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Kinematic Signature of a Rotating Bar Near a Resonance

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ABSTRACT

There have been several recent suggestions that the Milky Way has rotating bar-like features based on HI and star count data. In this paper, I show that such features cause distinctive stellar kinematic signatures near OLR and ILR. The effects of these resonances may be observable far from the peak density of the pattern and relatively nearby the solar position. The details of the kinematic signatures depend on the evolutionary history of the 'bar' and therefore velocity data, both systemic and velocity dispersion, may be used to probe the evolutionary history as well as the present state of the Galaxy.

Kinematic models for a variety of simple scenarios are presented. Models with evolving pattern speeds show significantly stronger dispersion signatures than those with static pattern speeds, suggesting that useful observational constraints are possible. The models are applied to the proposed rotating spheroid and bar models; we find: 1) none of these models chosen to represent the proposed large-scale rotating spheroid are consistent with the stellar kinematics; and 2) a Galactic bar with semimajor axis of $3\,\mathrm{kpc}$ will cause a large increase in velocity dispersion in the vicinity of OLR ($\sim 5\,\mathrm{kpc}$) with little change in the net radial motion and such a signature is suggested by K-giant velocity data. Potential future observations and analyses are discussed.

Subject headings: galaxy: kinematics, galaxy: structure, stellar dynamics

1. Introduction

Recently, several groups have suggested that the Milky Way may have one or more non-axisymmetric structures, such as stellar bars and triaxial spheroids (e.g. Blitz & Spergel

1991, hereafter BS; Weinberg 1992).[1] The presence of a bar was inferred from an analysis of IRAS source counts, while gas kinematics were used to deduce the existence and structure of a triaxial spheroid. In either case, the relationship between the non-axisymmetric density structure and the associated kinematic effects are well defined by theory. Briefly, a structure with a rotating pattern (as has been suggested both for the BS spheroid and for the bar), in a disk system with nearly circular orbits will generate three strong resonances: the inner Lindblad resonance (ILR), the outer Lindblad resonance (OLR), and a resonance at the location of corotation (CR). Orbits between ILR and CR will be elongated in a direction parallel to the bar while orbits outside OLR will be elongated perpendicular to the bar.

However, the impact of a bar on stellar orbits within a disk is not fully described by standard epicyclic theory particularly near resonances, where nonlinear effects may cause discontinuous changes in orbit morphology and trap apoapses into a narrow range of position angles. Furthermore, the standard theory assumes the existence of a static bar, whereas the kinematic consequences of a bar are inextricably linked to its past history. This paper shows that resonances between disk and bar potentials produce strong kinematic signatures. The simplest case is that where the bar grows adiabatically. The case of a bar with an evolving pattern speed is also considered. The results show that the integrated effect of bar-disk resonances on stellar kinematics allow the qualitative features of the evolution to be diagnosed.

The general approach is described in §2. The response near resonance is approximated by a non-linear Hamiltonian model which is exact to first order in epicyclic amplitude. The qualitative behavior of orbits near resonance is explored. Some readers may wish to skim §2 and proceed to §3, where the models are applied to an ensemble of orbits, and expressions for line of sight velocity and velocity dispersions are derived. In §4, these results are applied to the problems of a rotating spheroid and Galactic bar. Existing kinematic data are inconsistent with any of the rotating spheroid models developed here, but are in remarkable agreement with a bar ending at 3 kpc. Several suggestions for additional observations and analysis are offered in §4.3 and the results are summarized in §5.

2. Theory

If the force due to a rotating bar is a small perturbation in the region of interest, e.g., less than roughly 5% of the axisymmetric force, then the dominant response will be near

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[1]Throughout, I will call the general class of bisymmetric rotating disturbances 'bars'.

resonances. Standard epicyclic theory describes the motion near one of these locations as a coupled driven simple harmonic oscillator (SHO), whose amplitude diverges near resonance. In reality, however, non-linear terms detune the oscillator so that it is no longer a SHO. This section seeks a more appropriate model for the motion near resonance. Such a model must take into account that the effect of the resonance depends on the time history of the non-axisymmetric potential. As the amplitude and frequency of a rotating bar changes, its resonances sweep through phase space. As orbits are captured into libration and/or pass through resonance, they may significantly change in epicyclic amplitude and guiding center trajectory. For these reasons, the models presented below explicitly treat the bar potential in its time-dependent form.

2.1. Dynamical model

Let us begin with a flat axisymmetric galactic disk whose dynamics are then specified by the Hamiltonian function $H_o(\mathbf{I})$ where $\mathbf{I}$ are the actions. The action-angle variables are the natural choice to describe regular periodic motion (e.g. Goldstein 1950, Chap. 9). One action may be identified with the angular momentum and the other with the momentum of the radial motion. Expanding the bar perturbation in polar harmonics we may write

$$H = H_o + \sum_{m=-\infty}^{\infty} U_m(r) \exp\left[im\left(\phi - \int \Omega_b \, dt\right)\right]. \tag{1}$$

The quantity Ω_b is the pattern speed of the bar which may explicitly depend on time. Since the unperturbed Hamiltonian is cyclic in the conjugate angles, it proves convenient to expand the terms of the sum in equation (1) as a Fourier series in the unperturbed actions $\mathbf{I}$ and angles $\mathbf{w}$. In addition, since a bar-like perturbation is likely to be dominated by the quadrupole, let us restrict the sum in equation (1) to a single term ($m = 2$); other multipole terms may be treated similarly. We may write (c.f. Lynden-Bell & Kalnajs 1972, Tremaine & Weinberg 1984)

$$U_m(r) \exp\left[im\left(\phi - \int \Omega_b \, dt\right)\right] =$$
$$\sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} W_{l_1 l_2}^m(I_1, I_2) \exp\left[i\left(l_1 w_1 + l_2 w_2 - m \int dt \, \Omega_b\right)\right], \tag{2}$$

where

$$W_{l_1 l_2}^m(I_1, I_2) = \frac{1}{\pi} \int_0^{\pi} dw_1 \, \cos[l_1 w_1 - l_2(\phi - w_2)] U_m(r). \tag{3}$$

