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Infrared behavior of graviton-graviton scattering

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Abstract

The quantum effective theory of general relativity, independent of the eventual full theory at high energy, expresses graviton-graviton scattering at one loop order $\mathcal{O}(E^4)$ with only one parameter, Newton's constant. Dunbar and Norridge have calculated the one loop amplitude using string based techniques. We complete the calculation by showing that the $\frac{1}{d-4}$ divergence which remains in their result comes from the infrared sector and that the cross section is finite and model independent when the usual bremsstrahlung diagrams are included.

1 Introduction

The simplest low energy process in quantum gravity is graviton-graviton scattering. Although experimentally unobservable, this reaction forms an interesting theoretical laboratory that illustrates the workings of quantum gravity. If general relativity is the correct low energy classical theory of gravity, then its quantum theory forms an effective field theory capable of analyzing the low energy quantum effects. Graviton-graviton scattering is particularly useful in illustrating the logic of predictions in a quantum effective theory. Indeed, at one loop order this reaction provides a model-independent *quantum* prediction of general relativity.

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At tree level, the graviton-graviton scattering amplitude is simple in the helicity basis, although the calculation to obtain this result from the Einstein action is not so simple. With $+$ ($-$) representing helicity $+2$ (-2)¹, all tree amplitudes for $1+2 \rightarrow 3+4$ vanish except those related to $\mathcal{A}^{tree}(++; ++)$ by crossing and [1]

$$\mathcal{A}^{tree}(++; ++)=\frac{i}{4}\frac{\kappa^2 s^3}{tu} \quad (1)$$

Here $\kappa^2 = 32\pi G$ and $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ denote the usual Madelstam variables.

It is simple to show that graviton-graviton scattering should be finite and parameter independent at one-loop order [2]. In the effective low energy theory [3] gravitational effects are expanded in a derivative expansion with all terms satisfying general covariance

$$S_{grav} = \int d^4x \sqrt{g} \left[\frac{2}{\kappa^2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \dots + \mathcal{L}_{matter} \right] \quad (2)$$

Here $\kappa^2 = 32\pi G$ and G is Newton's constant, $c_{1,2}$ are unknown dimensionless parameters which contain information about the (presently unknown) ultimate high energy theory. A third covariant of order R^2 , $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ can be removed in four dimensional space-time through the use of the Bianchi identities. Since the curvature involves two derivatives of the metric, the Einstein action (the term with R) is seen to be of order E^2 , while R^2 and $R_{\mu\nu} R^{\mu\nu}$ are of order E^4 .

Loop diagrams obey a power-counting theorem [4, 5]. One loop diagrams formed from vertices given by the Einstein action yield effects at order E^4 – any process with more loops is higher order in the energy expansion. The ultraviolet divergences at one loop necessarily have the same structure as the local Lagrangian in Eqn. (2), which means that they must be proportional to R^2 or $R_{\mu\nu} R^{\mu\nu}$. Then, at this order, the ultraviolet divergences can be absorbed into renormalized values of the parameters $c_{1,2}$. These renormalized constants are unknown and will be different depending on the nature of the theory that forms the ultimate correct high energy theory which includes gravity. In this sense these parameters are model dependent. However, they do not contribute to the process of graviton-graviton scattering. At the order that we are working, the R^2 Lagrangians are applied to form vertices for on-shell amplitudes,

¹Note that in our notation crossing also requires one to flip the \pm sign for the affected gravitons. This implies in particular that $\mathcal{A}(-+; ++)$ must be a symmetric function of s, t and u .

which is to say that the equations of motion are satisfied for the external states. However, the equations of motion for the purely gravitational sector are $R_{\mu\nu} = 0$, and hence $R = 0$ also. Thus the effects of both of the R^2 terms in Eqn. (2) vanish in purely gravitational processes. It is this argument that tells us that graviton-graviton scattering is finite and independent of any unknown parameters at one loop order.

