1998

Weight for random quark masses

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Abstract

In theories in which the parameters of the low energy theory are not unique, perhaps having different values in different domains of the universe as is possible in some inflationary models, the fermion masses would be distributed with respect to some weight. In such a situation the specifics of the fermion masses do not have a unique explanation, yet the weight provides the visible remnant of the structure of the underlying theory. This paper introduces this concept of a weight for the distribution of masses and provides a quantitative estimate of it from the observed quarks and leptons. The weight favors light quark masses and appears roughly scale invariant ($\rho \sim 1/m$). Some relevant issues, such as the running of the weight with scale and the possible effects of anthropic constraints, are also discussed.

1 Basic ideas

Many of the parameters of the Standard Model, such as the quark masses and the weak mixing angles, appear without any obvious pattern. Perhaps the explanation for the specific values of these parameters is hidden in the physics at a deeper level. The goal of much of the work in particle physics has been to find the underlying theory, the golden Lagrangian, which by its structure explains the parameters of the Standard Model. If this is successful, it will indeed be satisfying.

However, another possibility also exists - that these parameters are in some sense random. This could potentially occur in various ways. For example, in some theories of inflation, different regions of the universe involve different parameters and perhaps even different low energy theories[1]. The dynamical fields which determine the properties of the low energy theory become fixed at different values in each domain, and subsequent inflation ensures that we live within only one of these domains. It is also conceivable in theories such as superstrings with different classically equivalent vacua, that different vacuum states could be selected in different regions of the universe. The moduli fields, whose vacuum expectation values determine the mass parameters of the low energy theory, are not fixed upon compactification[2]. Perhaps these fields are sampled in a random fashion rather than being determined uniquely by some mechanism. However, the general idea can at this time be considered distinct from the particular underlying theory. If some quasi-random mechanism is in fact at work, we should not expect to find a unique explanation for the values of the parameters in the Standard Model.

Given our present incomplete knowledge, it is not any more scientific to assume that there is a unique explanation for a single set of parameters which holds throughout the universe than it is to consider the possibility that the parameters may vary in different domains. Indeed, the history of science has taught us that we do not occupy a privileged position. In this spirit, it is fair to explore the possibility that our domain of the Universe and our particular parameters are not unique.

The quark masses do not appear strictly random either, as there are more light quarks than very massive ones. Indeed, we would not necessarily expect the masses to be equally distributed in such a multiple domain theory. The structure and dynamics of the fields which determine the low energy parameters may bias the distribution of possible parameters in various ways. The parameters would then be distributed randomly with respect to some weight. The potentially visible remnant of the underlying theory would not be the details of the individual mass values, but rather the weight by which they are distributed. If we can obtain an indication of the weight, perhaps we can use this to help determine the appropriate underlying theory.

The purpose of this paper is to provide an initial exploration of this idea of a weight for fermion masses, and provide an estimate of the weight from the observed masses. If we had available information from an ensemble of different domains or even from very many masses within a domain, the determination of the weight would be simple. Because we are aware of only one domain which contains six quark and three known lepton masses, we have quite limited data for this exercise. However, the fermion mass distributions are quite striking, and we can at least provide some quantification of this fact.

The weight is not an invariant concept, identical at all scales. Because the quark and lepton masses run under a change of scale, the weight extracted at different scales will also run. I give a discussion and quantification of this feature (Sec. 3), and present final masses at the weak interaction scale (Sec 4). The smoothing procedure to obtain information from a discrete spectrum is discussed in Sec 5. Attempts to increase the number of input parameters and to assess the uncertainty in this procedure are addressed in Sec. 6-8. Further information is contained in the Yukawa couplings that generate the weak mixing angles, although the diagonalization of quark mass matrices lead to a loss of some of this information(Sec 6). Lepton masses may also be relevant. Since, quarks and leptons run at different rates, there are also some complications in attempting to combine these into a single weight. This is explored using the effects of QCD interactions (Sec 7). The uncertainties are summarized in Sec 8, and then I try to provide a functional measure of the weight in Sec 9.

In a multiple domain theory, it is an obvious requirement that out of all the possible domains we must find ourselves in a domain with parameters amenable to the development of life[3,4,5]. This restricts the space of allowed parameters somewhat, most especially the parameters whose values might otherwise have to be fine-tuned. Weinberg[3] has estimated the tiny range of anthropically-allowed values of the cosmological constant from the requirement that the universe expand at a rate which allows galaxies to form.

My collaborators and I [4] have estimated the allowed range of value of the Higgs vacuum expectation value from the anthropic need for complex atoms to exist and be formed in the universe, and have suggested this as a possible explanation for the unnatural closeness of the weak scale and the QCD scale. If the fermion masses are also variable, there are anthropic constraints on these also[4], favoring having a light first generation. The subsequent generations appear to not have much impact on anthropic constraints. However, this issue does potentially complicate attempts to determine the weight, as I discuss later in the paper (Sec. 10). At this stage, I also discuss further considerations which may modify the procedure used to extract the weight appropriate for a given underlying theory. Some of these issues will be addressed more clearly if we are able to explore a specific underlying theory in an attempt to predict the observed weight.

2 Definition of the weight

The weight provides a normalized distribution function for the masses or Yukawa couplings. In an ensemble of domains similar to our own, the fraction of masses found at a value m within a range dm is defined to be

$$f(m) = \rho(m)\ dm \tag{1}$$

where $\rho(m)$ is the symbol for the weight. When we need to focus on the scale dependence of the weight we will include the scale μ as a subscript, i.e. $\rho_\mu(m)$, indicating that it is the form of ρ appropriate for that value of μ. By assumption, for a small number of masses the values of the masses will appear randomly distributed with respect to the weight $\rho(m)$.

The normalization of the weight is

$$1 = \int \rho(m)\ dm \quad . \tag{2}$$

The finiteness of the normalization imposes constraints on the low mass and high mass limits of $\rho(m)$. At small values of m, the weight cannot grow any faster than $\frac{1}{m}$ while for large values it must fall faster than $\frac{1}{m}$. From this it is clear that there is no pure power-law behavior that can hold for all values of m.

3 The running of the weight, quasi-fixed points, etc.

In the ultimate fundamental theory, the weight will be determined by some physics at a large energy scale, for example the GUT or Planck scales. In order for this to be compared with low-energy physics, it will need to be transformed to a low-energy scale. Since the quark masses run with the scale, the weight will similarly transform. This section discusses the nature of the scale dependence of the weight.

The renormalization group equations provide a continuous one-to-one flow for the masses. In changing from a scale μ_1 to a scale μ_2 a mass m_1 will flow to a value m_2. This defines a functional relationship between the masses at the two scales, $m_1 = m_1(m_2)$ or $m_2 = m_2(m_1)$. Similarly a small range of masses Δm_1 around the value m_1 will flow into a range Δm_2 around m_2. The magnitude of Δm_2 is generally not the same as that of Δm_1 as a range of masses can either grow closer or expand under the renormalization group transformation. The transformation of the weight follows from a conservation of probability. Since the transformation is continuous, the same fraction of masses that fall in the range Δm_1 will, after the rescaling, appear in the range Δm_2. From the definition of this fraction as $f = \rho\ \Delta m$, we have the condition

$$\rho_{\mu_2}(m_2)\Delta m_2 = \rho_{\mu_1}(m_1)\Delta m_1 \tag{3}$$

If we take the infinitesimal limit and define a "Jacobian"

$$J(m_2) = \frac{\partial m_1}{\partial m_2} \tag{4}$$

as a function of m_2, we have the transformation equation

$$\rho_{\mu_2}(m_2) = \rho_{\mu_1}(m_1(m_2))J(m_2)\ . \tag{5}$$

The normalization is obviously preserved under this rescaling.

As an example consider those masses where only the QCD gauge couplings are important in the scaling of the masses. This implies a linear scaling with a simple anomalous dimension

$$m(\mu) = m(\mu_0)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{d_m} \tag{6}$$

with

$$d_m = \frac{4}{11 - \frac{2N_F}{3}} \qquad (7)$$

In this situation we have the linear behavior

$$m_2 = m_1 \left(\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)}\right)^{d_m} \qquad (8)$$

$$J(m_2) = \left(\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)}\right)^{-d_m} \qquad (9)$$

$$\rho_{\mu_2}(m_2) = \rho_{\mu_1}\left(m_2 (\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)})^{-d_m}\right)\left(\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)}\right)^{-d_m} \qquad (10)$$

However, this simple form is only valid for small values of the masses, as we now discuss.

When the mass, or equivalently the Yukawa coupling, is large the running of the mass is influenced by the Yukawa coupling itself. When evolving from high energy to low, the influence of the Yukawa coupling tends to drive the mass to smaller values, while the gauge coupling will tend to evolve the mass to larger values. If the Yukawa coupling at the high energy scale is small enough, a finite evolution to lower energy will always be dominated by the gauge couplings, with the pattern discussed in the previous paragraph. However, for larger initial Yukawa couplings, the flow will approach a value where the effects of gauge and Yukawa couplings cancel. In particular all very large Yukawa couplings will quickly approach this quasi-fixed point[6,7]. The effect of this on the weight is that for reasonable distributions at high energy, the low-energy weight will have an upper cutoff at the quasi-fixed point.

