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Dispersive calculation of $B_7^{(3/2)}$ and $B_8^{(3/2)}$ in the chiral limit

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Abstract

We show how the isospin vector and axialvector current spectral functions $\rho_{V,3}$ and $\rho_{A,3}$ can be used to determine in leading chiral order the low energy constants $B_7^{(3/2)}$ and $B_8^{(3/2)}$. This is accomplished by matching the Operator Product Expansion to the dispersive analysis of vacuum polarization functions. The data for the evaluation of these dispersive integrals has been recently enhanced by the ALEPH measurement of spectral functions in tau decay, and we update our previous phenomenological determination. Our calculation yields in the NDR renormalization scheme and at renormalization scale $\mu = 2$ GeV the values $B_7^{(3/2)} = 0.55 \pm 0.07 \pm 0.10$ and $B_8^{(3/2)} = 1.11 \pm 0.16 \pm 0.23$ for the quark mass values $m_s + \hat{m} = 0.1$ GeV.

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I. INTRODUCTION

The recent KTeV and NA48 findings that $\epsilon'/\epsilon \simeq 20 \cdot 10^{-4}$ raise the important question whether a value so large can be consistent with Standard Model expectations. [1] One of the key quantities upon which the Standard Model prediction is based is the B-factor $B_8^{(3/2)}$. In this paper, we work in the chiral limit to obtain an analytic expression for $B_8^{(3/2)}$ (and also for $B_7^{(3/2)}$). Our results take the form of sum rules involving the difference $\rho_V - \rho_A$ of isospin-one spectral functions.

The constants $B_7^{(3/2)}(\mu)$ and $B_8^{(3/2)}(\mu)$ are defined in terms of the matrix elements

$$\langle 2\pi|\mathcal{Q}_i^{(3/2)}|K\rangle_\mu \equiv B_i^{(3/2)}(\mu)\ \langle 2\pi|\mathcal{Q}_i^{(3/2)}|K\rangle_\mu^{\rm vac} \qquad (i=7,8)\ \ , \tag{1}$$

where μ is the renormalization scale, $\mathcal{Q}_7^{(3/2)}$ and $\mathcal{Q}_8^{(3/2)}$ are the $\Delta I = 3/2$ electroweak penguin operators

$$\begin{aligned}\mathcal{Q}_7^{(3/2)} &\equiv \bar{s}_a\Gamma_{\rm L}^\mu d_a\left(\bar{u}_b\Gamma_\mu^{\rm R}u_b - \bar{d}_b\Gamma_\mu^{\rm R}d_b\right) + \bar{s}_a\Gamma_{\rm L}^\mu u_a\bar{u}_b\Gamma_\mu^{\rm R}d_b\ \ ,\\ \mathcal{Q}_8^{(3/2)} &\equiv \bar{s}_a\Gamma_{\rm L}^\mu d_b\left(\bar{u}_b\Gamma_\mu^{\rm R}u_a - \bar{d}_b\Gamma_\mu^{\rm R}d_a\right) + \bar{s}_a\Gamma_{\rm L}^\mu u_b\bar{u}_b\Gamma_\mu^{\rm R}d_a\ \ ,\end{aligned} \tag{2}$$

a, b are color labels and $\Gamma_{\rm L}^\mu \equiv \gamma^\mu(1+\gamma_5)$, $\Gamma_{\rm R}^\mu \equiv \gamma^\mu(1-\gamma_5)$. Analogous B-factors are defined for the other weak operators. The most important contributions to ϵ'/ϵ are the matrix elements of the penguin operator $\mathcal{Q}_6$ and the electroweak penguin operator $\mathcal{Q}_8^{3/2}$. As expressed in terms of the B-factors this is seen in the approximate numerical relation,

$$\frac{\epsilon'}{\epsilon} = 10\times 10^{-4}\left[(2.5B_6 - 1.3B_8^{(3/2)})\left(\frac{100\ {\rm MeV}}{m_s+m_d}\right)\right]\ \ , \tag{3}$$

when evaluated at $\mu = 2$ GeV in the $\overline{MS}$-NDR renormalization scheme. Alternatively, in terms of the operator matrix elements themselves one has

$$\frac{\epsilon'}{\epsilon} = 10\times 10^{-4}\left[-3.1\ {\rm GeV}^{-3}\cdot\langle\mathcal{Q}_6\rangle_0 - 0.51\ {\rm GeV}^{-3}\cdot\langle\mathcal{Q}_8^{(3/2)}\rangle_2\right] \tag{4}$$

where

$$\langle\mathcal{Q}_6\rangle_0 \equiv \langle(\pi\pi)_{I=0}|\mathcal{Q}_6|K^0\rangle \qquad \text{and} \qquad \langle\mathcal{Q}_8^{(3/2)}\rangle_2 \equiv \langle(\pi\pi)_{I=2}|\mathcal{Q}_8^{(3/2)}|K^0\rangle\ \ , \tag{5}$$

again defined in the NDR scheme at $\mu = 2$ GeV.

A kaon-to-pion weak matrix element can be analyzed by chiral methods and expressed as an expansion in momenta and quark masses. [2] Here we shall calculate the leading term in such an expansion, valid in the limit of exact chiral symmetry where the Goldstone bosons become massless. In fact, our analysis yields values for the relevant K-to-π matrix elements themselves. However, we shall transcribe this information into the equivalent form of B-factors in order to express our results in the more conventional form and thus allow comparison with other techniques. When considered in the chiral limit, the content of Eq. (1) reduces to

$$\lim_{p=0} \; B_7^{(3/2)}(\mu) = -\frac{3}{F_\pi^4} \cdot \frac{m_u+m_d}{m_\pi^2} \cdot \frac{m_u+m_s}{m_K^2} \; \langle \mathcal{O}_1 \rangle_\mu \quad ,$$
$$\lim_{p=0} \; B_8^{(3/2)}(\mu) = -\frac{1}{F_\pi^4} \cdot \frac{m_u+m_d}{m_\pi^2} \cdot \frac{m_u+m_s}{m_K^2} \; \left[\frac{1}{3} \langle \mathcal{O}_1 \rangle_\mu + \frac{1}{2} \langle \mathcal{O}_8 \rangle_\mu \right] \quad , \tag{6}$$

where $\mathcal{O}_1$, $\mathcal{O}_8$ are the local four-quark operators

$$\mathcal{O}_1 \equiv \bar{q}\gamma_\mu \frac{\tau_3}{2} q \; \bar{q}\gamma^\mu \frac{\tau_3}{2} q - \bar{q}\gamma_\mu\gamma_5 \frac{\tau_3}{2} q \; \bar{q}\gamma^\mu\gamma_5 \frac{\tau_3}{2} q \quad ,$$
$$\mathcal{O}_8 \equiv \bar{q}\gamma_\mu \lambda^A \frac{\tau_3}{2} q \; \bar{q}\gamma^\mu \lambda^A \frac{\tau_3}{2} q - \bar{q}\gamma_\mu\gamma_5 \lambda^A \frac{\tau_3}{2} q \; \bar{q}\gamma^\mu\gamma_5 \lambda^A \frac{\tau_3}{2} q \quad . \tag{7}$$