The power counting theorem is manifest in the one-loop results calculated by Dunbar and Norridge [7]. The one-loop amplitude is formed by using the lowest order tree amplitude twice in order to produce a loop diagram, and hence carries coupling constants $\kappa^4 \sim G_N^2$. Dimensionally this requires that the result carry four powers of the external energies. This is seen in the results:

$$\begin{aligned}
\mathcal{A}^{1-loop}(++; --) &= -i \frac{\kappa^4}{30720\pi^2} (s^2 + t^2 + u^2) \\
\mathcal{A}^{1-loop}(++; +-) &= -\frac{1}{3} \mathcal{A}^{1-loop}(++; --) \\
\mathcal{A}^{1-loop}(++; ++) &= \frac{\kappa^2}{4(4\pi)^{2-\epsilon}} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \mathcal{A}^{tree}(++; ++) \times (stu) \\
&\times \left[\frac{2}{\epsilon} \left(\frac{\ln(-u)}{st} + \frac{\ln(-t)}{su} + \frac{\ln(-s)}{tu} \right) + \frac{1}{s^2} f\left(\frac{-t}{s}, \frac{-u}{s}\right) \right. \\
&\quad \left. + 2 \left(\frac{\ln(-u)\ln(-s)}{su} + \frac{\ln(-t)\ln(-s)}{tu} + \frac{\ln(-t)\ln(-s)}{ts} \right) \right]
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
f\left(\frac{-t}{s}, \frac{-u}{s}\right) &= \frac{(t+2u)(2t+u)(2t^4 + 2t^3u - t^2u^2 + 2tu^3 + 2u^4)}{s^6} \left(\ln^2 \frac{t}{u} + \pi^2 \right) \\
&+ \frac{(t-u)(341t^4 + 1609t^3u + 2566t^2u^2 + 1609tu^3 + 341u^4)}{30s^5} \ln \frac{t}{u} \\
&+ \frac{1922t^4 + 9143t^3u + 14622t^2u^2 + 9143tu^3 + 1922u^4}{180s^4},
\end{aligned} \tag{4}$$

and all logarithms with negative arguments are understood to have a $-i\pi$ imaginary part. Note that this represents a real *tour-de-force*. Done in conventional field theory, the calculation is formidably difficult. It is a tribute to the string based techniques that the results are obtainable with less than Herculean effort. Indeed, after calculating the graviton loops, the authors write down the result for massless scalars, fermions and photons in the loops in just a few lines. However, the result is not tied

to the validity of string theory as a fundamental theory – the technique is simply an efficient way to calculate the results of usual (quantum) general relativity.

One notices that the one loop amplitude in Eqn. (3) contains a factor of $\frac{1}{\epsilon}$, i.e. it is *not* finite. At first, this seems to contradict the general reasoning given above. However, in the complete calculation of the physical process of graviton scattering, there will also be bremsstrahlung diagrams describing the radiation of soft gravitons off the external graviton lines. When calculated in a $d = 4 - 2\epsilon$ dimensional phase space these infrared effects also bring in a $\frac{1}{\epsilon}$ factor. If the divergence in Eqn. (3) is an infrared divergence, and if the effective field theory of gravity behaves as a proper effective field theory, then the infrared loop effects should be canceled against the soft radiation. While there are good reasons for believing that the gravitational effective field theory should be well behaved in the infrared, the long-standing doubts about quantum gravity make it worthwhile to check this property in the only complete calculation available. In Ref. [6] it was shown that the scattering of spin-0 fields is infrared finite even in the limit when their masses vanish. However, it was only conjectured there that the same is true for massless matter of higher spin (a situation similar to graviton-graviton scattering). One also notes that the scale of the logarithm is not defined. This is an indication that the calculation is incomplete. We will see that the scale in the logarithm comes from an infrared regulator for soft gravitons. Finally, part of our motivation comes from a minor quibble with the argument given above. In the effective Lagrangian we removed the $(R_{\mu\nu\alpha\beta})^2$ term by the use of an identity that is only valid in exactly four dimensions. Indeed, in any higher dimension the argument given would not apply, and the graviton scattering amplitude would contain a model dependent parameter. This means that in the quantum theory we can only be certain of the result if we use a regularization scheme that works in four dimensions. However, the only scheme that we know about that preserves the symmetries of general relativity is dimensional regularization, and it was that scheme used in Ref. [7]. While it is unlikely that the regularization scheme would lead to an extra divergence, we also want to confirm that the residual divergence is not an artifact of the ultraviolet regularization.

