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Spin half in classical general relativity

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Abstract.
It is shown that models of elementary particles in classical general relativity (geons) will naturally have the transformation properties of a spinor if the spacetime manifold is not time orientable. From a purely pragmatic interpretation of quantum theory this explains why spinor fields are needed to represent particles. The models are based entirely on classical general relativity and are motivated by the suggestion that the lack of a time-orientation could be the origin of quantum phenomena.

1. Introduction

Spin half is normally regarded as a non-classical property peculiar to quantum theory. Objects with spin-half are represented by spinor fields. The most distinctive, indeed the characteristic, property of a spinor is the way it transforms under a rotation. In particular the fact that a rotation of space coordinates by an angle of 2π is not an identity operation whereas a rotation by 4π is an identity operation. This work offers an explanation of why spinor fields are needed to represent fermions. The explanation is based entirely upon classical general relativity.

Many authors have explored the relation between general relativity and quantum theory, some attempt to apply quantum theory to curved spacetime and non-trivial spacetime topologies by adding quantum fields to the underlying manifold. Such an approach with spin-half particles requires the manifold to be endowed with a spinor field; Geroch [1] describes the constraints on the topology and orientability of spacetime in order to admit a spinor field. This work does not attempt to add or construct a spinor field on a curved spacetime. Instead it attempts to explain why we need to use spinor fields (on flat spacetime) to model some particles. The constraints imposed by Geroch do not apply.

In quantum theory, a wavefunction, or quantum field, is used to calculate the probabilistic results of experiments which detect a particle. This is certainly true. To go beyond this pragmatic view requires an interpretation of quantum theory (see for example Ballentine [2] or Isham [3] for a discussion). To regard the wavefunction itself as being the particle has substantial conceptual and practical problems. But if the particle and wavefunction are indeed different, then a model or description of an elementary particle is required. One way of modelling an elementary particle is as a region of space-time with non-trivial topology - a geon [4, 5] which can be described using classical general relativity. The wavefunction would still be the tool for calculating the results of experiments but its role would be that of a probability function. It is important to keep in mind this distinction between the particle itself and the quantum fields used to calculate the results of experiments.

The complex fields used to represent particles need to have, amongst other attributes, the same transformation properties as the particles that they represent. Particles with zero spin and spherical symmetry are represented by a complex scalar field. Spin-1 particles with the transformation properties of a vector are represented by vector fields. Note that vectors and spinors are defined by their transformation properties under rotations. An elementary particle (or indeed any object) may have the transformation properties of a vector - a pencil is a good approximation. The vector would describe its orientation, and a rotation of either the object or the coordinate system would be correctly modelled by the vector. By contrast the quantum mechanical wavefunction defines a vector (or scalar) at each point in space (a vector field) and the components would be complex numbers - this is what quantum theory requires to compute the probabilistic results of experiments. In a similar way there is a distinction

between a single object having the transformation properties of a spinor and a spinor field which is used by quantum mechanics to represent such an object.

Some authors have produced artificial models, parts of which, transform as a spinor see for example the cube within a cube described by Misner, Thorne and Wheeler[6] or the tethered rocks of Hartung [7]. This paper shows how an object with the transformation properties of a spinor can be constructed within classical general relativity. The work requires that the topology of spacetime within a fermion is non-trivial and is not time orientable. It would then follow that quantum theory in flat spacetime would need to model the dynamics of such a particle using spinor fields.

This paper relates to models of elementary particles as regions of spacetime with non-trivial topology. The term geon was used by Misner and Wheeler to describe particle-like structures with non-trivial spatial topology [5]. The idea was later extended by Hadley to regions which also had non-trivial causal structure called 4-geons [8]. The investigation of manifolds which are not time orientable is motivated by the suggestion that models of elementary particles based on such manifolds would give rise to quantum mechanical effects [8] and that a reversal of the time orientation is related both to topology change [9] and the measurement process in quantum theory. Therefore, the manifolds being examined have real significance quite apart from their transformation under rotations. The simplest example was described in the paper by Diemer and Hadley [10] and shown to display net charge from the source-free Maxwell equations. Hadley's gravitational explanation for quantum theory provides additional motivation for this work, but the results of this paper are interesting in their own right and do not depend upon the earlier work. The main result is a fascinating consequence of non-trivial causal structure in general relativity.