Consider the evolution of a large Yukawa coupling h under the influence of itself plus QCD interactions. (We here neglect the weak and electromagnetic effects.) Recall that within the Standard Model, the masses are related to the Yukawa couplings h_i via

$$m_i = \frac{h_i}{\sqrt{2}} v \qquad (11)$$

with $v = 246$ GeV being the Higgs vacuum expectation value. If μ_1 is the initial scale and $t = ln(\mu_1^2/\mu^2)$, the renormalization group equations for

$N_f = 6$ are[6,7]

$$\frac{dg_3^2}{dt} = \frac{7}{16\pi^2} g_3^4 \tag{12}$$

$$\frac{dh^2}{dt} = h^2 \left(\frac{1}{2\pi^2} g_3^2 - \frac{9}{32\pi^2} h^2 \right) \tag{13}$$

These equations have an exact solution[7]

$$h^2(t) = \left(\frac{\alpha_s(t)}{\alpha_s(0)} \right)^{2d_m} \frac{h^2(0)}{1 + \frac{9}{8\pi} \frac{h^2(0)}{\alpha_s(0)} \left((\frac{\alpha_s(t)}{\alpha_s(0)})^{\frac{1}{7}} - 1 \right)} \tag{14}$$

$$= \frac{2}{9} \frac{g_3^2(t)}{1 + (\frac{\alpha_s(0)}{\alpha_s(t)})^{\frac{1}{7}} \left(\frac{2}{9} \frac{g_3^2(0)}{h^2(0)} - 1 \right)} \tag{15}$$

The first form of this relation better illustrates the linear behavior in the small h limit, while the latter shows the quasi-fixed-point at $h^2/g_3^2 = 2/9$. When scaling from the GUT or Planck scales within the Standard Model, the quasi-fixed-point occurs at $m^* = 220$ GeV. This value would change if supersymmetry or other interactions occurred in the region between the GUT and weak scales.

The transformation of the weight follows directly from this form. Let us define

$$h(t) = b \frac{h(0)}{[1 + ah^2(0)]^{\frac{1}{2}}} \tag{16}$$

$$b(t) = \left(\frac{\alpha_s(t)}{\alpha_s(0)} \right)^{4/7} \tag{17}$$

$$a(t) = \frac{9}{2g_3^2(0)} \left[\left(\frac{\alpha_s(t)}{\alpha_s(0)} \right)^{\frac{1}{7}} - 1 \right] \tag{18}$$

We can invert this relation, obtaining

$$h(0) = \frac{h(t)}{[b^2 - ah^2(t)]^{\frac{1}{2}}} \tag{19}$$

Recalling that $h(t)$ differs from the mass $m(t)$ only by a constant (see Eq. (11)), we find the Jacobian

$$J(m) = \frac{b^2}{\left[b^2 - 2a\frac{m^2}{v^2}\right]^{\frac{3}{2}}} \ . \tag{20}$$

For small masses, this is equivalent to Eq. 9.

Let us explore the effects of rescaling and the quasi-fixed-point via an example. At a high scale, such as the Planck mass, we consider

$$\rho_{\mu=M_P}(h) = \frac{\lambda^{1-\delta}}{\Gamma(1-\delta)} \frac{1}{h^\delta} e^{-\lambda h} \ . \tag{21}$$

Finiteness constrains $\delta < 1$. When scaled down to the weak scale one obtains

$$\rho_{\mu=M_W}(m) = \frac{\lambda^{1-\delta}}{\Gamma(1-\delta)} \left(\frac{v}{\sqrt{2}m}\right)^\delta \frac{b^2}{\left[b^2 - 2a\frac{m^2}{v^2}\right]^{\frac{3-\delta}{2}}} e^{-\frac{\sqrt{2}\lambda m}{v\left[b^2-2a\frac{m^2}{v^2}\right]^{\frac{1}{2}}}} \ . \tag{22}$$

This is plotted in Fig. 1 and 2 for $\delta = 1/2$ and $\delta = 0.9$ with $\lambda = 1$. When scaled from the Planck mass to the weak scale, the parameters a and b have values $a = 7.9$ and $b = 3.2$. Note that while the high-energy distribution extends to large masses, once rescaled to low energy, there is a cut-off at the quasi-fixed-point. The integrated weight for large masses all appears at values close to the fixed point, leading to a peaking of the weight in this area. This indicates that even if the high-energy weight is small for large m, it is nevertheless likely that one or more of the masses will occur very close to the fixed point. It is tempting to feel that this has occurred in the case of the top quark.

4 Brief review of masses

The precise definition of quark masses involve many subtle issues, which are not presently fully resolved. Because of confinement in QCD, the quark masses can not be defined in the same way that we do for leptons. The attempts to provide clear values of the masses involves very interesting features. However, I will not dwell on many of these features since this paper uses the

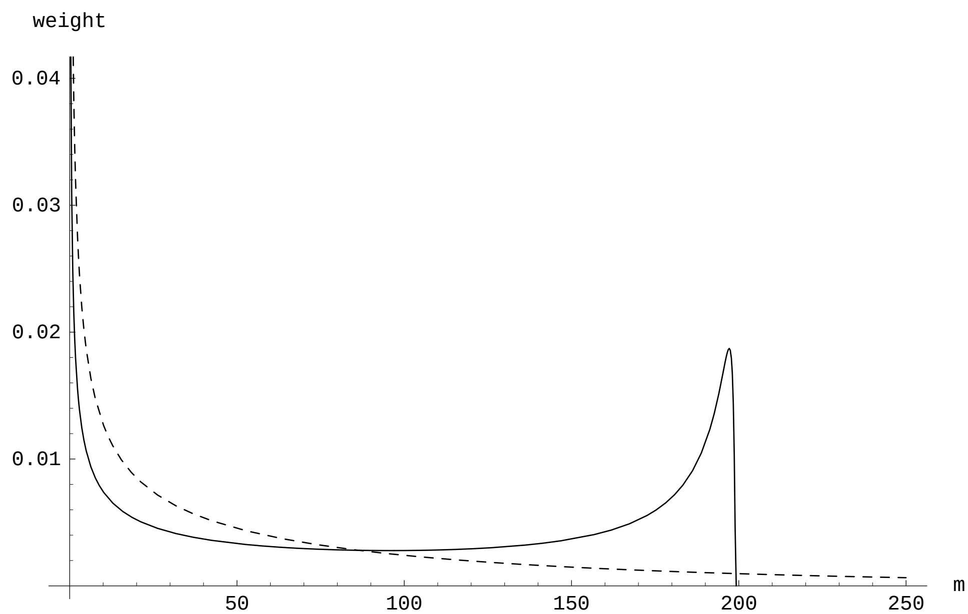

Figure 1: The effect of renormalization group rescaling on a possible weight function with $\delta = 1/2$. The dashed curve corresponds to a weight defined at the Planck scale and the solid curve is the same weight at the scale $\mu = M_W$.

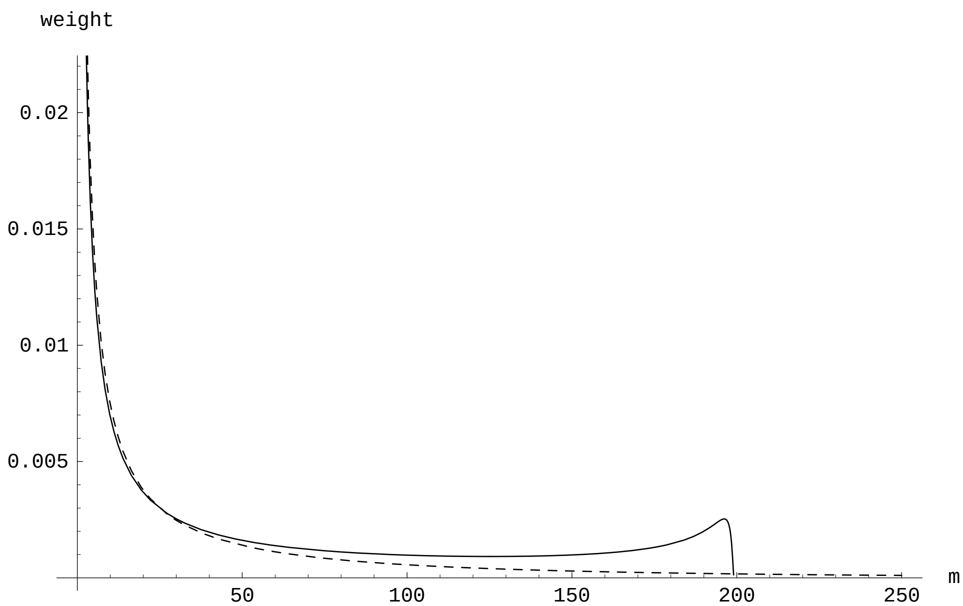

Figure 2: The same as Fig. 1, but with $\delta = 0.9$.

masses only in a relatively crude way. I will simply accept the uncertainties described in the review of Ref[8].