2 Soft gravitons in graviton-graviton scattering

We will explicitly calculate the divergences in the one-loop differential cross section for graviton-graviton scattering. We will find a complete cancellation of infrared divergences when we calculate the cross section up to $\mathcal{O}(\kappa^6)$, including the Bremsstrahlung graphs, as shown in Fig. 1.

$$\left| \kappa^2 \times \text{A} + \kappa^4 \times \text{A} + (\text{etc.}) \right|^2 +$$

$$+ \left| \kappa^3 \times \text{A} + (\text{etc.}) \right|^2$$

Figure 1: The expansion of the cross section in κ in graviton-graviton scattering. The quantity \mathcal{A}^{tree} represents the sum of all tree level diagrams. Solid lines represent hard gravitons, wavy lines are soft gravitons.

In this figure we explicitly show all factors of κ . The first term in the figure (and five additional graphs, not shown, with graviton exchanges between various pairs of external legs) is already included in the full one-loop scattering calculation and has an infrared divergence. This divergence is canceled by another divergence in the second term in the figure, a soft Bremsstrahlung process, which should be added as it is degenerate in energy with pure hard scattering. The second line in the figure shows that the actual cancellation occurs in the $\mathcal{O}(\kappa^6)$ terms because the leading $\mathcal{O}(\kappa^4)$ is tree level and infrared finite.

We will derive a general formula for the infrared divergences which uses the on-shell Born amplitude. The most convenient regularization procedure is dimensional regularization. We calculate the IR divergent part of the graviton radiation term in Fig. 1 and show [cf. Eqn. (26)] that to do so we only need to know the on-shell tree level amplitudes $\mathcal{A}^{tree}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. We always work in the helicity basis and $\lambda_i = \pm$

stands for the helicity of the i^{th} hard graviton. The only divergence occurs when the gravitons have helicity assignments $++$; $++$ (and in the cases related to this by crossing) and that is the only case when the tree level amplitude is nonvanishing. In the following we show [Eqn. (13)] that soft graviton radiation does not flip the hard graviton spins so that all IR divergences are proportional to the tree amplitude with the same helicity.

The amplitude with one soft graviton radiation is the sum of the four diagrams in Fig. 2. We first calculate the contribution from 2(a)

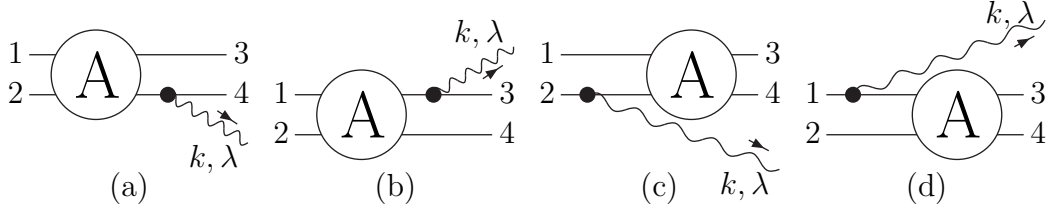


Figure 2: The four Feynman diagrams that contribute to soft graviton radiation at lowest order in hard graviton-graviton scattering.