Here it is shown that non time orientable manifolds present obstructions which prevent a physical (time parameterised) rotation being extended throughout the manifold in a non-trivial way. There must be some exempt points. This topological obstruction will, in general, prevent a 2π rotation (and any well behaved extension of it) from being an identity operation. This property of spinors is intimately related to the fact that the group of rotations, SO(3), is not simply connected. A result which can be seen in the famous scissors model of Dirac and the cube within a cube described by Misner, Thorne and Wheeler [6]. A description is given in references [11, 12] and [13] contains a proof.

Friedman and Sorkin [14] have previously considered other topological obstructions which prevent a rotational transformation being extended from the region of flat spacetime throughout the manifold (ie into the interior of the particle). On some 3-manifolds (characterised by Hendricks [15]) it is not possible to define a rotational vector field. In the context of quantum gravity, Friedman and Sorkin impose a wavefunction on such manifolds and consider the effects of a rotation on the asymptotically flat regions. They conclude that a wavefunction as required by a theory of quantum gravity could have a spinorial character. By contrast; this work invokes a different mechanism, it applies to manifolds which have a physical interest for other reasons. Furthermore,

these results arise from classical general relativity, they do not invoke any quantum theory of gravity, but they may help to explain quantum phenomena.

2. Line bundle models of spacetime

Although the analysis requires a clear distinction between time and space directions, the results can be obtained by applying topological arguments to a non-trivial line bundle over a 3-manifold - it is not necessary to define a metric explicitly.

A causal spacetime is modelled by a trivial line-bundle E over a 3-manifold, $\mathcal{M}$, $E \xrightarrow{\pi} \mathcal{M}$. The base manifold, $\mathcal{M}$, corresponds to space at $t = 0$. The evolution is given by a diffeomorphism, Q from $\mathcal{M} \times \mathbb{R}$ to the line bundle E such that $Q(\cdot, 0)$ is the zero section and $(Q(\cdot, t))(\mathcal{M})$ is the space at a time t. Under a transformation a fixed point in space would be mapped to a point on its own fibre. A non-trivial line-bundle over a 3-manifold is a convenient topological model of those non-time orientable spacetimes with the property that time is not orientable on any spacelike slice; the manifold described by Diemer and Hadley is an example.

A base 3-manifold endowed with a non-trivial line bundle would correspond to a spacetime that is not time orientable. If the line bundle is non-trivial then every section has some zero points. In general, the space of zero points, X, is a 2-dimensional surface. It is this topological property of sections of a non-trivial line bundle which gives rise to the results in this paper. This can be illustrated by considering sections of a möbius strip. The base manifold is a circle, S^1, which corresponds to a circular space - the time direction is everywhere normal to the base circle. Parts of the base manifold can be moved within the Möbius band, but an attempt to shift the entire circle up or down is not possible - there will be at least one point where the shifted circle crosses the base circle.

3. Models of particles

A particle in spacetime is modelled as an asymptotically flat 3-manifold with non-trivial topology and endowed with a line bundle where each fibre corresponds to the timeline of a point in space. A non-time-orientable manifold corresponds to a non-trivial line bundle.

The simplest example due to Diemer and Hadley[10] is RP3 with a point removed and endowed with a non-trivial line-bundle. The point removed corresponds to spatial infinity and a sphere S^2 enclosing the point corresponds to a sphere enclosing the particle. With the addition of a Faraday electromagnetic field 2-form, this construction can model a spherically symmetric electric monopole.

An alternative construction of the same spacetime manifold is as follows: a ball is removed from space (a cylinder from spacetime) and then joined up by identifying opposite points, but reversing the time direction. Externally this is spherically symmetric. The region of non-trivial topology can be surrounded by a sphere, S^2,

such that spacetime is topologically trivial and asymptotically flat outside the sphere.

4. Definitions of rotations

Rotations are well-defined in $\mathbb{R}^3$ as elements of SO(3). Every rotation has an axis, ζ, and is an element of the one parameter subgroup of rotations about this fixed axis. Rotations, $\rho_\zeta(\theta)$, by an angle θ about an axis ζ are smooth maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ which satisfy the following conditions:

Definition 4.1 (1-parameter subgroup conditions)

(i) $\rho_\zeta(0) = \mathrm{id}$
(ii) $\forall \theta, \phi \in \mathbb{R} \; \rho_\zeta(\theta + \phi) = \rho_\zeta(\theta)\rho_\zeta(\phi)$
(iii) ρ_ζ *is a smooth function*

In the following the subscript ζ will be omitted for clarity.

