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Chiral anomaly and $\eta - \eta'$ mixing*

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Abstract

We determine the $\eta - \eta'$ mixing angle via a procedure relatively independent of theoretical assumptions by simultaneously fitting η, η' reactions involving the anomaly—$\eta, \eta' \to \gamma\gamma,\ \pi^+\pi^-\gamma$. We extract reasonably precise renormalized values of the octet and singlet pseudoscalar decay constants F_8, F_0, as well as the mixing angle θ.

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1 Introduction

From a strictly theoretical perspective, there exists a significant difference between the octet pseudoscalar mesons—π,K,η_8—and their singlet counterpart - η_0.[1] The former are legitimate pseudo-Goldstone bosons whose masses vanish in the chiral limit while the latter is not due to the anomalous breaking of the axial U(3) symmetry down to SU(3). However, in the real world, this does not seem to make much difference. Indeed, the physical eigenstates—η, η'—are mixtures of octet and singlet components

$$\begin{aligned} \eta &= \eta_8 \cos\theta - \eta_0 \sin\theta \\ \eta' &= \eta_8 \sin\theta + \eta_0 \cos\theta \end{aligned} \tag{1}$$

and the mixing angle θ is an important quantity in confronting theoretical calculations with experimental results in these systems.[2]

The mixing angle can be evaluated in various ways but a standard procedure involves the diagonalization of the η, η' mass matrix, which at lowest order yields a mixing angle $\theta \approx -10°$. This can easily be seen by writing the mass matrix as

$$m^2 = \begin{pmatrix} m_8^2 & m_{08}^2 \\ m_{08}^2 & m_0^2 \end{pmatrix} \tag{2}$$

Employing the Gell-Mann-Okubo (GMO) relation to fix m_8^2 as[3]

$$m_8^2 = \frac{1}{3}(4m_K^2 - m_\pi^2) \tag{3}$$

and diagonalizing, one can determine m_{08}^2, m_0^2 and θ, in terms of $m_\eta, m_{\eta'}$, yielding

$$\theta = -9.4°, \quad m_{08}^2 = -0.44 m_K^2, \quad m_{\eta_0} = 0.95\ GeV \tag{4}$$

However, the GMO relation, Eq.3, is valid only at lowest order—$\mathcal{O}(p^2)$—in chiral perturbation theory. The inclusion of $\mathcal{O}(p^4)$ corrections to the relation results in significant changes. Characterizing these via

$$m_8^2 = \frac{1}{3}(4m_K^2 - m_\pi^2)(1+\delta) \tag{5}$$

a leading logarithm estimate[4]

$$\delta \approx \frac{-2\frac{m_K^4}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{(4\pi F_\pi)^2}}{4m_K^2 - m_\pi^2} \tag{6}$$

yields $\delta \approx 0.16$, which yields

$$\theta \cong -20°, \quad m_{08}^2 = -0.81 m_K^2, \quad m_0 = 0.90 GeV \tag{7}$$

suggesting a doubling of the mixing angle. A full one-loop chiral perturbation theory calculation confirms this finding.[5] It is also interesting that this solution is consistent with the assumptions of simple $U(3)$ symmetry wherein η_8, η_0 have the same wavefunction, leading to

$$m_{08}^2 \cong \frac{2\sqrt{2}}{3}\left(\frac{\hat{m} - m_s}{\hat{m} + m_s}\right) \simeq -0.9\ m_K^2 \tag{8}$$

Note that this result is strongly dependent upon chiral symmetry breaking effects on the GMO prediction.

An alternative and independent approach to the problem involves the use of the chiral anomaly, which is responsible for the well-known $\pi, \eta, \eta' \to \gamma\gamma$ decays.[6] In the case of $\pi^0 \to \gamma\gamma$, at lowest order—$\mathcal{O}(p^4)$—the anomalous (Wess-Zumino-Witten) chiral lagrangian predicts[7]

$$A_{\pi^0\to\gamma\gamma} = \frac{\alpha N_c}{3\pi\bar{F}}\ \epsilon^{\mu\nu\alpha\beta}\epsilon_{1\mu}\epsilon_{2\nu}k_{1\alpha}k_{2\beta} \tag{9}$$

where ϵ_i, k_i are the polarization, momenta of the outgoing photons, and $\bar{F}$ is the pion decay constant in the chiral limit. A leading log calculation of the chiral corrections reveals that the dominant effect is simply to replace $\bar{F}$ by its physical value $F_\pi = 92.4MeV$.[8] The resulting amplitude is guaranteed by general theorems to remain unchanged in higher chiral orders.[9] One then finds that the predicted amplitude

$$F_{\pi\gamma\gamma}(0) = \frac{\alpha N_c}{3\pi F_\pi} = 0.025\ GeV^{-1} \tag{10}$$

is in excellent agreement with the corresponding experimental value[10]

$$F_{\pi\gamma\gamma}(0) = (0.024 \pm 0.001)\ GeV^{-1} \tag{11}$$

thus providing the confidence that one may be able to analyze the η, η' decays with a similar precision.