In the above, $q = u, d, s$, τ_3 is a Pauli (flavor) matrix, $\{\lambda^A\}$ are the Gell Mann color matrices and the subscripts on $\mathcal{O}_1$, $\mathcal{O}_8$ refer to the color carried by their currents.[1] A chiral evaluation of $B_7^{(3/2)}$ and $B_8^{(3/2)}$ is thus seen (*cf* Eq. (6)) to depend upon $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \mathcal{O}_8 \rangle_\mu$. The $K^0 \to \pi\pi$ matrix elements are likewise recoverable in the chiral limit from the vacuum matrix elements,

$$\lim_{p=0} \; \langle (\pi\pi)_{I=2} | \mathcal{Q}_7^{(3/2)} | K^0 \rangle_\mu = -\frac{4}{F_\pi^3} \; \langle \mathcal{O}_1 \rangle_\mu \quad ,$$
$$\lim_{p=0} \; \langle (\pi\pi)_{I=2} | \mathcal{Q}_8^{(3/2)} | K^0 \rangle_\mu = -\frac{4}{F_\pi^3} \; \left[\frac{1}{3} \langle \mathcal{O}_1 \rangle_\mu + \frac{1}{2} \langle \mathcal{O}_8 \rangle_\mu \right] \quad . \tag{8}$$

The rest of this paper describes a calculational procedure to obtain analytic expressions for the vacuum matrix elements.

In Section II, we show how to extract $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \mathcal{O}_8 \rangle_\mu$ from the isospin-one vector and axialvector correlators by deriving sum rules for $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \mathcal{O}_8 \rangle_\mu$ in a momentum cutoff scheme. This renormalization is especially well suited for comparing theory to experiment. In Section III we introduce a four-quark operator $\mathcal{O}_{\Delta S=1}$, distinct from the familiar nonleptonic hamiltonian $\mathcal{H}_{\Delta S=1}$, whose kaon-to-pion matrix element is related to $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \mathcal{O}_8 \rangle_\mu$ in the chiral limit. We demonstrate consistency of this information with the sum rules of Section II. Section IV describes a procedure for obtaining $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \mathcal{O}_8 \rangle_\mu$ in $\overline{MS}$ renormalization, which is commonly used in lattice-theoretic simulations. Our final numerical results and concluding statements appear respectively in Sections V and VI.

II. DISPERSIVE ANALYSIS OF VACUUM POLARIZATION FUNCTIONS

In seeking values for $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \mathcal{O}_8 \rangle_\mu$, it is natural to study the vacuum polarization functions as these are also defined in terms of vacuum matrix elements of four-quark operators (but generally not all at the same spacetime point). Thus we consider the combination $\Pi_{V,3} - \Pi_{A,3}$ (the subscript '3' denotes the isospin flavor),

$$i \int d^4x \; e^{iq\cdot x} \langle 0 | T \left(V_3^\mu(x) V_3^\nu(0) - A_3^\mu(x) A_3^\nu(0) \right) | 0 \rangle$$
$$= (q^\mu q^\nu - q^2 g^{\mu\nu})(\Pi_{V,3} - \Pi_{A,3})(q^2) - q^\mu q^\nu \Pi_{A,3}^{(0)}(q^2) \quad . \tag{9}$$

[1] Throughout we denote vacuum expectation values as $\langle 0 | \mathcal{O} | 0 \rangle_\mu \equiv \langle \mathcal{O} \rangle_\mu$.

Associated with this correlator is the difference of spectral functions $\rho_{V,3} - \rho_{A,3}$,

$$[\Pi_{V,3} - \Pi_{A,3}]\,(Q^2) = \frac{1}{Q^4}\int_0^\infty ds\ \frac{s^2}{s+Q^2}\ \ [\rho_{V,3} - \rho_{A,3}]\,(s)\ \ , \tag{10}$$

where $Q^2 \equiv -q^2$. In writing this spectral relation, we have made use of the first and second Weinberg sum rules [9], which are both valid in the chiral limit

Due to the complexity of QCD, there exist no analytic expressions for the correlators and spectral functions that are valid over the entire energy domain. However, some crucial information is available. At low energies, $\rho_{V,3}$ and $\rho_{A,3}$ are determined from τ-lepton decays and from e^+e^- scattering. As one proceeds from the resonance region of nonperturbative physics to larger energies, the effect of individual channels becomes indistinguishable and perturbative QCD (pQCD) becomes operative. The boundary between nonperturbative and perturbative regions defines a scale $\Lambda \sim 2 \to 3$ GeV. In the pQCD domain, the leading-log behavior of $(\Pi_{V,3} - \Pi_{A,3})(Q^2)$ is given by [6]

$$Q^6(\Pi_{V,3} - \Pi_{A,3})(Q^2) \sim 2\pi\langle\alpha_s\mathcal{O}_8\rangle_\mu + \ln\left(\frac{Q^2}{\mu^2}\right)\left[\frac{8}{3}\langle\alpha_s^2\mathcal{O}_1\rangle_\mu - \langle\alpha_s^2\mathcal{O}_8\rangle_\mu\right] + \ldots\ \ . \tag{11}$$

This asymptotic relation will be of special value to our determination of $B_8^{(3/2)}$ since it contains information on the vacuum matrix element $\langle\mathcal{O}_8\rangle_\mu$ (*cf* Eq. (8)). The large-s behavior of the spectral functions can be inferred from the logarithmic term in Eq. (11) via continuation to the real q^2-axis,

$$(\rho_{V,3} - \rho_{A,3})(s) \sim \frac{1}{s^3}\left[\frac{8}{3}\langle\alpha_s^2\mathcal{O}_1\rangle_\mu - \langle\alpha_s^2\mathcal{O}_8\rangle_\mu\right] + \ldots\ \ . \tag{12}$$

Together, the spectral relations of Eq. (10) and Eq. (12) imply

$$\begin{aligned} Q^6(\Pi_{V,3} - \Pi_{A,3})(Q^2)\ \ \sim\ \ \ln\left(\frac{Q^2}{\Lambda^2}\right)\ \ \left[\frac{8}{3}\langle\alpha_s^2\mathcal{O}_1\rangle_\mu - \langle\alpha_s^2\mathcal{O}_8\rangle_\mu\right] \\ + \int_0^{\Lambda^2} ds\ s^2\ \ [\rho_{V,3} - \rho_{A,3}]\,(s) + \mathcal{O}(Q^{-2})\ \ . \end{aligned} \tag{13}$$

A. Correlators in d-Dimensions

Consider the definition of $\Pi_{V,3} - \Pi_{A,3}$ as expressed in d-dimensions,

$$\begin{aligned} \mu_{\rm d.r.}^{d-4} i \int d^dx\ e^{iq\cdot x}\langle 0|T\left(V_3^\mu(x)V_3^\nu(0) - A_3^\mu(x)A_3^\nu(0)\right)|0\rangle \\ = (q^\mu q^\nu - q^2 g^{\mu\nu})(\Pi_{V,3} - \Pi_{A,3})(q^2) - q^\mu q^\nu \Pi_{A,3}^{(0)}(q^2)\ \ . \end{aligned} \tag{14}$$

The energy scale $\mu_{\rm d.r.}$ ('d.r.' denotes dimensional regularization) has been introduced to maintain the proper dimensions away from $d = 4$. It is straightforward to invert Eq. (14) and we find

$$\langle 0|T\left(V_3^\mu(x)V_{\mu,3}(0) - A_3^\mu(x)A_{\mu,3}(0)\right)|0\rangle$$
$$= \frac{(d-1)\mu_{\rm d.r.}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dQ^2\ e^{-iq\cdot x}\ Q^d\left(\Pi_{V,3} - \Pi_{A,3}\right)(Q^2)\ \ . \tag{15}$$

Up to this point the procedure is well defined, as all quantities are finite-valued.