$$\mathcal{A}_{(a)}^{rad} = \mathcal{A}_{\mu\nu}^{tree}(1, 2; 3, k_4 + k) \frac{i}{(k_4 + k)^2 + i\epsilon} P_{\mu\nu, \mu'\nu'} i \frac{\kappa}{2} \tau_{\alpha\beta, \lambda\rho}^{\mu'\nu'} \overline{\epsilon_4^{\alpha\beta}} \overline{\epsilon^{\lambda\rho}} \quad (5)$$

where writing a number in the argument of the tree amplitude means putting those lines on shell and multiplying by the appropriate $\epsilon^{\mu\nu}$ polarization tensor. The matrices I and P denote

$$I_{\alpha\beta, \gamma\delta} = \frac{1}{2} (\eta_{\alpha\gamma} \eta_{\beta\delta} + \eta_{\alpha\delta} \eta_{\beta\gamma}) \quad (6)$$

and

$$P_{\alpha\beta, \gamma\delta} = I_{\alpha\beta, \gamma\delta} - \frac{1}{2} \eta_{\alpha\beta} \eta_{\gamma\delta}. \quad (7)$$

The gauge invariance of the tree level amplitudes implies, for $k \rightarrow 0$,

$$k_4^\mu \mathcal{A}_{\mu\nu}^{tree}(1, 2; 3, k_4 + k) = \mathcal{O}(k) \quad (8)$$

and also

$$\eta^{\mu\nu} \mathcal{A}_{\mu\nu}(1, 2; 3, k_4 + k) = \mathcal{O}(k). \quad (9)$$

These restrictions can be derived as follows. Gauge invariance implies that the on-shell amplitude is unchanged under shifting the polarization tensor by

$$\epsilon^{\mu\nu} \longrightarrow \epsilon^{\mu\nu} + (k^\mu \xi^\nu + \xi^\mu k^\nu - k \cdot \xi \eta^{\mu\nu}), \quad (10)$$

with any four-vector ξ^μ , a transformation that keeps $k_\mu \epsilon^{\mu\nu}$ zero. In order for an amplitude $A_{\mu\nu} \epsilon^{\mu\nu}$ to be invariant under such replacement, we need for any on shell momentum k

$$2\xi_\mu (k_\nu A^{\mu\nu}) = (k \cdot \xi) A_\mu^\mu. \quad (11)$$

This must hold for any ξ , hence we have Eqns. (8,9).

Figure 3: The graviton propagator and the triple graviton vertices in harmonic gauge. For an expression of $\tau_{\alpha\beta,\gamma\delta}^{\mu\nu}$ see Eqn. (12).

The graviton propagator and the triple graviton couplings are shown in Fig. 3

$$\begin{aligned} \tau_{\alpha\beta,\gamma\delta}^{\mu\nu}(k_1, k_2, k_3) = & \quad (12) \\ & P_{\alpha\beta,\gamma\delta} \left[k_2^\mu k_2^\nu + (k_2 - k_1)^\mu (k_2 - k_1)^\nu + k_1^\mu k_1^\nu - \frac{3}{2} \eta^{\mu\nu} k_1^2 \right] \\ & + 2 (k_1)_\lambda (k_1)_\sigma \left[I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta}^{\mu\nu} + I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta}^{\mu\nu} - I_{\alpha\beta}^{\lambda\mu} I_{\gamma\delta}^{\sigma\nu} - I_{\alpha\beta}^{\sigma\nu} I_{\gamma\delta}^{\lambda\mu} \right] \\ & + (k_1)_\lambda k_1^\mu \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\lambda\nu} + \eta^{\gamma\delta} I_{\alpha\beta}^{\lambda\nu} \right) + (k_1)_\lambda k_1^\nu \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\lambda\mu} + \eta_{\gamma\delta} I_{\alpha\beta}^{\lambda\mu} \right) \\ & - k_1^2 \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} + \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} \right) - \eta^{\mu\nu} k_1^\lambda k_1^\sigma \left(\eta_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} \right) \\ & + 2 k_1^\lambda \left(I_{\alpha\beta}^{\sigma\nu} I_{\gamma\delta,\lambda\sigma} (k_2 - k_1)^\mu + I_{\alpha\beta}^{\sigma\mu} I_{\gamma\delta,\lambda\rho} (k_2 - k_1)^\nu \right. \\ & \quad \left. - I_{\gamma\delta}^{\sigma\nu} I_{\alpha\beta,\lambda\sigma} k_2^\mu - I_{\gamma\delta}^{\sigma\mu} I_{\alpha\beta,\lambda\sigma} k_2^\nu \right) \\ & + k_1^2 \left(I_{\alpha\beta}^{\sigma\mu} I_{\gamma\delta,\sigma}^\nu + I_{\alpha\beta,\sigma}^\nu I_{\alpha\delta}^{\sigma\mu} \right) + \eta^{\mu\nu} k_1^\lambda (k_1)_\sigma \left(I_{\alpha\beta,\lambda\rho} I_{\gamma\delta}^{\rho\sigma} + I_{\gamma\delta,\lambda\rho} I_{\alpha\beta}^{\rho\sigma} \right) \\ & + \left(k_2^2 + (k_2 - k_1)^2 \right) \left(I_{\alpha\beta}^{\sigma\mu} I_{\gamma\delta,\sigma}^\nu + I_{\alpha\beta}^{\sigma\nu} I_{\gamma\delta,\sigma}^\mu - \frac{1}{2} \eta^{\mu\nu} P_{\alpha\beta,\gamma\delta} \right) \\ & - \left(k_2^2 \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} + (k_2 - k_1)^2 \eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} \right). \end{aligned}$$