In case of the $\eta, \eta' \to \gamma\gamma$ decays, one must include both, mixing as well as renormalization of the octet, singlet couplings, yielding the predicted amplitudes [1]

$$\begin{aligned} F_{\eta\gamma\gamma}(0) &= \frac{\alpha N_c}{3\sqrt{3}\pi F_\pi}\left(\frac{F_\pi}{F_8}\cos\theta - 2\sqrt{2}\frac{F_\pi}{F_0}\sin\theta\right) \\ F_{\eta'\gamma\gamma}(0) &= \frac{\alpha N_c}{3\sqrt{3}\pi F_\pi}\left(\frac{F_\pi}{F_8}\sin\theta + 2\sqrt{2}\frac{F_\pi}{F_0}\cos\theta\right) \end{aligned} \tag{12}$$

From the PDG one extracts experimental values[10]

$$\begin{aligned} F_{\eta\gamma\gamma}(0) &= 0.0249 \pm 0.0010\ GeV^{-1} \\ F_{\eta'\gamma\gamma}(0) &= 0.0328 \pm 0.0024\ GeV^{-1} \end{aligned} \tag{13}$$

In order to solve the system, however, we require an additional assumption since there are three unknowns—F_8, F_0, θ—and only two pieces of data. The usual approach in this case is to use the leading log prediction of one-loop chiral perturbation theory to predict[5]

$$\frac{F_8}{F_\pi} = \left[1 - \frac{m_K^2}{(4\pi F_\pi)^2}\ \ln\frac{m_K^2}{(4\pi F_\pi)^2} + \frac{m_\pi^2}{(4\pi F_\pi)^2}\right] \approx 1.30 \tag{14}$$

[1]Note that we implicitly assume here, as do other workers, that all $\mathcal{O}(m_\eta^2, m_{\eta'}^2/\Lambda_\chi^2)$ effects, where $\Lambda_\chi \sim 4\pi F_\pi$ is the chiral scale, are included in the renormalization of the pseudoscalar couplings F_8, F_0 and in the mixing angle θ. This does not have to be the case, but appears to be borne out by the consistency of the results obtained from treatments of differing manifestations of the anomaly, as we show below and as others have found.

and then solve for F_0, θ, yielding

$$\frac{F_0}{F_\pi} = 1.04, \quad \theta = -20^\circ \tag{15}$$

It is interesting to note that these results are quite consistent with those obtained from the one-loop analysis of the mass matrix—*i.e.* $\theta \approx -20^\circ$ and F_0/F_π consistent with the value—unity—which one would have if the singlet state and the pion were to have the same wavefunction.

While this agreement is satisfying, the extraction of these mixing parameters requires certain theoretical inputs, either Eq. 14 or Eq. 6, and it is interesting to inquire whether one can predict the mixing angle purely phenomenologically. As we shall show, the answer is yes, provided one utilizes the additional information available from the anomalous decays $\eta, \eta' \to \pi^+\pi^-\gamma$.[11]

In the next section then we show how the decays $\eta, \eta' \to \pi^+\pi^-\gamma$ can be analyzed in order to isolate the chiral anomaly. This involves a careful study of final state interactions and unitarity constraints in order to realistically extrapolate to zero four-momenta, as required by the anomaly. In the concluding section, we apply these results to evaluate θ, F_8, F_0 in an essentially model independent fashion.

2 Analysis of $\eta, \eta' \to \pi^+\pi^-\gamma$ Decays

Our goal is to use the experimental data on $\eta, \eta' \to \pi^+\pi^-\gamma$ in order to isolate the value of the anomaly in these decays. The resulting numbers can then be fit to the appropriate theoretical expressions, thus allowing extraction of the renormalized mixing angle and coupling constants. To this end, we define

$$A_{\eta,\eta' \to \pi\pi\gamma} = B_{\eta,\eta'}(s, s_{\pi\pi})\ \epsilon^{\mu\nu\alpha\beta}\epsilon_\mu k_\nu p_{+\alpha} p_{-\beta} \tag{16}$$

where $p_\pm$, k are the outgoing 4-momenta, ϵ_μ is the photon polarization vector, $s = m^2_{\eta,\eta'}$ and $s_{\pi\pi} = (p_+ + p_-)^2$. The chiral anomaly (*cf.* Figure 1(a)) requires

$$\begin{aligned} B_\eta(0,0) &= \frac{eN_c}{12\sqrt{3}\pi^2 F_\pi^3}\left(\frac{F_\pi}{F_8}\cos\theta - \sqrt{2}\frac{F_\pi}{F_0}\sin\theta\right) \\ B_{\eta'}(0,0) &= \frac{eN_c}{12\sqrt{3}\pi^2 F_\pi^3}\left(\frac{F_\pi}{F_8}\sin\theta + \sqrt{2}\frac{F_\pi}{F_0}\cos\theta\right) \end{aligned} \tag{17}$$

