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Abstract

We consider theoretically the possibility of observing unusual quantum fluid behavior in liquid ^{3}He and solutions of ^{3}He in ^{4}He systems confined to nano-channels. In the case of pure ballistic flow at very low temperature conductance will be quantized in units of $2m^2/h$. We show that these steps should be sensitive to increases in temperature. We also use a random scattering matrix simulation to study flow with diffusive wall scattering. Universal conductance fluctuations analogous to those seen in electron systems should then be observable. Finally, we consider the possibility of the cross-over to a one-dimensional system at sufficiently low temperature where the system could form a Luttinger liquid.

Keywords: liquid ^{3}He, solutions of ^{3}He in liquid^{4}He, nanotubes, quantized conductance, universal conductance fluctuations, Luttinger liquid

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INTRODUCTION

At very low temperatures, the thermal motion of the ^{3}He atoms in the liquid becomes very small as the de Broglie wavelength becomes comparable with the distance between the atoms and so quantum effects become significant enough to dictate the macroscopic properties of the liquid. In a recent paper[1], one of us noted that degenerate dilute solutions of ^{3}He in liquid ^{4}He might be cooled by flow through an array of nano-channels whose diameters are comparable to the de Broglie wave length of the channel. Aligning the Fermi energy of a container of fermions with the lowest allowed band in the channel would allow only the hot gases through the channel, thereby cooling the remaining gas in the container. The presence of such separated bands in a channel immediately suggests further possible unusual behavior. For fluids confined over such small scales, one may legitimately ask whether or not the physics of the system, for example, the flow properties, are still the same as at larger scale, or if perhaps a breakdown of the fluid mechanics might occur. Indeed recently Sato et al[2] considered the feasibility of seeing quantized conductance in dilute solutions when there was pure ballistic flow. Recent advances in materials research have opened up the possibilities to design and engineer single, or arrays of pores with diameters of tens of nanometer in ceramic membranes[3], and down to only $\sim$1 nm in tailor-made carbon nanotubes membranes[4]. These materials provide a radial length scale small enough to quantize the transverse motion of the helium atoms so that a realization of quantum-limited ballistic flow of helium, similar to that observed in the electronic transport of quantum wires[5], truly seems within experimental reach.

We consider here possible experiments on pure ^{3}He as well as dilute solutions. We first examine the temperature dependence of the conductance quantization for pure ballistic flow. We then investigate the effect of disorder and how wall scattering in diffusive flow causes the onset of "universal" conductance fluctuations in the mass flow. In the ultra-low temperature limit $T \rightarrow 0$, we also argue that one-dimensional quantum fluids should crossover to an entirely new exotic type of quantum matter known theoretically as a Luttinger liquid[6]. Such liquid should possess entirely different flow properties and excitation spectra owing to its 'spin-mass' separated ground state[7].

This study presented here constitutes only a preliminary analysis to illustrate the richness of flow properties that one might expect in experiments on helium in nanoscaled cylindrical

pores. More thorough investigations at the theoretical and experimental level are in progress.

QUANTIZED CONDUCTANCE

The de Broglie wave length λ_{dB} gives the scale of the diameters of the nanochannels necessary for seeing unusual quantum effects; the Fermi temperature T_F tells us what temperature we need for degeneracy. For pure ^{3}He $\lambda_{dB} \approx 0.7$ nm, $T_F \approx 2$ K while for a 1% solution of ^{3}He in liquid ^{4}He, $\lambda_{dB} \approx 3$ nm and $T_F \approx 120$ mK. Moreover, the interparticle mean free path is in the micron range because it contains the factor $(T_F/T)^2$. In each case a dilution refrigerator is able to produce adequate degeneracy. Ref. [2] considered the possibility of using such arrays of channels to see the quantization of the conductance. Unfortunately dispersion in the channel size can smooth out the quantization steps. In the present study we will ignore this problem and assume that technology will soon provide the feasibility of detecting flow from a single channel or a very small number of channels. Initially we will neglect any interparticle scattering and many-body interaction; the only scattering we will consider is that from the walls.