To obtain a relation for $\langle\mathcal{O}_1\rangle_\mu$, we need to evaluate Eq. (15) in the limit of $x \to 0$. However, the asymptotic condition of Eq. (11) implies that unless the integral on the right-hand-side of Eq. (15) is regularized, it will diverge as $x \to 0$. There are a number of ways to perform the regularization, and we shall consider two particularly useful approaches — first a momentum space cutoff directly below and then $\overline{MS}$ renormalization in Sect. IV. We shall distinguish vacuum matrix elements in the two schemes by means of the superscripts '(c.o.)' for momentum-cutoff and '($\overline{MS}$)' for modified minimal subtraction.

B. Two Sum Rules in Momentum-space Cutoff Renormalization

Let us remove the divergence which occurs for $d = 4$ in Eq. (15) by cutting off the Q^2-integral at $Q^2 = \mu^2$, where μ is the renormalization scale and for convenience we set $\mu_{\rm d.r.} = \mu$. It is valid to take $d = 4$ in this case since the integral is finite. We find

$$\langle\mathcal{O}_1\rangle_\mu^{\rm (c.o.)} = \frac{3}{16\pi^2}\int_0^{\mu^2} dQ^2\ Q^4(\Pi_{V,3} - \Pi_{A,3})(Q^2)\ \ . \tag{16}$$

Using Eq. (10) to express this relation in terms of spectral functions, we arrive immediately at the following sum rule,

$$\frac{16\pi^2}{3}\langle\mathcal{O}_1\rangle_\mu^{\rm (c.o.)} = I_1 \equiv \int_0^\infty ds\ s^2\ln\left(\frac{s+\mu^2}{s}\right)\ \left[\rho_{V,3} - \rho_{A,3}\right](s)\ \ . \tag{17}$$

It is equally straightforward to derive a sum rule for $\langle\alpha_s\mathcal{O}_8\rangle_\mu^{\rm (c.o.)}$. We first set $Q^2 = \mu^2$ in Eq. (11) to obtain

$$\langle\alpha_s\mathcal{O}_8\rangle_\mu^{\rm (c.o.)} = \frac{\mu^6}{2\pi}(\Pi_{V,3} - \Pi_{A,3})(\mu^2)\ \ . \tag{18}$$

Because the variable Q^2 is constrained in Eq. (11) to lie in the range where pQCD makes sense, the same must be true for the scale μ. Then by combining Eq. (18) with Eq. (10), we obtain the sum rule

$$2\pi\langle\alpha_s\mathcal{O}_8\rangle_\mu^{\rm (c.o.)} = I_8 \equiv \int_0^\infty ds\ s^2\frac{\mu^2}{s+\mu^2}\ \left[\rho_{V,3} - \rho_{A,3}\right](s)\ \ . \tag{19}$$

Despite their apparent similarity, it is important to understand that there is a basic difference between the sum rules for $\langle\mathcal{O}_1\rangle_\mu^{\rm (c.o.)}$ and $\langle\alpha_s\mathcal{O}_8\rangle_\mu^{\rm (c.o.)}$. The former is obtained rather directly by taking the $x \to 0$ limit of Eq. (15) and using a cutoff in momentum to regularize the procedure. However, the latter rests upon assuming the dominance of the leading Q^{-6} term in the OPE of Eq. (11). This assumption becomes increasingly questionable as μ is lowered to energies just above the resonance region. It leads to an uncertainty in the value of $\langle\alpha_s\mathcal{O}_8\rangle_\mu^{\rm (c.o.)}$ which is not present in $\langle\mathcal{O}_1\rangle_\mu^{\rm (c.o.)}$. We postpone discussion to Sect. V regarding numerical evaluation of the integrals I_1, I_8 appearing in Eqs. (17),(19).

III. KAON-TO-PION MATRIX ELEMENTS OF A LEFT-RIGHT OPERATOR

A distinct but equivalent path to learn about $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \mathcal{O}_8 \rangle_\mu$ is to perform a chiral analysis of the kaon-to-pion matrix elements themselves. However, the usual (V-A)×(V-A) weak hamiltonian $\mathcal{H}_{\Delta S=1}$ would be of no help in the chiral limit since its K-to-pi matrix elements vanish there. Instead we introduce a (V-A)×(V+A) nonleptonic operator $\mathcal{O}_{\Delta S=1}$ defined as

$$\begin{aligned}
\mathcal{O}_{\Delta S=1} &\equiv \frac{g_2^2}{8} \int d^4x \; \mathcal{D}_{\mu\nu}(x, M_W^2) \; J^{\mu\nu}(x) \quad , \\
J^{\mu\nu}(x) &\equiv \frac{1}{2} T \left[\bar{d}(x)\gamma^\mu(1+\gamma_5)u(x) \; \bar{u}(0)\gamma^\nu(1-\gamma_5)s(0) \right] \\
&= \frac{1}{2} T \left[(V^\mu_{1-i2}(x) + A^\mu_{1-i2}(x)) \; \left(V^\nu_{4+i5}(0) - A^\nu_{4+i5}(0) \right) \right] \quad ,
\end{aligned} \tag{20}$$

where $\mathcal{D}_{\mu\nu}$ is the W-boson propagator and V^μ_a, A^μ_a $(a = 1, \ldots 8)$ are the flavor-octet vector, axialvector currents. Operators similar to $\mathcal{O}_{\Delta S=1}$ have received some previous attention in the literature. [3,4] The LR chiral structure of $\mathcal{O}_{\Delta S=1}$ ensures the survival of the K-to-pi matrix element $\mathcal{M}(p) = \langle \pi^-(p) | \mathcal{O}_{\Delta S=1} | K^-(p) \rangle$ in the $p \to 0$ limit, where we obtain

$$\mathcal{M} \equiv \lim_{p=0} \mathcal{M}(p) = \frac{g_2^2}{16 F_\pi^2} \int d^4x \; \mathcal{D}(x, M_W^2) \; \langle 0 | T \left(V^\mu_3(x) V_{\mu,3}(0) - A^\mu_3(x) A_{\mu,3}(0) \right) | 0 \rangle \quad . \tag{21}$$

A. Leading-log Analysis of QCD Corrections

In the following, we perform a leading-log calculation of QCD corrections to the chiral matrix element $\mathcal{M}$. This leads naturally to renormalization group equations (RGE) for the quantities $\langle \mathcal{O}_1 \rangle_\mu$ and $\langle \alpha_s \mathcal{O}_8 \rangle_\mu$.