The ratios of light quark masses are better determined than the absolute magnitude. The ratios follow from the masses of pseudoscalar mesons when analyzed using chiral methods[9]. To specify the magnitude, one must specify a specific renormalization condition. This is presently only possible in model-dependent methods, leading to at least a factor of two uncertainty in the scale of the masses. We use

$$m_u = 4\,^{+4}_{-2}\ MeV \tag{23}$$

$$m_d = 8\,^{+7}_{-4}\ MeV \tag{24}$$

$$m_s = 150\,^{+150}_{-50}\ MeV \tag{25}$$

These values are intended to be estimates in a mass-independent renormalization scheme such as $\bar{M}S$ at a scale of order 1 GeV.

Heavy quark masses can be defined to some level of precision in the context of Heavy Quark Effective Theory (HQET)[10]. Here one can define either a pole mass or a running mass evaluated at the scale of the mass itself. For the running masses we use

$$m_c(m_c) = 1.4 \pm 0.2\ GeV \tag{26}$$

$$m_b(m_b) = 4.3 \pm 0.2\ GeV \tag{27}$$

$$m_t(m_t) = 166 \pm 5\ GeV. \tag{28}$$

In describing the weight, we need to transform the masses to a common scale. I will primarily use M_W as this scale. Running the masses to this value yields

$$m_u(M_W) = 2.2\,^{+2.2}_{-1.1}\ MeV \tag{29}$$

$$m_d(M_W) = 4.4\,^{+4}_{-2}\ MeV \tag{30}$$

$$m_s(M_W) = 80\,^{+80}_{-30}\ MeV \tag{31}$$

$$m_c(M_W) = 0.81 \pm 0.12\ GeV \tag{32}$$

$$m_b(M_W) = 3.1 \pm 0.2\ GeV \tag{33}$$

$$m_t(M_W) = 170 \pm 5\ GeV \tag{34}$$

In this calculation I used $\alpha_s(M_W) = 0.115$ and included the changes in N_F at charm and beauty thresholds. The set of masses are equivalent to a set of dimensionless Yukawa couplings. Recall that the top quark mass is equivalent to $h_t = 1$.

5 Smoothing the quark distribution

With only six quarks our insight into the weight is necessarily limited. In particular, there is nothing that we can do that can give much information at large values of the mass. The single very heavy quark, the top quark, tells us that the weight cannot vanish at large mass, nor be exponentially suppressed. However it is not possible to use the top quark to say anything detailed about the shape of the weight. The situation is marginally better at low mass. Viewed globally there is a striking clustering of the masses at low values. This requires the weight to be peaked at low mass. If we assume a smooth functional form for the weight, we will be able to compare it with some of the features of the observed masses. So we should be aware that our limited information on the weight exists almost entirely in the region where most masses are, i.e. around and below a GeV.

In a very crude form, we can consider binning the masses to give a rough estimate of the weight. This is illustrated in Fig. 3.

In order to compare with the physical masses in a more quantitative fashion we need some smoothing scheme which is able to encode the information contained in the discrete values of the masses, yet which can be compared to possible smooth trial functions describing the weight. I will use two such schemes. One involves an integral with some similarity to the Hilbert transform

$$H(z) = \int_0^\infty dm \; \frac{z\rho(m)}{m+z} \tag{35}$$

The second involves the Laplace transform

$$L(z) = \int_0^\infty dm \; \rho(m) \; e^{-m/z} \tag{36}$$

These transforms are constrained by the normalization condition to both be equal to unity at $z = \infty$ and are mainly sensitive to masses smaller in magnitude than the parameter z. I will refer to these functions as "transformed weights". The transformed weights H(z) and L(z) turn out contain quite similar information, and I will display L(z) only occasionally in what follows.

For the experimental side we use

$$\rho_{exp}(m) = \frac{1}{6}\Sigma_{i=1}^{6}\delta(m - m_i) \tag{37}$$

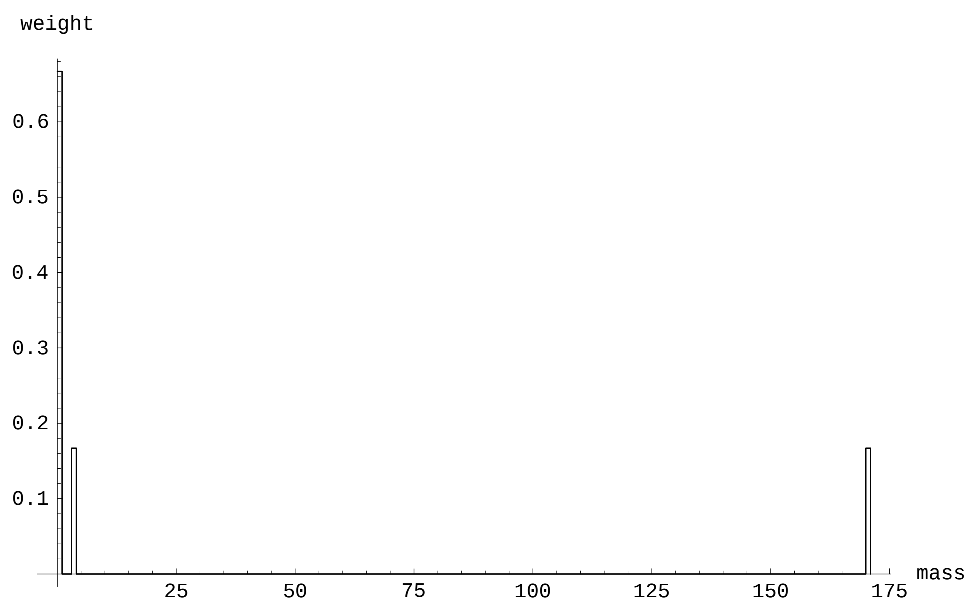

Figure 3: A rough characterization of the weight, obtained by binning the quark masses in 1 GeV bins.

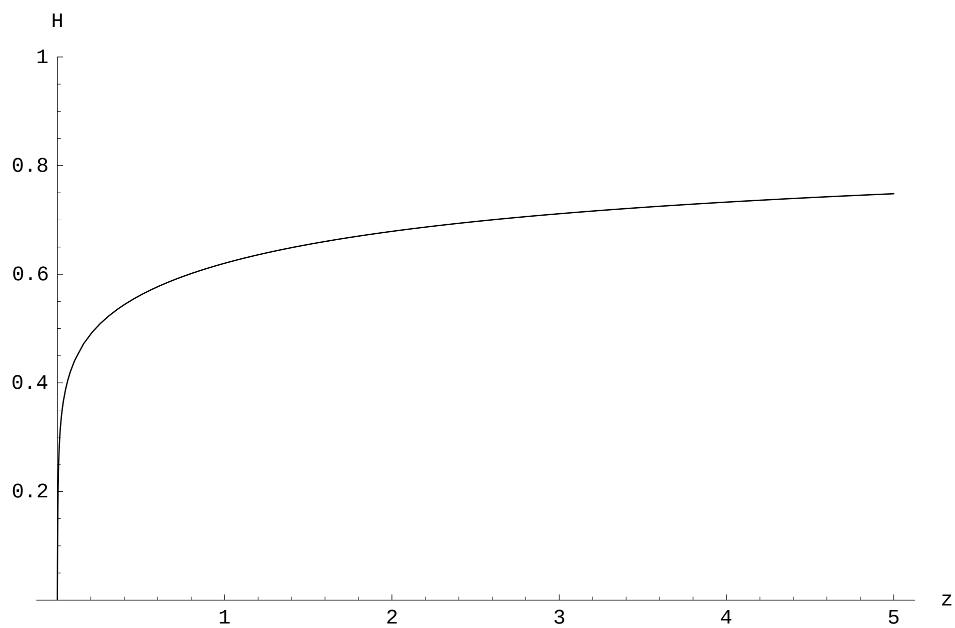

Figure 4: The "experimental" transformed weight H formed from the quark masses defined at the scale $\mu = M_W$.

