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SEMIINFINITE FLAGS. II. LOCAL AND GLOBAL INTERSECTION COHOMOLOGY OF QUASIMAPS' SPACES.

BORIS FEIGIN, MICHAEL FINKELBERG, ALEXANDER KUZNETSOV, AND IVAN MIRKOVIĆ

To Dmitri Borisovich Fuchs on the occasion of his 60th birthday

1. INTRODUCTION

1.1. This paper is a sequel to [FM]. We will make a free use of notations, conventions and results of *loc. cit.*

One of the main results of the present work is a computation of local $\mathcal{IC}$ stalks of the Schubert strata closures in the spaces $\mathcal{Z}^\alpha$. We prove that the generating functions of these stalks are given by the *generic* (or *periodic) Kazhdan-Lusztig polynomials*, see the Theorem 6.9. We understand that this result was known to G.Lusztig for a long time, cf. [L3] §11. His proof was never published though, and as far as we understand, it differs from ours: for example we never managed to find a direct geometric proof of the key property [L1] 11.1.(iv) of Lusztig's R-polynomials.

Our proof uses the standard convolution technique. The only nonstandard feature is the check of pointwise purity of the $\mathcal{IC}$ sheaves involved (Theorem 6.3). Usually one proceeds by finding global transversal slices. We were not able to find the good slices, and instead reduced the proof to the purity properties of $\mathcal{IC}$ sheaves on the affine Grassmannian.

1.2. The central result of this work is the geometric construction of the universal enveloping algebra $U(\mathfrak{n}_+^L)$ of the nilpotent subalgebra of the Langlands dual Lie algebra $\mathfrak{g}^L$ (Theorem 4.10). This construction occupies the section 2. The geometric incarnation $\mathcal{A}$ of $U(\mathfrak{n}_+^L)$ naturally acts on the global intersection cohomology $\oplus_{\alpha \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$ of all Quasimaps' spaces, and this action extends to the geometrically defined $\mathfrak{g}^L$-action (section 4).

In case $\mathbf{G} = SL_n$ such action was constructed in [FK], and the present paper grew out of attempts to generalize the results of *loc. cit.* to the case of arbitrary simple $\mathbf{G}$. For $\mathfrak{g} = \mathfrak{sl}_n$ the geometric construction of $U(\mathfrak{n}_+^L)$ given in *loc. cit.* used the Laumon resolution $\mathcal{Q}_L^\alpha$ of the Quasimaps' spaces $\mathcal{Q}^\alpha$, and provided $U(\mathfrak{n}_+^L)$ with the geometrically defined Poincaré-Birkhoff-Witt basis. The present construction also provides $U(\mathfrak{n}_+^L)$ with a special basis numbered by the irreducible components of the semiinfinite orbits' intersections in the affine Grassmannian. We want to stress that the two bases are *different*: the latter one looks more like a canonical basis of [L4].

Another advantage of the new construction of $U(\mathfrak{n}_+^L)$ is its local nature. It allows one to define a $\mathfrak{g}^L$-action on the global cohomology of Quasimaps' spaces with coefficients in sheaves more general than just $\mathcal{IC}(\mathcal{Q}^\alpha)$. Namely, given a perverse sheaf $\mathcal{F} \in \mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$ on the affine Grassmannian we defined in [FM] its *convolution* $\mathbf{c}_{\mathcal{Q}}^\alpha(\mathcal{F})$ — a perverse sheaf on $\mathcal{Q}^{\eta+\alpha}$. In section 7 we construct the $\mathfrak{g}^L$-action on $\oplus_{\alpha \in Y} H^\bullet(\mathcal{Q}^{\eta+\alpha}, \mathbf{c}_{\mathcal{Q}}^\alpha(\mathcal{F}))$. In case $\mathcal{F}$ is $\mathbf{G}[[z]]$-equivariant we conjecture that the resulting $\mathfrak{g}^L$-module is *tilting*. As in [FK], this conjecture is motivated by an analogy with the semiinfinite cohomology of quantum groups.

Recall that for a $\mathbf{G}[[z]]$-equivariant sheaf $\mathcal{F}$ on the affine Grassmannian the action of $\mathfrak{g}^L$ on its global cohomology was constructed in [MV]. The relation between the various $\mathfrak{g}^L$-actions on global cohomology will be discussed in a separate paper.

1.3. Let us list the other points of interest in this paper.

In section 3 we compute the stalks of $\mathcal{IC}(\mathcal{Z}^\alpha)$ (Theorem 3.1 and Corollary 3.1.1). Their generating functions are expressed in terms of Lusztig's q-analogue $\mathcal{K}^\alpha(t)$ of Kostant partition function. In case $\mathbf{G} = SL_n$ this result was proved in [Ku] using the Laumon resolution of the Quasimaps' space $\mathcal{Q}^\alpha$. The proof in general case uses the *Beilinson-Drinfeld* incarnation of $\mathcal{Z}^\alpha$ (see [FM] §6). Formally, Theorem 3.1 is just a particular case of the Theorem 6.9 computing $\mathcal{IC}$-stalks of the general Schubert strata closures in $\mathcal{Z}^\alpha$. But the argument goes the other way around: we deduce 6.9 as a rather formal corollary of 3.1. It is well known that the generating function of $\mathcal{IC}$ stalks of $\mathbf{G}[[z]]$-orbits' closures in the affine Grassmannian is also given by $\mathcal{K}^\alpha(t)$ in the stable range (see [L2]). This coincidence is explained in section 3: though the local singularities of $\mathcal{Z}^\alpha$ and $\overline{\mathcal{G}}_\eta$ *are different*, their "skeleta" (intersections of semiinfinite orbits in $\overline{\mathcal{G}}_\eta$, and the *central fiber* in $\mathcal{Z}^\alpha$) *are the same*.

In section 4 we prove that $\oplus_{\alpha \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$ carries a natural pure Tate Hodge structure and compute its generating function (Theorems 4.1, 4.1.1).

In section 5 we prove that the adjacency order on the set of Schubert strata in $\mathcal{Z}$ is equivalent to Lusztig's order (see [L1]) on the set of *alcoves*. The proof uses a map $\pi : \mathcal{Q}^\alpha_K \longrightarrow \mathcal{Q}^\alpha$ from Kontsevich's space of *stable maps* to Drinfeld's Quasimaps' space. This map is constructed in the Appendix. The construction is just an application of Givental's "Main Lemma" (see [G]). His proof of the "Main Lemma" has not satisfied all of its readers, so we include the complete proof into the Appendix.

In section 7 we collect various conjectures on the structure of $\mathfrak{g}^L$-modules of geometric origin. For a mysterious reason these conjectures involve tilting modules — either over $\mathfrak{g}^L$ itself, or over the related quantum group. The conjecture 7.5 was recently proved in [FKM]. The conjecture 7.8 may be viewed as a description of an "automorphic sheaf" on the moduli space of $\mathbf{G}$-torsors corresponding to the trivial $\mathbf{G}^L$-local system on $\mathbb{P}^1$. Finally, in 7.7 we propose a direct geometric construction of the $\mathfrak{n}^L_\pm$-action on $H^\bullet(\mathcal{G}, \mathcal{F})$ for a $\mathbf{G}[[z]]$-equivariant perverse sheaf $\mathcal{F}$ on the affine Grassmannian $\mathcal{G}$. Surprisingly enough, no direct construction of $\mathfrak{n}^L_\pm$-action has been found so far (cf. [MV] for the direct construction of the action of the dual Cartan $\mathfrak{h}^L \subset \mathfrak{g}^L$).

1.4. We are deeply grateful to V.Lunts and L.Positselsky whose explanations helped us at the moments of despair. We are very much obliged to R.Bezrukavnikov who found a serious gap in the original version of geometric construction of $U(\mathfrak{n}^L_+)$. It is a pleasure to thank S.Arkhipov, D.Gaitsgory, and V.Ostrik for the inspiring discussions around the tilting conjectures in section 7. We are very much obliged to D.Gaitsgory for the careful reading of this paper, and for suggesting a lot of drastic simplifications of the arguments and proofs of the conjectures therein. They will all appear in his forthcoming publication.

2. LOCAL EXT-ALGEBRA $\mathcal{A}$

2.1. Throughout this paper C will denote a genus zero curve $\mathbb{P}^1$ with the marked points $0, \infty$. The complement $C - \infty$ is the affine line $\mathbb{A}^1$. The *twisting* map $\sigma_{\beta,\gamma} : \mathcal{Q}^\beta \times C^\gamma \longrightarrow \mathcal{Q}^{\beta+\gamma}$ defined in [FM] 3.4.1 restricts to the map $\varsigma_{\beta,\gamma} : \mathcal{Z}^\beta \times \mathbb{A}^\gamma \longrightarrow \mathcal{Z}^{\beta+\gamma}$. We denote its image by $\partial_\gamma \mathcal{Z}^{\beta+\gamma}$. The map $\varsigma_{\beta,\gamma} : \mathcal{Z}^\beta \times \mathbb{A}^\gamma \longrightarrow \partial_\gamma \mathcal{Z}^{\beta+\gamma}$ is finite. Moreover, in case $\beta = 0$, the space $\mathcal{Z}^\beta$ is just a point, and the map $\varsigma_{0,\gamma} : \mathbb{A}^\gamma \longrightarrow \partial_\gamma \mathcal{Z}^\gamma$ is an isomorphism. We will identify $\partial_\gamma \mathcal{Z}^\gamma$ with $\mathbb{A}^\gamma$ via this map.

