(0)}-4L_6^{(0)}+64L_7^{(0)}+\frac{64}{3}L_8^{(0)}\right]\\
&+\frac{512}{3}L_6^{(0)}\left[L_6^{(0)}-12L_7^{(0)}-4L_8^{(0)}\right]\\
&-\frac{512}{9}L_8^{(0)}\left[16L_7^{(0)}+5L_8^{(0)}\right]\\
&+\frac{1}{\pi^2}\left[-\frac{1}{108}L_1^{(0)}-\frac{29}{216}L_2^{(0)}-\frac{5}{108}L_3^{(0)}\right]\\
&+\frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[-\frac{1049}{1492992\pi^2}+\frac{1}{18}L_1^{(0)}+\frac{1}{18}L_2^{(0)}+\frac{1}{36}L_3^{(0)}\right.\\
&-\frac{1}{9}L_4^{(0)}-\frac{10}{81}L_5^{(0)}+\frac{7}{27}L_6^{(0)}+\frac{16}{27}L_7^{(0)}+\frac{17}{54}L_8^{(0)}\\
&\left.-\frac{67}{497664\pi^2}\log\frac{M_\eta^2}{\mu^2}-\frac{5}{6144\pi^2}\log\frac{M_K^2}{\mu^2}+\frac{37}{27648\pi^2}\log\frac{M_\pi^2}{\mu^2}\right]\\
&+\frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\left[-\frac{1}{2048\pi^2}+L_1^{(0)}+\frac{1}{4}L_2^{(0)}+\frac{1}{4}L_3^{(0)}-\frac{5}{3}L_4^{(0)}\right.\\
&\left.+\frac{1}{9}L_5^{(0)}+\frac{7}{3}L_6^{(0)}-12L_7^{(0)}-\frac{85}{18}L_8^{(0)}+\frac{95}{18432\pi^2}\log\frac{M_\pi^2}{\mu^2}\right]\\
&+\frac{1}{\pi^4}\log\frac{M_K^2}{\mu^2}\left[-\frac{11}{4096}+\frac{1}{6144}\log\frac{M_K^2}{\mu^2}+\frac{1}{2048}\log\frac{M_\pi^2}{\mu^2}\right]\Big)\\
+M_\pi^4M_K^2\Big(&-\frac{2597}{221184\pi^4}-\frac{263\ C}{248832\pi^4}\\
&+\frac{128}{3}L_5^{(0)}\left[-L_4^{(0)}+\frac{2}{3}L_5^{(0)}+2L_6^{(0)}-16L_7^{(0)}-4L_8^{(0)}\right]\\
&+\frac{256}{3}L_8^{(0)}\left[-L_4^{(0)}+2L_6^{(0)}-16L_7^{(0)}-8L_8^{(0)}\right]\\
&+\frac{1}{\pi^2}\left[\frac{1}{9}L_1^{(0)}+\frac{11}{18}L_2^{(0)}+\frac{2}{9}L_3^{(0)}\right]\\
&+\frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\left[-\frac{373}{27648\pi^2}-\frac{1}{6}L_4^{(0)}+\frac{5}{18}L_5^{(0)}+\frac{1}{3}L_6^{(0)}-\frac{16}{3}L_7^{(0)}\right.\\
&\left.-\frac{7}{3}L_8^{(0)}+\frac{5}{1536\pi^2}\log\frac{M_\pi^2}{\mu^2}-\frac{7}{2304\pi^2}\log\frac{M_K^2}{\mu^2}\right]\\
&+\frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[\frac{757}{124416\pi^2}-\frac{2}{3}L_1^{(0)}-\frac{2}{3}L_2^{(0)}-\frac{1}{3}L_3^{(0)}+\frac{5}{6}L_4^{(0)}\right.\\
&+\frac{22}{27}L_5^{(0)}-\frac{19}{9}L_6^{(0)}-\frac{16}{3}L_7^{(0)}-\frac{70}{27}L_8^{(0)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{97}{13824\pi^2}\log\frac{M_K^2}{\mu^2}+\frac{11}{10368\pi^2}\log\frac{M_\eta^2}{\mu^2}\Bigg] \qquad \text{(B24)}\\
&+\frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\left[\frac{7}{1536\pi^2}-4L_1^{(0)}-L_2^{(0)}-L_3^{(0)}+\frac{9}{2}L_4^{(0)}\right.\\
&-\frac{8}{9}L_5^{(0)}-5L_6^{(0)}+\frac{52}{3}L_7^{(0)}+\frac{68}{9}L_8^{(0)}\\
&\left.\left.-\frac{53}{6912\pi^2}\log\frac{M_\eta^2}{\mu^2}-\frac{1}{2304\pi^2}\log\frac{M_\pi^2}{\mu^2}\right]\right)\\
+M_\pi^2M_K^4&\left(\frac{6887}{331776\pi^4}-\frac{883\ C}{82944\pi^4}\right.\\