We need to extend the usual definition so that it applies to manifolds with a non-trivial topology and to manifolds which are not time-orientable. For physical reasons we restrict attention to asymptotically flat manifolds, and in the asymptotically flat region we require correspondence with rotations defined in $\mathbb{R}^3$.

Definition 4.2 (A Rotation on an Asymptotically Flat Manifold) *A rotation, $R(\theta)$, by an angle θ, on an Asymptotically Flat Manifold, $\mathcal{M}$, is an element of a 1-parameter subgroup of diffeomorphisms from $\mathcal{M}$ to itself which satisfies the conditions 4.1 above and $\forall \theta \in \mathbb{R}$ converges to a rotation, $\rho_\zeta(\theta)$, in $\mathbb{R}^3$ for large x:*

$$\mathrm{dist}(\rho_\zeta(\theta)x, R(\theta)x) \to 0 \qquad \text{as} \qquad |x| \to \infty \tag{1}$$

The definition is very wide, the form of the transformation is only tightly prescribed in the asymptotic regions. In particular $R(\theta)$ is not an isometry, it is any smooth extension of the familiar form of a rotation.

Remark 4.3 *A rotation always exists - consider two concentric spheres S_a^2 and S_b^2 where S_b^2 encloses S_a^2 which in turn encloses the region of non-trivial topology:*

$$R(\theta) = \begin{cases} \mathrm{id} = R(0) & \text{inside } S_a^2 \\ \rho(\theta) & \text{outside } S_b^2 \\ \rho(f(r)\theta) & \text{in between} \end{cases} \tag{2}$$

where $f(r)$ is any smooth function of r such that $f(r) = 0$ when $r = r_a$ and $f(r) = 1$ when $r = r_b$.

Remark 4.4 *Rotations are not uniquely defined by definition 4.2. Any smooth function $f(r)$ with the properties above can be used to construct a rotation $R(\theta)$ which satisfies 4.2 and has the same asymptotic form.*

A rotation is an element of a 1-parameter subgroup of diffeomorphisms acting on $\mathcal{M}$. The rotation and the subgroup define a path in $\mathcal{M}$ from x to $R(\theta)x$: $\gamma_\theta(\lambda) = \{R(\lambda\theta)x | x \in \mathcal{M}, \lambda \in [0,1]\}$, This definition has no notion of time. The parameter, λ has no physical significance, the rotation is not an operation that can be physically implemented. Of specific concern in this paper is the notion of a physical rotation - a rotation which defines a worldline parameterised by time. This gives an operation that is physically relevant, an object or a 3-manifold can be transformed from an initial state at $t = 0$ to a rotated state at a later time $t = 1$:

Definition 4.5 (Physical Rotation on a time-orientable manifold) *A physical rotation, $\mathcal{R}(\theta)$ on a time-orientable manifold is a map from $\mathcal{M}$, identified with the zero section, to the image of the unit section of the line bundle:*

$$\mathcal{R}(\theta)(x, 0) \rightarrow (R(\theta)x, 1) \tag{3}$$

where $\mathcal{R}(\theta)$ also satisfies the 1-parameter subgroup conditions 4.1. The projection of a physical rotation onto the base manifold gives a rotation of an asymptotically flat manifold defined earlier:

$$\pi \circ \mathcal{R}(\theta) = R(\theta) \tag{4}$$

A physical rotation and its subgroup define a curve in the line bundle E from $(x, 0)$ to $(R(\theta)x, 1)$, $\chi_\theta(t) = \{(R(t\theta)x, t) | x \in \mathcal{M}, t \in [0,1]\}$, which is the worldline of each point $x \in \mathcal{M}$. The projection of the worldline is equal to the path in $\mathcal{M}$ defined earlier: $\pi \circ \chi_\theta(t) = \gamma_\theta(t)$. Although a physical rotation is conceptually different, there is a one to one correspondence between physical rotations and rotations of an asymptotically flat manifold defined earlier. Similarly the curves, $\gamma_\theta(\lambda)$, and world lines, $\chi_\theta(t)$ are in one to one correspondence.