Since the W-boson propagator in Eq. (21) acts as a cutoff for contributions with $|x| \geq M_W^{-1}$, we consider the leading term of the following operator product expansion,

$$V^\mu_3(x) V^\nu_3(0) - A^\mu_3(x) A^\nu_3(0) = V^\mu_3(0) V^\nu_3(0) - A^\mu_3(0) A^\nu_3(0) + \mathcal{O}(x) \quad . \tag{22}$$

Evaluation of the spacetime integral in Eq. (21) is straightforward,

$$\int d^4x \; \mathcal{D}_{\mu\nu}(x, M_W^2) = \frac{g_{\mu\nu}}{M_W^2} \quad , \tag{23}$$

so that the matrix element specified at energy scale M_W becomes

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2} F_\pi^2} \langle \mathcal{O}_1 \rangle_{M_W} \quad . \tag{24}$$

In order to express this vacuum matrix element at some lower energy μ, we must take QCD radiative corrections into account. The effect of these will be to mix $\mathcal{O}_1$ with $\mathcal{O}_8$. The result of mixing at one-loop order is

$$\begin{bmatrix}\mathcal{O}_1\\ \mathcal{O}_8\end{bmatrix} \rightarrow \begin{bmatrix}\mathcal{O}_1\\ \mathcal{O}_8\end{bmatrix} + \frac{\alpha_s}{4\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\begin{bmatrix}0 & 3/2\\ 16/3 & 7\end{bmatrix}\begin{bmatrix}\mathcal{O}_1\\ \mathcal{O}_8\end{bmatrix} \quad , \tag{25}$$

where $\alpha_s \equiv g_3^2/4\pi$ is the QCD fine structure constant and $\mu < M_W$. Using standard techniques [2], we use the renormalization group (RG) to provide a summation of the leading-log dependence over the range from M_W down to μ,

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\left[c_1(\mu)\langle\mathcal{O}_1\rangle_\mu + c_8(\mu)\langle\mathcal{O}_8\rangle_\mu\right] \quad , \tag{26}$$

where

$$\begin{aligned} c_1(\mu) &= \frac{1}{9}\left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{8/9} + 8\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-1/9}\right] \quad , \\ c_8(\mu) &= \frac{1}{6}\left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{8/9} - \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-1/9}\right] \quad , \end{aligned} \tag{27}$$

with

$$\alpha_s(\mu) = \left[1 + 9\frac{\alpha_s(\mu)}{4\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\right]\alpha_s(M_W) \quad . \tag{28}$$

An expansion of Eq. (26) through second order in $\alpha_s(\mu)$ gives

$$\begin{aligned} \mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\Big[&\langle\mathcal{O}_1\rangle_\mu + \frac{3}{8\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\langle\alpha_s\mathcal{O}_8\rangle_\mu \\ &+ \frac{3}{32\pi^2}\ln^2\left(\frac{M_W^2}{\mu^2}\right)\left(\frac{8}{3}\langle\alpha_s^2\mathcal{O}_1\rangle_\mu - \langle\alpha_s^2\mathcal{O}_8\rangle_\mu\right)\Big] \quad . \end{aligned} \tag{29}$$

Let us gain some feeling for the numbers involved. The minimum value of renormalization scale considered in this paper is $\mu_0 = 2$ GeV. Adopting this scale and taking $\alpha_s(M_W) = 0.119$ and $\alpha_s(\mu_0) = 0.334$ [5], we find for the RG coefficients in Eq. (26),

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\Big[1.071\langle\mathcal{O}_1\rangle_{\mu_0} + 0.268\langle\mathcal{O}_8\rangle_{\mu_0}\Big] \quad , \tag{30}$$

whereas the coefficients in the perturbative expression of Eq. (29) become

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\Big[\langle\mathcal{O}_1\rangle_{\mu_0} + 0.88\langle\alpha_s\mathcal{O}_8\rangle_{\mu_0} - 0.52\left(\langle\alpha_s^2\mathcal{O}_8\rangle_{\mu_0} - \frac{8}{3}\langle\alpha_s^2\mathcal{O}_1\rangle_{\mu_0}\right)\Big] \quad . \tag{31}$$

Finally, the condition of scale independence for the matrix element $\mathcal{M}$,

$$\mu^2\frac{\partial}{\partial\mu^2}\,\mathcal{M} = 0 \quad , \tag{32}$$

leads directly to the renormalization group equations

$$\mu^2 \frac{\partial}{\partial \mu^2} \langle \mathcal{O}_1 \rangle_\mu = \frac{3}{8\pi} \langle \alpha_s \mathcal{O}_8 \rangle_\mu \quad , \tag{33}$$

$$\mu^2 \frac{\partial}{\partial \mu^2} \langle \alpha_s \mathcal{O}_8 \rangle_\mu = \frac{1}{4\pi} \left[\frac{16}{3} \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu - 2 \langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu \right] \quad . \tag{34}$$

To summarize — the above operator-product analysis involves computing radiative corrections perturbatively to one-loop order in QCD (*cf* Eq. (25)) and retaining only the dependence on leading logarithms in the evolution from scale M_W down to scale μ. The value of μ cannot be taken too small, otherwise the perturbative framework breaks down.

B. Verification of the Operator Product Expansion

Despite the explicit difference between the procedures of Sect. II and that carried out directly above, the two are equivalent. In particular, we can show that the $\langle \mathcal{O}_1 \rangle_\mu$ sum rule of Eq. (17) and the $\langle \mathcal{O}_8 \rangle_\mu$ sum rule of Eq. (19) reproduce the OPE to the leading log level. This verifies both the derivations that we provided and gives a direct insight into the workings of the OPE.

Consider a partition of $\mathcal{M}$ characterized by the scale μ,

$$\mathcal{M} = \mathcal{M}_<(\mu) + \mathcal{M}_>(\mu) \quad , \tag{35}$$

where $\mathcal{M}_<(\mu)$ and $\mathcal{M}_>(\mu)$ are dependent respectively on contributions with $Q < \mu$ and $Q > \mu$. Also, in addition to maintaining the requirement that μ lie in the pQCD domain, we further constrain it to obey $\mu \ll M_W$. We then obtain

$$\begin{aligned} \mathcal{M}_<(\mu) &= \frac{3 G_F M_W^2}{32\sqrt{2}\pi^2 F_\pi^2} \int_0^{\mu^2} dQ^2 \; \frac{Q^4}{Q^2 + M_W^2} \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2) \right] \\ &= \frac{3 G_F}{32\sqrt{2}\pi^2 F_\pi^2} \int_0^{\mu^2} dQ^2 \; Q^4 \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2) \right] + \mathcal{O}(\mu^2/M_W^2) \end{aligned} \tag{36}$$

and

$$\mathcal{M}_>(\mu) = \frac{3 G_F M_W^2}{32\sqrt{2}\pi^2 F_\pi^2} \int_{\mu^2}^{\infty} dQ^2 \; \frac{Q^4}{Q^2 + M_W^2} \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2) \right] \quad . \tag{37}$$

Upon inserting the large-Q form of Eq. (11) into Eq. (37), we obtain

$$\begin{aligned} \mathcal{M}_>(\mu) = \frac{G_F}{\sqrt{2} F_\pi^2} \Bigg[& \frac{3}{8\pi} \ln \left(\frac{M_W^2}{\mu^2} \right) \langle \alpha_s \mathcal{O}_8 \rangle_\mu \\ & - \frac{3}{32\pi^2} \ln^2 \left(\frac{M_W^2}{\mu^2} \right) \left(\langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu - \frac{8}{3} \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu \right) \Bigg] \quad . \end{aligned} \tag{38}$$

Comparison of Eq. (29) with Eq. (38) yields the relation,

$$\mathcal{M}_<(\mu) = \frac{G_F}{2\sqrt{2} F_\pi^2} \langle \mathcal{O}_1 \rangle_\mu \quad . \tag{39}$$

We see that the operators that we originally defined independently of the weak interaction are in fact the ones that appear in the Operator Product Expansion. Of course, this is to be expected, but it provies an explicit pedagogical demonstration of the nature of the OPE.