A natural definition of the mass conductance, analogous to that used in the case of an electric current, is given by[2]

$$G = \frac{I}{\Delta\mu/m} \tag{1}$$

where I is the mass current, m the particle mass, and $\Delta\mu$ the difference in chemical potential between the ends of the channel. The mass current is taken to be

$$I = \frac{2m}{h} \sum_n \int_{-\infty}^{\infty} dp_z \frac{p_z}{m} \left[n(\epsilon_z + \epsilon_n - \mu - \Delta\mu) - n(\epsilon_z + \epsilon_n - \mu) \right] \mathcal{T}(\epsilon_z) \tag{2}$$

where p_z is the longitudinal momentum, and we have written the energy in the channel as a longitudinal part $\epsilon_z = p_z^2/2m$, plus ϵ_n, the discrete transverse energies in the channel. The Fermi distribution is

$$n(\epsilon - \mu) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}. \tag{3}$$

The quantity $\mathcal{T}(\epsilon_z)$ is the transition probability for the scatterers in the channel, and $\beta = 1/k_b T$. In Eq. (2) we have assumed the temperature in the channel to be uniform; any gradient can easily be incorporated into the formula. Eq. (2) is just the Landauer formula[8] for the conductance. We have assumed $\mathcal{T}$ for each band is just a function of ϵ_z. Expanding

the Fermi distributions in small $\Delta\mu$ we find

$$G = \frac{2m^2}{h} \sum_n \int_0^\infty dz \frac{e^{z-\alpha_n}}{(e^{z-\alpha_n}+1)^2} \mathcal{T}(k_b T z) \tag{4}$$

where $z = \beta\epsilon_z$ and $\alpha_n = \beta(\mu - \epsilon_n)$. Note that if $z < 0$, the transmission probability vanishes.

For extremely low temperature the derivative of the Fermi distribution is sharply peaked around $z = \alpha_n$ so we can write

$$G = \frac{2m^2}{h} \sum_n \mathcal{T}(\epsilon_F - \epsilon_n) \tag{5}$$

where ϵ_F is the Fermi energy. The states in a cylindrical nano-channel are denoted by (n, m, k_z) where n and m are the radial and azimuthal quantum numbers and k_z is the longitudinal wave number. We assume that the states of different m are all degenerate and that a band is characterized by $(n, k_z^{(n)})$ and a degeneracy factor. If an atom of energy E enters a channel from a reservoir whose Fermi energy matches a state in the Nth band, then it can be in a superposition of bands $(1, k_z^{(1)})$ to $(N, k_z^{(N)})$ such that $E = \epsilon_n + \hbar^2 k_z^{(n)2}/2m$. If there is no scattering (that is, there is only specular scattering at the walls and the latter simply provide boundary conditions for the wave functions), then particles in each k_z state stay there and do not exchange from one band to another. The transmission probability for any band is just a step function

$$\mathcal{T}(\epsilon_z) = \Theta(\epsilon_z). \tag{6}$$

The conductance is then simply

$$G = \frac{2m^2}{h} N. \tag{7}$$

This function is plotted in Fig. 1. The conductance is clearly quantized in units of $2m^2/h$. Double steps occur because of a two-fold degeneracy of some states.

Suppose now we turn up the temperature while still assuming that particles in the channel maintain their k_z values and do not elastically scatter to other bands. Eq. (2) can be written as

$$\begin{aligned} G &= \frac{2m^2}{h} \sum_n \int_0^\infty dz \frac{e^{z-\alpha_n}}{(e^{z-\alpha_n}+1)^2} \Theta(z) \\ &= \frac{2m^2}{h} \sum_n \frac{1}{(e^{-\alpha_n}+1)}. \end{aligned} \tag{8}$$

The states at the Fermi energy feeding into the channel are not sharp and the steps are rounded as shown in Figs. 2 and 3. The steps are already reduced substantially at $T/T_F = 0.1$. The steps are completely gone by $T/T_F = 0.5$.