Putting together Eqns. (5 – 12), we arrive at a simplified expression

$$\mathcal{A}_{(a)\text{IR}}^{\text{rad}} = -\kappa \frac{k_4^\mu \overline{\epsilon_{\mu\nu}(k, \lambda)} k_4^\nu}{(k_4 + k)^2 + i\epsilon} \mathcal{A}^{\text{tree}}(1, 2, 3, 4) + \frac{\kappa \mathcal{O}(k)}{k \cdot k_4 + i\epsilon} \quad (13)$$

where the “IR” index emphasizes that we keep only the leading term when $k \rightarrow 0$. We observe indeed that the $\mathcal{O}\left(\frac{1}{k}\right)$ term is proportional to the Born amplitude without flipping any of the hard particles’ spins.

Now we add on the contribution from Fig. 2b,c,d. The result is

$$\mathcal{A}_{\text{IR}}^{\text{rad}} = -\kappa \mathcal{A}^{\text{tree}}(1, 2, 3, 4) \sum_{n=1}^4 \frac{k_n^\mu \overline{\epsilon_{\mu\nu}(k, \lambda)} k_n^\nu}{(k_n + \eta_n k)^2 + i\epsilon} + \frac{\kappa \mathcal{O}(k)}{k \cdot k_n}. \quad (14)$$

Next we need to square this amplitude and sum over the soft graviton spin:

$$\sum_{\lambda} |\mathcal{A}_{\text{IR}}^{\text{rad}}|^2 = \kappa^2 |\mathcal{A}^{\text{tree}}|^2 \sum_{i,j=1}^4 \frac{k_i^\mu k_i^\nu \Pi_{\mu\nu,\alpha\beta}(k) k_j^\alpha k_j^\beta}{(k_i + \eta_i k)^2 (k_j + \eta_j k)^2}, \quad (15)$$

where the sum over graviton polarization tensors is

$$\Pi^{\mu\nu,\alpha\beta}(k) \equiv \sum_{\pm} \epsilon_{\pm}^{\mu}(k) \epsilon_{\pm}^{\nu}(k) \overline{\epsilon_{\pm}^{\alpha}(k)} \overline{\epsilon_{\pm}^{\beta}(k)} = \frac{1}{2} \left(\Pi^{\mu\alpha} \Pi^{\nu\beta} + \Pi^{\mu\beta} \Pi^{\nu\alpha} - \Pi^{\mu\nu} \Pi^{\alpha\beta} \right) \quad (16)$$

and

$$\Pi^{\mu\nu}(k) = k^{\mu} \lambda^{\nu} + k^{\nu} \lambda^{\mu} - (k \cdot \lambda) \eta^{\mu\nu} \quad (17)$$

with an arbitrary vector λ , *same* for all terms in the sum, chosen as $\lambda^{\mu} = (1, \mathbf{0})$.