C. Sum Rules and RG Relations

To complete the chain of logic, we demonstrate consistency of the spectral function sum rules for $\langle \mathcal{O}_1 \rangle_\mu^{(\mathrm{c.o.})}$ and $\langle \mathcal{O}_8 \rangle_\mu^{(\mathrm{c.o.})}$ with the corresponding renormalization group relations obtained previously from the operator product expansion (*cf* Eqs. (33),(34)). The RG equation for $\langle \mathcal{O}_1 \rangle_\mu^{(\mathrm{c.o.})}$ is immediately recovered upon differentiating the sum rule of Eq. (17) and making use of Eq. (19),

$$\mu^2 \frac{\partial}{\partial \mu^2} \langle \mathcal{O}_1 \rangle_\mu^{(\mathrm{c.o.})} = \frac{3}{8\pi} \cdot \frac{\mu^2}{2\pi} \int_0^\infty ds \; \frac{s^2}{s+\mu^2} \; \left[\rho_{V,3} - \rho_{A,3}\right](s) = \frac{3}{8\pi} \langle \alpha_s \mathcal{O}_8 \rangle_\mu^{(\mathrm{c.o.})} \quad . \tag{40}$$

To derive the RG relation for $\langle \mathcal{O}_8 \rangle_\mu^{(\mathrm{c.o.})}$ we start with the sum rule of Eq. (19),

$$\mu^2 \frac{\partial}{\partial \mu^2} \langle \mathcal{O}_8 \rangle_\mu^{(\mathrm{c.o.})} = \frac{\mu^2}{2\pi} \frac{\partial}{\partial \mu^2} \int_0^\infty ds \; \frac{\mu^2}{s+\mu^2} \; s^2 \left[\rho_{V,3} - \rho_{A,3}\right](s) \quad . \tag{41}$$

The integral in the above is seen to be $\mu^6 \left(\Pi_{V,3}(\mu^2) - \Pi_{A,3}(\mu^2)\right)$. We replace it using the aymptotic expression of Eq. (13) to obtain

$$\begin{aligned} \mu^2 \frac{\partial}{\partial \mu^2} \langle \mathcal{O}_8 \rangle_\mu^{(\mathrm{c.o.})} = \frac{\mu^2}{2\pi} \frac{\partial}{\partial \mu^2} \Bigg[\ln\left(\frac{\mu^2}{\Lambda^2}\right) \; \left[\frac{8}{3} \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu^{(\mathrm{c.o.})} - \langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu^{(\mathrm{c.o.})}\right] \\ + \int_0^{\Lambda^2} ds \; s^2 \; \left[\rho_{V,3} - \rho_{A,3}\right](s) + \mathcal{O}(\mu^{-2}) \Bigg] \quad , \end{aligned} \tag{42}$$

from which the RG relation of Eq. (34) follows directly.

IV. $\overline{MS}$ RENORMALIZATION

The work of the preceding sections was based on a momentum-space cutoff renormalization scheme, which is useful in yielding sum rules directly related to experimental data. At the same time, however, it is distinct from the more standard $\overline{MS}$ prescription. In this Section, we demonstrate how to relate the two approaches.

A. Short Distance Analysis

Let us reconsider Eq. (15). We can show how the cutoff renormalization is related to dimensional regularization by keeping the high-Q^2 part of the integral in Eq. (15) and analyzing it in terms of an ϵ-expansion. We divide the integral into integration ranges below and above μ^2. For the part of integral with Q^2 below μ^2, we can let $d \to 4$ and recover exactly the cutoff integrand of Eq. (16),

$$\langle \mathcal{O}_1 \rangle_\mu^{(\mathrm{d.r.})} = \langle \mathcal{O}_1 \rangle_\mu + \frac{(d-1)\mu^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)} \int_{\mu^2}^\infty dQ^2 \; Q^d (\Pi_{V,3} - \Pi_{A,3})(Q^2) \tag{43}$$

The asymptotic tail can be analysed in d dimensions. This introduces scheme dependence depending on which method is used to define Dirac algebra away from four dimensions, *e.g.*

the naive dimensional regularization (NDR) and t'Hooft-Veltman (HV) schemes in which γ_5 is respectively anticommuting and commuting. [7] We find

$$\frac{1}{3}(d-1)Q^d \cdot (\Pi_{V,3} - \Pi_{A,3})(Q^2) = 2\pi\alpha_s \langle \mathcal{O}_8 \rangle_\mu \; Q^{d-6} \left[1 + \left(d_s + \frac{1}{4} \right) \epsilon \right] + \mathcal{O}(\alpha_s^2) \quad . \tag{44}$$

where $\epsilon \equiv 4 - d$ and $d_s \equiv d_{\rm scheme}$ is associated with the loop integration and scheme-dependence. The values of d_s in the NDR and HV schemes are

$$d_s = \begin{cases} -5/6 & \text{(NDR)} \\ 1/6 & \text{(HV)} \end{cases} . \tag{45}$$

The $Q^2 > \mu^2$ integral can then be performed with the result

$$\langle \mathcal{O}_1 \rangle_\mu^{\rm (d.r.)} = \langle \mathcal{O}_1 \rangle_\mu + \frac{3}{16\pi^2} \left[\frac{2}{4-d} - \gamma + \ln 4\pi + \frac{3}{2} + 2d_s \right] \langle \alpha_s \mathcal{O}_8 \rangle_\mu \quad . \tag{46}$$

The $\overline{MS}$ prescription is a subcase of dimensional regularization in which the terms $2/(4-d) - \gamma + \ln 4\pi$ in Eq. (46) are removed in the renormalization procedure. This gives our desired relation in a given scheme,

$$\langle \mathcal{O}_1 \rangle_\mu^{\overline{\rm MS}} = \langle \mathcal{O}_1 \rangle_\mu^{\rm (c.o.)} + \frac{3\alpha_s}{8\pi} \left(\frac{3}{2} + 2d_s \right) \langle \mathcal{O}_8 \rangle_\mu \quad . \tag{47}$$

To derive an analogous relation between $\langle \mathcal{O}_8 \rangle_\mu^{\overline{\rm MS}}$ and $\langle \mathcal{O}_8 \rangle_\mu^{\rm (c.o.)}$ we employ the leading behavior of correlators and spectral functions in the $\overline{\rm MS}$ renormalization prescription, which has been calculated using the NDR scheme, [6]

$$\begin{aligned} Q^6 (\Pi_{V,3} - \Pi_{A,3})(Q^2) \;\sim\; & 2\pi \langle \alpha_s \mathcal{O}_8 \rangle_\mu^{(\overline{\rm MS})} \\ + \left[2 + \frac{8}{3} \ln \left(\frac{Q^2}{\mu^2} \right) \right] \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu^{(\overline{\rm MS})} & + \left[\frac{119}{12} - \ln \left(\frac{Q^2}{\mu^2} \right) \right] \langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu^{(\overline{\rm MS})} \end{aligned} \tag{48}$$

and

$$(\rho_{V,3} - \rho_{A,3})(s) \sim \frac{1}{s^3} \left[\frac{8}{3} \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu^{(\overline{\rm MS})} - \langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu^{(\overline{\rm MS})} \right] + \ldots \quad . \tag{49}$$