Note, however, that the chiral anomaly, strictly speaking, only constrains the form factors $B_{\eta,\eta'}(s, s_{\pi\pi})$ at zero four momentum—$B_{\eta,\eta'}(0,0)$—while the experimental input occurs at $s = m^2_{\eta,\eta'}$, $s_{\pi\pi} \geq 4m_\pi^2$. One indication of this fact is that, using the simple energy-independent form given in Eq. 17 to calculate the decay rate, one obtains for the η-channel the value $\Gamma_{\eta-\pi\pi\gamma} = 35.7$ eV compared to the experimental rate of $\Gamma^{\rm exp}_{\eta-\pi\pi\gamma} = 64 \pm 6$ eV. For the η' channel things are even worse—$\Gamma_{\eta'-\pi\pi\gamma} = 61 \pm 5$ KeV while the theoretical value arising from use of the simple anomaly amplitude is a factor of twenty less! We conclude that in order to extract values for the anomaly in these transitions, it is absolutely essential to correctly model the energy dependence of the amplitude in the physical region.

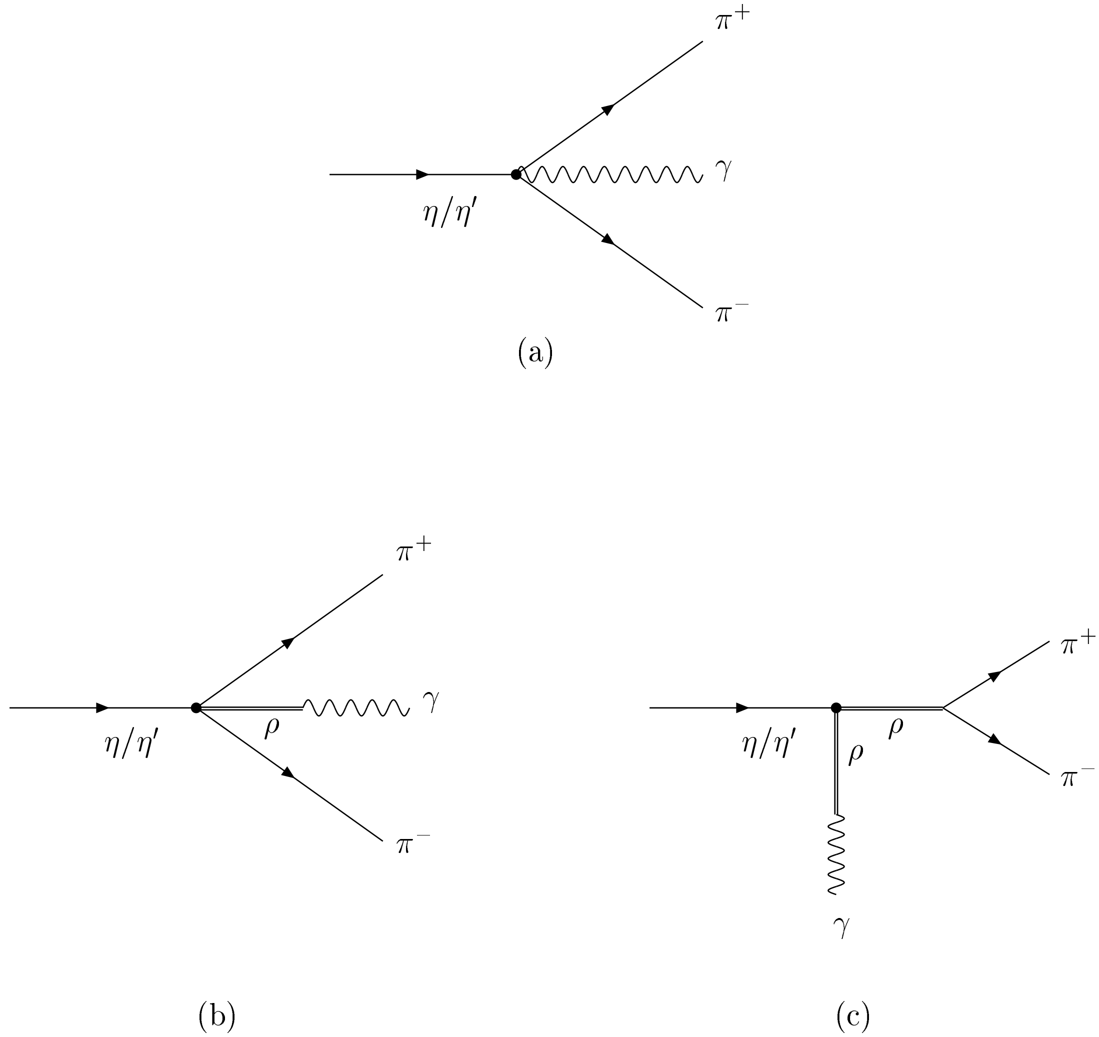

Figure 1: Shown are contact (a) and VMD (b,c) contributions to $\eta, \eta' \to \pi^+\pi^-\gamma$ decay.