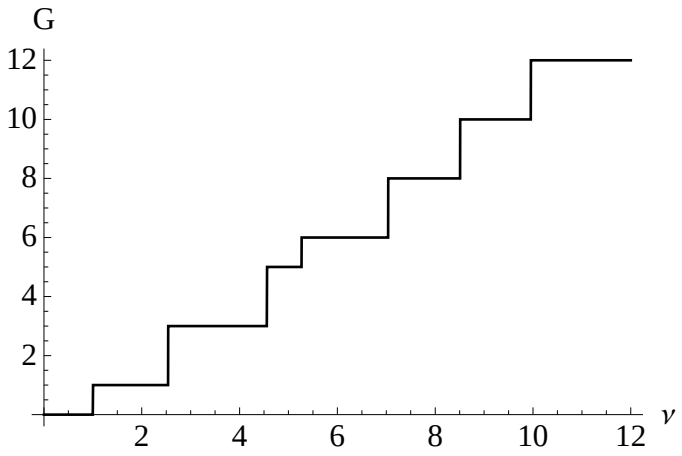

FIG. 1: Conductance versus $\nu = \epsilon_F/\epsilon_0$ (where ϵ_0 is the lowest band edge) for pure ballistic flow at $T = 0$.

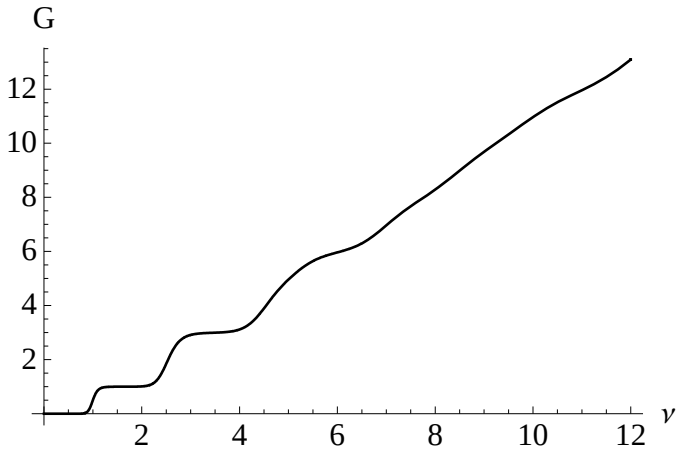

FIG. 2: Conductance versus ν at $T/T_F = 0.05$ for ballistic flow.

CONDUCTANCE FLUCTUATIONS

The above analysis considered only ballistic flow. However wall scattering can cause diffusive flow. We return here to the $T = 0$ situation. An atom entering the channel is in a superposition of bands $(1, k_z^{(1)})$ to $(N, k_z^{(N)})$ as above. Backscattering is possible so that the state at the entrance (left end) to the channel is

$$\psi_{left} = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}, \tag{9}$$

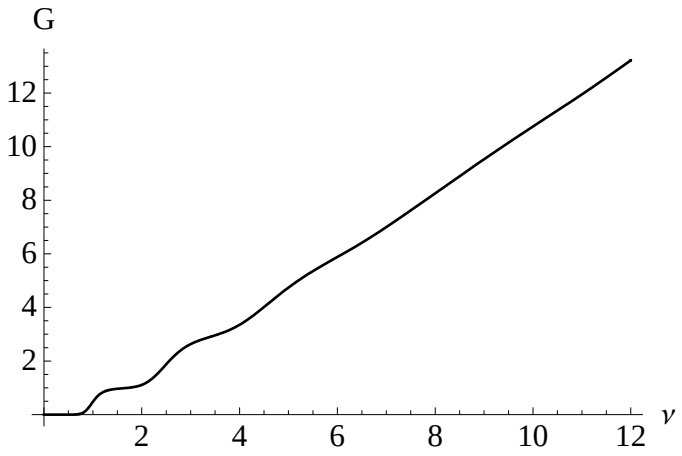

FIG. 3: Conductance versus ν at $T/T_F = 0.1$ for ballistic flow.

where $\phi_{i+} = \{(1, k_z^{(1)}), \cdots, (N, k_z^{(N)})\}$ and $\phi_- = \{(1, -k_z^{(1)}), \cdots, (N, -k_z^{(N)})\}$. That is, ψ_{left} is a $2N$ dimensional vector. This state has N scattering channels. An elastic scattering of the particle from $(n, k_z^{(n)})$ can be to any of the N bands as long as energy remains the same. If $\epsilon_0 < E < \epsilon_1$, only one channel is available and we have simple 1D scattering.