Doing the algebra in the above formula gives us

$$\sum_{\lambda} |\mathcal{A}_{\text{IR}}^{\text{rad}}|^2 = \frac{\kappa^2 |\mathcal{A}^{\text{tree}}|^2}{4k^2} \sum_{ij} \eta_i \eta_j E_i E_j \frac{(\cos \gamma_{ij} - \cos \alpha_i \cos \alpha_j)^2 - \frac{1}{2} \sin^2 \alpha_i \sin^2 \alpha_j}{(1 - \cos \alpha_i)(1 - \cos \alpha_j)}, \quad (18)$$

where k now stands for the energy of the soft gluon (not the four-momentum), and γ_{ij} is the angle between the $(d-1)$ -dimensional momenta of the hard gravitons, α_i is the angle between the i^{th} hard and the soft gravitons; E_i is the CM energy of the i^{th} graviton and $\eta_i = +1 (-1)$ for incoming (outgoing) hard gravitons.

At this point we make a comment on how dimensional regularization works. In the one-loop amplitude we find a $\frac{1}{\epsilon}$ divergence in dimensional regularization. As pure gravity is one-loop finite and all the divergences in one-loop graviton-graviton scattering come from the pure gravity part only, all of this $\frac{1}{\epsilon}$ should be of infrared origin and consequently be canceled by the square of the amplitude \mathcal{A}^{rad} . However,

\mathcal{A}^{rad} itself is a tree level amplitude which does *not* diverge; the canceling $\frac{1}{\epsilon}$ factor comes from the phase space integral. One might wonder then why we are not getting too much divergence: the leading term is $\frac{1}{k^2}$, so the phase space integral introduces

$$\int \frac{d^{d-1}k}{k k^2} \quad (19)$$

which is logarithmically divergent. In the same time the angular integration is also divergent and we find that dimensional regularization *does not handle correctly* an integral of the type

$$\oint \frac{d\Omega_{d-2}(\mathbf{n})}{(1 - \cos \alpha)^2} \sim B(1 - \epsilon, -1 - \epsilon) \quad (20)$$

(here α is the angle between the direction of \mathbf{n} and a fixed direction.) The above Euler function is

$$B(1 - \epsilon, -1 - \epsilon) = \frac{\Gamma(1 - \epsilon)\Gamma(-1 - \epsilon)}{\Gamma(-2\epsilon)} \rightarrow -2 + \mathcal{O}(\epsilon) \quad (21)$$

finite, although the integral includes a severe collinear singularity. Fortunately, in our case, we will not encounter this problem: the spins “conspire” so that there is an additional angular factor which takes away all collinear singularities in this integral. In other models, however, like one with elementary massless scalars, this might be a problem which requires further treatment.

Now we calculate the differential cross section in $d = 4 - 2\epsilon$ dimensions. We focus on the infrared region only, integrating up to a cutoff $\Lambda \ll \sqrt{s}$ and neglecting momenta of order Λ and above. Such soft graviton radiation should (and will) be sufficient to cancel the IR divergences due to one-loop integrals. In particular, we do not consider hard collinear gravitons. The divergences due to hard collinear graviton radiation (i.e. when one of the α_i 's is small) are not canceled by loops. However, these divergences are all proportional to Λ so can be unambiguously separated from the soft divergences. Some rather tedious algebra leads to an integral over the direction \mathbf{m} of the soft Bremsstrahlung graviton

$$\begin{aligned} \frac{d\sigma_{\text{IR}}^{rad}}{d\Omega_{d-1}(\mathbf{n})} &= \frac{\kappa^2 |\mathcal{A}^{tree}|^2}{(2\pi)^{3d-7} 2^{2d+2}} \sum_{ij} \eta_i \eta_j \int_0^\Lambda \frac{dk}{k} k^{d-4} \\ &\times \oint d\Omega_{d-1}(\mathbf{m}) \frac{(\cos \gamma_{ij} - \cos \alpha_i \cos \alpha_j)^2 - \frac{1}{2} \sin^2 \alpha_i \sin^2 \alpha_j}{(1 - \cos \alpha_i)(1 - \cos \alpha_j)}. \end{aligned} \quad (22)$$