This problem is not unique, of course, to the $\eta, \eta' \to \pi\pi\gamma$ system and has been addressed previously in the extraction of the anomaly in $\gamma \to 3\pi$ from the Primakoff-effect data of Antipov et al.[12] In this case, one also has a clear prediction of the chiral anomaly.[13] Writing

$$A_{\gamma\to 3\pi} = F_{3\pi}(s,t,u)\ \epsilon^{\mu\nu\alpha\beta}\epsilon_\mu p_{+\nu} p_{-\alpha} p_{0\beta} \tag{18}$$

where $s = (p_+ + p_-)^2, t = (p_+ + p_0)^2, u = (p_- + p_0)^2$, the chiral stricture demands

$$F_{3\pi}(0,0,0) = \frac{eN_c}{12\pi F_\pi^3} = 9.7\ GeV^{-3} \tag{19}$$

The effects of p-wave pi-pi interactions at low and moderate energies are known to be reasonably described by vector dominance.[14] In particular, the models described in ref.[15], which incorporate vector dominance and chiral symmetry, when applied to the $\gamma \to 3\pi$ reaction, provide a form

$$F_{3\pi}(s,t,u) = -\frac{1}{2}F_{3\pi}(0,0,0)\left[1 - \frac{m_\rho^2}{m_\rho^2 - s} - \frac{m_\rho^2}{m_\rho^2 - t} - \frac{m_\rho^2}{m_\rho^2 - u}\right] \tag{20}$$

which matches on to the anomaly at zero four-momentum but also offers a plausible extension into the physical region. This form must be modified, of course, in order to confront real data since unitarity demands the presence of branch cuts and consequent imaginary components in the form factor. This is clear from a one-loop chiral perturbation theory calculation, which yields[16]

$$F_{3\pi}(s,t,u)=F_{3\pi}(0,0,0)\left[1+\frac{3m_\pi^2}{2m_\rho^2}+\frac{m_\pi^2}{24\pi^2F_\pi^2}\left(\frac{3}{4}\ln\frac{m_\rho^2}{m_\pi^2}+F(s)+F(t)+F(u)\right)\right] \tag{21}$$

where

$$F(s)=\begin{cases}(1-\frac{s}{4m_\pi^2})\sqrt{\frac{s-4m_\pi^2}{s}}\ln\frac{1+\sqrt{\frac{s-4m_\pi^2}{s}}}{-1+\sqrt{\frac{s-4m_\pi^2}{s}}}-2 & s>4m_\pi^2\\ 2(1-\frac{s}{4m_\pi^2})\sqrt{\frac{4m_\pi^2-s}{s}}\tan^{-1}\sqrt{\frac{s}{4m_\pi^2-s}}-2 & s\le 4m_\pi^2\end{cases} \tag{22}$$

Here we note that the imaginary component of the function $F(s)$ is given in terms of the energy-dependent width of the rho meson via

$$\frac{m_\pi^2}{24\pi^2F_\pi^2}\mathrm{Im}F(s)=\frac{1}{m_\rho}\Gamma_\rho(s) \tag{23}$$

where

$$\Gamma_\rho(s)=\frac{g_{\rho\pi\pi}^2 s}{48\pi m_\rho}(1-\frac{4m_\pi^2}{s})^{3/2} \tag{24}$$

This one-loop form is no doubt appropriate in the near threshold region. However, once $s_{\pi\pi}\geq 10m_\pi^2$ or so, one does not anticipate that a simple one-loop description will be adequate. In order to address these difficulties, the use of an N/D form has been suggested.[17] In this approach, one utilizes the Omnes function, which encodes information concerning the pi-pi interaction[18]

$$D_1(s)=exp\left(-\frac{s}{\pi}\int_{4m_\pi^2}^\infty\frac{ds'\;\delta_1(s')}{s'(s'-s-i\epsilon)}\right) \tag{25}$$

where $\delta_1(s)$ is the $(l=1)$ p-wave pi-pi phase shift at center of mass energy $\sqrt{s}$. There are two ways to proceed at this point.

i) One can use the experimental phase shifts, with some assumptions made about their asyptotic form. In our case we took the values quoted by Froggatt and Peterson,[19] which are given up to $\sqrt{s}=1$ GeV and assumed a constant value after that. We label the Omnes function obtained in this way as $D_1^{\exp}(s)$.

ii) One can employ a simple analytic form[20]

$$D_1(s)=1-\frac{s}{m_\rho^2}-\frac{s}{96\pi^2F_\pi^2}ln\frac{m_\rho^2}{m_\pi^2}-\frac{m_\pi^2}{24\pi^2F_\pi^2}F(s) \tag{26}$$

which has been shown to provide an approximate description of the empirical $\pi\pi$ p-wave phase shifts in the low energy region.[21] We label the Omnes function obtained via this procedure as $D_1^{\rm anal}(s)$.