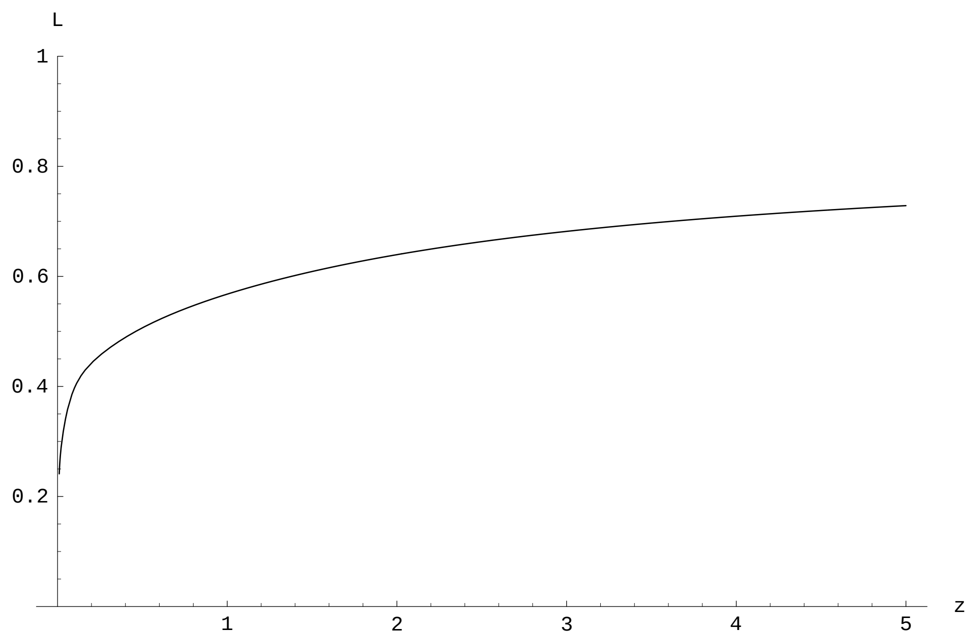

Figure 5: The “experimental” transformed weight L.

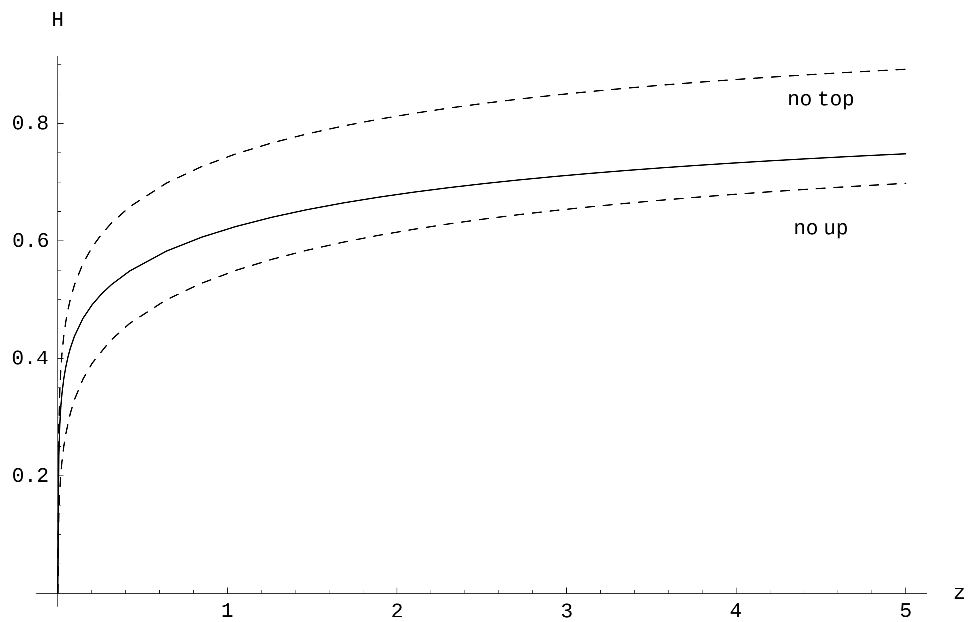

Figure 6: The transformed weights obtained by discarding either the up quark or the top quark, compared to the full result (solid curve).

The resultant transforms are displayed in Fig 4,5.

Here and in subsequent sections, I wish to provide some estimates of the uncertainty in the transformed weights. A minimal such estimate is obtained by considering what would occur if we had knowledge of only five quarks instead of six. Removing the information on the mass of a strange or charmed quark does relatively little to change the transformed weight. The extremes occur if we discard either the up or the top quark masses. These shifts in the form of H(z) are illustrated in Fig. 6. The effects of the error bars from the experimental determination of the masses is much smaller than the effect of removing one mass from the distribution.

6 The CKM weak mixing matrix

The known quark masses emerge from the diagonalization of the matrix of Yukawa couplings. This diagonalization also produces the elements of the weak mixing matrix. Therefore the CKM elements also contain information on the distribution of the Yukawa couplings. In this section, I consider this information, primarily as an indication of the uncertainty in the weight.

There is a serious loss of information in the diagonalization process which hinders our use of the CKM elements. The original mass matrices of the charge 2/3 and $-1/3$ quarks are diagonalized via

$$V_L^{(u)\dagger} M_0^{(u)} V_R^{(u)} = m^{(u)} \tag{38}$$
$$V_L^{(d)\dagger} M_0^{(d)} V_R^{(d)} = m^{(d)} \tag{39}$$

with the resultant CKM matrix formed from the product of the two left-handed rotations

$$V_{CKM} = V_L^{(u)\dagger} V_L^{(d)}. \tag{40}$$

We lose any information contained in the right-handed rotations and any common features of the up and down type left-handed rotations. Thus we cannot reconstruct the original Yukawa matrices except in an arbitrary choice of basis. (Indeed, the choice of basis can only be specified with reference to some physics beyond the Standard Model.)

In order to provide two simple estimates of the uncertainties in the distribution of Yukawa couplings, I will rotate the CKM elements either into the matrix of up-type quarks or into the matrix of down-type quarks. Specifically I consider the elements of

$$M_{u,test} = V_{CKM}^{\dagger} m^{(u)} = \begin{pmatrix} 0.002 & 0.18 & 1.19 \\ 0.0005 & 0.78 & 6.8 \\ 7 \cdot 10^{-6} & 0.032 & 170 \end{pmatrix} \tag{41}$$

or of

$$M_{d,test} = V_{CKM} m^{(d)} = \begin{pmatrix} 0.004 & 0.018 & 0.01 \\ 0.001 & 0.077 & 0.12 \\ 3 \cdot 10^{-5} & 0.003 & 3.1 \end{pmatrix} \tag{42}$$

In the numbers quoted, I do not include CP violation and use the PDG central values[8]. For the purpose of this exercise, I will consider all elements to be

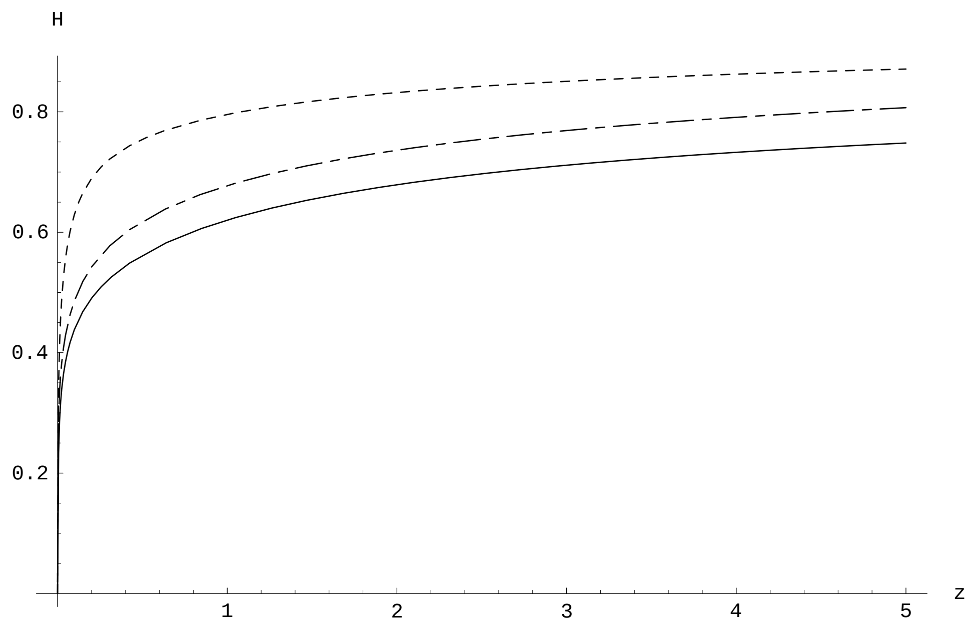

Figure 7: The transformed weight (solid curve) compared to the corresponding result formed by transferring the information in the CKM elements into the up-quark mass matrix(dot-dashed curve) and down-quark mass matrix(dashed curve).

independent. In each case, I also include the diagonal masses of the other type quark to be included in the weight, resulting in 12 input parameters per case. This leads to the transformed weights shown in Fig 7.

We see that including this increased amount of data does not significantly change the weight.