The right-hand-side of equation (2) is a Fourier series in the angle variables w_j whose Fourier coefficients $W^m_{l_1 l_2}(I_1, I_2)$ are given by equation (3). Furthermore, since the Galactic disk is rotationally supported with relatively small radial excursion, we may represent the unperturbed orbits in the epicyclic limit. For definiteness, we take the rotation curve to be flat with rotation velocity V_{rot}. With these limits, the orbital quantities and the action-angle variables become:

$$r = R + a \sin w_1, \tag{4}$$

$$I_1 = \frac{V_{rot}}{\sqrt{2}} \frac{a^2}{R}, \tag{5}$$

$$I_2 = V_{rot} R, \tag{6}$$

$$\Omega_1 = \frac{\sqrt{2} V_{rot}}{R}, \tag{7}$$

$$\Omega_2 = \frac{V_{rot}}{R}, \tag{8}$$

$$w_1 = \Omega_1 t + w_{10}, \tag{9}$$

$$w_2 = \phi - \frac{\Omega_2}{\Omega_1} \frac{2a}{R} \cos w_1, \tag{10}$$

where quantities I_1 and I_2 are the radial and azimuthal actions with corresponding angles w_j and frequencies Ω_j, R is the guiding center of the trajectory and w_{10} describes the radial phase at $t = 0$. Using the above relations, we may now explicitly evaluate equation (3):

$$
\begin{aligned}
W^2_{l_1 l_2} = {}& \delta_{l_2 2} \exp[-im\Omega_b t] \exp[-i\pi/2] \frac{a}{R} \times \\
& \left\{ \delta_{l_1 0} \frac{2\Omega_2}{\Omega_1} U_2(R) + \delta_{|l_1| 1} \operatorname{sgn}(l_1) \frac{R}{2} \left. \frac{dU_2}{dr} \right|_R \right\} \\
& + c.c. + \mathcal{O}(a^2),
\end{aligned}
\tag{11}
$$

where δ_{ij} is the Kronecker delta. The epicyclic approximation is consistent as long as the epicyclic amplitude a remains significantly smaller than R throughout the evolution and this condition must be checked explicitly. If desired, equation (11) may be explicitly rewritten in terms of the actions using equations (4)–(10). Substituting into equation (1) and (2) yields the governing Hamiltonian in terms of the action-angle variables.

We are interested in the effects near a particular resonance defined by a commensurability between frequencies:

$$l_1 \Omega_1 + l_2 \Omega_2 - m\Omega_b = 0. \tag{12}$$

It follows that the angle $w_s \equiv l_1 w_1 + l_2 w_2 - m \int dt\, \Omega_b$ is very slowly varying near resonance. To make the evolution near resonance explicit, we can rewrite the Hamiltonian (eq. 1)

with w_s as one angle. To do this, we may define the generating function of the canonical transformation (2nd kind, Goldstein 1950) as follows:

$$S = \left(l_1 w_1 + l_2 w_2 - m \int dt\, \Omega_b \right) J_s + w_1 J_f, \tag{13}$$

which gives the following new set of variables:

$$J_f = I_1 - \frac{l_1}{l_2} I_2, \tag{14}$$

$$J_s = \frac{1}{l_2} I_2, \tag{15}$$

$$w_f = w_1, \tag{16}$$

$$w_s = l_1 w_1 + l_2 w_2 - \int dt\, \Omega_b, \tag{17}$$

Since w_s is slowly changing relative to the second "fast" angle near resonance, relative to the timescale of the slow variable, the motion in the fast variable is adiabatic. Physically, the slow variable correspond to the precession of the orbit in the frame of the rotating disturbance and the fast variable correspond to the motion of the particle around its orbit (e.g. Lynden-Bell & Kalnajs 1972). We exploit the disparity in slow and fast frequency to average over the motion in w_f and rewrite the Hamiltonian in terms of the slow variables alone. This is the so-called "averaging principle" (e.g. Arnold 1980, Lichtenberg & Lieberman 1983).

We may derive a simplified model describing near-resonant behavior by expanding the averaged Hamiltonian, $\bar{H}$, about the value of the slow action at resonance:

$$\begin{aligned}
\bar{H} &= [H_o(J_{s,res}) - m\Omega_b(t) J_{s,res}] + [l_1\Omega_1 + l_2\Omega_2 - m\Omega_b(t)]|_{res}\, (J_s - J_{s,res}) + \\
&\quad \frac{1}{2} \left.\frac{\partial^2 H_o}{\partial J_s^2}\right|_{res} (J_s - J_{s,res})^2 + W_{l_1 l_2}^2\big|_{res} \cos(w_s + w_r),
\end{aligned} \tag{18}$$

where the subscripts "res" denote values at resonance. In equation (18), the first term on the RHS is independent of J_s and w_s and the second term is zero at the resonance. In general Ω_b is a function of time, as indicated. Therefore, if we define t_{res} to be the time at which the resonance occurs, $l_1\Omega_1 + l_2\Omega_2 - m\Omega_b(t_{res}) = 0$, then equation (18) becomes

$$\begin{aligned}
\bar{H} &= [H_o(J_{s,res}) - m\Omega_b(t) J_{s,res}] - m[\Omega_b(t) - \Omega_b(t_{res})](J_s - J_{s,res}) + \\
&\quad \frac{1}{2} \left.\frac{\partial^2 H_o}{\partial J_s^2}\right|_{res} (J_s - J_{s,res})^2 + W_{l_1 l_2}^2(\mathbf{I}) \cos(w_s + w_r).
\end{aligned} \tag{19}$$

We will further simplify the 2nd term by assuming that any time dependence in Ω_b is slow compared to the evolution of a particular orbit and write $\Omega_b(t) - \Omega_b(t_{res}) = \dot{\Omega}_b(t - t_{res})$.