Using either of these forms, and postulating an N/D form of the $\gamma \to 3\pi$ amplitude as

$$F_{3\pi}(s,t,u) = -\frac{1}{2}A_{3\pi}(0)[1-(\frac{m_\rho^2}{m_\rho^2-s}+\frac{m_\rho^2}{m_\rho^2-t}+\frac{m_\rho^2}{m_\rho^2-u})] \times \left(\frac{1-\frac{s}{m_\rho^2}}{D_1(s)}\right)\left(\frac{1-\frac{t}{m_\rho^2}}{D_1(t)}\right)\left(\frac{1-\frac{u}{m_\rho^2}}{D_1(u)}\right) \quad (27)$$

one can see that Eq. 27 matches onto the one-loop chiral result in the limit of low energies *and* onto the vector dominance form when unitarity inspired logarithms are dropped.

Whether such an N/D form accurately describes the data for $\gamma \to 3\pi$ awaits the arrival of sufficiently precise information from CEBAF and CERN. However, it certainly appears to satisfy the various criteria which nature demands, and suggests the treatment of the related $\eta, \eta' \to \pi^+\pi^-\gamma$ decay amplitudes in a parallel fashion.

In the case of the $\eta, \eta' \to \pi^+\pi^-\gamma$ decays we can proceed similarly. In this case, the one-loop chiral perturbation theory calculation gives

$$B_\eta^{1-\mathrm{loop}}(s,s_{\pi\pi}) = B_\eta(0,0)[1+\frac{1}{32\pi^2F_\pi^2}((-4m_\pi^2+\frac{1}{3}s_{\pi\pi})\ln\frac{m_\pi^2}{m_\rho^2}+\frac{4}{3}F(s_{\pi\pi})-\frac{20}{3}m_\pi^2+\frac{3}{2m_\rho^2}s_{\pi\pi}] \quad (28)$$

while vector dominance (*cf.* Figure 1(b,c))yields

$$B_{\eta,\eta'}(s,s_{\pi\pi}) = B_{\eta,\eta'}(0,0)[1+\frac{3}{2}\frac{s_{\pi\pi}}{m_\rho^2-s_{\pi\pi}}] \quad (29)$$

Certainly, in order to treat the decay of the η', one must go further and include unitarity effects via final state interactions. One very simple approach is to include the (energy-dependent) width of the rho-meson in the propagator via

$$\frac{s_{\pi\pi}}{m_\rho^2-s_{\pi\pi}} \to \frac{s_{\pi\pi}}{m_\rho^2-s_{\pi\pi}-im_\rho\Gamma_\rho(s_{\pi\pi})} \quad (30)$$

This use of vector width-modified vector dominance already makes an important difference from the simple anomaly—tree level—results (especially in the case of the η'), changing the predicted decay widths from the values 35 eV and 3 KeV quoted above to the much more realistic numbers

$$\Gamma_{\eta-\pi\pi\gamma} = 62.3eV, \qquad \Gamma_{\eta'-\pi\pi\gamma} = 67.5KeV \quad (31)$$

if the parameters

$$F_8/F_\pi = 1.3, \qquad F_0/F_\pi = 1.04, \qquad \theta = -20^\circ \quad (32)$$

are employed. However, this approach does not reproduce the one-loop chiral form in the low energy limit.

In order to determine a form for the final state interactions which matches onto both the one-loop chiral correction *and* to the vector dominance result in the appropriate limits, we postulate an N/D structure, as in the related $\gamma \to 3\pi$ case—

$$B_{\eta-\pi\pi\gamma}(s,s_{\pi\pi}) = B_{\eta-\pi\pi\gamma}(0,0)\left[1-c+c\frac{1+as_{\pi\pi}}{D_1(s_{\pi\pi})}\right] \quad (33)$$

where for the Omnes function we use one of the two forms itemized above and a, c are free parameters to be determined. In order to reproduce the coefficient of the $F(s_{\pi\pi})$ function, which contains the rho width, we require $c = 1$. On the other hand, matching onto the VMD result at $\mathcal{O}(p^6)$ can be achieved by the choice $a = 1/2m_\rho^2$. Thus in the case of the η the form is completely determined. Since the η' spectrum is closely related and is dominated by the presence of the rho we shall assume an identical form for the η' case. Using these forms we can then calculate the decay widths assuming the theoretical values for the anomaly. Using the parameters given in Eq. 32 one finds, for example,

$$\begin{aligned} i) D_1^{\mathrm{exp}}(s) \qquad & \Gamma_{\eta-\pi\pi\gamma} = 65.7eV, \quad \Gamma_{\eta'-\pi\pi\gamma} = 66.2KeV \\ ii) D_1^{\mathrm{anal}}(s) \qquad & \Gamma_{\eta-\pi\pi\gamma} = 69.7eV, \quad \Gamma_{\eta'-\pi\pi\gamma} = 77.8KeV \end{aligned} \tag{34}$$