7 Leptons

It is tempting to also combine information of the lepton masses with that of quark masses. Certainly the overall impression is the same, with a preponderance of light fermions. However, we should be cautious in this procedure

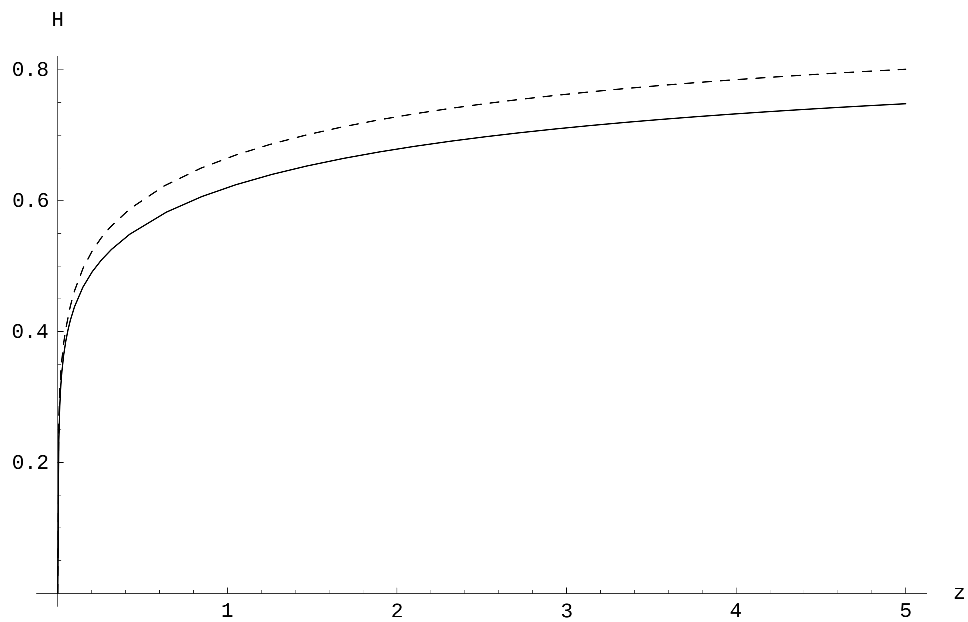

Figure 8: The transformed weight formed from the quark masses and the lepton masses (dashed curve), compared to that from quarks alone (solid curve).

as there are certainly some different considerations for leptons compared to quarks. In the spirit of exploring the uncertainties in the weight, in this section we combine quark and lepton information. We will see that within the significant uncertainties the quark and lepton distributions are consistent.

Initially let us blindly add the lepton to the quark information (i.e. yielding 9 elements of data) This is shown in Fig. 8. (I neglect the electroweak running of masses as this uncertainty is far below the real uncertainty in this whole procedure.) Leptons produce only a minor change in the transformed weight.

However, if the masses arise at some higher energy scale, this is unlikely to be the correct procedure. The quark and lepton masses should be compared at the high scale rather than the electroweak scale. This of course cannot be done without knowledge of the underlying theory. To estimate this effect,

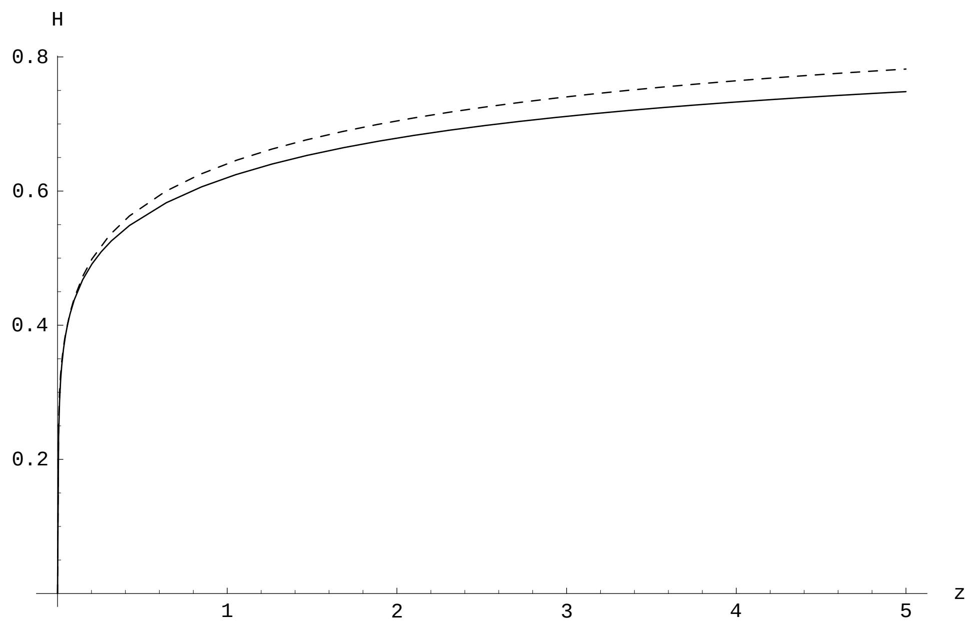

Figure 9: The transformed weight formed from quark masses and lepton masses which have been rescaled as described in the text (dashed curve) compared to that from quarks alone (solid curve).

I have done the following. First, imagine that all masses are scaled to a typical grand unification scale, $M \sim 10^{16}$GeV. Then all masses are scaled back to the W scale as if they were quarks. This gives a plausible common distribution, which can be compared to the result of the previous section. The net effect is to increase the lepton input values by a factor of two. This change does not drastically modify the resulting distribution. See Fig. 9. The lepton information is consistent with that of the quarks.

Neutrinos may or may not have a mass in the Standard Model. However, if they have a non-zero mass, present indications are that the values are so small that they are unlikely to be standard Dirac masses. The "see-saw" mechanism naturally explains such small masses and would be the favored explanation should present indications be confirmed. This mechanism would indicate that we should not combine neutrino mass information together with

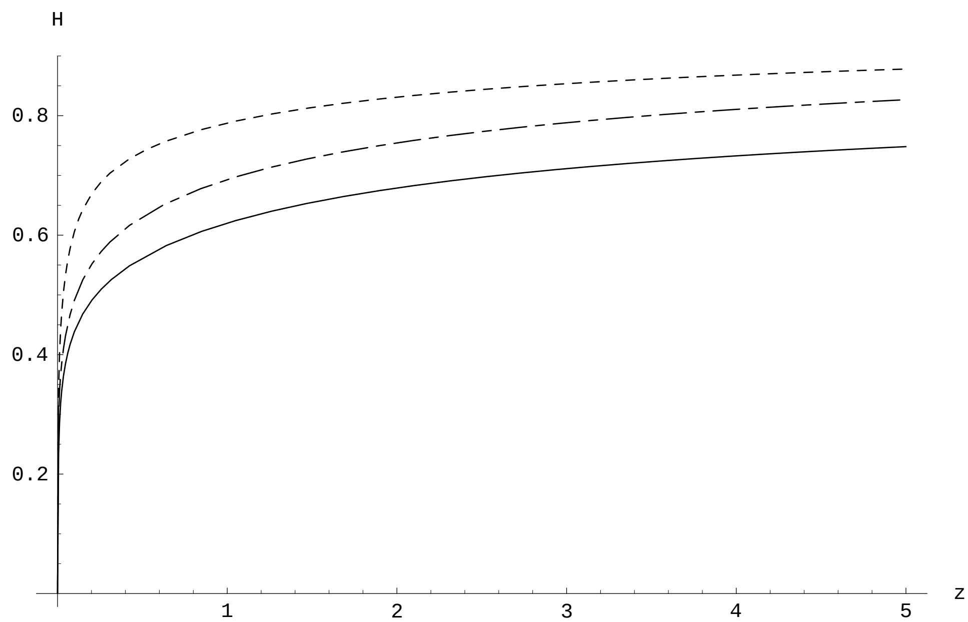

Figure 10: A summary of all inputs (quarks, leptons and CKM elements), with the CKM elements in the up-quark mass matrix (dot-dashed curve) and the down-quark mass matrix (dashed curve).

the other masses in discussing the weight.

Finally, we combine the CKM, (rescaled) lepton and quark information in order to obtain the estimate in Fig. 10.