Definition. $\mathcal{A}_\alpha := \mathrm{Ext}^\bullet_{\mathcal{Z}^\alpha}(\mathcal{IC}(\partial_\alpha \mathcal{Z}^\alpha), \mathcal{IC}(\mathcal{Z}^\alpha)); \ \mathcal{A} := \oplus_{\alpha \in \mathbb{N}[I]} \mathcal{A}_\alpha$.

Here the $\mathrm{Ext}^{\bullet}_{\mathcal{Z}^{\alpha}}$ is taken in the constructible derived category on $\mathcal{Z}^{\alpha}$. *A priori* $\mathcal{A}_{\alpha}$ is a graded vector space. In this section we will show that it is concentrated in the degree $|\alpha|$, and we will define a structure of cocommutative $\mathbb{N}[I]$-graded bialgebra on $\mathcal{A}$.

2.2. To unburden the notations we will denote the $\mathcal{IC}$ sheaf $\mathcal{IC}(\mathcal{Z}^{\alpha})$ by $\mathcal{IC}^{\alpha}$ (see [FM] 10.7.1). We denote the closed embedding of $\mathbb{A}^{\alpha} = \partial_{\alpha}\mathcal{Z}^{\alpha}$ into $\mathcal{Z}^{\alpha}$ by $\mathfrak{s}_{\alpha}$. Then $\mathcal{A}_{\alpha} = \mathrm{Ext}^{\bullet}_{\mathcal{Z}^{\alpha}}(\mathcal{IC}(\partial_{\alpha}\mathcal{Z}^{\alpha}), \mathcal{IC}^{\alpha}) = \mathrm{Ext}^{\bullet}_{\partial_{\alpha}\mathcal{Z}^{\alpha}}(\mathcal{IC}(\partial_{\alpha}\mathcal{Z}^{\alpha}), \mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha}) = H^{\bullet}(\mathbb{A}^{\alpha}, \mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha})[-|\alpha|]$ since $\mathcal{IC}(\partial_{\alpha}\mathcal{Z}^{\alpha}) = \underline{\mathbb{C}}[|\alpha|]$.

Theorem. a) $H^{\bullet}(\mathbb{A}^{\alpha}, \mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha})$ is concentrated in degree 0;

b) $\dim H^{0}(\mathbb{A}^{\alpha}, \mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha}) = \mathcal{K}(\alpha)$ where $\mathcal{K}(\alpha)$ stands for the Kostant partition function.

The proof of the Theorem will occupy the subsections 2.3–2.5.

2.3. We denote the closed embedding of the origin 0 into $\mathbb{A}^{\alpha}$ by ι_{α}. Since $\mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha}$ is constructible with respect to the diagonal stratification of $\mathbb{A}^{\alpha}$, we have the canonical isomorphism $H^{\bullet}(\mathbb{A}^{\alpha}, \mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha}) = \iota^{*}_{\alpha}\mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha}$.

Recall the projection $\pi_{\alpha} : \mathcal{Z}^{\alpha} \longrightarrow \mathbb{A}^{\alpha}$ (see [FM] 7.3). We denote the *central fiber* $\pi^{-1}_{\alpha}(0)$ by $\mathcal{F}^{\alpha}$, and we denote its closed embedding into $\mathcal{Z}^{\alpha}$ by ι_{α}.

The Cartan group $\mathbf{H}$ acts on $\mathcal{Z}^{\alpha}$ contracting it to the fixed point set $(\mathcal{Z}^{\alpha})^{\mathbf{H}} = \partial_{\alpha}\mathcal{Z}^{\alpha}$. The projection π_{α} is $\mathbf{H}$-equivariant. Hence the canonical morphism $\mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha} \longrightarrow \pi_{\alpha!}\mathcal{IC}^{\alpha}$ of sheaves on $\mathbb{A}^{\alpha}$ is an isomorphism. By the proper base change we have the canonical isomorphism $\iota^{*}_{\alpha}\pi_{\alpha!}\mathcal{IC}^{\alpha} = \pi_{\alpha!}\iota^{*}_{\alpha}\mathcal{IC}^{\alpha} = H^{\bullet}_{c}(\mathcal{F}^{\alpha}, \iota^{*}_{\alpha}\mathcal{IC}^{\alpha})$.

Combining all the above isomorphisms, we conclude that $H^{\bullet}(\mathbb{A}^{\alpha}, \mathfrak{s}^{!}_{\alpha}\mathcal{IC}^{\alpha}) = H^{\bullet}_{c}(\mathcal{F}^{\alpha}, \iota^{*}_{\alpha}\mathcal{IC}^{\alpha})$.

2.4. In this subsection we prove that $H^{\bullet}_{c}(\mathcal{F}^{\alpha}, \iota^{*}_{\alpha}\mathcal{IC}^{\alpha})$ is concentrated in nonpositive degrees. To this end we study the intersection of the *fine stratification* of $\mathcal{Z}^{\alpha}$ (see [FM] 8.4.1) with the central fiber $\mathcal{F}^{\alpha}$. Recall the description of the central fiber given in *loc. cit.* 6.4.1, 6.4.2 in terms of semiinfinite orbits in the affine Grassmannian: $\mathcal{F}^{\alpha} = \overline{T}_{-\alpha} \cap S_{0}$.

2.4.1. *Lemma.* The intersection of $\mathcal{F}^{\alpha}$ with a fine stratum $\overset{\circ}{\mathcal{Z}}{}^{\gamma} \times (\mathbb{C}^{*})^{\beta-\gamma}_{\Gamma}$, $\gamma \leq \beta \leq \alpha, \Gamma \in \mathfrak{P}(\beta - \gamma)$, is nonempty iff $\gamma = \beta$. In this case $\mathcal{F}^{\alpha} \cap \overset{\circ}{\mathcal{Z}}{}^{\beta} \simeq T_{-\beta} \cap S_{0}$.

Proof. Evident. $\square$

2.4.2. *Corollary.* Let $\mathcal{S}$ be a stratum of the fine stratification of $\mathcal{Z}^{\alpha}$. Then either $\mathcal{S} \cap \mathcal{F}^{\alpha} = \emptyset$, or $\dim(\mathcal{S} \cap \mathcal{F}^{\alpha}) = \frac{1}{2}\dim \mathcal{S}$.

2.4.3. According to the above Lemma, we have $\mathcal{F}^{\alpha} = \sqcup_{\beta \leq \alpha}(\overset{\circ}{\mathcal{Z}}{}^{\beta} \cap \mathcal{F}^{\alpha})$ — a partition into locally closed subsets. The restriction of $\mathcal{IC}^{\alpha}$ to $\overset{\circ}{\mathcal{Z}}{}^{\beta} \cap \mathcal{F}^{\alpha}$ is concentrated in the degrees $\leq -\dim \overset{\circ}{\mathcal{Z}}{}^{\beta} = -2|\beta|$. Moreover, by the definition of $\mathcal{IC}$ sheaf, if $\beta < \alpha$, the above inequality is strict. Since $\dim(\overset{\circ}{\mathcal{Z}}{}^{\beta} \cap \mathcal{F}^{\alpha}) = |\beta|$ we conclude that $H^{\bullet}_{c}(\overset{\circ}{\mathcal{Z}}{}^{\beta} \cap \mathcal{F}^{\alpha}, \mathcal{IC}^{\alpha}|_{\overset{\circ}{\mathcal{Z}}{}^{\beta} \cap \mathcal{F}^{\alpha}})$ is concentrated in nonpositive degrees, and if $\beta < \alpha$ it is concentrated in strictly negative degrees. Applying the Cousin spectral sequence associated with the partition $\mathcal{F}^{\alpha} = \sqcup_{\beta \leq \alpha}(\overset{\circ}{\mathcal{Z}}{}^{\beta} \cap \mathcal{F}^{\alpha})$ we see that $H^{\bullet}_{c}(\mathcal{F}^{\alpha}, \iota^{*}_{\alpha}\mathcal{IC}^{\alpha})$ is concentrated in nonpositive degrees.

2.4.4. The above spectral sequence shows also that $H_c^0(\mathcal{F}^\alpha, \iota_\alpha^* \mathcal{IC}^\alpha) = H_c^0(\overset{\circ}{\mathcal{Z}}{}^\alpha \cap \mathcal{F}^\alpha, \mathcal{IC}^\alpha|_{\overset{\circ}{\mathcal{Z}}{}^\alpha \cap \mathcal{F}^\alpha}) = H_c^0(\overset{\circ}{\mathcal{Z}}{}^\alpha \cap \mathcal{F}^\alpha, \underline{\mathbb{C}}[2|\alpha|]) = H_c^{2|\alpha|}(\overset{\circ}{\mathcal{Z}}{}^\alpha \cap \mathcal{F}^\alpha), \mathbb{C}) = H_c^{2|\alpha|}(T_{-\alpha} \cap S_0)$. The latter term has a canonical basis of irreducible components of $T_{-\alpha} \cap S_0$. According to [MV], the number of irreducible components equals $\mathcal{K}(\alpha)$. This completes the proof of 2.2 b).