The k integral has a $\frac{1}{\epsilon}$ infrared divergence. All divergences that are collinear and infrared simultaneously should come from the second integral. However, we observe that the numerator in the angular integral vanishes when the denominator does, actually canceling out the singularity. This fact is necessary to allow us to consistently separate collinear divergences from soft ones. In order to find the divergent part, we need to calculate only the leading term in

$$\mathcal{F}^{(0)}(\gamma) + \epsilon \mathcal{F}^{(1)}(\gamma) + \dots = \oint d\Omega_{d-1}(\mathbf{m}) \frac{(\cos \gamma_{ij} - \cos \alpha_i \cos \alpha_j)^2 - \frac{1}{2} \sin^2 \alpha_i \sin^2 \alpha_j}{(1 - \cos \alpha_i)(1 - \cos \alpha_j)}. \quad (23)$$

Substituting this into Eqn. (22) we find

$$\begin{aligned} \frac{d\sigma_{\text{IR}}^{\text{rad}}}{d\Omega_{d-1}(\mathbf{n})} &= -\frac{\kappa^2 |\mathcal{A}^{\text{tree}}|^2}{(2\pi)^{5-6\epsilon} 2^{11-4\epsilon}} \sum_{ij} \mathcal{F}^{(0)}(\gamma_{ij}) \eta_i \eta_j \\ &\times \left[\frac{1}{\epsilon} - 2 \ln \Lambda + 6 \ln(2\pi) + 4 \ln 2 + \frac{\mathcal{F}^{(1)}(\gamma_{ij})}{\mathcal{F}^{(0)}(\gamma_{ij})} \right]. \end{aligned} \quad (24)$$

The result in four dimensions is

$$\mathcal{F}^{(0)}(\gamma) = 4\pi \left[\frac{3 + \cos \gamma}{6} - (1 - \cos \gamma) \ln \frac{2}{1 - \cos \gamma} \right]. \quad (25)$$

With this, we finally find for the cross section

$$\begin{aligned} \frac{d\sigma_{\text{IR}}^{\text{rad}}}{d\Omega} &= -\frac{\kappa^2 |\mathcal{A}^{\text{tree}}|^2}{2^7 (2\pi)^4} \left(\frac{t}{s} \ln \frac{-t}{s} + \frac{u}{s} \ln \frac{-u}{s} \right) \\ &\times \left[\frac{1}{\epsilon} - 2 \ln \Lambda + 6 \ln(2\pi) + 4 \ln 2 + \frac{\sum_{ij} \eta_i \eta_j \mathcal{F}^{(1)}(\gamma_{ij})}{\sum_{ij} \eta_i \eta_j \mathcal{F}^{(0)}(\gamma_{ij})} \right] + \mathcal{O}\left(\frac{\Lambda}{\sqrt{s}}\right). \end{aligned} \quad (26)$$

We have found that the infrared divergent part is indeed proportional to the square of the Born amplitude. Because in the $(++; --)$ and $(++; +-)$ helicity cases there is no IR divergence to cancel, the vanishing of the Born terms makes sure none emerges in the radiative process.