&+128L_4^{(0)}\left[-4L_4^{(0)}-L_5^{(0)}+16L_6^{(0)}-24L_7^{(0)}-\frac{26}{3}L_8^{(0)}\right]\\
&+256L_5^{(0)}\left[L_6^{(0)}-\frac{16}{3}L_7^{(0)}-\frac{32}{9}L_8^{(0)}\right]\\
&+512L_6^{(0)}\left[-4L_6^{(0)}+12L_7^{(0)}+\frac{13}{3}L_8^{(0)}\right]\\
&+\frac{8192}{3}L_8^{(0)}\left[2L_7^{(0)}+L_8^{(0)}\right]\\
&+\frac{1}{\pi^2}\left[-\frac{4}{9}L_1^{(0)}-\frac{11}{18}L_2^{(0)}-\frac{17}{72}L_3^{(0)}\right]\\
&+\frac{1}{\pi^2}\log\frac{M_K^2}{\mu^2}\left[\frac{59}{1728\pi^2}+\frac{4}{3}L_1^{(0)}+\frac{1}{3}L_2^{(0)}+\frac{7}{12}L_3^{(0)}\right.\\
&-\frac{1}{3}L_4^{(0)}-\frac{11}{9}L_5^{(0)}-\frac{2}{3}L_6^{(0)}+\frac{40}{3}L_7^{(0)}+\frac{74}{9}L_8^{(0)}\\
&\left.-\frac{1}{96\pi^2}\log\frac{M_\pi^2}{\mu^2}+\frac{139}{27648\pi^2}\log\frac{M_K^2}{\mu^2}\right]\\
&+\frac{1}{\pi^2}\log\frac{M_\pi^2}{\mu^2}\left[\frac{8}{3}L_4^{(0)}+\frac{16}{9}L_5^{(0)}-\frac{16}{3}L_6^{(0)}-\frac{16}{3}L_7^{(0)}-\frac{16}{3}L_8^{(0)}\right]\\
&+\frac{1}{\pi^2}\log\frac{M_\eta^2}{\mu^2}\left[-\frac{193}{10368\pi^2}+\frac{8}{3}L_1^{(0)}+\frac{8}{3}L_2^{(0)}+\frac{4}{3}L_3^{(0)}\right.\\
&-2L_4^{(0)}-\frac{40}{27}L_5^{(0)}+\frac{52}{9}L_6^{(0)}+\frac{128}{9}L_7^{(0)}+\frac{176}{27}L_8^{(0)}\\
&\left.\left.+\frac{1}{108\pi^2}\log\frac{M_\pi^2}{\mu^2}-\frac{73}{3456\pi^2}\log\frac{M_K^2}{\mu^2}-\frac{1}{384\pi^2}\log\frac{M_\eta^2}{\mu^2}\right]\right)\\
+M_K^6&\left(-\frac{2401}{248832\pi^4}+\frac{6923\ C}{186624\pi^4}+\frac{1}{\pi^2}\left[\frac{16}{27}L_1^{(0)}+\frac{34}{27}L_2^{(0)}+\frac{19}{54}L_3^{(0)}\right]\right.\\
&+\frac{512}{3}L_4^{(0)}\left[-4L_4^{(0)}-\frac{11}{3}L_5^{(0)}+16L_6^{(0)}+12L_7^{(0)}+14L_8^{(0)}\right]\\
&+\frac{1024}{27}L_5^{(0)}\left[-4L_5^{(0)}+33L_6^{(0)}+30L_7^{(0)}+34L_8^{(0)}\right]\\
&-\frac{2048}{3}L_6^{(0)}\left[4L_6^{(0)}+6L_7^{(0)}+7L_8^{(0)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4096}{9} L_8^{(0)} \left[7 L_7^{(0)} + 5 L_8^{(0)} \right] \\
&+ \frac{1}{\pi^2} \log \frac{M_K^2}{\mu^2} \left[- \frac{59}{864\pi^2} - \frac{16}{3} L_1^{(0)} - \frac{4}{3} L_2^{(0)} - \frac{7}{3} L_3^{(0)} + 8 L_4^{(0)} + \frac{40}{9} L_5^{(0)} \right. \\
&\left. - \frac{32}{3} L_6^{(0)} - 8 L_7^{(0)} - \frac{104}{9} L_8^{(0)} - \frac{31}{6912\pi^2} \log \frac{M_K^2}{\mu^2} + \frac{7}{288\pi^2} \log \frac{M_\eta^2}{\mu^2} \right] \\
&+ \frac{1}{\pi^2} \log \frac{M_\eta^2}{\mu^2} \left[\frac{515}{23328\pi^2} - \frac{32}{9} L_1^{(0)} - \frac{32}{9} L_2^{(0)} - \frac{16}{9} L_3^{(0)} + \frac{16}{9} L_4^{(0)} \right. \\
&\left. \left. + \frac{64}{81} L_5^{(0)} - \frac{160}{27} L_6^{(0)} - \frac{256}{27} L_7^{(0)} - \frac{128}{27} L_8^{(0)} + \frac{1}{486\pi^2} \log \frac{M_\eta^2}{\mu^2} \right] \right) \ ,
\end{aligned}
$$