However definition 4.5 cannot be applied to a spacetime which is not time orientable (a non-trivial line bundle) because it assumes the existence of a nowhere vanishing section (the unit section) which implies that a global trivialisation of the line bundle exists. We refine definition 4.5 so that it applies to any asymptotically flat manifold including those which are not time orientable. The definition requires the replacement on the unit section of the line bundle by another section ϕ :

Definition 4.6 (Physical Rotation) *A physical rotation, $\mathcal{R}(\theta)$ is a map from $\mathcal{M}$, identified with the zero section, to the image of a section ϕ of the line bundle:*

$$\mathcal{R}(\theta)(x, 0) \quad \rightarrow (R(\theta)x, \phi(x)) \tag{5}$$
$$\text{when } \phi(x) = 0 \quad R(\theta)x = x \ \forall\theta \tag{6}$$
$$\text{as } |x| \rightarrow \infty \quad \phi(x) \rightarrow 1 \tag{7}$$

Clearly a physical rotation exists and is not unique. Equation 4 is still valid but there is no longer a one to one correspondence between physical rotations and rotations of an asymptotically flat manifold due to equation 6.

This definition is obviously of physical relevance. A real rotation in the laboratory is well-defined in an almost flat region of spacetime. The internal structure of a particle is unknown and the structure of spacetime within a particle is also unknown. This definition places few constraints on the internal structure of a particle and accommodates both a spacetime with non-trivial topology as well as the existence of extraneous fields on a flat spacetime.

In general a *Physical Rotation* is not an isometry, it does not preserve distances except in the asymptotic region. It is defined as a general extension of a rotation from the asymptotic region to the whole manifold. An isometric extension would not normally be possible. An analogy would be a large rubber disk, fixed in the middle with two or more nails. The outer rim could be rotated, near the outer rim the rotation would be an isometry. The transformation created by rotating the rim, can be extended to the whole disk, but it is not an isometry throughout the disk.

This construction agrees with the normal definition of a rotation in the asymptotically flat region, it can be extended to the whole of a time orientable manifold with $\phi(x) = 1 \forall x$. It can also be extended to the whole manifold even when it is not time orientable, but for some points $\phi(x) = 0$. Definition 4.6 and equation 6 distinguish two types of fixed points under a rotation.

Definition 4.7 (fixed point) *x is a fixed point of the rotation if* $\mathcal{R}(\theta)(x, 0) \rightarrow (x, \phi(x))$

These points correspond to the axes of a rotation. But there are also exempt points:

Definition 4.8 (exempt point) *x is an exempt point of the rotation if* $\mathcal{R}(\theta)(x, 0) \rightarrow (x, 0)$

We denote by X the set of all exempt points. The name exempt conforms to the definition of Hartung[7], where he considers the rotation of a tethered object, the object rotates but the other end of the tether is an exempt point in the sense of being a point to which the rotation is not applied.

As already pointed out, rotations are not uniquely specified by this definition. The rotation is tightly defined in the asymptotic region where it maps $(x, 0) \rightarrow (R(\theta)x, \phi(x)) \approx (\rho(x), 1)$, but there are many ways of extending the section of the line bundle and many ways of extending the rotation transformation. Consequently the exempt points are not determined by the rotation in the asymptotic region, but depend upon the way in which the section is extended. On a manifold which is not time-orientable, time is not orientable around at least one class of non-contractable closed curves in $\mathcal{M}$ and therefore every such curve must have at least one exempt point. Consequently, the space of exempt points is at least two dimensional (otherwise a small distortion of the curve could be made which would not have an exempt point) and the proof in [13] applies.

5. Spin-half

The following results apply to the specific particle model described above and also to the more general case where the particle is a region of spacetime with non-trivial topology surrounded by a world tube $S^2 \times \mathbb{R}$ such that any region spanning the S^2 does not admit a time orientation. This is sufficient to ensure that any section of the line bundle has exempt points.

The fact that the exempt points form a 2-dimensional surface ensures that this model is analogous to the tethered rocks of Hartung [7] and the scissors trick of Dirac [11]. It follows that a 2π rotation cannot be an identity operation but 4π rotation can be:

$$\forall \mathcal{R}(0) \qquad (x,0) \rightarrow (x, \phi(x)) \tag{8}$$

$$\forall \mathcal{R}(2\pi) \; \exists x \text{ st}: \; (x,0) \rightarrow (R(2\pi)x, \phi(x)) \neq (x, \phi(x)) \tag{9}$$

$$\exists \mathcal{R}(4\pi) \text{ st}: \; \forall x \; (x,0) \rightarrow (R(4\pi)x, \phi(x)) = (x, \phi(x)) \tag{10}$$

It follows that $\mathcal{R}(0)$, the identity transformation, is diffeomorphic to $\mathcal{R}(4\pi)$ relative to $\{X, S^2\}$ and that $\mathcal{R}(2\pi)$ is not. This is also a necessary, but not sufficient condition, for $\mathcal{R}(4\pi)$ to be an isometry and for $\mathcal{R}(2\pi)$ not to be an isometry. A 2π rotation could be an isometry if the particle had an internal symmetry - which is not possessed by the simple monopole example.