The character of the solutions to the Hamiltonian given by equation (19) depends on the functional behavior of $W_{l_1 l_2}^2$ on J_s (cf. eq. [3]). If $l_1 = 0$, which gives the corotation resonance, then $W_{l_1 l_2}^2$ be will a constant to lowest order and equation (18) then corresponds to the Hamiltonian of a simple pendulum. If $l_1 = \pm 1$, then $W_{l_1 l_2}^2 \propto a \propto \sqrt{I_1}$. This is not a pendulum but the qualitative properties of the solution are similar. Let us consider the corotation case explicitly. If the bar-like perturbation grows slowly over time, then W_{02}^2 slowly increases from zero. Physically, this corresponds to a pendulum with an increasing gravitational constant (e.g. Yoder 1979). Initially, the pendulum will be uniformly rotating around it pivot like a propeller. The action, the integral of the angular momentum around the trajectory, is an adiabatic invariant and is conserved as the gravitational force grows. However, if the gravity becomes strong enough the bob will not be able to make it over the pivot. At this transition point, the trajectory has infinite period (the pendulum bob can just make it to the unstable equilibrium in an infinite time) which breaks the adiabatic invariant. After the transition, the motion is no longer rotation but swinging or *librating* and the action is again conserved but with a different value. The new value of the action depends on the rate of growth at the transition. This transition point is often referred to as the resonance, but the motion does not correspond to unbounded amplitude in the sense of a forced SHO but zero frequency in the sense of equation (12). The transitional or *critical* trajectory divides the motion into regions of rotation and libration.

If $l_1 = \pm 1$, the qualitative properties of the solution are similar: there can be both rotational and librational regions of phase-space along with trajectories of infinite period giving rise to jumps in action.

2.2.   Evolution Through Resonance: Example

Although the dynamical model has been reduced to one degree of freedom, and for example the motion may vanish in that dimension, the orbit itself remains similar to a circular orbit. The slow angle describes the position of the apocenter relative to the bar and is correlated with the variation of slow action in the presence of the bar. A change in slow action changes both the epicyclic amplitude and guiding center radius. As an example, Figure 1 shows the radius and angle of apocenter relative to the bar position angle, ϕ_{apo}, for an orbit trapped into libration at ILR. The orbit is initially a simple rosette with an r.m.s. radial velocity of 1/10 its tangential velocity. As the bar strength grows, the orbit passes through the critical trajectory which causes a large jump in the epicyclic radius. Notice that ϕ_{apo} is restricted to a narrow range of position angles relative to the bar as expected

for libration. Also, the angle of the apocenter is correlated with the epicyclic amplitude and "lingers" near $\pm\pi/2$; the orbit is an oval oriented orthogonal to the bar. As ϕ_{apo} swings quickly through zero, the orbit is either nearly circular or very elongated. The plot of this orbit in the bar frame is shown in Figure 2 (for simplicity only 1/4 of the orbit shown in Figure 1 is displayed). At $t \approx 0$, $\phi_{apo} \approx 0$ and the orbit is rather eccentric. The apocenter angle quickly swings up to and lingers near $\pi/2$. If the plot continued, the orbit would appear more and more circular and the apocenter would then rapidly swing to $\approx -\pi/2$ and become more elongated, and then the cycle would repeat. This orbit responds strongly to the bar perturbation because it is very slowly precessing in the bar frame. The same orbit viewed from the inertial frame is shown in Figure 3; it appears to be an unremarkable rosette orbit.

Figures 4 and 5 are similar to Figure 1 but for rotating orbits with initial guiding centers at $R = 1.0$ and 2.5 respectively. The orbit at $R = 1.0$ passed through the critical trajectory and the one with $R = 1.7$ did not; note the large difference in the amplitude of radial oscillation. As R decreases, the orbit does cross the critical trajectory (at $R \approx 1.6$) and the amplitude of radial oscillation doubles. The existence of critical behavior causes a radially well-demarcated band of resonance-induced kinematic changes in the stellar disk which may be observed. The graphs of these orbits in rotation are similar to a rosette even in the bar's rotating frame. However, the position angle remains correlated with epicyclic amplitude even though ϕ_{apo} is takes on all angles roughly evenly. The amplitude is strongest near $\phi_{apo} = 0, \pi$ and causes the observed kinematics for orbits near resonance to vary with respect to the bar position angle.

The location and size of the band where resonant effects are significant depends on the history of the bar perturbation. For example, if the pattern speed changes as the bar evolves, the band may be wider with a different fraction of librating to rotating orbits. In fact, the relative fraction of librating orbits is very different depending on the sign of $\dot{\Omega}_b$. The mathematical details of the model Hamiltonian (eqs. 19, A1) and its solution are discussed in Appendices A and B. Quantitative observational predictions for a variety of Galactic scenarios are discussed in §3.

3. Application to observed velocities

In this section, we will describe the general features of the kinematic signatures near resonance. We begin by determining the model from §2 (and Appendix A) for two specific scenarios: the response of the Galactic disk to a rotating Galactic bar near OLR and to

a rotating triaxial spheroid near ILR. In both cases, the models either have constant or decreasing pattern speeds and are chosen to illustrate the sensitivity of the kinematic signatures to the history of the evolution. We will compare these models with Galactic observations in §4.

The Galactic rotation curve is assumed to be everywhere flat with value V_{rot}. This is a fair approximation for radii of interest: $R \gtrsim 4\,\mathrm{kpc}$ (e.g. Schneider & Terzian 1983, Fich et al. 1989). We take the potential of the triaxial spheroid as estimated by BS, use the outer solution to Poisson's equation for the bar, and assume that the non-axisymmetric perturbation is dominated by its quadrupole term. In order to compute the velocity signature near both the ILR and OLR, we need $dU_2(r)/dr$ (cf. 3 and 11). One finds

$$\frac{dU_2(r)}{dr} = \begin{cases} \epsilon \dfrac{V_{rot}^2}{R_b} \dfrac{2r/R_b}{1+(r/R_b)^5} & \text{bar;} \\[2em] \epsilon \dfrac{3}{4} \dfrac{V_{rot}^2}{R_b} \left(\dfrac{r}{R_b}\right)^\gamma & \text{spheroid,} \end{cases} \tag{20}$$

where ϵ is the strength of the bar relative to the axisymmetric restoring force at the characteristic radius R_b. BS estimate γ to be $2 \leq -\gamma \leq 2.5$. The units and choice of parameters parameters for these models are listed in Table 1. A given pattern speed Ω_b determines the radial location of the resonance. However, as the bar evolves, the pattern speed may change and therefore the locations of the resonances may change. We describe the initial and final locations as R_i and R_f. The parameters of the models we will discuss are presented in Table 2.