8 Summary of estimated uncertainties

The considerations above have described various inputs into the "experimental" determination of the transformed weight. The number of inputs has ranged from 5 to 15. The greatest downward variation comes from the removal of the up quark from the distribution. The upper boundary comes from the rotation of the CKM elements into the down-quark mass matrix. All other inputs are consistent with the underlying distribution within this

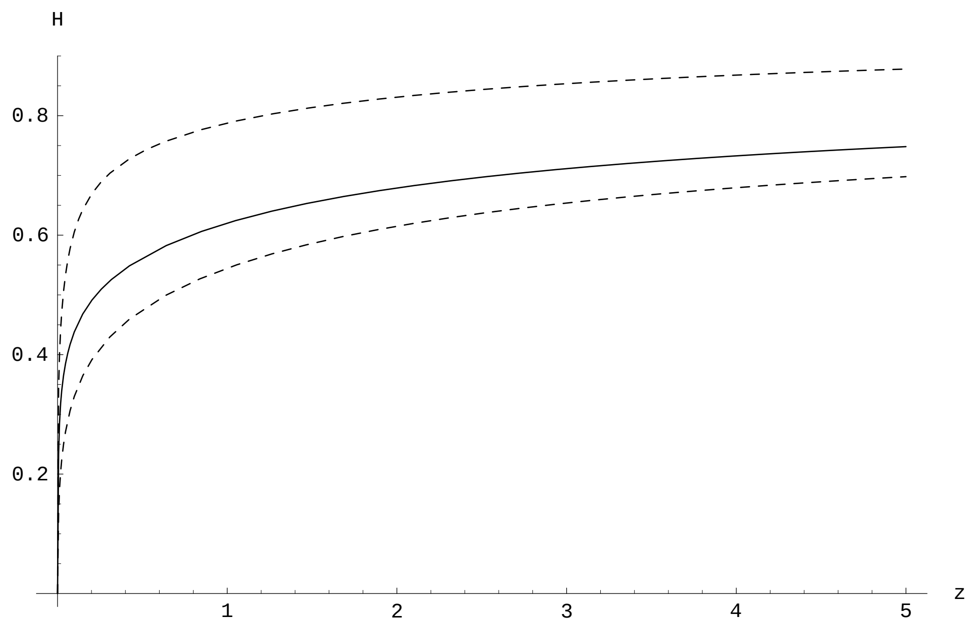

Figure 11: The transformed weight (solid curve) and limits of the range of uncertainty (dashed curves) as described in the text.

uncertainty.

The resulting uncertainty is shown in Fig. 11. Despite the generous nature of the uncertainty in the transformed weight, it will turn out that the weight itself is reasonably constrained.

9 Phenomenology of the weight

The function describing the weight needs to be peaked at low energy, yet extend out to high mass in order to accommodate the existence of the top quark. The latter requirement eliminates purely exponential forms which would have a scale of order 1 GeV, and hence no significant probability at multi-GeV masses. Power law forms will be seen to provide acceptable distributions, and we will explore some variants of these.

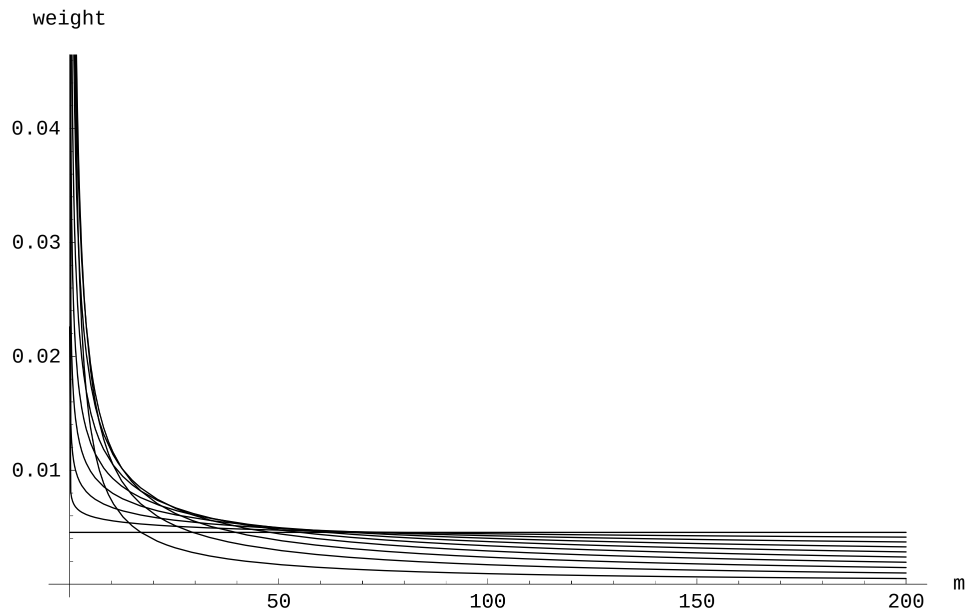

Figure 12: Sample power-law weights, with powers ranging from $\delta = 0$ to $\delta = 0.9$

Consider a trial weight of a pure power behavior combined with a cutoff at the quasi-fixed point

$$\rho_1(m) = \frac{N}{m^\delta}\Theta(m^* - m) \tag{43}$$

with a normalization constraint $N = (1-\delta)/m^{*(1-\delta)}$. I will use the cutoff at $m^* = 220$ GeV. Here we are constrained by $\delta < 1$ for the distribution to be integrable at low mass. This weight is shown in Fig. 12 for the values $\delta = 0, 0.1, 0.2, ..., 0.9$.

The simplest consideration is the use of the median value of the distribution, $\hat{m}$, defined by

$$\frac{1}{2} = \int_0^{\hat{m}} \rho(m)\ dm \tag{44}$$

On the average, half of the quark masses should appear below $\hat{m}$. For this simple example

$$\hat{m} = \frac{m^*}{2^{\frac{1}{(1-\delta)}}} \tag{45}$$

Even a crude estimate

$$0.05\ MeV \leq \hat{m} \leq 5\ GeV \tag{46}$$

leads to a rather stringent constraint on the power δ,

$$0.82 \leq \delta \leq 0.955 \tag{47}$$

with higher values of δ corresponding to lower values of $\hat{m}$.

Similar constraints follow from even a rough look at the transformed weight. In Fig 13 are shown the transformed weights for these distributions for the same range of δ. Again, only higher values of δ are allowed. The physics behind this is clear - if δ is small we would expect to populate larger values of the masses, and hence would see the variation in H(z) occur at larger values of z. A search for the optimal value of δ leads to the trial with $\delta = 0.91$ shown in Fig 14. A similar comparison of the same trial weight with the transform L(z) yields an equally reasonable form, shown in Fig. 15. Of course, given the uncertainties described above the specific value of this exponent should not be taken too seriously. Within the scope of this class of trial functions, I estimate that $\delta = 0.82$ to $\delta = 0.95$ spans the uncertainties described in the previous sections.

A second class of trial functions consists of the form given in Eq. 22 from the renormalization group scaling down from high energies. This differs primarily in the region of the high mass cut-off. Here a similar power provides a similarly good fit, as shown in Fig 16 for $\delta = 0.92$. This confirms the expectation that the high energy form of $\rho(m)$ is not significantly constrained by the data.

The closeness of these exponents to unity suggests that we try to accommodate a behavior $\rho(m) \sim 1/m$. Surely a weight function of this inverse power is a more pleasing result than one with a non-integral power. However, a pure power-law with $\delta = 1$ is not integrable at the low energy end. We can form a normalizable weight if we include a low-energy cutoff.

$$\rho_3(m) = \frac{N}{m}\theta(m - m_{MIN})\theta(m^* - m) \tag{48}$$

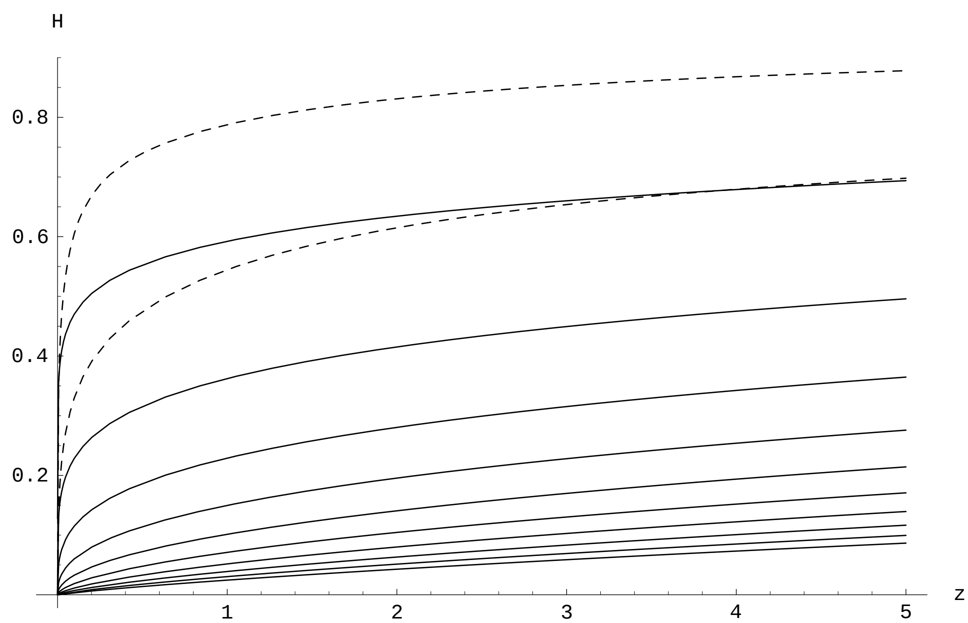

Figure 13: The transformed weights corresponding to power-law weights, with powers ranging from $\delta = 0$ (bottom) to $\delta = 0.9$ (top). Also shown is the range of uncertainty in the transformed weight from Fig. 11.