There is a tendency then for the numbers obtained via the analytic form of the Omnes function to be somewhat too high.

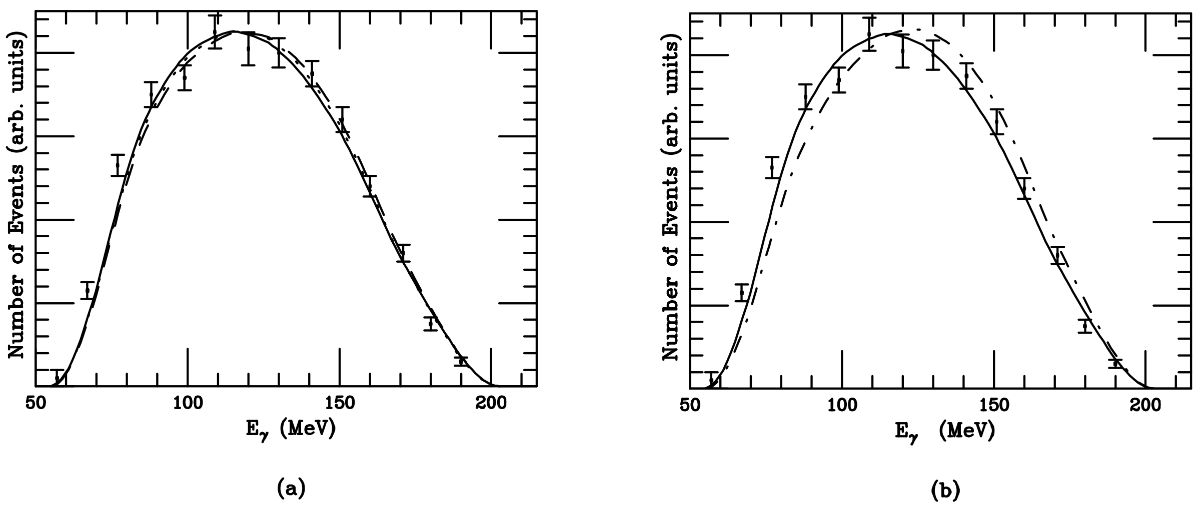

Figure 2: Shown is the photon spectrum in $\eta \to \pi^+\pi^-\gamma$ from Gormley et al.[26] as well as various theoretical fits. In the first figure, the dashed line represents the (width-modified) VMD model. The (hardly visible) dotted line and the solid line represent the final state interaction ansatz Eq.33 with use of the analytic and experimental version of the Omnes function respectively. The second figure shows the experimental Omnes function result (solid line) compared with the one-loop result (dotdash line).

We can also compare the predicted spectra with the corresponding experimentally determined values. As shown in Figure 2, we observe that the experimental spectra are well fit in the η case in terms of both the N/D or the VMD forms, but that the one-loop chiral expression does not provide an adequate representation of the data.[22] In the case of the corresponding η' decay the results are shown in Figure 3, wherein we observe that either the unitarized VMD or the use of N/D_1^{exp} provides a reasonable fit to the data (we get χ^2/dof=32/17 and 20/17, respectively), while the use of the analytic form for the Omnes function yields a predicted spectrum (χ^2/dof=104/17) which is slightly too low on the high energy end. However, for both η and η' we see that our simple ansatz—Eq.33—provides a very satisfactory representation of the decay spectrum.