Then by setting $Q = \mu$ in Eq. (48) and combining the result with Eq. (49) we find

$$\mu^2 \int_0^\infty ds \; \frac{s^2}{s+\mu^2} \; [\rho_{V,3} - \rho_{A,3}](s) = 2\pi\alpha_s \left[\left(1 + \frac{119\alpha_s}{24\pi} \right) \langle \mathcal{O}_8 \rangle_\mu^{(\overline{\rm MS})} + \frac{\alpha_s}{\pi} \langle \mathcal{O}_1 \rangle_\mu^{(\overline{\rm MS})} \right] \quad . \tag{50}$$

Comparison with Eq. (47) then implies that the NDR matrix element is given to first order in α_s by

$$\langle \mathcal{O}_8 \rangle_\mu^{(\overline{\rm MS})} = \left(1 - \frac{119\alpha_s}{24\pi} \right) \langle \mathcal{O}_8 \rangle_\mu^{\rm (c.o.)} - \frac{\alpha_s}{\pi} \langle \mathcal{O}_1 \rangle_\mu \quad . \tag{51}$$

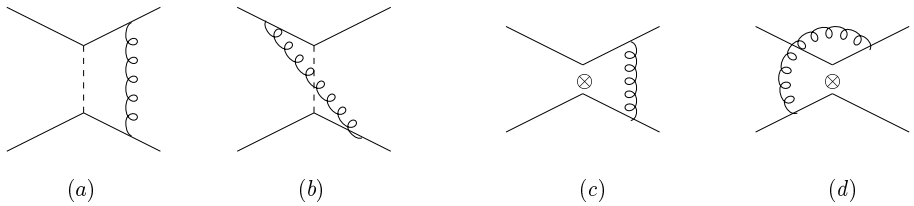

FIG. 1. Some QCD corrections: full theory (a)-(b), effective theory (c)-(d).

B. $\overline{\text{MS}}$ Matching at One Loop

In this section we perform the matching at one loop and verify the scheme independence of the result. The effective operator $\mathcal{O}^{(\text{eff})}_{\Delta S=1}$, is expressed in terms of local operators $\mathcal{O}_1$ and $\mathcal{O}_8$,

$$\mathcal{O}^{(\text{eff})}_{\Delta S=1} = \frac{G_F}{2\sqrt{2}} \left[c_1(\mu)\mathcal{O}_1 + c_8(\mu)\mathcal{O}_8 \right] \quad . \tag{52}$$

Determination of the coefficients c_1 and c_8 proceeds in two steps: first calculate QCD radiative corrections in both the full and effective theories, and then 'match' the two calculations. We shall carry out this procedure at one-loop order of the QCD radiative corrections. Since $c_1 = 1 + \mathcal{O}(\alpha_s^2)$ and $c_8 = \mathcal{O}(\alpha_s)$, this will yield a determination of c_8. For definiteness, we consider the free scattering of zero momentum quarks and adopt a common quark mass m to serve as the infrared cutoff.

In the full theory, evaluations of the one-loop radiative corrections like those displayed in Figs. 1(a)-(b) are finite and yield

$$\mathcal{M} = \frac{G_F}{2\sqrt{2}} \left[\langle qq|\mathcal{O}_1|qq\rangle_{\text{tree}} + \frac{3\alpha_s}{8\pi} \left(\ln \frac{M_W^2}{m^2} - 1 \right) \langle qq|\mathcal{O}_8|qq\rangle_{\text{tree}} + \mathcal{O}(\alpha_s^2) \right] \quad . \tag{53}$$

The analogous calculation in the effective theory is divergent and must be regularized. We employ dimensional regularization which introduces the scheme dependence mentioned above. Our calculation of amplitudes like those in Figs. 1(c)-(d) gives

$$\mathcal{M} = \frac{G_F}{2\sqrt{2}} \Big[\langle qq|\mathcal{O}_1|qq\rangle_{\text{tree}} + \frac{3\alpha_s}{8\pi} \left(1 + d_s\epsilon + d_\ell\epsilon\right) \left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\mu^2} \right] \langle qq|\mathcal{O}_8|qq\rangle_{\text{tree}} + c_8 \langle qq|\mathcal{O}_8|qq\rangle_{\text{tree}} \Big] \quad , \tag{54}$$

After the removal of the divergent term in $\overline{MS}$ renormalization, we compare the full theory and the effective theory to identify the coefficient function as

$$c_8(\mu) = \frac{3\alpha_s}{8\pi} \left[\ln \frac{M_W^2}{\mu^2} - \frac{3}{2} - 2d_s \right] \quad . \tag{55}$$

When $\mathcal{O}^{(\text{eff})}_{\Delta S=1}$ is applied to our problem of vacuum matrix elements, we have the amplitude

$$\mathcal{M} = \frac{G_F}{2\sqrt{2}F_\pi^2} \left[c_1(\mu)\langle\mathcal{O}_1\rangle_\mu^{\overline{MS}} + c_8(\mu)\langle\mathcal{O}_8\rangle_\mu^{\overline{MS}} \right] \quad . \tag{56}$$

Given our previous identification of the scheme dependent operator O_1 in Eq. (47), it can be seen that the scheme dependence cancels between that of the matrix element and of the coefficient in the operator product expansion.

V. NUMERICAL ANALYSIS AND UNCERTAINTIES

We base our numerical determination of $\langle \mathcal{O}_8 \rangle_\mu^{(\mathrm{c.o.})}$ and $\langle \mathcal{O}_1 \rangle_\mu^{(\mathrm{c.o.})}$ respectively on the sum rules in Eq. (19) and Eq. (17). This involves calculation of the integrals I_8 and I_1, which contain the combination of spectral functions $\rho_{V,3} - \rho_{A,3}$,

$$I_i = \int_0^\infty ds \; K_i(s,\mu^2) \; [\rho_{V,3} - \rho_{A,3}](s) \qquad (i = 1, 8) \quad , \tag{57}$$

with

$$K_8 = s^2 \frac{\mu^2}{s+\mu^2} \; , \qquad K_1 \equiv s^2 \ln\left(\frac{s+\mu^2}{s}\right) \quad . \tag{58}$$

As such, I_8 and I_1 belong to the family of spectral integrals which include the DMO sum rule [8], the two Weinberg sum rules [9] and the sum rule for the pion electromagnetic mass splitting [10]. The kernels occurring in these 'classical' sum rules are

$$K_{\mathrm{DMO}} = 1/s \; , \quad K_{\mathrm{W1}} = 1 \; , \quad K_{\mathrm{W2}} = s \; , \quad K_{\mathrm{em}} = s \ln \; s \quad . \tag{59}$$