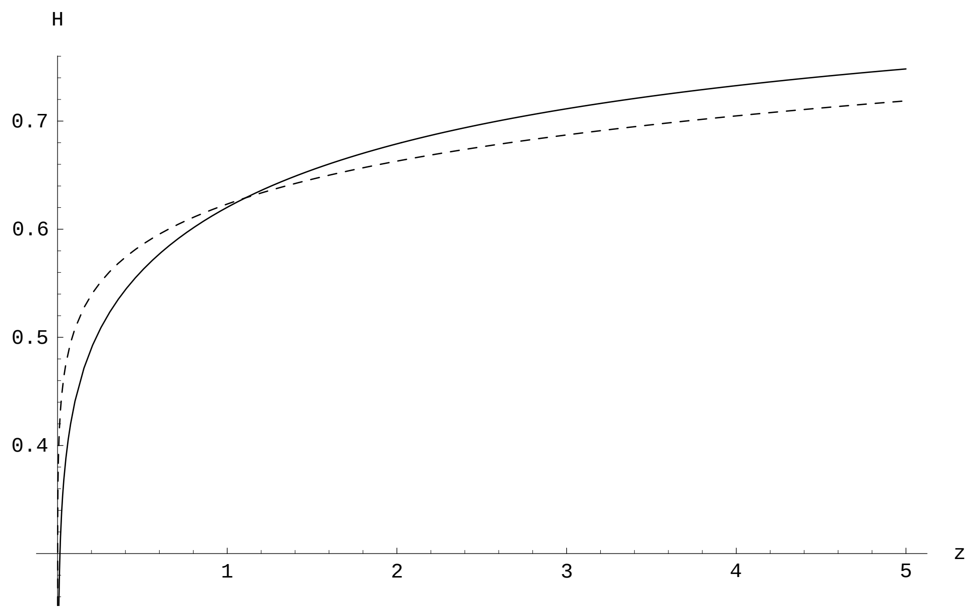

Figure 14: A comparison of the experimental transformed weight H with a power-law fit (dashed curve) with $\delta = 0.91$.

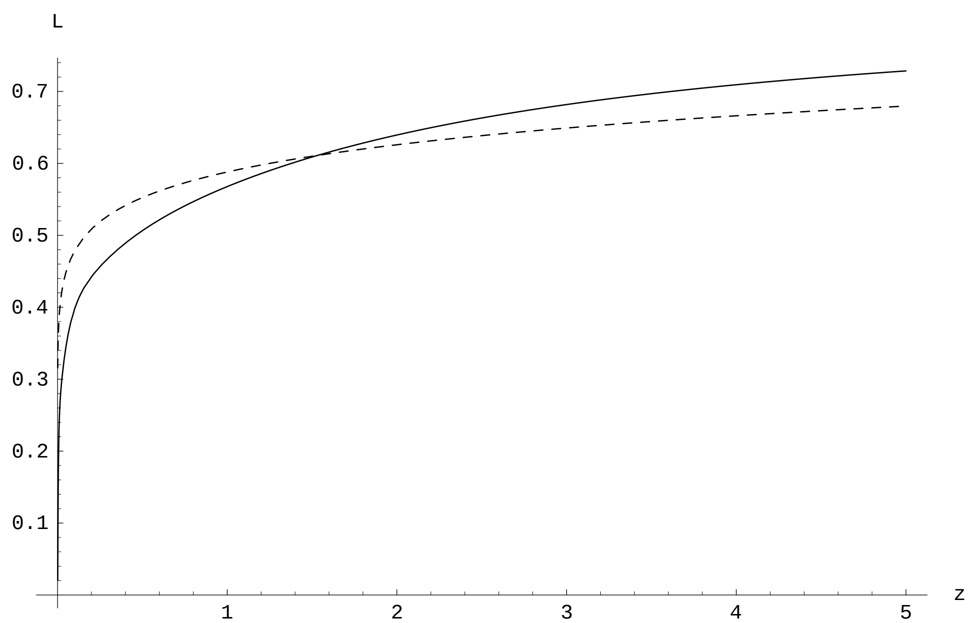

Figure 15: The same as Fig. 14 but for the transformed weight L.

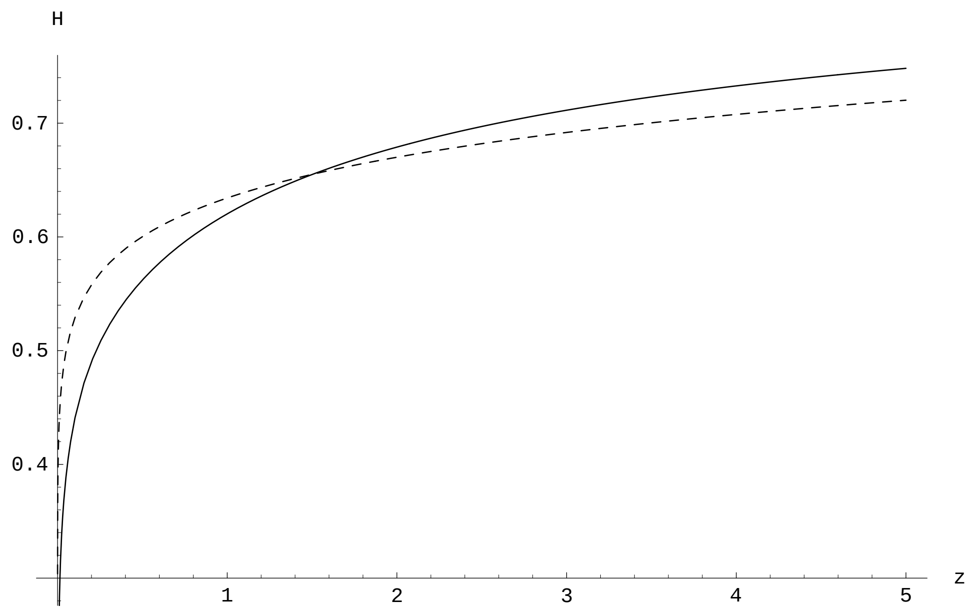

Figure 16: A comparison of the transformed weight with a form obtained by scaling from high energy (dashed curve), as described in the text. Here the power is $\delta = 0.92$.

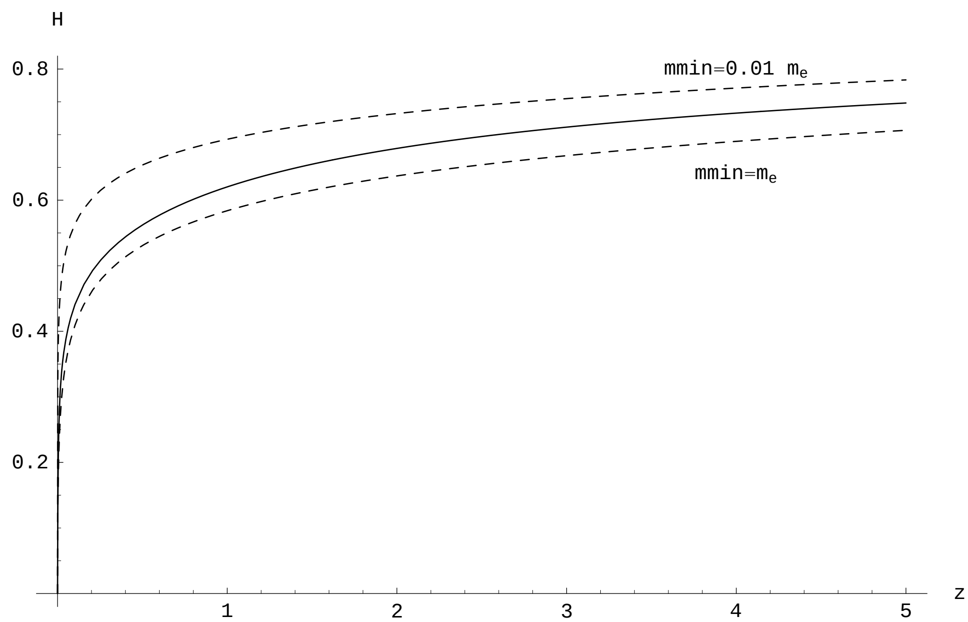

Figure 17: The transformed weight compared with scale-invariant forms with cut-offs m_e and $m_e/100$.

with $N = 1/ln(m^*/m_{MIN})$. At this stage the origin of the low-energy cutoff is unexplained; however, fortunately m_{MIN} enters only logarithmically. Fig 17 displays these forms for values $m_{MIN} = m_e$ and $m_e/100$. We see that these forms in fact do very well at describing the transformed weight within the intrinsic uncertainty.

For this form of the weight, the median value of the distribution is determined by the endpoints in the simple form

$$\hat{m} = \sqrt{m^* \; m_{MIN}} \tag{49}$$

For $m_{MIN} = m_e$, this equals $\hat{m} = 0.34$ GeV, a reasonable value.