The Landauer formula[8] for multichannel conductance is

$$G = \frac{2m^2}{h} \sum_{a,b} T_{ab} \tag{10}$$

where T_{ab} is the transmission coefficient for scattering from channel a to channel b.

If we assume that a scattering center on the wall has random position and strength then a method that simply uses a series of completely random scattering matrices gives a suitable simulation. Such random matrix theory has been used extensively to treat conductance in electron systems.[9]. If the state after a scattering event is

$$\psi' = \begin{pmatrix} \phi'_+ \\ \phi'_- \end{pmatrix}. \tag{11}$$

Then the transfer matrix M satisfies

$$\psi' = M\psi \tag{12}$$

or

$$\begin{pmatrix} \phi'_+ \\ \phi'_- \end{pmatrix} = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \tag{13}$$

where each M_{ij} is an $N \times N$ matrix. The S matrix, also a $2N \times 2N$ matrix that we write as

$$S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}, \tag{14}$$

connects outgoing (on the left of the equation) and incoming waves (on the right of the equation) according to

$$\begin{pmatrix} \phi_- \\ \phi'_+ \end{pmatrix} = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi'_- \end{pmatrix}. \tag{15}$$

where t, t', r, and t' are $N \times N$ matrices. Suppose we have a two-channel system with just one scattering site so that the final state has $\phi'_- = 0$. That is, after the scattering one has only the transmitted wave. Then we have $\phi_- = r\phi_+$ and $\phi'_+ = t\phi_+$; r represents reflection and t transmission. In detail for the two-channel system we have

$$\begin{pmatrix} \phi'_{+1} \\ \phi'_{+2} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} \phi_{+1} \\ \phi_{+2} \end{pmatrix}. \tag{16}$$

If a particle starts in channel 1 ($\phi_{+2} = 0$), the probability of scattering to final channel 1 is $|t_{11}|^2$ and the probability of scattering to final channel 2 is $|t_{21}|^2$. The total probability of scattering out of channel 1 is $T_1 = |t_{11}|^2 + |t_{21}|^2$. The total conductance for an incoherent mixture of all incoming states is then

$$G = \frac{2m^2}{h} \sum_{a,b} |t_{ab}|^2 = \frac{2m^2}{h} \mathrm{Tr}[t^\dagger t]. \tag{17}$$

The same result holds for many channels.

The S matrix is unitary and symmetric if we impose current conservation and time-reversal invariance, which also impose conditions on the M matrix. We want a whole series of randomly chosen transfer matrices because the transfer matrix for P consecutive scatterings satisfies

$$M_{tot} = M_1 M_2 \cdots M_P. \tag{18}$$

Because S and M both connect $\phi_{\pm a}$ and $\phi'_{\pm a}$ we can relate the elements of one to the other to find M. We then use

$$t = (M_{--})^{-1}. \tag{19}$$

It is not difficult to construct these random matrices. Some results are shown in the figures. In Fig. 4, there is only a single scattering. Steps are clearly established depending on

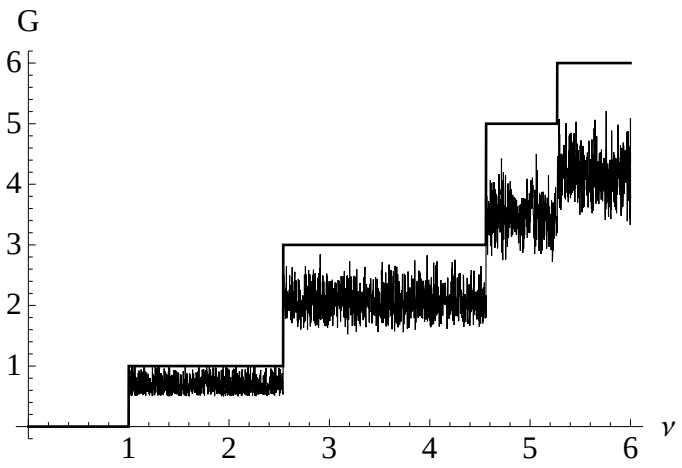

FIG. 4: Conductance versus ν for one scatterer. The regular stepped line is the conductance with pure ballistic flow.

the number of channels. On the other hand, there are fluctuations in the conductance having amplitude close to one conductance unit; these are the "universal conductance fluctuations" observed in electron systems.[10]

In Fig. 5 and Fig. 6 we increase the number of scatterers to three and ten, respectively. The step sizes are reduced in each case, as one might expect from an increase in backscattering. All steps maintain the one unit fluctuation value except the single-channel case, which is strongly reduced for three scatterers and has disappeared for ten. One-dimensional motion has reached full localization for 10 scatterers. In that case the step structure has almost disappeared as well.