2.5. It remains to show that $H^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ is concentrated in nonnegative degrees, or dually, that $H_c^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha)$ is concentrated in nonpositive degrees. To this end we study the intersection of the fine stratification of $\mathcal{Z}^\alpha$ with $\mathbb{A}^\alpha = \partial_\alpha \mathcal{Z}^\alpha$.

2.5.1. *Lemma.* The intersection of $\partial_\alpha \mathcal{Z}^\alpha$ with a fine stratum $\overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)_\Gamma^{\beta - \gamma}$, $\gamma \leq \beta \leq \alpha, \Gamma \in \mathfrak{P}(\beta - \gamma)$, is nonempty iff $\gamma = 0$. In this case $\partial_\alpha \mathcal{Z}^\alpha$ contains the stratum $\overset{\circ}{\mathcal{Z}}{}^0 \times (\mathbb{C}^*)_\Gamma^\beta \simeq (\mathbb{C}^*)_\Gamma^\beta$.

Proof. Evident. □

2.5.2. If $\Gamma = \{\{\gamma_1, \dots, \gamma_k\}\}$ is a partition of β then $\dim(\mathbb{C}^*)_\Gamma^\beta = k$, and due to the factorization property [FM] 7.3, the stalk of $\mathcal{IC}^\alpha$ at (any point in) the stratum $(\mathbb{C}^*)_\Gamma^\beta$ is isomorphic to $\mathcal{IC}_\Gamma^{\alpha - \beta} \simeq \mathcal{IC}_{\{\{\alpha - \beta\}\}}^0 \otimes \bigotimes_{r=1}^k \mathcal{IC}_{\{\{\gamma_r\}\}}^0$ (notations of *loc. cit.* 10.7.1, 10.7.2).

Lemma. The restriction of $\mathcal{IC}^\alpha$ to the stratum $(\mathbb{C}^*)_\Gamma^\beta$ is concentrated in the degrees $\leq -2k$ (here k is the number of elements in the partition Γ).

Proof. In view of the above product formula, it suffices to check that for any $r = 1, \dots, k$ the simple stalk $\mathcal{IC}_{\{\{\gamma_r\}\}}^0$ is concentrated in degree ≤ -2. This is the stalk of $\mathcal{IC}^{\gamma_r}$ at the one-dimensional (diagonal) stratum $(\mathbb{C}^*)_{\{\{\gamma_r\}\}}^{\gamma_r} \subset \mathcal{Z}^{\gamma_r}$. By the definition of $\mathcal{IC}$ sheaf, its restriction to any nonopen l-dimensional stratum is concentrated in degrees $< -l$. This completes the proof of the Lemma. □

2.5.3. We consider the partition of $\partial_\alpha \mathcal{Z}^\alpha$ into locally closed subsets: $\partial_\alpha \mathcal{Z}^\alpha = \sqcup (\mathbb{C}^*)_\Gamma^\beta$. According to the above Lemma, the restriction of $\mathcal{IC}^\alpha$ to a k-dimensional stratum is concentrated in degrees $\leq -2k$. Applying the Cousin spectral sequence associated with this partition we conclude, exactly as in subsection 2.4, that $H_c^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha)$ is concentrated in nonpositive degrees. Dually, $H^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ is concentrated in nonnegative degrees. This completes the proof of the Theorem 2.2. □

2.5.4. *Corollary.* $\mathcal{A}_\alpha = \mathrm{Ext}_{\mathcal{Z}^\alpha}^\bullet(\mathcal{IC}(\partial_\alpha \mathcal{Z}^\alpha), \mathcal{IC}^\alpha)$ is concentrated in the degree $|\alpha|$.

Proof. See the beginning of 2.2. □

2.6. In the rest of this section we construct the multiplication map $\mathcal{A}_\beta \otimes \mathcal{A}_\alpha \longrightarrow \mathcal{A}_{\beta + \alpha}$. We start with the following Lemma.

2.6.1. *Lemma.* $\mathcal{IC}(\partial_\alpha \mathcal{Z}^{\alpha + \beta}) = (\varsigma_{\beta, \alpha})_* \mathcal{IC}(\mathcal{Z}^\beta \times \mathbb{A}^\alpha) = (\varsigma_{\beta, \alpha})_* (\mathcal{IC}^\beta \boxtimes \underline{\mathbb{C}}[|\alpha|])$.

Proof. The map $\varsigma_{\beta, \alpha}$ is finite and generically one-to-one. □

2.6.2. *Remark.* In fact, $\varsigma_{\beta, \alpha}$ is the normalization map, but we do not need nor prove this fact.

2.6.3. We denote by $\mathcal{IC}_\alpha^\beta$ the sheaf $\mathcal{IC}(\partial_\alpha \mathcal{Z}^{\alpha+\beta})$. In particular, $\partial_0 \mathcal{Z}^\beta = \mathcal{Z}^\beta$, and $\mathcal{IC}_0^\beta = \mathcal{IC}^\beta$.

Consider the restriction of the map $\varsigma_{\beta+\alpha,\gamma} : \mathcal{Z}^{\beta+\alpha} \times \mathbb{A}^\gamma \longrightarrow \partial_\gamma \mathcal{Z}^{\gamma+\beta+\alpha}$ to $\partial_\alpha \mathcal{Z}^{\beta+\alpha} \times \mathbb{A}^\gamma$. Evidently, $\varsigma_{\beta+\alpha,\gamma}(\partial_\alpha \mathcal{Z}^{\beta+\alpha} \times \mathbb{A}^\gamma) = \partial_{\alpha+\gamma} \mathcal{Z}^{\gamma+\beta+\alpha}$. The map $\varsigma_{\beta+\alpha,\gamma}$ restricted to $\partial_\alpha \mathcal{Z}^{\beta+\alpha} \times \mathbb{A}^\gamma$ is finite, so the semisimple perverse sheaf $(\varsigma_{\beta+\alpha,\gamma})_* \mathcal{IC}(\partial_\alpha \mathcal{Z}^{\beta+\alpha} \times \mathbb{A}^\gamma) = (\varsigma_{\beta+\alpha,\gamma})_*(\mathcal{IC}_\alpha^\beta \boxtimes \underline{\mathbb{C}}[|\gamma|])$ contains the direct summand $\mathcal{IC}_{\alpha+\gamma}^\beta$ with multiplicity one. Let

$$\mathcal{IC}_{\alpha+\gamma}^\beta \xrightarrow{e} (\varsigma_{\beta+\alpha,\gamma})_*(\mathcal{IC}_\alpha^\beta \boxtimes \underline{\mathbb{C}}[|\gamma|]) \xrightarrow{f} \mathcal{IC}_{\alpha+\gamma}^\beta$$

denote the corresponding inclusion and projection.

2.7. We include the spaces $\mathcal{A}_\alpha$ into the wider framework. For $\alpha, \beta, \gamma \in \mathbb{N}[I]$ we define

$$\mathcal{A}_\beta^{\alpha,\gamma} := \mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma}}(\mathcal{IC}_{\alpha+\beta}^\gamma, \mathcal{IC}_\alpha^{\beta+\gamma})$$

In particular, $\mathcal{A}_\beta = \mathrm{Ext}^\bullet_{\mathcal{Z}^\beta}(\mathcal{IC}_\beta^0, \mathcal{IC}_0^\beta) = \mathcal{A}_\beta^{0,0}$.

Now for $\delta \in \mathbb{N}[I]$ we construct the *stabilization map* $\tau_{\beta,\delta}^{\alpha,\gamma} : \mathcal{A}_\beta^{\alpha,\gamma} \longrightarrow \mathcal{A}_\beta^{\alpha,\gamma+\delta}$, and the *costabilization map* $\xi_{\beta,\delta}^{\alpha,\gamma} : \mathcal{A}_\beta^{\alpha,\gamma} \longrightarrow \mathcal{A}_\beta^{\alpha+\delta,\gamma}$.

2.7.1. We choose an open subset $U' \xhookrightarrow{j'} \mathcal{Z}^{\alpha+\beta+\gamma+\delta}$ (resp. $U \xhookrightarrow{j} \mathcal{Z}^{\alpha+\beta+\gamma}$) such that $U' \cap \partial_{\alpha+\beta} \mathcal{Z}^{\alpha+\beta+\gamma+\delta} = W' := \overset{\circ}{\mathcal{Z}}{}^{\gamma+\delta} \times \mathbb{A}^{\alpha+\beta}$ (resp. $U \cap \partial_{\alpha+\beta} \mathcal{Z}^{\alpha+\beta+\gamma} = W := \overset{\circ}{\mathcal{Z}}{}^\gamma \times \mathbb{A}^{\alpha+\beta}$) (the open subset formed by all the quasimaps of defect *exactly* $\alpha + \beta$).