3.1. Evolution of an ensemble

The goal is to compute the velocity along any given line of sight incorporating effects induced by the bar. The previous section (§2) outlines the evolution of a particular orbit. In order to compute the line-of-sight velocity at a particular point we need to average over the entire evolved ensemble given by the initial distribution function. Here, we assume a Schwarzschild distribution with a velocity dispersion σ_r. Unfortunately, since the behavior near resonance is non-linear, a closed form solution is not possible. However as described in Appendix A, the post-evolution actions may be determined as a function of the original actions using a simple look-up table.

For discussion, we will consider initial ensembles described by a constant value of I_2 and a distribution of I_1 consistent with the Schwarzschild distribution. For each I_1, all

Table 1: Model parameters

Model	Parameter	Value	Comment
Galaxy[a]			
	R_{LSR}	1.0	radius of LSR
	V_{rot}	1.0	speed of LSR (flat rotation curve)
	σ_r	0.1	radial velocity dispersion in units of V_{rot}
Bar at OLR			
	R_b	0.3	characteristic radius of bar
	ϵ	0.2[b]	ratio non-axisymmtric to Galaxy force at bar end
	R_{OLR}	0.64	location of resonance ($\approx 10.4\,\mathrm{kpc}$)
Spheroid at ILR			
	R_b	1.0	spheroid scale factor
	ϵ	0.02	ratio of non-axisymmetric to Galaxy force at LSR
	γ	-2.0	exponent of quadrupole force powerlaw
	R_{ILR}	1.3	location of resonance ($\approx 5.1\,\mathrm{kpc}$)

[a]defines sysem of units

[b]ratio of non-axisymmetric to axisymmetric force at the solar circle is 1%

phases w_1 and w_2 are equally represented. In the epicyclic limit, these ensembles may be described by the initial value of the guiding center. Note that individual members of the post-evolution ensemble may be distributed in guiding center radii and therefore ensemble averages do not strictly represent the value at a point in space. Also, a local patch of the disk contains orbits from a distribution of guiding center radii. Nonetheless, the ensemble evolution gives some indication of the expected kinematic signatures since the epicyclic amplitudes are relatively small. In addition, this definition of an ensemble does greatly simplify the calculation and provides insight. Although a tractable extension of the results presented here, the derivation of the space-localized evolved distribution requires models for a large fraction of the entire disk and gives less insight into the evolutionary mechanism. Such large-scale models will be useful for comparing with a large spatially distributed kinematic data set (see §4.3).

The radial and tangential velocities of the ensemble observed along a line of sight at angle ϕ to the bar may then be written:

$$V_r = \left\langle \int dJ_f \int \frac{dw_f}{2\pi} f(I_1, I_2) \left(\pm \Omega_1(R) a \cos w_1 \right) \delta \left(w_s - w_s(w_1, w_2, \phi) \right) \right\rangle, \qquad (21)$$

Table 2: Run parameters

Run	Resonance	R_i[a]	R_f[b]
A	ILR	1.3	1.3
B		1.1	1.3
C		1.0	1.3
D		0.9	1.3
I	OLR	0.625	0.625
J		0.625	0.75
K		0.625	0.875
L		0.625	1.0

[a]initial resonance location

[b]final resonance location

$$V_t = \left\langle \int dJ_f \int \frac{dw_f}{2\pi} f(I_1, I_2) \left(V_{rot} \pm \Omega_2(R) a \sin w_1 \right) \delta \left(w_s - w_s(w_1, w_2, \phi) \right) \right\rangle, \quad (22)$$

where $\pm$ is for ILR and OLR respectively, $\langle \rangle$ indicates the ensemble average at fixed I_2, and $w_s(w_1, w_2, \phi)$ is the value of w_s for a given w_1 and w_2 from the look-up table. Similar expressions may be derived for the velocity dispersions, $\sigma_t = \sqrt{V_r^2 - (V_r)^2}$ and $\sigma_t = \sqrt{V_t^2 - (V_t)^2}$, where

$$V_r^2 = \left\langle \int dJ_f \int \frac{dw_f}{2\pi} f(I_1, I_2) \left(\Omega_1(R) a \cos w_1 \right)^2 \delta \left(w_s - w_s(w_1, w_2, \phi) \right) \right\rangle, \quad (23)$$

$$V_t^2 = \left\langle \int dJ_f \int \frac{dw_f}{2\pi} f(I_1, I_2) \left(V_{rot} \pm \Omega_1(R) a \sin w_1 \right)^2 \delta \left(w_s - w_s(w_1, w_2, \phi) \right) \right\rangle. \quad (24)$$

Initially with no perturbation, the distribution is independent of w_s and therefore $V_r = 0, V_t = V_{rot}$. The post-evolution trajectories depend on the initial phase but if the original distribution is phase mixed than the new distribution should be phase mixed for sufficiently slow evolution, $\left| \frac{\dot{\omega}_b}{\Omega_b^2} \right|$. Therefore, the distribution of w_1 and w_2 for the final orbit may then be derived by sampling the final orbit at equal time intervals or, equivalently, by assuming an flat distribution in the angle conjugate to the post-evolution action:

$$J_{final} = \frac{1}{2\pi} \oint dw_s J_s \quad (25)$$

In order to determine the observable quantities (eqns. 21–24), we must convert back to local variables. Near the ILR or OLR, a librating orbit will still have a well-defined

guiding center trajectory instantaneously, but the apses will be confined to a small range of position angles. If we observe in a frame rotating at the pattern speed, the trajectories will be streaming forward (backward) near the ILR (OLR) on almost closed orbits. However, as long as the post-resonant epicyclic amplitude remains relatively small, the trajectory is instantaneously close to a valid epicycle.

3.2.   Velocity signature near resonance

Using equations (21) and (22), we may predict the line-of-sight velocity near resonance for any given scenario. We will be begin with a detailed investigation of the signature near ILR and discuss specific scenarios below.

3.2.1.   Results at particular radii for a bar with constant pattern speed

The line-of-sight velocity and dispersion in an inertial frame fixed at the Galactic center for Model A (see Table 1) are summarized in Figures 6–8. The angle ϕ describes difference between the bar's position angle and the viewing angle.[2] The guiding centers for the ensembles are chosen for a range of discrete values of R. Figure 6 (7) describes the velocity signature for ensembles with guiding centers inside (outside) R_{res}. The sign of the changes are as expected:

1. $R < R_{res}$.   Inside of ILR, the apses will tend to be antialigned with the bar and orbits drift forward in the bar's frame; therefore, in the first quadrant relative to the bar the mean outward velocity will be positive.