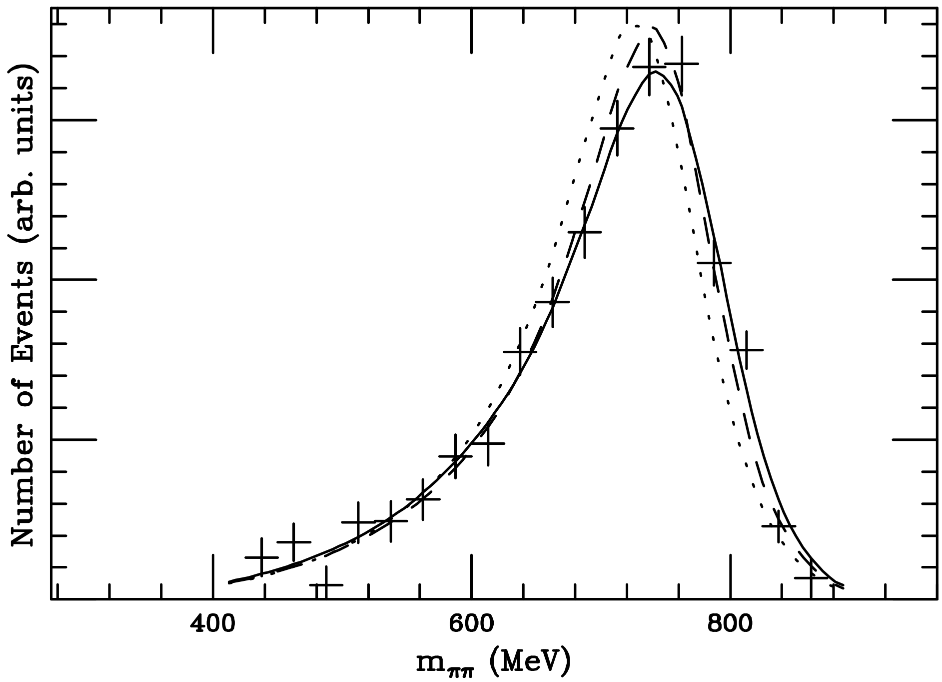

Figure 3: Shown is the photon spectrum in $\eta' \to \pi^+\pi^-\gamma$ from Abele et al.[24] as well as various theoretical fits. As in Figure 2, the dashed line represents the (width-modified) VMD model. The dotted and solid lines represent the final state interaction ansatz Eq.33 with use of the analytic and experimental version of the Omnes function, respectively. Here the curves have been normalized to the same number of events.

3 Evaluation of $\eta - \eta'$ mixing parameters

Our conclusion in the last section was that if the mixing angle and pseudoscalar coupling constants were given values consistent with present theoretical and experimental leanings, then the predicted widths and spectra of both $\eta, \eta \to \pi^+\pi^-\gamma$ are basically consistent with experimental values. Our goal in this section is to go the other way, however. That is, using the assumed N/D forms for the decay amplitude, and treating the pseudoscalar decay constants F_8, F_0 as well as the $\eta - \eta'$ mixing angle θ as free parameters, we wish to inquire as to how well they can be constrained purely from the experimental data on $\eta, \eta' \to \gamma\gamma$ and $\eta, \eta' \to \pi^+\pi^-\gamma$ decays, with reasonable assumptions made about the final state interaction effects in these two channels.

On theoretical grounds, one is somewhat more confident about the extraction of the threshold amplitude in the case of the lower energy $\eta \to \pi^+\pi^-\gamma$ system. Indeed, in this case the physical region extends only slightly into the tail of the rho unlike the related η' decay wherein the spectrum extends completely over the resonance so that there exists considerable sensitivity to details of the shape. Thus a first approach might be to utilize only the two-photon decays together with the $\eta \to \pi^+\pi^-\gamma$ width in order to determine the three desired parameters. In this fashion one finds the results shown in Table 1. We observe that the results are in agreement, both with each other and with the chiral symmetry expectations—$F_8/F_\pi \sim 1.3$, $F_0/F_\pi \sim 1$, and $\theta \sim -20°$. However, the uncertainties obtained in this way are uncomfortably high.

	F_8/F_π	F_0/F_π	θ
VMD	1.28 ± 0.24	1.07 ± 0.48	$-20.3^\circ \pm 9.0^\circ$
$\mathrm{N/D}_1^{\mathrm{anal}}$	1.49 ± 0.29	1.02 ± 0.42	$-22.6^\circ \pm 9.6^\circ$
$\mathrm{N/D}_1^{\mathrm{exp}}$	1.37 ± 0.26	1.02 ± 0.45	$-21.2^\circ \pm 9.3^\circ$

Table 1: Values of the renormalized pseudoscalar coupling constants and the $\eta - \eta'$ mixing angle using the $\eta, \eta' - \gamma\gamma$ and $\eta - \pi\pi\gamma$ amplitudes in a three parameter fit.

	F_8/F_π	F_0/F_π	θ
VMD	1.28 ± 0.20	1.07 ± 0.04	$-20.8^\circ \pm 3.2^\circ$
$\mathrm{N/D}_1^{\mathrm{anal}}$	1.48 ± 0.24	1.09 ± 0.03	$-24.0^\circ \pm 3.0^\circ$
$\mathrm{N/D}_1^{\mathrm{exp}}$	1.38 ± 0.22	1.06 ± 0.03	$-22.0^\circ \pm 3.3^\circ$

Table 2: Values of the renormalized pseudoscalar coupling constants and of the $\eta - \eta'$ mixing angle obtained from a maximum likelihood analysis using the $\eta, \eta' - \gamma\gamma$ and $\eta, \eta' - \pi\pi\gamma$ amplitudes.