We have $\mathrm{Ext}^\bullet_{U'}(\mathcal{IC}_{\alpha+\beta}^{\gamma+\delta}, \mathcal{IC}_\alpha^{\beta+\gamma+\delta}) = \mathrm{Ext}^\bullet_{W'}(\mathcal{IC}_{\alpha+\beta}^{\gamma+\delta}, \varsigma^!_{\gamma+\delta,\alpha+\beta} \mathcal{IC}_\alpha^{\beta+\gamma+\delta})$, and $\mathrm{Ext}^\bullet_U(\mathcal{IC}_{\alpha+\beta}^\gamma, \mathcal{IC}_\alpha^{\beta+\gamma}) = \mathrm{Ext}^\bullet_W(\mathcal{IC}_{\alpha+\beta}^\gamma, \varsigma^!_{\gamma,\alpha+\beta} \mathcal{IC}_\alpha^{\beta+\gamma})$, where $\varsigma_{\gamma,\alpha+\beta}$ stands for the finite map $\mathcal{Z}^\gamma \times \mathbb{A}^{\alpha+\beta} \to \partial_{\alpha+\beta} \mathcal{Z}^{\alpha+\beta+\gamma} \hookrightarrow \mathcal{Z}^{\alpha+\beta+\gamma}$.

Now by the Lemma 2.6.1 and the factorization property, there exists a unique complex of sheaves $\mathcal{M}$ on $\mathbb{A}^{\alpha+\beta}$ such that $(\varsigma^!_{\gamma+\delta,\alpha+\beta} \mathcal{IC}_\alpha^{\beta+\gamma+\delta})|_{W'} = \underline{\mathbb{C}}[2|\gamma+\delta|] \boxtimes \mathcal{M}$, and $(\varsigma^!_{\gamma,\alpha+\beta} \mathcal{IC}_\alpha^{\beta+\gamma})|_W = \underline{\mathbb{C}}[2|\gamma|] \boxtimes \mathcal{M}$. As $\mathcal{IC}_{\alpha+\beta}^\gamma|_W = \underline{\mathbb{C}}[2|\gamma|] \boxtimes \underline{\mathbb{C}}[|\alpha+\beta|]$, we get the restriction map $j^* : \mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma}}(\mathcal{IC}_{\alpha+\beta}^\gamma, \mathcal{IC}_\alpha^{\beta+\gamma}) \to \mathrm{Ext}^\bullet_{\overset{\circ}{\mathcal{Z}}{}^\gamma \times \mathbb{A}^{\alpha+\beta}}(\mathcal{IC}(\overset{\circ}{\mathcal{Z}}{}^\gamma) \boxtimes \mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \mathcal{IC}(\overset{\circ}{\mathcal{Z}}{}^\gamma) \boxtimes \mathcal{M})$. We can project the right hand side to the summand $\mathrm{Id} \otimes \mathrm{Ext}^\bullet_{\mathbb{A}^{\alpha+\beta}}(\mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \mathcal{M})$. The resulting map from $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma}}(\mathcal{IC}_{\alpha+\beta}^\gamma, \mathcal{IC}_\alpha^{\beta+\gamma})$ to $\mathrm{Ext}^\bullet_{\mathbb{A}^{\alpha+\beta}}(\mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \mathcal{M})$ will be also denoted by j^*.

Multiplying with $\mathrm{Id} \in \mathrm{Ext}^0_{\mathcal{Z}^{\gamma+\delta}}(\mathcal{IC}^{\gamma+\delta}, \mathcal{IC}^{\gamma+\delta})$, we get an embedding from $\mathrm{Ext}^\bullet_{\mathbb{A}^{\alpha+\beta}}(\mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \mathcal{M})$ to $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\gamma+\delta} \times \mathbb{A}^{\alpha+\beta}}(\mathcal{IC}^{\gamma+\delta} \boxtimes \mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \mathcal{IC}^{\gamma+\delta} \boxtimes \mathcal{M})$.

The key observation is that there exists a unique morphism $c : \mathcal{IC}^{\gamma+\delta} \boxtimes \mathcal{M} \to \varsigma^!_{\gamma+\delta,\alpha+\beta} \mathcal{IC}_\alpha^{\beta+\gamma+\delta}$ extending the isomorphism $(\varsigma^!_{\gamma+\delta,\alpha+\beta} \mathcal{IC}_\alpha^{\beta+\gamma+\delta})|_{W'} = \underline{\mathbb{C}}[2|\gamma+\delta|] \boxtimes \mathcal{M}$ from W' to $\mathcal{Z}^{\gamma+\delta} \times \mathbb{A}^{\alpha+\beta}$. In effect, let $\mathbf{IC}_\alpha^{\beta+\gamma+\delta}$ be the mixed Hodge counterpart of $\mathcal{IC}_\alpha^{\beta+\gamma+\delta}$. It is a pure Hodge module of weight $w = 2|\beta+\gamma+\delta| + |\alpha|$. Then $(\varsigma^!_{\gamma+\delta,\alpha+\beta} \mathbf{IC}_\alpha^{\beta+\gamma+\delta})|_{W'} = \underline{\mathbb{Q}}[2|\gamma+\delta|] \boxtimes \mathbf{M}$ for some Hodge module $\mathbf{M}$ on $\mathbb{A}^{\alpha+\beta}$. It follows from the Proposition 3.5 below that $\mathbf{M}$ is a pure (hence semisimple) Hodge complex of weight $|\alpha+2\beta|$. Thus $\mathbf{IC}^{\gamma+\delta} \boxtimes \mathbf{M}$ is a subquotient of the direct sum of cohomology of the mixed Hodge complex $\varsigma^!_{\gamma+\delta,\alpha+\beta} \mathbf{IC}_\alpha^{\beta+\gamma+\delta}$. The latter complex has weight $\geq w$, while $\mathbf{IC}^{\gamma+\delta} \boxtimes \mathbf{M}$ is pure of weight w. The desired morphism c is constructed by induction in the cohomology degree using the Proposition 5.1.15 of [BBD].

Finally, we define

$$\tau_{\beta,\delta}^{\alpha,\gamma} : \mathcal{A}_\beta^{\alpha,\gamma} = \mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma}}(\mathcal{IC}_{\alpha+\beta}^\gamma, \mathcal{IC}_\alpha^{\beta+\gamma}) \longrightarrow \mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma+\delta}}(\mathcal{IC}_{\alpha+\beta}^{\gamma+\delta}, \mathcal{IC}_\alpha^{\beta+\gamma+\delta}) = \mathcal{A}_\beta^{\alpha,\gamma+\delta}$$

as the following composition: $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma}}(\mathcal{IC}^\gamma_{\alpha+\beta}, \mathcal{IC}^{\beta+\gamma}_\alpha) \xrightarrow{j^*} \mathrm{Ext}^\bullet_{\mathbb{A}^{\alpha+\beta}}(\mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \mathcal{M}) \xrightarrow{\mathrm{Id}\otimes?}$ $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\gamma+\delta}\times\mathbb{A}^{\alpha+\beta}}(\mathcal{IC}^{\gamma+\delta}\boxtimes\mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \mathcal{IC}^{\gamma+\delta}\boxtimes\mathcal{M}) \xrightarrow{c} \mathrm{Ext}^\bullet_{\mathcal{Z}^{\gamma+\delta}\times\mathbb{A}^{\alpha+\beta}}(\mathcal{IC}^{\gamma+\delta}\boxtimes\mathcal{IC}(\mathbb{A}^{\alpha+\beta}), \varsigma^!_{\gamma+\delta,\alpha+\beta}\mathcal{IC}^{\beta+\gamma+\delta}_\alpha) =$ $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma+\delta}}(\mathcal{IC}^{\gamma+\delta}_{\alpha+\beta}, \mathcal{IC}^{\beta+\gamma+\delta}_\alpha)$. Quite obviously, the result does not depend on a choice of U and U'.

2.7.2. To define the costabilization map $\xi^{\alpha,\gamma}_{\beta,\delta} : \mathcal{A}^{\alpha,\gamma}_\beta \longrightarrow \mathcal{A}^{\alpha+\delta,\gamma}_\beta$ we note that $\mathcal{A}^{\alpha,\gamma}_\beta :=$ $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma}}(\mathcal{IC}^\gamma_{\alpha+\beta}, \mathcal{IC}^{\beta+\gamma}_\alpha) = \mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma}\times\mathbb{A}^\delta}(\mathcal{IC}^\gamma_{\alpha+\beta}\boxtimes\underline{\mathbb{C}}[|\delta|], \mathcal{IC}^{\beta+\gamma}_\alpha\boxtimes\underline{\mathbb{C}}[|\delta|])$ maps by $(\varsigma_{\alpha+\beta+\gamma,\delta})_*$ to $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma+\delta}}((\varsigma_{\alpha+\beta+\gamma,\delta})_*(\mathcal{IC}^\gamma_{\alpha+\beta}\boxtimes\underline{\mathbb{C}}[|\delta|]), (\varsigma_{\alpha+\beta+\gamma,\delta})_*(\mathcal{IC}^{\beta+\gamma}_\alpha\boxtimes\underline{\mathbb{C}}[|\delta|]))$.