2. $R > R_{res}$.   Outside of ILR the apses will tend to be aligned with the bar and therefore, in the first quadrant relative to the bar the mean outward velocity will be negative.

The large values of V_r at small R ($R \lesssim 0.6$) are not a significant feature of the resonance but due to the increasing amplitude of the spheroid quadrupole (cf. eq. 20). Far outside the R_{res}, V_r drops quickly (cf. Fig. 7) due to the decreasing spheroid strength.

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[2]Angles and rotations are defined in the mathematical sense, as if viewing from the SGP.

3.2.2. Capture into libration and critical-crossing trajectories

The picture is more complex near resonance as seen in Figure 8. The "kick" received from the evolving bar as the orbit evolves through the critical trajectory accounts for the large amplitude of the curves near $R \approx 1.4$ even though these trajectories are initially outside the resonance. A significant population of librating orbits causes the oscillation in V_r with ϕ.

Figures 9 and 10 show the fractional contribution to the line-of-sight velocities from non-critical, critical and librating trajectories for $R = 0.8$ and 1.4 (cf. Figs. 6 and 8). For $R = 0.8$ few orbits have passed through the critical trajectory and 15% are in libration. For $R = 1.4$ nearly all trajectories have both passed through the critical trajectory as the potential evolved and are in libration. For $R \lesssim 0.9$ or $R \gtrsim 1.6$, no trajectories will have passed through the critical one and received a jump in action; those at small R will have a large fraction of trajectories in libration (cf. Fig. 2) and those at large R will have none.[3]

The initial ensemble has orbits evenly distributed in radial and azimuthal phases and velocity dispersion or epicyclic energy chosen according to the Schwarzschild distribution. However, the response for critical-crossing trajectories depends on the slow phase, w_s, at the crossing. To better understand the spatial morphology of these orbits after the bar has evolved, we may select an ensemble of orbits all with the same epicyclic energy σ_r^2 and average over all phase but at fixed w_s to get a "mean" orbit. This may be thought of as a time-averaged trace of the particular orbit at the final (fixed) bar strength and frequency. Figure 11 shows the mean orbits computed from 10 initially equally spaced w_s for $R = 0.6, 0.9, 1.1, 1.3, 1.5$. The bar major axis is along $\phi = 0$. For all but $R = 1.3$, the mean orbits are nearly phase independent and the 10 trajectories are simultaneous. For $R \approx 1.3$, capture into libration depends on initial phase (cf. Fig. 10) resulting in different guiding center radii. The librating orbits are captured into the antialigned orientation and have the largest epicyclic amplitudes in Figure 11. The $R = 0.6$ orbits are also all in libration which gives them more elongation than, say, the case with $R = 0.9$ which has no librating orbits.

3.2.3. Results for a bar with decreasing pattern speed

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[3]For an example of the number fraction of orbits for each ensemble at R that have crossed the critical trajectory and/or been trapped into libration, see Fig. 16.

Figure 12 shows the mean orbits for Model B whose pattern speed decreases by 17% (see Table 1). Since the fraction of orbits that are captured into libration and cross the critical trajectory depends on the evolutionary history and not simply the final state, we find quantitatively different results. The guiding center $R = 0.9$ is broken up into two distinct groups. The inner "boxy" mean trajectories have been captured into libration and the outer trajectories have not. All have crossed the critical trajectory. The librating trajectories are antialigned with the bar and the rotating trajectories are aligned with the bar. Guiding centers with $R \gtrsim 1.7$, which are non-critical and in rotation, only depend on the final value of Ω_b and not on its history. Overall, the changing pattern speed in Model B has increased the fraction of libration and critical trajectory-crossing orbits over Model A. As we will see below, these differences are reflected in observable kinematic signatures and may be used to limit evolutionary hypotheses.

3.2.4.   Expectations at a specified position angle

The dispersion and velocity profiles along particular lines of sight are shown in Figures 13 and 14 for Models A and B. These plots represent cuts at constant ϕ through (e.g.) Figures 6–8. We select $\phi = -45°$, anticipating comparison with a rotating spheroid whose position angle is in the 4th quadrant in Galactic longitude. Note that the overall amplitude and radial breadth of the feature is larger in the case of Model B than A. This trend is found for most models; the larger the variation of the model parameters with time, the more change in the trajectories. Therefore a model which both grows in strength and evolves in pattern speed shows a stronger response (Model B) than one which only grows in strength (Model A). Although the graph of V_r vs. ϕ is smooth for individual trajectories, after evolution the curves may overlap, especially near resonance, and produce a distinct signature along particular lines of sight. For example the dip in V_r at $R = 1.1$ in Figure 14 is caused by the superposition of librating and non-librating orbits (cf. Fig. 12). The response at constant perturbation strength increases dramatically with $\Delta\Omega_b$; some of the orbits near ILR in Models C and D are perturbed so strongly after evolution the epicyclic approximation is invalid. The observed lack of anomalously large velocities and dispersions expected from these distorted orbits near the solar position would suggest that a rotating spheroid must either have an extremely stable pattern speed or small amplitude.

Figure 15 shows the relative contribution to the line-of-sight velocity orbits which have become critical or captured into libration in Model A. A large fraction of all the orbits between 1 and 1.5 passed through the critical trajectory as the rotating perturbation grows

in strength. We see that the fraction of librating orbits follows the run of velocity dispersion in Figure 13. At small R, the relative strength of the perturbation increases and the orbits find themselves in the libration zone without having been critical.

To summarize, the strongest perturbations are caused by the non-linear response and are in a band about the formal location of the resonance.

4.   Application to the Galaxy

Resonances (ILR or OLR) may lead to pronounced kinematic signatures in the line-of-sight velocity and velocity dispersion (§3). In this section, we discuss two cases in detail: a triaxial spheroid and standard stellar bar.

4.1.   ILR models and implications for a rotating triaxial spheroid

In order to explain the motion of the LSR inferred from the asymmetry in the HI l-v diagram, Blitz and Spergel (1991) postulate a rotating triaxial spheroid with $\Omega_b = 6\,\mathrm{km\,s^{-1}/\,kpc}$. The parameters in Models A—D are chosen in accord with these parameters, although there was no attempt to tune the models to reproduce a particular LSR velocity.