This random matrix method seems to capture many of the effects we expect experimentally. However this method is so generic that it says nothing specific, for example, about the scattering being just at the walls of the channels. Alternative techniques more specific to helium are being investigated.

LUTTINGER LIQUIDS?

Based on simple estimates of the transverse quantum zero-point motion, we argue that in the regime of nanometric pore size and at sufficiently low temperatures, the quantum flow properties may not be described by a fluid with independent particles, as used above, but

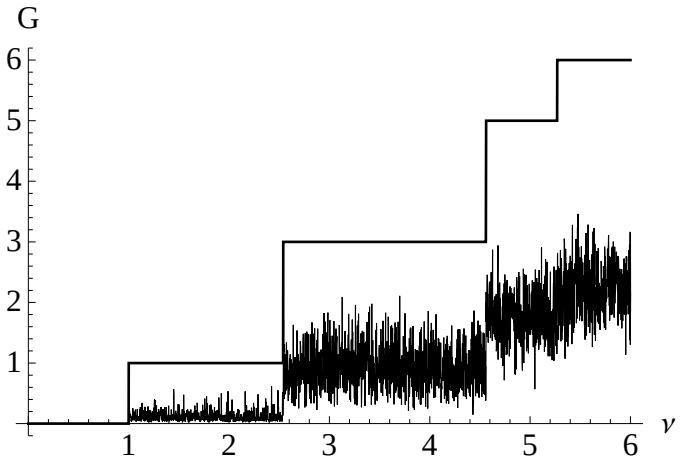

FIG. 5: Conductance versus ν for 3 scatterers.

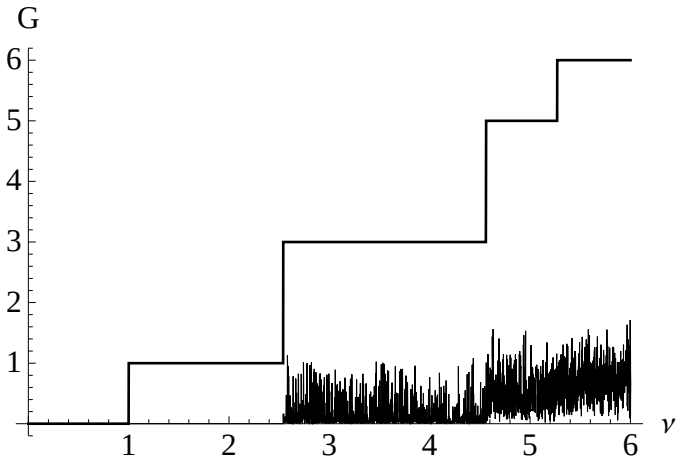

FIG. 6: Conductance versus ν for ten scatterers.

can enter a new kind of low-dimensional ^{3}He ground state with experimentally unknown excitation spectra and transport properties. In this regime, the ^{3}He Fermi system should reach the strongly interacting one-dimensional limit for which a breakdown of the Landau Fermi liquid picture is expected[6] and the system becomes a Luttinger liquid. This expected crossover to the one-dimensional regime should be considered in a theory describing the transport of ^{3}He particles inside very narrow channels. The ideal gas approximation is no longer valid in this situation and interactions are vital to the existence of this phase.