then the counterterm amplitude,

$$
\begin{aligned}
&F_\pi^4 M_{\eta,\mathrm{CT}}^{(6)2} = \frac{2}{9} M_\pi^6 \left(6\tilde{B}_1 + 2\tilde{B}_2 \right. \\
&\left. +16\tilde{B}_3 - 12\tilde{B}_4 + 9\tilde{B}_5 - 9\tilde{B}_6 - 12\tilde{B}_7 - 16\tilde{B}_8 - 2\tilde{B}_9 \right) \\
&+\frac{8}{9} M_\pi^4 M_K^2 \left(-10\tilde{B}_1 - 3\tilde{B}_2 \right. \\
&\left. -24\tilde{B}_3 + 12\tilde{B}_4 + 9\tilde{B}_5 + +20\tilde{B}_7 + 24\tilde{B}_8 + 3\tilde{B}_9 \right) \\
&+\frac{8}{9} M_\pi^2 M_K^4 \left(20\tilde{B}_1 + 36\tilde{B}_3 - 6\tilde{B}_4 - 27\tilde{B}_5 + 27\tilde{B}_6 - 40\tilde{B}_7 - 36\tilde{B}_8 \right) \\
&+\frac{32}{9} M_K^6 \left(-4\tilde{B}_1 + 4\tilde{B}_2 \right. \\
&\left. -4\tilde{B}_3 + 6\tilde{B}_4 + 9\tilde{B}_5 + 9\tilde{B}_6 + 8\tilde{B}_7 + 4\tilde{B}_8 - 4\tilde{B}_9 \right) \ ,
\end{aligned} \tag{B25}
$$

and finally the YZ contribution,

$$
\begin{aligned}
&F_\pi^4 M_{\eta,\mathrm{YZ}}^{(6)2} = \\
&-\frac{M_\pi^4}{6} S_{\mathrm{YZ}}(M_\eta^2, M_\eta^2, M_\pi^2) - \frac{(16M_K^2 - 7M_\pi^2)^2}{486} S_{\mathrm{YZ}}(M_\eta^2, M_\eta^2, M_\eta^2) \\
&-\frac{1}{8} R_{\mathrm{YZ}}(M_\eta^2; M_\pi^2; M_K^2; \frac{2}{3}(M_\pi^2 - M_K^2)) \\
&-\frac{1}{8} R_{\mathrm{YZ}}(M_\eta^2; M_\eta^2; M_K^2; \frac{2}{3}(3M_K^2 - M_\pi^2)) \ .
\end{aligned} \tag{B26}
$$

4. Relation to $\overline{MS}$ Renormalization

Our formulae for the hypercharge polarization functions $\hat{\Pi}_8^{(1,0)}$ displayed earlier in this appendix refer to the $\overline{\lambda}$ renormalization used throughout this paper. To obtain corresponding expressions in the $\overline{MS}$ renormalization of Ref. [17], it suffices to follow the discussion in Sect. VI-C for the isospin polarization functions.

However some additional analysis is required in order to compare our $\overline{\lambda}$-subtracted decay constant and mass formulae to the $\overline{MS}$ scheme. We shall omit detailed derivation and simply display the results, as the procedure mirrors that used to obtain Eq. (112). What is needed is the following set of relations between the $\overline{\lambda}$-subtracted constants $\{\tilde{B}_\ell\}$ and the associated $\overline{MS}$ quantities $\tilde{B}_\ell^{\overline{\rm MS}}$ for $\ell = 1, \ldots, 9$,

$$\tilde{B}_\ell = \tilde{B}_\ell^{\overline{\rm MS}} - \Delta\tilde{B}_\ell \quad , \tag{B27}$$

where

$$\Delta\tilde{B}_\ell = \frac{C}{(4\pi)^4}\left(-\frac{175}{576}, \frac{19}{96}, \frac{691}{5184}, \frac{43}{192}, \frac{10}{27}, -\frac{10}{81}, -\frac{167}{288}, \frac{371}{1296}, -\frac{9}{32}\right) \quad . \tag{B28}$$

As expected from our previous discussion, to obtain the masses and decay constants in the $\overline{MS}$ renormalization of Ref. [17] one needs only make the replacements $\tilde{B}_\ell \to \tilde{B}_\ell^{\overline{\rm MS}}$ and omit all dependence on the constant C.

Figure Captions

Fig. 1 Lowest-order graphs for the axialvector propagator.

Fig. 2 One-loop graphs.

Fig. 3 Generic corrections to the axialvector propagator.

Fig. 4 Two-loop 1PI non-sunset graphs.

Fig. 5 The two-loop 1PI sunset graph.

Fig. 6 Two-loop 1PI vertex graphs.

Fig. 7 Two-loop 1PI self-energy graphs.