Each of the models describes a possible evolutionary trend in the rotating spheroid. Model A with constant pattern speed shows a discontinuity in V_r around the resonant radius at $R = 1.3$ corresponding to $10.4\,\mathrm{kpc}$ scaled to Galactic units (Fig. 19). This model would imply an outward motion of the LSR of about $20\,\mathrm{km\,s^{-1}}$ (similar to BS's estimate of $14\,\mathrm{km\,s^{-1}}$ and differing because of the nonlinear response). Outside of $R \approx 14\,\mathrm{kpc}$, V_r would be unchanged and therefore the outer Galaxy would appear to be approaching the Sun. Note that there is a strong peak predicted in the line-of-sight velocity dispersion about the resonance location.

Model B–D have decreasing pattern speeds. In the case of (e.g.) Model B, the resonance moves from $R = 8.8\,\mathrm{kpc}$ to its final position at $R = 10.4\,\mathrm{kpc}$. A larger measure of orbits have been perturbed by the resonance giving a broad increase in velocity dispersion and smearing and shifting the line-of-sight velocity profile as seen in Figure 14. The amplitude of V_r is a factor of 2 larger in this case. An evolving pattern speed produces an observable signature with smaller spheroid amplitudes than assumed by BS, suggesting that velocity kinematics may be an even more sensitive probe of asymmetry than previously assumed.

Metzger and Schechter (1992) found that carbon stars in the direction of the Galactic anticenter appear to be systematically receding from the LSR (cf. Figure 19). This is also consistent with the K-giant data from Lewis & Freeman (1989). A naive interpretation suggests that the net stellar motion opposes the net gas motion. On the other hand, the motions might be better explained by a spheroid with $\phi \approx +45°$ giving rise to an inward

LSR motion. However, this counters the original motivation by Blitz and Spergel (1992) to explain the asymmetry in the HI l–v diagrams. Regardless, rotating spheroid models predict a strong jump in velocity dispersion near the OLR and provides a method to predict (or limit) their amplitude.

4.2. Implication for the Inner Bar

As emphasized in previous sections, ILR ($l_1 = -1, l_2 = 2$) is closely related to OLR ($l_1 = 1, l_2 = 2$). The governing equations (19, A1) are identical and the physical explanations above apply, with appropriate changes in sign.

Consistent with Weinberg (1992), we assume a bar ending at corotation at 3 kpc, major-axis position angle of 45°, and quadrupole strength of 20% of the axisymmetric background force at the end of the bar, corresponding to a strong stellar bar. This gives an OLR at ≈ 5 kpc or $R = 0.625$ in model units. Four models, I–L, consistent with these parameters are described in Table 1. Model I has a static pattern speed and all others have a decreasing pattern speed. Figure 17 shows Model I. The LSR motion is unchanged by the perturbation but the line-of-sight velocity increases toward the OLR with a factor of ~ 2.5 jump in line-of-sight velocity dispersion about the OLR.

We expect a moderate bar to lose angular momentum as it evolves (Weinberg 1985, Little & Carlberg 1991, Hernquist & Weinberg 1992). If this torque causes the pattern speed to decrease, the position of the OLR will increase, causing a large increase in velocity dispersion in the vicinity of OLR (~ 5 kpc) with little change in the net radial motion. Figure 18 shows the results of Model J whose OLR moves from 5 to 6 kpc. The radial velocity peak broadens and decreases in amplitude by a factor of 2 while the dispersion broadens and increases by a factor of two. Figure 20 shows Model J scaled to $R_{LSR} = 8$ kpc $V_{rot} = 220$ km s^{-1} (solid line and open circles) together with the Lewis & Freeman velocities (open squares and error bars); the observations suggest the predicted signature for the stellar bar. There has been no attempt to fit the model to the data other than selecting Model J from I–L.

Radakrishnan & Sarma (1980) estimate a radial dispersion of gas clumps in the direction of the center of 5 km s^{-1} and a systemic velocity of < 1 km s^{-1} based on the HI absorption spectrum of Sgr A. Although one expects the gas dispersion to be lower, in general, than the stellar velocity dispersion, the low systemic velocity is consistent with the predicted and observed trends in stellar kinematics: the perturbation to the line-of-sight velocity is small while the large dispersion is due to intersecting librating orbits.

4.3. Future Observations and Analyses

For an axisymmetric Galaxy, we expect $V_r = 0$ for all R in the direction of the center or anticenter, and since these measurements are natural diagnostics for asymmetry, they have received the most attention. However, the models described in previous sections predict distinct features in the velocity dispersion as well as the systemic velocity because of jumps in action and capture into libration near the resonances (cf. Figs. 6–8 and 11–12). These features allow for explicit testing of the various rotating spheroid and bar models using a spatially distributed sample of kinematic tracers. The theoretical models described above (Table 1) are easy to compute, allowing a wide variety of cases to be tested. Existing K-giant, carbon star, Mira and Cepheid variable data may permit precise testing of the various bar hypotheses using the standard statistical estimators (e.g. likelihood) and models constructed for full annuli of the Galactic disk rather than individual guiding center radii. This work is in progress.

5. Summary

The major conclusions are as follows:

1. A rotating pattern, such as a bar or spheroid, causes a distinctive stellar kinematic signature near primary resonances (OLR and ILR). The amplitude is larger than one would predict from the mean field of the static potential perturbation alone because of nonlinearity of the resonant response. These resonances may be far from the peak density of the pattern and relatively nearby the solar position for both a triaxial spheroid and Galactic bar, raising the possibility for direct observation.

2. Near OLR or ILR, a fraction of the trajectories for a given guiding center will be trapped into libration; the probability of trapping depends on phase. Overlapping librating and non-librating trajectories increase the velocity dispersion. Velocity dispersion measurements may be as useful as the systemic velocities in testing for the existence of a rotating disturbance. For example, a moderate strength Galactic bar ending at $3\,\mathrm{kpc}$ may easily produce an increased velocity dispersion at OLR ($\sim 5\,\mathrm{kpc}$) of $50\,\mathrm{km\,s^{-1}}$ or larger.