In order to ameliorate this problem, we have also done a maximum likelihood fit including the $\eta' - \pi\pi\gamma$ decay rate, yielding the results shown in Table 2. We observe that the central values stay fixed but that the error bars are somewhat reduced. The conclusions are the same, however—substantial renormalization for $F_8 \sim 1.3F_\pi$, almost none for $F_0 \sim F_\pi$, and a mixing angle $\theta \sim -20^\circ$. These numbers appear nearly invariant, regardless of the approach.

4 Conclusion

Before summarizing the results of our above analysis, it should certainly be emphasized that we are not the first to undertake the program of isolating the anomaly from the $\eta, \eta' - \pi\pi\gamma$ data. Indeed, there has been considerable work in this regard, both on the theoretical side [11, 23] as well as experimentally, including the most recently published $\eta' \to \pi^+\pi^-\gamma$ data.[24] The recent analysis of ref.[23] leads to results quite different from ours in both the mixing angle as well as the renormalization of the pseudoscalar couplings. On the other hand, ref.[24] (at least for the model labeled M_1) finds a somewhat smaller mixing angle ($\theta \sim -16^\circ$) and pseudoscalar renormalization ($F_8/F_0 \sim 1.1$).

However, there is an important difference between these analyses and our own. In refs.[23] and [24], the decay amplitude is written in terms of a piece due to the anomaly (parametrized by $E_X, X = \eta, \eta'$) and a component due to the rho pole (parametrized by $F_X, X = \eta, \eta'$). The ratio E_X/F_X is then fitted, through a minimization procedure, to produce the experimental spectrum and partial widths. In our analysis, the parameters of the two pieces are fixed *a priori* to reproduce the results of one-loop chiral perturbation theory [16] (by fixing $c = 1$) and VMD [15] (by fixing $a = 1/2m_\rho^2$). (Indeed a recent analysis of other I=1 $\pi^+\pi^-$ processes found that only within a model such as ref.[15], which links chiral symmetry and VMD, could the data be fit consistently [25]). However, following the picture of ref.[15], we do not include a non-resonant coupling in $\gamma\pi^+\pi^-$ (as considered in models M_1 and M_3 of [23]). Our successful fits of the experimental data speak for themselves.

In previous treatments of the η, η' system via the anomaly, which have omitted $\eta, \eta' - \pi\pi\gamma$ constraints, the mixing angle θ has generally been determined only at the cost of theoretical assumptions about the renormalization of the octet pseudoscalar coupling constant F_8 with respect to F_π. We have in this paper asked whether it is possible to obtain the mixing angle in a fashion relatively independent of such theoretical assumptions by simultaneously fitting $\eta, \eta' \to \gamma\gamma$ as well as $\eta, \eta' \to \pi^+\pi^-\gamma$ decays. As shown above, the answer is affirmative. However, one must incorporate *some* sort of model for the final state interactions of the pi-pi system in order to extrapolate down to zero four-momentum where the anomaly obtains. We have argued that the $N/D_1^{\exp}$ form given in Eq. 33 is reasonable both on theoretical grounds—matching both the requirements of VMD and of low energy chiral symmetry—and via successful fitting of the experimental spectra. Using this form and the PDG values for $\eta, \eta' \to \gamma\gamma$ and $\eta, \eta' \to \pi\pi\gamma$ amplitudes we have obtained values

$$F_8/F_\pi = 1.38 \pm 0.22 \qquad F_0/F_\pi = 1.06 \pm 0.03 \qquad \theta = -22.0^\circ \pm 3.3^\circ \tag{35}$$

which are quite consistent with those obtained in previous analyses which required assumptions about chiral symmetry breaking. One can then assess these results in two different ways. Although it is our contention that the assumptions made above concerning pion-pion interactions are relatively model-independent and that the numbers given thereby in Eq. 35 are quite solid, one could also take a contrary view that the forms utilized for final state interactions *do* require critical dynamical assumptions. In this case, however, we would argue that via three quite different routes

i) mass matrix analysis including GMO breaking

ii) $\eta, \eta' \to \gamma\gamma$ analysis with assumptions made about F_8/F_π

iii) simultaneous analysis of $\eta, \eta' \to \gamma\gamma$ and $\eta, \eta' \to \pi^+\pi^-\gamma$ with (minimal) assumptions concerning final state pi-pi interactions

one finds virtually the same value of the mixing angle—$\theta \simeq -20^\circ$—and for pseudoscalar couplings—$F_8/F_\pi \sim 1.3, F_0/F_\pi \sim 1.0$. In any case, we would assert that these values are now strongly (and independently) confirmed from within the chiral anomaly sector.

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