This happenstance is most fortunate as the integrals defined by the kernels in Eq. (59) form a powerful set of constraints for any evaluation of I_8 and I_1. Using an updated form of our earlier study [11] of chiral sum rules, we find for renormalization scale $\mu = 2$ GeV the values

$$I_8 = -(0.30 \pm 0.04) \cdot 10^{-2} \; , \qquad I_1 = -(0.42 \pm 0.06) \cdot 10^{-2} \; , \tag{60}$$

At the higher renormalization scale, $\mu = 4$ GeV, we obtain

$$I_8 = -(0.43 \pm 0.06) \cdot 10^{-2} \; , \qquad I_1 = -(0.97 \pm 0.12) \cdot 10^{-2} \; . \tag{61}$$

A. Uncertainties from Data Analysis

The error bars quoted above correspond to our estimate of the uncertainty in the sum rules due to imprecision in our knowledge of the spectral functions. Before proceeding let us describe how these were arrived at, and assess other sources of uncertainty. The data at lower values of s are extremely well known, and they introduce very little uncertainty compared to other sources which we are concerned with. The high energy tail of $\rho_V - \rho_A$ is small above $s = 5$ GeV^2. In the $\mu = 2$ GeV integrals, there remains essentially no sensitivity to this high

energy tail once the constraints are imposed[2]. It is in the matching of these two regions that one encounters the greatest uncertainties. Fortunately, the four integral constraints described above are very stringent and allow us to limit the uncertainties in this region. We have used several methods to construct spectral functions which match the data within error bars and yet satisfy our sum rule constraints. These give variations in our integrals of under 6%. In addition, we have considered the situation where the asymptotic form of the spectral function is reached only on the average, with a damped oscillating term that provides deviations from the average. Since our sum rules are equivalent to transforming back to euclidiean Q^2, these oscillations give exponentially suppressed effects at large μ^2 once integrated. Again the constraints are very powerful in further limiting this effect, and our studies lead us to increase the uncertainty in the fit to 10% to account for this form of variation.

We also must account for the fact that the data and the input into the constraints are measured in a world where m_π^2 is not zero, yet we are interested in the result in the chiral limit. This introduces corrections of order m_π^2/m_ρ^2 which is of order 3%. In fact, since we know some of the physics involved in passing to the chiral limit, we could attempt to make a realistic correction for the extrapolation to the chiral limit. However, since this would appear to introduce some model dependence into our procedure, we prefer to simply include the uncertainty as an error bar. In practice, the effect which has the most sensitivity for our results is the constraint of the pion electromagnetic mass difference, since the kernel $K_{\rm em}$ bears the greatest resemblance to K_1 and K_8. Work on the pion and kaon electromagnetic mass differences indicates that the quark mass corrections are somewhat larger than average. Therefore, to be conservative we triple the canonical error estimate, leading us to quote a 9% uncertainty for the extrapolation to the chiral limit. We have added this in quadrature to the statistical uncertainty to arrive at the error bars cited above.

B. Uncertainty from the Operator Product Expansion

Finally, we need to address the fact that it has become common to cite matrix elements at a scale $\mu = 2$ GeV, which is a rather low scale for perturbative QCD to be fully in the asymptotic region. In fact, our method can be used for any μ, and we can check if the asymptotic QCD behavior is obtained. For example, the renormalization group equations relate the μ-dependence to the magnitudes of the operator matrix elements. Although one of the relations (Eqs. (33),(40)) is automatically satisfied, we explicitly showed above that the second holds only if μ is large enough, *i.e.* that it is well into the region where the asymptotic tail of the spectral functions becomes applicable. It is easy to see from the data alone that this is not the case at $\mu = 2$ GeV. Another way to state the same result is that

[2]At higher values of μ there occurs more sensitivity to the asymptotic tail, and it is the tail that describes the logarithmic running of the O_8 matrix element. While we have a good handle on the size of the $1/s^3$ component of the tail, we know little about the $1/s^4$ component. However as long as the $1/s^4$ portion is not much larger than the $1/s^3$ behavior for $s > 5$ GeV^2, its effect is within our quoted error bars.

there remain power corrections in the sum rule, although the renormalization group states that the running with μ should be only logarithmic. We believe (because of the generality of our framework) that this issue must also be present in the lattice results, and we urge the evaluation of lattice matrix elements at larger values of μ.

We do see such non-asymptotic behavior in our results. Actually, for $\mathcal{O}_1$ our method of cutting off the high frequency modes of the current-current product is in accord with Wilson's original idea of the definition of a scale-dependent matrix element. Therefore, our sum rule for $\langle \mathcal{O}_1 \rangle_\mu^{(\mathrm{c.o.})}$ can be treated as a *definition* of this amplitude at any scale μ, even if that scale is not yet asymptotic. For $\mathcal{O}_8$, however, there is some uncertainty as to an ideal definition in the non-asymptotic region. For example, the RG relation of Eqs. (33),(40) requires that we use exactly our definition, yet this only is foolproof if the RGEs are fully valid. Equivalently, if this matrix element is defined via the coefficient of Q^{-6} in the vacuum polarization, there can be order Q^{-8} corrections remaining if one works in the non-asymptotic region. We see evidence of such power corrections. Moreover, attempting to discard the Q^{-8} effects leads to a larger matrix element. At $\mu = 4$ GeV, the corrections are rather modest, in line with other uncertainties that we have described. However, at $\mu = 2$ GeV, these non-asymptotic effects represent a significant intrinsic uncertainty. *We have taken these into account by combining two evaluations, one obtained by evaluating the sum rule at $\mu = 4$ GeV and using the RGE to transform down to $\mu = 2$ GeV, and the other by direct evaluation of the sum rule at the lower scale. We average these two and assign the difference as an independent error bar.* The error bar is chosen such that a one-sigma variation reproduces the full range between the two methods of evaluation. We do this for both matrix elements. The quoted error bar at $\mu = 4$ GeV is scaled down from the $\mu = 2$ GeV values by a factor of four, as appropriate for quadratic power corrections.

C. Conversion to $\overline{MS}$ Renormalization

We now transform to the $\overline{MS}$ matrix elements. The results of our direct evaluation at $\mu = 2$ GeV leads to the matrix elements

$$\begin{aligned} \langle \mathcal{O}_8 \rangle_{2\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(0.67 \pm 0.09) \cdot 10^{-3}\ \mathrm{GeV}^6 \ , \\ \langle \mathcal{O}_1 \rangle_{2\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(0.70 \pm 0.10) \cdot 10^{-4}\ \mathrm{GeV}^6 \ , \end{aligned} \tag{62}$$

where we have taken $\alpha_s(2\ \mathrm{GeV}) \simeq 0.334$. When we evaluate the integrals at $\mu = 4$ GeV and use the RGE to rescale back to $\mu = 2$ GeV, we instead obtain

$$\begin{aligned} \langle \mathcal{O}_8 \rangle_{2\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(1.29 \pm 0.15) \cdot 10^{-3}\ \mathrm{GeV}^6 \ , \\ \langle \mathcal{O}_1 \rangle_{2\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(1.02 \pm 0.10) \cdot 10^{-4}\ \mathrm{GeV}^6 \ , \end{aligned} \tag{63}$$

which is a measure of the potential non-asymptotic corrections found at low values of μ. As explained above, this leads us to quote our result as

$$\begin{aligned} \langle \mathcal{O}_8 \rangle_{2\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(0.98 \pm 0.13 \pm 0.23) \cdot 10^{-3}\ \mathrm{GeV}^6 \ , \\ \langle \mathcal{O}_1 \rangle_{2\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(0.86 \pm 0.10 \pm 0.16) \cdot 10^{-4}\ \mathrm{GeV}^6 \ , \end{aligned} \tag{64}$$