In the $(++; ++)$ helicity case we need to use the cross section formula

$$\left(\frac{d\sigma[gg \rightarrow gg]}{d\Omega} \right)_{\text{nonrad}} = \frac{2\text{Re}(\overline{\mathcal{A}^{\text{tree}}} \mathcal{A}^{1\text{-loop}})}{(2\pi)^2 2^5 s} \quad (27)$$

in order to calculate the $\mathcal{O}(\kappa^6)$ contribution to the cross section (see Fig. 1.) Using the Dunbar-Norridge [7] 1-loop amplitude, Eqn. (3), we find the $\mathcal{O}(\kappa^6)$

contribution to the cross section for the $2 \rightarrow 2$ process:

$$\begin{aligned} \left(\frac{d\sigma(++; ++)}{d\Omega} \right)_{nonrad} &= \frac{\kappa^2 |\mathcal{A}^{tree}|^2}{2^7 (2\pi)^4} \times \\ &\times \left\{ \left(\frac{t}{s} \ln \frac{-t}{s} + \frac{u}{s} \ln \frac{-u}{s} \right) \left(\frac{1}{\epsilon} + \ln 4\pi - \ln s - \gamma \right) \right. \\ &\quad \left. + \left[\ln \frac{-t}{s} \ln \frac{-u}{s} + \frac{tu}{2s^2} f \left(\frac{-t}{s}, \frac{-u}{s} \right) \right] \right\}. \end{aligned} \quad (28)$$

We observe that the $\frac{1}{\epsilon}$ divergence cancels when we add together Eqns. (26) and (28). The finite term in Eqn. (28) contains an undetermined scale due to the logarithm of s . The occurrence of such a scale is a common feature of dimensional regularization in the presence of infinities. This scale is provided by the ‘‘ultraviolet’’ cutoff in the radiative cross section. Our final result for the sum of the cross sections is

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega} \right)_{tree} + \left(\frac{d\sigma}{d\Omega} \right)_{rad.} + \left(\frac{d\sigma}{d\Omega} \right)_{nonrad.} &= \\ &= \frac{\kappa^4 s^5}{2048\pi^2 t^2 u^2} \left\{ 1 + \frac{\kappa^2 s}{16\pi^2} \left[\ln \frac{-t}{s} \ln \frac{-u}{s} + \frac{tu}{2s^2} f \left(\frac{-t}{s}, \frac{-u}{s} \right) \right. \right. \\ &\quad \left. \left. - \left(\frac{t}{s} \ln \frac{-t}{s} + \frac{u}{s} \ln \frac{-u}{s} \right) \left(3 \ln(2\pi^2) + \gamma + \ln \frac{s}{\Lambda^2} + \frac{\sum_{ij} \eta_i \eta_j \mathcal{F}^{(1)}(\gamma_{ij})}{\sum_{ij} \eta_i \eta_j \mathcal{F}^{(0)}(\gamma_{ij})} \right) \right] \right\}. \end{aligned} \quad (29)$$

In this form all divergences are canceled and all logarithms are dimensionless. There is a logarithmic dependence on the scale where we cut off the non-infrared radiative gravitons. We remind the reader that the finite functions $f \left(\frac{-t}{s}, \frac{-u}{s} \right)$ and $\frac{\sum_{ij} \eta_i \eta_j \mathcal{F}^{(1)}(\gamma_{ij})}{\sum_{ij} \eta_i \eta_j \mathcal{F}^{(0)}(\gamma_{ij})}$ are respectively given in Eqn. (4) and can be extracted from Eqn. (23).

3 Conclusions

This has been the first explicit investigation of the infrared properties of a one loop amplitude in quantum gravity. We have achieved our goal in demonstrating that the effective theory of gravitation is not plagued by infrared divergences, its soft divergences even cancel in the case of one-loop graviton-graviton scattering, and also

demonstrated that, similarly to the case of QED, summation over degenerate states in the final state suffices to get a final and sensible cross section.

The result for graviton-graviton scattering to one-loop order is beautiful and significant because it forms a low energy theorem for quantum gravity. No matter what the high energy theory of gravity may turn out to be, and independent of the massive particles in the theory, as long as the low energy limit leads to general relativity the scattering rate must have the model independent form shown in Eqn (29). As expected, the quantum effective field theory of general relativity is well-behaved in the infrared.

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