According to 2.6.3, the first argument of the latter Ext contains the canonical direct summand $e\mathcal{IC}^\gamma_{\alpha+\delta+\beta}$, while the second argument of the latter Ext canonically projects by f to $\mathcal{IC}^{\beta+\gamma}_{\alpha+\delta}$. Thus, the latter Ext canonically projects to $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta+\gamma+\delta}}(\mathcal{IC}^\gamma_{\alpha+\delta+\beta}, \mathcal{IC}^{\beta+\gamma}_{\alpha+\delta}) = \mathcal{A}^{\alpha+\delta,\gamma}_\beta$. Composing this projection with the above map $(\varsigma_{\alpha+\beta+\gamma,\delta})_*$ we obtain the desired map $\xi^{\alpha,\gamma}_{\beta,\delta} : \mathcal{A}^{\alpha,\gamma}_\beta \longrightarrow \mathcal{A}^{\alpha+\delta,\gamma}_\beta$.

2.8. Proposition. For $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{N}[I]$ we have

a) $\tau^{\alpha,\gamma+\delta}_{\beta,\epsilon} \circ \tau^{\alpha,\gamma}_{\beta,\delta} = \tau^{\alpha,\gamma}_{\beta,\delta+\epsilon} : \mathcal{A}^{\alpha,\gamma}_\beta \longrightarrow \mathcal{A}^{\alpha,\gamma+\delta+\epsilon}_\beta$;

b) $\xi^{\alpha+\delta,\gamma}_{\beta,\epsilon} \circ \xi^{\alpha,\gamma}_{\beta,\delta} = \xi^{\alpha,\gamma}_{\beta,\delta+\epsilon} : \mathcal{A}^{\alpha,\gamma}_\beta \longrightarrow \mathcal{A}^{\alpha+\delta+\epsilon,\gamma}_\beta$;

c) $\xi^{\alpha,\gamma+\delta}_{\beta,\epsilon} \circ \tau^{\alpha,\gamma}_{\beta,\delta} = \tau^{\alpha+\epsilon,\gamma}_{\beta,\delta} \circ \xi^{\alpha,\gamma}_{\beta,\epsilon} : \mathcal{A}^{\alpha,\gamma}_\beta \longrightarrow \mathcal{A}^{\alpha+\epsilon,\gamma+\delta}_\beta$.

Proof. Routine check. $\square$

2.9. Definition. Let $a \in \mathcal{A}_\alpha = \mathcal{A}^{0,0}_\alpha$, $b \in \mathcal{A}_\beta = \mathcal{A}^{0,0}_\beta$. We define the product $a \cdot b$ as the following composition:

$$a \cdot b := \tau^{0,0}_{\alpha,\beta}(a) \circ \xi^{0,0}_{\beta,\alpha}(b) \in \mathcal{A}^{0,0}_{\alpha+\beta} = \mathcal{A}_{\alpha+\beta}$$

The associativity of this multiplication follows immediately from the Proposition 2.8.

2.10. To reward the patient reader who has fought his way through the above notation, we repeat here the definition of $\tau^{\gamma,\delta}_{\alpha,\beta}$ in the particular case of $\gamma = \delta = 0$. This is the only case needed in the definition of multiplication (the general case is needed to prove the associativity), and the definition simplifies somewhat in this case.

So we are going to define the map

$$\tau^{0,0}_{\alpha,\beta} : \mathcal{A}_\alpha = \mathrm{Ext}^\bullet_{\mathcal{Z}^\alpha}(\mathcal{IC}(\partial_\alpha\mathcal{Z}^\alpha), \mathcal{IC}^\alpha) \longrightarrow \mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta}}(\mathcal{IC}(\partial_\alpha\mathcal{Z}^{\alpha+\beta}), \mathcal{IC}^{\alpha+\beta})$$

First of all, tensoring with $\mathrm{Id} \in \mathrm{Ext}^0_{\mathcal{Z}^\beta}(\mathcal{IC}^\beta, \mathcal{IC}^\beta)$, we get the map from $\mathrm{Ext}^\bullet_{\mathcal{Z}^\alpha}(\mathcal{IC}(\partial_\alpha\mathcal{Z}^\alpha), \mathcal{IC}^\alpha) = \mathrm{Ext}^\bullet_{\mathbb{A}^\alpha}(\mathcal{IC}(\mathbb{A}^\alpha), \mathfrak{s}^!_\alpha\mathcal{IC}^\alpha)$ to $\mathrm{Ext}^\bullet_{\mathcal{Z}^\beta\times\mathbb{A}^\alpha}(\mathcal{IC}^\beta\boxtimes\mathcal{IC}(\mathbb{A}^\alpha), \mathcal{IC}^\beta\boxtimes\mathfrak{s}^!_\alpha\mathcal{IC}^\alpha)$.

Now the same argument as in 2.7.1 shows that there is a unique morphism c from $\mathcal{IC}^\beta\boxtimes\mathfrak{s}^!_\alpha\mathcal{IC}^\alpha$ to $\varsigma^!_{\beta,\alpha}\mathcal{IC}^{\alpha+\beta}$ extending the factorization isomorphism from the open part $\overset{\circ}{\mathcal{Z}}{}^\beta\times\mathbb{A}^\alpha$ of $\mathcal{Z}^\beta\times\mathbb{A}^\alpha$. Thus c induces the map from $\mathrm{Ext}^\bullet_{\mathcal{Z}^\beta\times\mathbb{A}^\alpha}(\mathcal{IC}^\beta\boxtimes\mathcal{IC}(\mathbb{A}^\alpha), \mathcal{IC}^\beta\boxtimes\mathfrak{s}^!_\alpha\mathcal{IC}^\alpha)$ to $\mathrm{Ext}^\bullet_{\mathcal{Z}^\beta\times\mathbb{A}^\alpha}(\mathcal{IC}^\beta\boxtimes\mathcal{IC}(\mathbb{A}^\alpha), \varsigma^!_{\beta,\alpha}\mathcal{IC}^{\alpha+\beta})$.

The latter space equals $\mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta}}((\varsigma_{\beta,\alpha})_*(\mathcal{IC}^\beta\boxtimes\mathcal{IC}(\mathbb{A}^\alpha)), \mathcal{IC}^{\alpha+\beta}) = \mathrm{Ext}^\bullet_{\mathcal{Z}^{\alpha+\beta}}(\mathcal{IC}(\partial_\alpha\mathcal{Z}^{\alpha+\beta}), \mathcal{IC}^{\alpha+\beta})$.

Finally, we define $\tau^{0,0}_{\alpha,\beta}$ as the composition of above maps.

2.11. We close this section with a definition of comultiplication $\Delta : \mathcal{A}_\alpha \longrightarrow \oplus_{\beta+\gamma=\alpha}\mathcal{A}_\beta \otimes \mathcal{A}_\gamma$.

2.11.1. We choose two disjoint open balls $\Omega, \Upsilon \subset \mathbb{C} = \mathbb{A}^1$. Then $\Omega^\beta \times \Upsilon^\gamma \subset \mathbb{A}^{\beta+\gamma} = \mathbb{A}^\alpha$. According to the factorization property of $\mathcal{Z}^\alpha$, we have $(\mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)|_{\Omega^\beta \times \Upsilon^\gamma} = \mathfrak{s}_\beta^! \mathcal{IC}^\beta|_{\Omega^\beta} \boxtimes \mathfrak{s}_\gamma^! \mathcal{IC}^\gamma|_{\Upsilon^\gamma}$. We define the comultiplication component $\Delta_{\beta,\gamma} : \mathcal{A}_\alpha \longrightarrow \mathcal{A}_\beta \otimes \mathcal{A}_\gamma$ as the following composition: $\mathcal{A}_\alpha = H^0(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) \longrightarrow H^0(\Omega^\beta, \mathfrak{s}_\beta^! \mathcal{IC}^\beta) \otimes H^0(\Upsilon^\gamma, \mathfrak{s}_\gamma^! \mathcal{IC}^\gamma) = H^0(\mathbb{A}^\beta, \mathfrak{s}_\beta^! \mathcal{IC}^\beta) \otimes H^0(\mathbb{A}^\gamma, \mathfrak{s}_\gamma^! \mathcal{IC}^\gamma) = \mathcal{A}_\beta \otimes \mathcal{A}_\gamma$.

Quite evidently, the result does not depend on a choice of Ω and Υ. This comultiplication is manifestly cocommutative and coassociative. It also commutes with the multiplication defined in 2.9, so it defines the structure of cocommutative bialgebra on $\mathcal{A}$.

2.11.2. Consider the main diagonal stratum $\mathbb{A}^1 \overset{u_\alpha}{\hookrightarrow} \mathbb{A}^\alpha$. We will denote by $U^\alpha \overset{j}{\hookrightarrow} \mathbb{A}^\alpha$ the open inclusion of the complement $U^\alpha = \mathbb{A}^\alpha - \mathbb{A}^1$. We say that a cohomology class $a \in H^0(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ *is supported on the main diagonal* if $0 = j^* a \in H^0(U^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$. The following criterion will prove very useful in section 4.

Lemma. $a \in \mathcal{A}_\alpha$ is supported on the main diagonal iff a is primitive, i.e. $\Delta(a) = 1 \otimes a + a \otimes 1$.