This gives a physical explanation of why it is appropriate to model a fermion as a tethered object. This model, with non-trivial causal structure, was originally constructed to explain how the effects of quantum theory could arise within classical general relativity. The spin half effects (and also the appearance of electric charge [10]) are simply a consequence of using general relativity to explain quantum effects.

6. Conclusion

On the relation to quantum theory, we take a pragmatic view of quantum theory *that it is a scheme for predicting the probabilistic distribution of the outcomes of measurements.* It is clear that a particle must be represented by a mathematical object with the appropriate transformation properties under rotations. The particle modelled here has the transformation properties of a spinor, so any field theory which attempts to model the behaviour under rotations will necessarily use spinor fields.

This approach contrasts with that of Sorkin, who considered a wavefunction imposed upon a non-trivial manifold. That would be meaningless when the constructions given can display quantum mechanical effects by themselves. Here the wavefunction is defined in $\mathbb{R}^4$ and is a means of mapping the evolution of non-trivial causal structure onto a conventional spacetime.

The previous sections modelled a single free particle. Hartung extends the idea of a tethered rock to two or more tethered rocks and concludes that an exchange of particles is equivalent to a 2π rotation of one of the particles. However his analysis assumes that

two tethered rocks is equivalent to a pair of rocks tethered to each other. With the model presented here, such an equivalence is not apparent.

In general relativity, the *non-gravitational* energy, momentum and angular momentum can be derived from the energy momentum tensor which is defined at every point of spacetime. However the contribution of the gravitational field itself to the total energy, momentum and angular momentum cannot be defined locally and hence the total values cannot be defined locally (see for example [6]). Well defined global values exist in asymptotically flat manifolds such as the one described here. This example has a total angular momentum of zero. The asymptotic form of the metric is independent of the time orientability. This must be the case because a non-zero value would give a well-defined spin direction independently of measurement which is contrary both to quantum theory and experiment. Indeed a zero angular momentum for an unpolarised electron is just what one would expect from the Bohm interpretation of quantum mechanics.

In conclusion, the transformation properties of fermions have been modelled using classical general relativity. A 4-geon model of a particle, originally constructed to explain quantum effects, is shown to exhibit spin-half transformation properties. The well-known fact that fermions transform like tethered objects is therefore explained with an established classical theory - general relativity. It is particularly significant that the explanation arises naturally, indeed almost inevitably, as a consequence of the proposed gravitational explanation of quantum mechanics [8, 9].

Acknowledgments

References

[1] Geroch R 1968 *J. Math. Phys.* **9** 1739–1744
[2] Ballentine L E 1989 *Quantum Mechanics.* Prentice Hall, New York
[3] Isham C J 1995 *Lectures on Quantum Theory - Mathematical and Structural Foundations.* Imperial College Press, London.
[4] Einstein A and Rosen N 1935 *Phys. Rev.* **48** 73–77
[5] Misner C W and Wheeler J A 1957 *Ann. Phys.* **2** 525–603
[6] Misner C W, Thorne K S, and Wheeler J A 1973 *Gravitation.* W H Freedman and Sons
[7] Hartung R W 1979 *Am. J. Phys.* **47** 900–910
[8] Hadley M J 1997 *Found. Phys. Lett.* **10** 43–60
[9] Hadley M J 1999 *Int. J. Theor. Phys.* **38** 1481–1492
[10] Diemer T and Hadley M J 1999 *Class. Quantum Grav.* **16** 3567–3577
[11] Feynman R P and Weinberg S 1987 *Elementary Particles and the Laws of Physics.* Cambridge University Press
[12] Penrose R and Rindler W 1984 *Spinors and Space-time* Cambridge University Press
[13] Newman M H A 1942 *J. Lond. Math. Soc.* **17** 173–177
[14] Friedman J L and Sorkin R D 1982 *Gen. Rel. Grav.* **14** 615–620
[15] Hendriks H 1977 *Bull. Soc. Math. de France Mem* **53** 81–96