The weights with $\rho(m) \sim 1/m$ can be described as “scale invariant” in the following senses. In the first place, there is no scale (other than the endpoints) in the shape of $\rho(m)$, and the normalization constant is dimensionless and independent of the overall scale. In addition, under any linear rescaling of the masses such as

$$m_2(\mu_2) = \left(\frac{\alpha_s(\mu_2)}{\alpha_s(mu_1)}\right)^{d_m} \quad m_1(\mu_1) \tag{50}$$

the transformation rule of Eq. 5 tells us that this weight will remain unchanged (again, aside from the endpoints), since

$$\begin{aligned} \rho_\mu(m) &= \rho_{\mu_1}\left(m\left(\frac{\alpha_s(\mu_)}{\alpha_s(mu_1)}\right)^{-d_m}\right)\left(\frac{\alpha_s(\mu_)}{\alpha_s(mu_1)}\right)^{-d_m} && (51)\\ &= \frac{1}{m\left(\frac{\alpha_s(\mu_)}{\alpha_s(mu_1)}\right)^{-d_m}}\left(\frac{\alpha_s(\mu_)}{\alpha_s(mu_1)}\right)^{-d_m} && (52)\\ &= \frac{1}{m}. && (53)\end{aligned}$$

It is tempting to speculate that this scale invariant form could approximate an IR fixed point for some class of weights initially defined at high energy.

The best fit functional form for the weight is shown in Fig 18 and 19. It is the scale-invariant form with $m_{MIN} = m_e/4$, and tracks the central "experimental" curve quite closely. However, the specifics of this form need not be taken too seriously given the uncertainties inherent in this problem. Many of the trial functions shown above are compatible within the uncertainty. However, all of these forms are strikingly close to $\delta = 1$, suggesting that this is the key to the structure of the mass spectrum.

10 Further comments and summary

The previous sections have contained a first consideration of the "experiment" and phenomenology of the weight function for quark masses. The hope is that this weight is the visible remnant of the fundamental theory in situations where the specific values of the masses are themselves not unique.

There are some considerations which could be important in trying to predict the weight function. In certain cases, it may prove that a slightly modified "experimental" description is most relevant for a given theory. The procedure used here is specific to the weight in a domain with 3 generations of

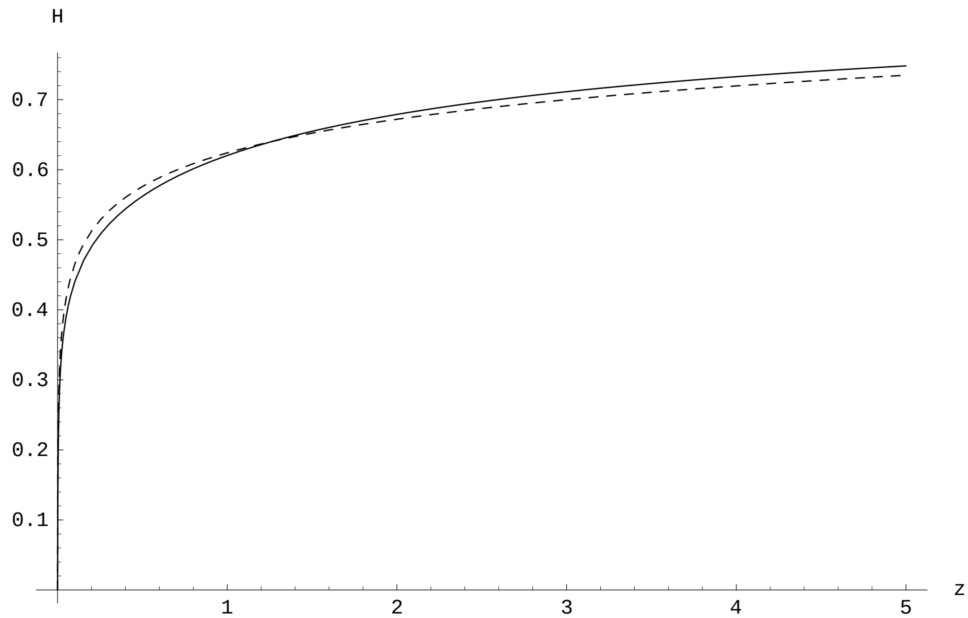

Figure 18: The best description of the transformed weight, obtained with the scale-invariant form using $m_{MIN} = 0.4 m_e$.

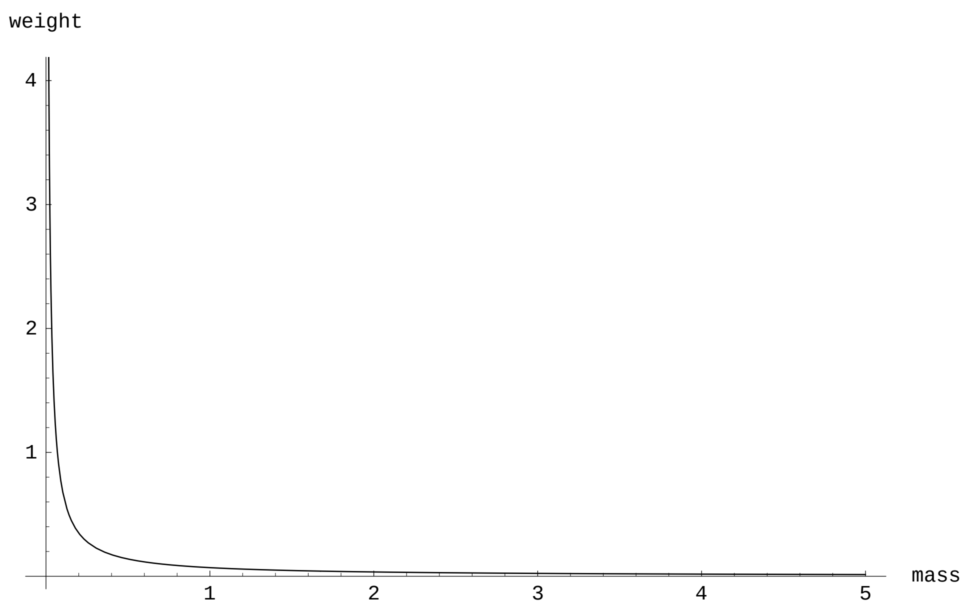

Figure 19: The scale invariant weight describing the best match to the quark mass spectrum, corresponding to Fig. 18.

fermions in an $SU(3) \times SU(2) \times U(1)$ theory with the observeredcalues of the gauge couplings. These other features may potentially also be variable, and the weight function could be different for other situations. In the case that the Higgs vacuum expectation value is also variable, it is simple to convert $\rho(m)$ into a distribution of the Yukawa couplings. While there are a few subtle features of transforming the Yukawa distribution to other scales, these are small in comparison with the intrinsic uncertainty in the distribution.

The procedure used above implicitly assumes that it is the quark masses themselves that are independently distributed with respect to some weight. In specific models, this may not be the case. For example in models with an intrinsic hierarchy between different Yukawa couplings generated through radiative corrections, it may turn out that the smallness of some masses is the result of high powers of a gauge coupling rather than a consequence of the weight itself. In this case, a different procedure to extract the weight for the appropriate random variables would need to be employed.

Of more serious concern could be a bias introduced by anthropic considerations. In multiple domain theories, it is a natural requirement that out of all the possible domains we could only find ourselves in a domain that has the ingredients relevant for life. Without too much anthro-centric reasoning, it seems plausible that this requirement implies the need for complex chemicals (i.e. more elements than simply hydrogen). In [4], this was argued to require that at least some of the quark masses must be small compared to the QCD scale. This forms a bias for low masses, and would shift the shape of the weight function. The "experimental" weight function must then be understood to be subject to this constraint.

The issues described above are best treated in the context of a theory in which we try to predict the weight. Depending on the dynamics of this hypothetical theory, we would best know which variables are quasi-random and over what range. Within the parameter space of the theory, we can impose constraints on other parameters and explore the distribution of masses subject to those constraint. It is even possible that we could estimate the effects of the anthropic bias, producing a weight function subject to the constraint that complex chemicals are able to be formed. It would be interesting to explore these issues even in the context of a toy model.

In theories where the parameters are variable in different domains of the universe, many of the standard questions that we address in particle physics appear in a different light. In some cases, such as the attempt to understand the scale of electroweak symmetry breaking, the form of analysis has a quite different character[4]. For quasi-random quark masses, we might fear that there is in this case no longer anything that we can do phenomenologically with the masses. This paper has been an attempt to extract the remnant of the underlying theory which survives even in the case where the specific values of the masses are not unique. The result is intriguingly close to a scale-invariant weight (see Figs. 18 and 19), and this encodes the observed bias for small quark masses. This form of theory is relatively new, and it is not clear how much it will be developed in the future. To the extent that these theories are studied further, the weight for fermion masses will be a fundamental input that has the potential to test the theories.

Acknowledgments

I would like to thank Gene Golowich, Dave Seckel and especially Steve Barr for conversations and communications which have helped shape my thoughts on these unusual topics.

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