Proof. a) If $a \in \mathcal{A}_\alpha$ is supported on the main diagonal then, obviously, $\Delta_{\beta,\gamma}(a) = 0$ when $\beta > 0 < \gamma$.

b) The converse follows by the Mayer-Vietoris type argument since U^α may be covered by the open subsets of the type $\Omega^\beta \times \Upsilon^\gamma$, $\beta > 0 < \gamma$. $\square$

3. Simple $\mathcal{IC}$-stalks

3.1. Let $\check{\mathcal{R}}^+ \subset Y$ denote the set of positive coroots.

For $\alpha \in \mathbb{N}[I]$ the following q-analogue of the Kostant partition function $\mathcal{K}(\alpha)$ was introduced in [L2]:

$$\mathcal{K}^\alpha(t) := \sum_\kappa t^{-|\kappa|}$$

Here the sum is taken over the set of partitions of α into a sum of positive coroots $\check{\theta} \in \check{\mathcal{R}}^+$, and for a partition κ the number of elements in κ is denoted by $|\kappa|$. Finally, t is a formal variable of degree 2. We have $\mathcal{K}(\alpha) = \mathcal{K}^\alpha(1)$.

In this section we prove the following Theorem:

Theorem. $\mathcal{K}^\alpha(t)$ is the generating function of the simple stalk $\mathcal{IC}_{\{\{\alpha\}\}}^0$ (see [FM] 10.7.1), that is (notations of 2.3),

a) for odd k we have $H^k \iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha = 0$;

b) $\dim H^{2k} \iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha$ equals the coefficient $\mathcal{K}_k^\alpha$ of t^k in $\mathcal{K}^\alpha(t)$.

3.1.1. *Corollary.* The generating function of the stalk $\mathcal{IC}_\Gamma^0$, $\Gamma = \{\{\gamma_1, \ldots, \gamma_k\}\}$ (see [FM] 10.7.1) equals $\prod_{r=1}^k \mathcal{K}^{\gamma_r}(t)$.

The Corollary follows immediately from the above Theorem and the factorization property. In its turn, the Corollary (along with the Proposition 10.7.3 of [FM]) implies the Parity vanishing conjecture 10.7.4 of *loc. cit.*

3.2. We start with the proof of 3.1 a). Recall that the Cartan group $\mathbf{H}$ acts on $\mathcal{Z}^\alpha$ contracting it to the fixed point set $(\mathcal{Z}^\alpha)^{\mathbf{H}} = \partial_\alpha \mathcal{Z}^\alpha = \mathbb{A}^\alpha$. Moreover, if $\mathbb{C}^*$ is the group of automorphisms of $C = \mathbb{P}^1$ preserving $0, \infty$, then $\mathbf{H} \times \mathbb{C}^*$ acts on $\mathcal{Z}^\alpha$ contracting it to the fixed point set $(\mathcal{Z}^\alpha)^{\mathbf{H} \times \mathbb{C}^*} = \mathfrak{s}_\alpha \circ \iota_\alpha(0)$. Applying the Corollary 14.3 of [BL] we see that the equivariant cohomology $H_{\mathbf{H} \times \mathbb{C}^*}^\bullet(\mathcal{Z}^\alpha, \mathcal{IC}^\alpha)$ is isomorphic to $H_{\mathbf{H} \times \mathbb{C}^*}^\bullet(\cdot) \otimes H^\bullet(\mathcal{Z}^\alpha, \mathcal{IC}^\alpha)$, and $H_{\mathbf{H}}^\bullet(\mathcal{Z}^\alpha, \mathcal{IC}^\alpha) \cong H_{\mathbf{H}}^\bullet(\cdot) \otimes H^\bullet(\mathcal{Z}^\alpha, \mathcal{IC}^\alpha) = H_{\mathbf{H}}^\bullet(\cdot) \otimes \iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha$.

On the other hand, according to the Theorem B.2 of [EM], the natural map

$$H_{\mathbf{H}}^\bullet(\cdot) \otimes H^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) = H_{\mathbf{H}}^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) \longrightarrow H_{\mathbf{H}}^\bullet(\mathcal{Z}^\alpha, \mathcal{IC}^\alpha)$$

is an isomorphism after certain localization in $H_{\mathbf{H}}^\bullet(\cdot)$.

Now $H_{\mathbf{H}}^\bullet(\cdot)$ is concentrated in even degrees, and the above localization is taken with respect to a homogeneous multiplicative subset. By the Theorem 2.2, $H^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ is concentrated in degree 0, hence the (localized) $H_{\mathbf{H}}^\bullet(\cdot) \otimes H^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) = H_{\mathbf{H}}^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ is concentrated in even degrees, hence the (localized) $H_{\mathbf{H}}^\bullet(\mathcal{Z}^\alpha, \mathcal{IC}^\alpha) \cong H_{\mathbf{H}}^\bullet(\cdot) \otimes H^\bullet(\mathcal{Z}^\alpha, \mathcal{IC}^\alpha) = H_{\mathbf{H}}^\bullet(\cdot) \otimes \iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha$ is concentrated in even degrees, hence $\iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha$ is concentrated in even degrees. This completes the proof of the Theorem 3.1 a).

3.2.1. *Corollary of the proof.* $\dim(\iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha) = \chi(\iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha) = \mathcal{K}(\alpha)$.

3.3. To prove 3.1 b) we need to introduce and recall some notation.

3.3.1. As we have seen in 2.4.2, the fine stratification of $\mathcal{Z}^\alpha$ induces a partition of the central fiber $\mathcal{F}^\alpha$ into the locally closed subsets: $\mathcal{F}^\alpha = \sqcup_{\beta \le \alpha}(\overset{\circ}{\mathcal{Z}}{}^\beta \cap \mathcal{F}^\alpha) =: \sqcup_{\beta \le \alpha} \overset{\circ}{\mathcal{F}}{}^\beta$. Moreover, $\overset{\circ}{\mathcal{F}}{}^\beta \simeq T_{-\beta} \cap S_0$. The closed embedding of the smallest point stratum $\overset{\circ}{\mathcal{F}}{}^0$ into $\mathcal{F}^\alpha$ will be denoted by $\mathfrak{s}_\alpha$. Recall that the closed embedding $\mathcal{F}^\alpha \hookrightarrow \mathcal{Z}^\alpha$ is denoted by ι_α. Certainly, we have $\iota_\alpha^* \mathfrak{s}_\alpha^* \mathcal{IC}^\alpha = \mathfrak{s}_\alpha^* \iota_\alpha^* \mathcal{IC}^\alpha$.

3.3.2. We will use the machinery of *mixed Hodge modules*. The sheaf $\mathcal{IC}^\alpha$ has its mixed Hodge counterpart $\mathbf{IC}^\alpha$ — the irreducible pure Hodge module on $\mathcal{Z}^\alpha$ of weight $2|\alpha|$. The following Theorem is a strong version of 3.1.

Theorem. a) for odd k we have $H^k \mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha = 0$;

b) $H^{2k} \mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$ is the direct sum of $\mathcal{K}_k^\alpha$ copies of the Tate Hodge structure $\mathbb{Q}(k + |\alpha|)$.

3.3.3. Recall that for a dominant $\eta \in Y^+$ we have a $\mathbf{G}(\mathcal{O})$-orbit $\mathcal{G}_\eta$ and its closure $\overline{\mathcal{G}}_\eta$ in the affine Grassmannian $\mathcal{G}$. Recall that w_0 is the longest element in the Weyl group $\mathcal{W}_f$ of $\mathbf{G}$. For a sufficiently dominant $\eta \in Y^+$, the difference $\eta - w_0(\alpha) =: \vartheta$ is also dominant, and $\mathcal{G}_\eta \subset \overline{\mathcal{G}}_\vartheta$. Let $\mathbf{IC}(\overline{\mathcal{G}}_\vartheta)_\eta$ denote the stalk of the Hodge module $\mathbf{IC}(\overline{\mathcal{G}}_\vartheta)$ at any point in $\mathcal{G}_\eta$. G.Lusztig has proved the following theorem in [L2] (in fact, he used the language of Weil sheaves instead of Hodge modules):

Theorem. Let $\eta \in Y^+$ be sufficiently dominant. Then

a) for odd k we have $H^k(\mathbf{IC}(\overline{\mathcal{G}}_\vartheta)_\eta[-2|\eta|]) = 0$;

b) $H^{2k}(\mathbf{IC}(\overline{\mathcal{G}}_\vartheta)_\eta[-2|\eta|])$ is the direct sum of $\mathcal{K}_k^\alpha$ copies of the Tate Hodge structure $\mathbb{Q}(k + |\alpha|)$.

3.3.4. Comparing 3.3.3 and 3.3.2 we see that the Theorems 3.3.2 and 3.1 follow from the following

Theorem. $\mathbf{IC}(\overline{\mathcal{G}}_\vartheta)_\eta[-2|\eta|] \cong \mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$.

The rest of this section is devoted to the proof of this Theorem.