3. The details of the signature depend on the evolutionary history of the bar; a changing pattern speed or perturbation strength change the details of the response. The

fraction of orbits both trapped into libration and strongly perturbed by passing through resonance depends sensitively on the change in pattern speed; a 20% change in pattern speed may increase the velocity dispersion by a factor of 2–3. Therefore, kinematic data may be used to probe the evolutionary history as well as the present state of the Galaxy.

4. Blitz & Spergel (1991) suggest that the LSR motion may be explained by a large-scale rotating spheroid. It has been recently pointed out (Metzger & Schechter 1992) that the stellar kinematics are inconsistent with this simple picture. Various possible evolutionary scenarios are explored but none allow the gas and stellar kinematics to be simply understood with the proposed rotating spheroid model. In addition to LSR motion, a rotating non-axisymmetric spheroid will produce a velocity dispersion increase near ILR which should be observable.

5. I have predicted the kinematic signature that might be found for a Galactic bar with semimajor axis of 3 kpc such as inferred by Weinberg (1992). We expect a moderate bar to lose angular momentum as it evolves (Weinberg 1985, Little & Carlberg 1991, Hernquist & Weinberg 1992). If this torque causes the pattern speed to decrease, the position of the OLR will increase. This will cause a large increase in velocity dispersion in the vicinity of OLR (~ 5 kpc) with little change in the net radial motion. Such a signature is suggested by K-giant velocity data (Lewis & Freeman 1989) and consistent with HI gas data (Radakrishnan & Sarma 1980).

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A. Computational Method

Here, we develop and efficient computational form for equation (19). In the cases of interest, $l_1 = \pm 1$ and the potential coefficient $W^2_{l_1 l_2} \propto a \propto \sqrt{I_1}$. In terms of actions, $a \propto \sqrt{J_s + J_f}$ which makes equation (19) numerically inconvenient. However, it is straightforward to make a canonical transformation to a new set of variables where $\mathcal{J} \equiv J_s + J_f$ keeping J_f constant. The transformed Hamiltonian becomes:

$$\bar{H}' = \frac{1}{2}G\mathcal{J}^2 - G\mathcal{J}_o\mathcal{J} - \dot{\Omega}_b(t - t_{res})\mathcal{J} - F\sqrt{\mathcal{J}}\cos(\theta), \tag{A1}$$

where $G \equiv \partial^2 H_o/\partial J_s^2|_{res}$, $F \equiv dU_2/dr|_{res}$, $\mathcal{J}_o \equiv J_{s,res} + J_f$, $\theta = w_s + \pi/2$, and constant terms have been dropped. Both the coefficients of the $\mathcal{J}$ and $\sqrt{\mathcal{J}}$ terms are in general time-dependent in our model. However, as long as $\dot{\Omega}$ and $\dot{F}$ are very small, the trajectory remains near a solution with fixed coefficients for many dynamical times.

Numerical solution to the equations of motion generated by equation (A1) are further complicated by the $1/\sqrt{\mathcal{J}}$ term in the equations of motion. This singularity in the coordinate system (I, θ) may be removed by a transformation to a rectangular coordinate system, (x, y). This may be affected with the generating function $S_1(y, \theta) = y^2 \cot\theta/2$ (1st kind, Goldstein 1950) which gives the transformation $x = \sqrt{2\mathcal{J}}\cos\theta, y = \sqrt{2\mathcal{J}}\sin\theta$. The new Hamiltonian is then:

$$\bar{H}'' = \frac{1}{8}G(x^2 + y^2)^2 - \frac{1}{2}\left[G\mathcal{J}_o - \dot{\Omega}_b(t - t_{res})\right](x^2 + y^2) - \frac{F}{\sqrt{2}}f(t)x. \tag{A2}$$

The function $f(t)$ is chosen to slowly and smoothly vary from 0 at $t = T_{min}$ to 1 for $t > T_{max}$. To compute the response of an orbit to the adiabatically growing potential, $\mathcal{J}_o$ is first computed from the given initial values of I_1 and I_2. The equations of motion are then integrated until $t > T_{max}$ and the new value J_s is computed from $\mathcal{J}$. Since we are only interested in the ensemble average, it is sufficient to tabulate $\mathcal{J}(\mathcal{J}_o, \theta)$ for given values of G and F. For a logarithmic background potential, the quantity $G = -2(2 \pm \sqrt{2})/R_{res}^2$ for OLR and ILR at R_{res} respectively. In practice, I choose

$$f(t) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{t - T_{max}}{\tau}\right)\right] \tag{A3}$$

and increase τ with appropriate choice of T_{min} and T_{max} to verify that the evolution is adiabatic.

Away from the resonance the solutions will scale with F. However, near the resonance, measure of orbits that cross the critical trajectory will depend on F. In fact, as long as

the orbit has not become critical or captured into libration then the response should scale linearly and this has been verified numerically. Here, we are explicitly interested in the nonlinear effects of the resonances which requires us to solve the model for each value of F and G of interest.

B.   Evolution Through Resonance: technical discussion

Since the orbit morphology near resonance is governed by equation (A1) or (A2), the dynamical consequences by be inferred from one-dimensional solutions directly. To reiterate, the appearance of the orbit changes with the evolving bar but the actions are adiabatically invariant except near critical trajectories. As the trajectory passes through the critical trajectory, the action may change discontinuously by a finite amount but then remain invariant thereafter as in the case of the pendulum. The critical trajectory, divides the phase plane into distinct regions. The probability of a particular trajectory entering one or the other depends on the phase at the critical trajectory. Hamiltonian functions of the form equation (A1) have been studied in some detail (e.g. Henrard & Lemaitre 1983, Lichtenberg & Lieberman 1983), and here we give an example of these graphical methods for Model I (cf. Tables 1 and 2).

Figure 21 shows some possible trajectories in this Hamiltonian where the x and y axes are $\sqrt{2\mathcal{J}}\cos\theta$ and $\sqrt{2\mathcal{J}}\sin\theta$ respectively. Physically, the radius $\sqrt{2\mathcal{J}}$ is proportional to the epicyclic radius and the phase angle θ describes the rotation or libration about the resonance. The critical points (equilibria) have $\theta = 0, \pi$. In the case of Figure 21, the only (stable) critical point has $\sqrt{2\mathcal{J}} = -0.156$ and no other critical trajectory. The corresponding trajectory is a closed oval in the frame rotating with the pattern and often is called a *periodic orbit.* Around this critical point the orbits are in libration; the angle of the apocenter is confined to a range smaller than $[0, 2\pi]$. For sufficiently large $\sqrt{2\mathcal{J}}$, the trajectory includes the origin and the trajectory circulates: the angle of the apocenter precesses through 2π radians. As the strength of the bar grows the critical point shifts further to the left and the measure of librating trajectories grow. A librating trajectory has its phase at apocenter "trapped" opposite to the position angle of the bar in this case.