The first error bar corresponds to the uncertainty in the evaluation of the sum rule whereas the second is the potential non-asymptotic uncertainty defined above. Note that the two matrix elements differ by an order of magnitude. The related $K^0 \to \pi\pi$ matrix elements are then

$$\begin{aligned} \langle(\pi\pi)_{I=2}|\mathcal{Q}_7^{(3/2)}|K^0\rangle_{2\ \mathrm{GeV}} &= (0.43 \pm 0.05 \pm 0.08)\ \mathrm{GeV}^3 \quad , \\ \langle(\pi\pi)_{I=2}|\mathcal{Q}_8^{(3/2)}|K^0\rangle_{2\ \mathrm{GeV}} &= (2.58 \pm 0.37 \pm 0.47)\ \mathrm{GeV}^3 \quad . \end{aligned} \tag{65}$$

In the NDR scheme with $\mu = 2$ GeV this translates into the following B-factor determinations,

$$\begin{aligned} B_7^{(3/2)}[\mathrm{NDR}, \mu = 2\ \mathrm{GeV}] \left(\frac{0.1\ \mathrm{GeV}}{m_s + \hat{m}}\right)^2 &= 0.55 \pm 0.07 \pm 0.10 \quad , \\ B_8^{(3/2)}[\mathrm{NDR}, \mu = 2\ \mathrm{GeV}] \left(\frac{0.1\ \mathrm{GeV}}{m_s + \hat{m}}\right)^2 &= 1.11 \pm 0.16 \pm 0.23 \quad . \end{aligned} \tag{66}$$

Note that the combination of B-factor and quark masses is 'physical', appearing in the formula for ϵ'/ϵ. The comparison of these results with some lattice evaluations is hampered by the fact that our evaluation is of the full matrix elements, while most lattice calculations are of the B-factors directly [12]. If large values of quark masses are used, our results are larger that other estimates, yet for the currently favored smaller quark masses the results are not inconsistent. There is one recent lattice evaluation which provides absolute matrix elements which we can compare to. The Rome group [14] quotes the matrix element in in the the quenched approximation using $K \to \pi$ matrix elements plus the chiral relation between $K \to \pi$ and $K \to \pi\pi$. When the meson masses are taken as the kaon mass they find, in the NDR scheme at $\mu = 2$ GeV,

$$\begin{aligned} \langle(\pi\pi)_{I=2}|\mathcal{Q}_7^{(3/2)}|K^0\rangle_{2\ \mathrm{GeV}} &= (0.22 \pm 0.04)\ \mathrm{GeV}^3 \quad , \\ \langle(\pi\pi)_{I=2}|\mathcal{Q}_8^{(3/2)}|K^0\rangle_{2\ \mathrm{GeV}} &= (1.02 \pm 0.10)\ \mathrm{GeV}^3 \quad . \end{aligned} \tag{67}$$

The quoted error does not include estimates of the effect of quenching nor the extrapolation to the continuum limit. Their results seem to be systematically smaller than ours.

Finally at $\mu = 4$ GeV we have the vacuum matrix elements,

$$\begin{aligned} \langle\mathcal{O}_8\rangle_{4\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(1.63 \pm 0.20 \pm 0.06) \cdot 10^{-3}\ \mathrm{GeV}^6 \ , \\ \langle\mathcal{O}_1\rangle_{4\ \mathrm{GeV}}^{(\overline{\mathrm{MS}})} &= -(1.71 \pm 0.20 \pm 0.04) \cdot 10^{-4}\ \mathrm{GeV}^6 \quad . \end{aligned} \tag{68}$$

The corresponding $K \to \pi\pi$ matrix elements are

$$\begin{aligned} \langle(\pi\pi)_{I=2}|\mathcal{Q}_7^{(3/2)}|K^0\rangle_{4\ \mathrm{GeV}} &= (0.85 \pm 0.10 \pm 0.02)\ \mathrm{GeV}^3 \\ \langle(\pi\pi)_{I=2}|\mathcal{Q}_8^{(3/2)}|K^0\rangle_{4\ \mathrm{GeV}} &= (4.34 \pm 0.56 \pm 0.15)\ \mathrm{GeV}^3 \end{aligned} \tag{69}$$

and for the B-factors we obtain,

$$\begin{aligned} B_7^{(3/2)}[\mathrm{NDR}, \mu = 4\ \mathrm{GeV}] \left(\frac{0.1\ \mathrm{GeV}}{m_s + \hat{m}}\right)^2 &= 1.10 \pm 0.13 \pm 0.03 \quad , \\ B_8^{(3/2)}[\mathrm{NDR}, \mu = 4\ \mathrm{GeV}] \left(\frac{0.1\ \mathrm{GeV}}{m_s + \hat{m}}\right)^2 &= 1.87 \pm 0.25 \pm 0.07 \quad . \end{aligned} \tag{70}$$

That $B_7^{(3/2)}$ has a large variation with μ is expected from the RGE of Eq. (34), given our previous result that the vacuum matrix element of $\mathcal{O}_8$ is much larger than that of $\mathcal{O}_1$.

VI. CONCLUDING COMMENTS

The method that we have described has the virtue of being a fully rigorous framework. Moreover, the input data is largely taken from experiment, and hence represents an evaluation that is model independent. Besides the direct comparison with the results with lattice calculations, there may also be other lessons in this calculation. Since in our method the matix elements are evaluated by constructing the Euclidean vacuum polarization function, lattice calculations may also be able to directly follow many of the steps in our procedure, and thereby test their methods in more detail. By explicitly studying the product of currents at non-zero values of the spatial separation, the matrix elements can be evaluated without some of the operator mixing problems that occur on the lattice when using local operators. Moreover, by studying vacuum matrix elements as well as hadronic matrix elements, the chiral relations can be checked on the lattice. Finally, we recall the lesson, discussed above, that power-law corrections still appear to exist at $\mu = 2$ GeV, especially in the O_8 matrix element. This raises the concern that when one is working at such a low value of μ, there may be significant corrections even in lattice evaluations. Certainly, use of $\mu < 1$ GeV, as occurs in many model dependent evaluations, appears extremely dubious.

The values displayed above are based on working in the chiral limit of massless quarks. One must, however, add to these the chiral corrections. Work has begun on this important problem. [15] Nevertheless, it is interesting to look at the phenomenological consequences of the results reported in this paper.. While we cannot give a full evaluation of ϵ'/ϵ because we have not evaluated the contribution of B_6, we can give the contribution arising from the electroweak penguin,

$$\left(\frac{\epsilon'}{\epsilon}\right)_{B_8} = (-12 \pm 3) \cdot 10^{-4} \quad . \tag{71}$$

The effect of B_6 is expected to be positive, and needs to be almost three times larger than that of B_8 if the Standard Model is to account for the experimental value.

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