3.4. We consider the intersection of semiinfinite orbits $\overline{T}_{w_0\eta-\alpha} \cap S_{w_0\eta}$ in the affine Grassmannian $\mathcal{G}$. It lies in the closure $\overline{\mathcal{G}}_\vartheta$, and the translation by the Cartan loop $w_0\eta(z) \in \mathbf{G}((z))$ identifies it with $\overline{T}_{-\alpha} \cap S_0 \simeq \mathcal{F}^\alpha$. Moreover, the partition of $\overline{T}_{w_0\eta-\alpha} \cap S_{w_0\eta}$ into intersections with $\mathbf{G}(\mathcal{O})$-orbits $\mathcal{G}_{\eta-w_0\beta}$, $0 \le \beta \le \alpha$, corresponds under this identification to the partition $\mathcal{F}^\alpha = \sqcup_{0 \le \beta \le \alpha} \overset{\circ}{\mathcal{F}}{}^\beta$. We denote the locally closed embedding $\mathcal{F}^\alpha \simeq \overline{T}_{w_0\eta-\alpha} \cap S_{w_0\eta} \hookrightarrow \overline{\mathcal{G}}_\vartheta$ by $\mathbf{i}_\alpha$. Thus, $\mathbf{IC}(\overline{\mathcal{G}}_\vartheta)_\eta = \mathfrak{s}_\alpha^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)$.

3.4.1. *Conjecture.* $\mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)[-2|\eta|] \cong \iota_\alpha^* \mathbf{IC}^\alpha$.

This conjecture would imply the Theorem 3.3.4. Unfortunately, we cannot prove this conjecture at the moment. Instead of it, we will prove its version in the K-group of mixed Hodge modules.

3.5. Proposition. a) $\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$ is a pure Hodge complex of weight $2|\alpha|$;

b) $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha$ is a pure Hodge structure of weight $2|\alpha|$.

Proof. a) The natural map $H^\bullet(\mathcal{Z}^\alpha, \mathbf{IC}^\alpha) \longrightarrow \mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$ is an isomorphism since $\mathbf{IC}^\alpha$ is smooth along the fine stratification. Now $\mathbf{IC}^\alpha$ is pure of weight $2|\alpha|$, and $H^\bullet(?)$ increases weights, while both $\iota_\alpha^*(?)$ and $\mathfrak{s}_\alpha^*(?)$ decrease weights.

b) We already know that $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha = H^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathbf{IC}^\alpha)$, and both $H^\bullet(?)$ and $\mathfrak{s}_\alpha^!(?)$ increase weights. On the other hand, $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha = \pi_{\alpha!} \iota_\alpha^* \mathbf{IC}^\alpha$, and both $\pi_{\alpha!}(?)$ and $\iota_\alpha^*(?)$ decrease weights. $\square$

3.6. The above Proposition implies that to prove the Theorems 3.3.4 and 3.3.2 it suffices to identify the classes of $[\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha]$ and $[\mathbf{IC}(\overline{\mathcal{G}}_\vartheta)_\eta]$ in the K-group of mixed Hodge structures $K(MHM(\cdot))$. We will proceed by induction in α. The first step $\alpha = 0$ being trivial, we may suppose that our Theorems are established for all $\beta < \alpha$.

Let us denote the complement $\mathcal{F}^\alpha - \overset{\circ}{\mathcal{F}}{}^0$ by U, and let us denote its open embedding into $\mathcal{F}^\alpha$ by j. By factorization, $\iota_\alpha^* \mathbf{IC}^\alpha|_{\overset{\circ}{\mathcal{F}}{}^\beta}$ is the constant Hodge module with the stalk $\mathfrak{s}_{\alpha-\beta}^* \iota_{\alpha-\beta}^* \mathbf{IC}^{\alpha-\beta}[2|\beta|]$. Thus, by induction, we know that the classes of $[j^* \iota_\alpha^* \mathbf{IC}^\alpha]$ and $[j^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)]$ in the K-group of mixed Hodge modules $K(MHM(U))$ coincide.

Let us consider the following exact triangles of mixed Hodge complexes:

$$\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha \longrightarrow \mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha \longrightarrow \mathfrak{s}_\alpha^* j_* j^* \iota_\alpha^* \mathbf{IC}^\alpha \longrightarrow \mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha[1] \longrightarrow \ldots \tag{1}$$

$$\mathfrak{s}_\alpha^! \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta) \longrightarrow \mathfrak{s}_\alpha^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta) \longrightarrow \mathfrak{s}_\alpha^* j_* j^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta) \longrightarrow \mathfrak{s}_\alpha^! \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)[1] \longrightarrow \ldots \tag{2}$$

3.6.1. We already know that $\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$ lives in the even negative degrees, and $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha$ lives in degree 0. Hence the differential $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha \longrightarrow \mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$ vanishes (since the category of mixed Hodge structures has the cohomological dimension 1), and $\mathfrak{s}_\alpha^* j_* j^* \iota_\alpha^* \mathbf{IC}^\alpha = \mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha[1] \oplus \mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$.

3.6.2. By [MV] we know that $\mathfrak{s}_\alpha^! \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)$ lives in degree 0 (and has dimension $\mathcal{K}(\alpha)$). Now the same argument as above proves that $\mathfrak{s}_\alpha^* j_* j^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta) = \mathfrak{s}_\alpha^! \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)[1] \oplus \mathfrak{s}_\alpha^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)$.

Moreover, by [MV] we know that $\mathfrak{s}_\alpha^! \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)$ is a direct summand of $H^\bullet(\overline{\mathcal{G}}_\vartheta, \mathbf{IC}(\overline{\mathcal{G}}_\vartheta))$, and hence it is a pure Tate Hodge module of weight $2|\alpha|$. We deduce that $\mathfrak{s}_\alpha^* j_* j^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)$ is a Tate Hodge module.

3.6.3. By the induction assumption, the classes of $[\mathfrak{s}_\alpha^* j_* j^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)]$ and $[\mathfrak{s}_\alpha^* j_* j^* \iota_\alpha^* \mathbf{IC}^\alpha]$ in $K(MHM(\cdot))$ coincide. So they both lie in the subgroup $K(TMHM(\cdot))$ of Tate Hodge modules. We have $[\mathfrak{s}_\alpha^* j_* j^* \iota_\alpha^* \mathbf{IC}^\alpha] = [\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha] - [\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha]$, and no cancellations occur in the RHS. In effect, all graded parts of $\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha$ have weights $< 2|\alpha|$ (since they live in negative degrees), while $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha$ is pure of weight $2|\alpha|$ (see 3.5). We deduce that both $[\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha]$ and $[\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha]$ lie in $K(TMHM(\cdot))$. Thus $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha$ is a sum of $\mathcal{K}(\alpha)$ copies of $\mathbb{Q}(|\alpha|)$, hence $\mathfrak{s}_\alpha^! \iota_\alpha^* \mathbf{IC}^\alpha \cong \mathfrak{s}_\alpha^! \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)$. It follows that $[\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha] = [\mathfrak{s}_\alpha^* \mathbf{i}_\alpha^* \mathbf{IC}(\overline{\mathcal{G}}_\vartheta)]$. This completes the proof of the Theorem 3.3.4 along with the Theorems 3.3.2 and 3.1. $\square$

3.7. Remark. It is easy to describe the semisimple complex $\mathfrak{s}^!_\alpha \mathcal{IC}^\alpha$ on $\mathbb{A}^\alpha$. For a Kostant partition $\kappa \in \mathfrak{K}(\alpha)$ let $\mathbb{A}^\alpha_\kappa \subset \mathbb{A}^\alpha$ denote the closure of the corresponding diagonal stratum. It is known that the normalization $\tilde{\mathbb{A}}^\alpha_\kappa \xrightarrow{N^\kappa} \mathbb{A}^\alpha_\kappa$ is isomorphic to an affine space. We have $\mathfrak{s}^!_\alpha \mathcal{IC}^\alpha \simeq \bigoplus_{\kappa \in \mathfrak{K}(\alpha)} N^\kappa_* \underline{\mathbb{C}}$.

4. $\mathfrak{g}^L$-ACTION ON THE GLOBAL $\mathcal{IC}$-COHOMOLOGY

4.1. We compute the global Intersection Cohomology of $\mathcal{Q}^\alpha$ following §2 of [FK]. First we introduce the open subset $\ddot{\mathcal{Q}}^\alpha \subset \mathcal{Q}^\alpha$ formed by all the quasimaps ϕ such that the defect of ϕ does not meet $\infty \in C$. Twisting by the multiples of ∞ we obtain the partition into locally closed subsets $\mathcal{Q}^\alpha = \sqcup_{\beta \leq \alpha} \ddot{\mathcal{Q}}^\beta$. The evaluation at ∞ defines the projection $p_\alpha : \ddot{\mathcal{Q}}^\alpha \longrightarrow \mathbf{X}$ which is a locally trivial fibration with a fiber isomorphic to $\mathcal{Z}^\alpha$.

By the Poincaré duality and Theorem 3.1 the generating function of $H^\bullet_c(\mathcal{Z}^\beta, \mathcal{IC}^\beta)$ is given by $\mathcal{K}^\beta(t^{-1})$. By the parity vanishing, the spectral sequence of fibration $p_\beta : \ddot{\mathcal{Q}}^\beta \longrightarrow \mathbf{X}$ degenerates and provides us with the isomorphism of graded vector spaces $H^\bullet_c(\ddot{\mathcal{Q}}^\beta, \mathcal{IC}(\mathcal{Q}^\beta)) \cong H^\bullet(\mathbf{X}, \underline{\mathbb{C}}[\dim \mathbf{X}]) \otimes H^\bullet_c(\mathcal{Z}^\beta, \mathcal{IC}^\beta)$. By the factorization property, $\mathcal{IC}(\mathcal{Q}^\alpha)|_{\ddot{\mathcal{Q}}^\beta} \simeq \mathcal{IC}(\mathcal{Q}^\beta) \otimes \mathcal{IC}^0_{\{\{\alpha - \beta\}\}}$. Finally, the parity vanishing implies the degeneration of the Cousin spectral sequence associated with the partition $\mathcal{Q}^\alpha = \sqcup_{\beta \leq \alpha} \ddot{\mathcal{Q}}^\beta$.