As an idealized case, let's consider the transport of pure ^{3}He inside a nanochannel with a diameter $d \sim 1$ nm and of length L such that $L \gg d$. If the fluid is effectively confined

radially in a cylindrical nanochannel, the fluid will be in the one-dimensional limit when the occupied states are all in the lowest band. The degree of occupation is measured by the one-dimensional Fermi energy, given by

$$T_{F1D} = \frac{\hbar^2}{2m_3 k_B}\left(\frac{\pi n_{1D}}{2}\right)^2, \tag{20}$$

where the number density n_{1D} is the reciprocal of the interatomic distance. A characteristic temperature measuring the separation between bands is of order

$$T^\star_{1D} = \frac{\hbar^2}{m_3 d^2 k_B}. \tag{21}$$

We expect a crossover to the one-dimensional regime in nanochannels to require $T_{F1D} < T^\star_{1D}$.

Assuming an interatomic distance of ~ 1 nm, we estimate $T_{F1D} \sim 200$ mK and $T^\star_{1D} \sim 160$ mK for $d = 1$ nm. With T_{F1D} and $T^\star_{1D}$ comparable, reaching the 1D liquid in the pure case is a possiblity if the temperature is low enough. In addition, a new unidimensional state of matter will arise only when the ^{3}He atoms are in the strongly correlated regime attained at temperature $T \ll T_{F1D}$, say, at $T \lesssim 0.1 T_{F1D} \sim 20$ mK. The conditions needed may be within experimental reach for a few nanometer hole in the pure case.

For a 1% concentration of isotopic solution the one-dimensional Fermi temperature is of order 10 mK, and $T^\star_{1D} \sim 17$ mK for $d = 3$ nm. The crossover around ~ 1 mK is within the regime of the cooling technique by filtering discussed by Ref. [1]. The use of isotopic helium solutions may allow one to adjust the parameters into the needed ranges.

The excitation spectrum of the one-dimensional fermionic system differs entirely from a Fermi liquid, and, for electrons, this state has also been known to give rise to interesting power law transport behaviour. Depending on whether the interactions are attractive or repulsive, the 1D-quenched ^{3}He system could lead to interesting new physics analogous to that of one-dimensional electronic system such as, for example, the highly sought-after spin-charge separation for which there are now important experimental hints[11]. Having both pure ^{3}He and dilute solutions available allows the possibility of a considerable alteration of the interatomic interactions leading to differing physical properties.

CONCLUSION

We have shown that many of the interesting effects known for electron systems can be studied experimentally and theoretically in ^{3}He in nanochannels. We have provided

examples of quantum-limited mass flow giving rise to a quantization of the conductance. We have also calculated its temperature dependence and have shown the appearance of conductance fluctuations when diffusive scattering takes place. We have also argued that a very interesting new state of matter, a Luttinger liquid, which may have quite new transport properties and excitations, could perhaps form in ^{3}He confined to nano-channels. While this theoretical study has been of a preliminary nature, we present it in the hopes of stimulating further discussion of this interesting system.

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[1] W.J. Mullin and N. Kalechofsky, Phys. Rev. Lett **97**, 010801 (2006).

[2] Y. Sato, B.H. Eom and R. Packard, J. Low Temp. Phys. **141**, 99 (2005).

[3] Synkera Technologies, Inc. http://www.synkera.com/products/nanomemb.htm.

[4] J. K. Holt, H. G. Park, Y. Wang, M. Stadermann, A. B. Artyukhin, C. P. Grigoropoulos, A. Noy, and O. Bakajin. Science **312**, 5776 (2006).

[5] Wees, J. van *et.al.* Phys. Rev. Lett. **60**, 848 (1988).

[6] J.M. Luttinger, J. of Math. Phys., **4**, 1154 (1963); F.D.M. Haldane, J. Phys. C- Sol. State Phys., **14**, 2585 (1981).

[7] I. Affleck, private communication.

[8] For a review see, C. W. J. Beenakker and H. van Houten, Solid State Phys. **44**, 1 (1991).

[9] C. W. J. Beenakker, Rev. Mod. Phys, 69, 731 (1997).

[10] See for example, P. A. Lee, A. D. Stone, and H. Fukuyama, Phys. Rev. B **35**,1622 (1985).

[11] O.M. Auslaender, H. Steinberg, A. Yacoby, Y. Tserkovnyak, B. I. Halperin, K. W. Baldwin, L. N. Pfeiffer, and K. W. West, Science **308**, 88 (2005).