In the case of Figure 22, the guiding center is inside the resonance initially and there are three critical points, one at the center of the librating orbit (stable), one at the center of the tiny loop (stable), and one at the × (unstable). The critical trajectory, which terminates at the unstable point, divides the phase space into three regions. As the potential grows, a trajectory may switch region, ending up either in rotation or trapped into libration. Here,

the trapping causes a jump in epicyclic amplitude. and changes the apparent line-of-sight velocity signature as a function of viewpoint. Note that the fraction of librating orbits is not symmetric on either side of the resonance due to the topological differences. In addition, trajectories in the small loop will librate with small amplitude along the position angle of the bar while trajectories like the kidney-shaped one shown exhibit large amplitude librations orthogonal to the position angle of the bar. In the case of models with changing pattern speeds, the phase space topologies are similar to Figures 21–22 although the size of the librating regions, for example, may change. Diagrams of this sort allow quantitative comparison of evolutionary scenarios without constructing observable quantities.

Fig. 1.— Librating orbit near ILR after bar evolution; the strength and pattern speed are constant. The quadrupole force is 2% of the axisymmetric force at $R = 1$. The resonance is at $R = 1.3$ and the initial guiding center is at $R = 1.35$.

Fig. 2.— Graph of the orbit described in Fig. 1 as seen from the bar frame. The first (last) 30 time units are shown as a solid (dashed) line. The bar is oriented along the x-axis.

Fig. 3.— Graph of the orbit as seen from the inertial frame. The bar rotates in this frame at frequency Ω_b.

Fig. 4.— As in Fig. 1 but for a rotating orbit with initial guiding center is at $R = 1.0$.

Fig. 5.— As in Fig. 1 but for a rotating orbit with initial guiding center is at $R = 2.5$, outside the resonance. Note that the run of ϕ_{apo} is retrograde with time in this case (outside resonance) and prograde in Fig. 4 (inside resonance).

Fig. 6.— The line-of-sight radial velocity for different guiding center radii. The quantity V_r is relative to the galaxy's inertial frame (not the LSR!) and the angle ϕ describes the angle between the bar's position and the line of sight. Each curve represents the mean velocity for an ensemble of orbits with a Schwarzschild velocity distribution with radial dispersion of $0.1V_{rot}$ and initial guiding center trajectory R. The resonant radius is $R_{res} = 1.3$ (coresponding to $10.4\,\mathrm{kpc}$ if $R_{LSR} = 8\,\mathrm{kpc}$).

Fig. 7.— As in Fig. 6 but shows large guiding center radii ($R > R_{res}$).

Fig. 8.— As in Fig. 6 but shows guiding center radii near the resonance ($R = 1.3$).

Fig. 9.— Shows the relative contribution to V_r for $R = 0.8$ from orbits which have received a kick in action (critical-crossing) and from those in libration. The total is shown as a heavy solid line.

Fig. 10.— As in Fig. 9 but for $R = 1.4$. Note that near the resonance (at $R \approx 1.3$), nearly all trajectories are homoclinic crossing.

Fig. 11.— Mean trajectories for Model A and initial guiding center radii as labeled. Ten different initial phases are shown for each guiding center. For all but $R = 1.3$, the curves are independent of initial phase and are coincident. The bar position angle is $0°$.

Fig. 12.— As in Fig. 12 but for Model B. For all cases but $R = 1.7$, the curves are no longer phase independent. In the case of initial guiding center $R = 1.1$, the trapping depends on initial phase (the orbits elongated along $\phi = 90°$ and $270°$ are trapped.)

Fig. 13.— Line-of-sight radial velocity V_r, and dispersion σ_r for Model A at $\phi = -45°$. The symbols show evaluations of V_r and may be thought of as a cut at ϕ in Figures 6–8; the connecting lines are provided to guide the eye.

Fig. 14.— As in Fig. 13 but for Model B. The dip at $R = 1.1$ is caused by the superposition of both trapped and untrapped orbits.

Fig. 15.— As in the lower panel of Fig. 13 but showing the individual contributions to V_r from post-critical orbits (solid) and librating orbits (dashed). The total (dotted) is shown for comparison. The resonance is at $R = 1.3$.

Fig. 16.— Shows the fraction of orbits that have crossed the critical trajectory (solid) or have been trapped into libration (dashed) for the ensemble shown in Fig. 15.

Fig. 17.— As in Fig. 13 but for Model I.

Fig. 18.— As in Fig. 13 but for Model J.

Fig. 19.— Comparison of Model A scaled to $V_{rot} = 220\,\mathrm{km\,s^{-1}}$ and $R_{LSR} = 8\,\mathrm{kpc}$ compared with Lewis and Freeman's K-giant and Metzger and Schechter's carbon star velocity data.

Fig. 20.— Lewis and Freeman's K-giant velocity data compared with Model J.

Fig. 21.— Shows trajectories for a family of orbits with guiding center radius corresponding to 0.7 if the OLR radius is 0.64. The resonance location is fixed and the bar strength $|\epsilon|$ slowly grows to its maximum of 0.2. The quantity $\sqrt{2\mathcal{J}}$ is proportional to the epicylic radius a so small radii in the figure correspond to nearly circular orbits. There is only one critical point and no critical trajectory. If the bar were very weak, the trajectories would have constant $\mathcal{J}$ and be circles in this plot.

Fig. 22.— As in Fig. 21 but with guiding center radius inside the OLR (corresponding to 0.475). In this case there are three critical points (one at the center of each small loop and one at the ×) and a critical trajectory.

Fig. 23.— Location of critical points for the models described in Figs. 21 and 22 in units of the solar radius, $R = 1$. The vertical dotted line shows the point at which the bifurcation occurs. The two new critical points appear at the point marked by the open dot. The upper locus is (dashed) is unstable.