Combining all the above equalities we arrive at the formula for the generating function of $H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$. To write it down in a neat form we will consider the generating function $P_{\mathbf{G}}(t)$ of $\oplus_{\alpha \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$. To record the information on α we will consider this generating function as a formal cocharacter of $\mathbf{H}$ with coefficients in the Laurent polynomials in t. Formal cocharacters will be written multiplicatively, so that the cocharacter corresponding to α will be denoted by e^α. Finally, for the reasons which will become clear later (see Proposition 4.9), we will make the following rescaling. We will attach to $H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$ the cocharacter $e^{\alpha + 2\check{\rho}}$.

With all this in mind, the Poincaré polynomial $P_{\mathbf{G}}(t)$ of $\oplus_{\alpha \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$ is calculated as follows:

Theorem.
$$P_{\mathbf{G}}(t) = \frac{e^{2\check{\rho}} t^{-\frac{1}{2}\dim \mathbf{X}} \sum_{w \in \mathcal{W}_f} t^{l(w)}}{\prod_{\check{\theta} \in \check{\mathcal{R}}^+}(1 - t e^{\check{\theta}})(1 - t^{-1} e^{\check{\theta}})}$$
where $l(w)$ stands for the length of $w \in \mathcal{W}_f$, and $2\check{\rho}$ stands for the sum of positive coroots $\check{\theta} \in \check{\mathcal{R}}^+$. □

4.1.1. The above argument along with the Theorem 3.3.2 establishes also the following Theorem:

Theorem. a) For odd k we have $H^{k - 2|\alpha| - \dim \mathbf{X}}(\mathcal{Q}^\alpha, \mathbf{IC}(\mathcal{Q}^\alpha)) = 0$;

b) $H^{2k - 2|\alpha| - \dim \mathbf{X}}(\mathcal{Q}^\alpha, \mathbf{IC}(\mathcal{Q}^\alpha))$ is the direct sum of a few copies of the Tate Hodge module $\mathbb{Q}(k)$. □

4.2. In the rest of this section we equip $\oplus_{\alpha \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$ with the structure of the Langlands dual Lie algebra $\mathfrak{g}^L$-module with the character $\frac{|\mathcal{W}_f| e^{2\check{\rho}}}{\prod_{\check{\theta} \in \check{\mathcal{R}}^+} (1 - e^{\check{\theta}})^2}$.

We start with the following Proposition describing the subspace $\mathrm{Prim}(\mathcal{A})$ of the primitive elements of bialgebra $\mathcal{A}$.

Proposition. a) If $\alpha \in \mathbb{N}[I]$ is not a positive coroot, that is $\alpha \notin \check{\mathcal{R}}^+$, then $\mathcal{A}_\alpha$ does not contain primitive elements;

b) $\dim(\mathrm{Prim}(\mathcal{A}) \cap \mathcal{A}_\alpha) \leq 1$ for any $\alpha \in \mathbb{N}[I]$.

Proof. We will use the Criterion 2.11.2. Consider the long exact sequence of cohomology (notations of 2.11.2):

$$\ldots \longrightarrow H^\bullet(\mathbb{A}^1, u_\alpha^! \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) \longrightarrow H^\bullet(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) \longrightarrow H^\bullet(U^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) \longrightarrow \ldots$$

Applying Poincaré duality to the Theorem 3.1 we see that $u_\alpha^! \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha$ is the direct sum of constant sheaves living in nonnegative degrees, and $H^0(u_\alpha^! \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ is nonzero iff $\alpha \in \check{\mathcal{R}}^+$. In this case $H^0(u_\alpha^! \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) = \underline{\mathbb{C}}$. We deduce that $H^0(\mathbb{A}^1, u_\alpha^! \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ is nonzero iff $\alpha \in \check{\mathcal{R}}^+$, and in this case $\dim H^0(\mathbb{A}^1, u_\alpha^! \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) = 1$. Now an element $a \in H^0(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$ is supported on the main diagonal iff it comes from $H^0(\mathbb{A}^1, u_\alpha^! \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha)$. This completes the proof of the Proposition. □

4.2.1. *Remark.* Later on we will see that in fact the converse of 4.2 a) is also true (see 4.10).

4.3. Definition. a) $\mathbf{a}$ is the Lie algebra formed by the primitive elements $\mathrm{Prim}(\mathcal{A})$;

b) $\mathfrak{a} \subset \mathbf{a}$ is the Lie subalgebra generated by $\oplus_{i \in I} \mathcal{A}_i \subset \mathrm{Prim}(\mathcal{A})$.

4.4. We are going to prove that $\mathfrak{a} = \mathbf{a} \cong \mathfrak{n}_+^L$ — the nilpotent subalgebra of the Langlands dual Lie algebra $\mathfrak{g}^L$. We choose generators $e_i \in \mathcal{A}_i - 0$, and start with the following Proposition:

Proposition. The generators e_i satisfy the Serre relations of $\mathfrak{n}_+^L$.

Proof. The corresponding commutator has the weight lying out of $\check{\mathcal{R}}^+$, and hence vanishes by 4.2 a). □

4.4.1. *Remark.* In fact, there is a preferred choice of generators e_i. Note that $\mathcal{Z}^i \cong \mathbb{A}^2$, and $\partial_i \mathcal{Z}^i \cong \mathbb{A}^1$. So $\mathcal{IC}^i$ contains a subsheaf $\underline{\mathbb{Z}}[2]$, and $H^0(\mathbb{A}^1, \mathfrak{s}_i^! \mathcal{IC}^i) \supset H^0(\mathbb{A}^1, \mathfrak{s}_i^! \underline{\mathbb{Z}}[2]) = \mathbb{Z}$. The element $1 \in \mathbb{Z}$ in the RHS corresponds to the canonical element e_i in the LHS. We will use this choice of e_i in what follows.

4.5. We construct the action of $\mathcal{A}$ on $\oplus_{\alpha \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$ closely following 2.6–2.10.

4.5.1. First we construct the *stabilization map* $\mathfrak{t}_{\alpha,\beta} : \mathcal{A}_\alpha \longrightarrow \mathrm{Ext}_{\mathcal{Q}^{\alpha+\beta}}^{|\alpha|}(\mathcal{IC}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}), \mathcal{IC}(\mathcal{Q}^{\alpha+\beta}))$ where $\partial_\alpha \mathcal{Q}^{\alpha+\beta}$ stands for the image of the twisting map $\sigma_{\beta,\alpha} : \mathcal{Q}^\beta \times C^\alpha \longrightarrow \mathcal{Q}^{\alpha+\beta}$ (see [FM] 3.4.1).

To this end we note that the sheaf $\mathfrak{s}_\alpha^! \mathcal{IC}^\alpha$ on $\mathbb{A}^\alpha$ extends smoothly to the sheaf $\mathfrak{F}_\alpha$ on C^α, and $H^0(\mathbb{A}^\alpha, \mathfrak{s}_\alpha^! \mathcal{IC}^\alpha) = H^0(C^\alpha, \mathfrak{F}_\alpha) = \mathrm{Ext}_{C^\alpha}^{|\alpha|}(\mathcal{IC}(C^\alpha), \mathfrak{F}_\alpha)$.

Furthermore, tensoring with $\mathrm{Id} \in \mathrm{Ext}^0(\mathcal{IC}(\mathcal{Q}^\beta), \mathcal{IC}(\mathcal{Q}^\beta))$, we get the map from $\mathcal{A}_\alpha = \mathrm{Ext}_{C^\alpha}^{|\alpha|}(\mathcal{IC}(C^\alpha), \mathfrak{F}_\alpha)$ to $\mathrm{Ext}_{\mathcal{Q}^\beta \times C^\alpha}^{|\alpha|}(\mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \mathcal{IC}(C^\alpha), \mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \mathfrak{F}_\alpha)$.

Now the same argument as in 2.7.1 shows that there is a unique morphism c from $\mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \mathfrak{F}_\alpha$ to $\sigma_{\beta,\alpha}^! \mathcal{IC}(\mathcal{Q}^{\alpha+\beta})$ extending the factorization isomorphism from an open part $\overset{\circ}{\mathcal{Q}}{}^\beta \times \mathbb{A}^\alpha \subset \mathcal{Q}^\beta \times C^\alpha$. Thus c induces the map from $\mathrm{Ext}_{\mathcal{Q}^\beta \times C^\alpha}^{|\alpha|}(\mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \mathcal{IC}(C^\alpha), \mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \mathfrak{F}_\alpha)$ to $\mathrm{Ext}_{\mathcal{Q}^\beta \times C^\alpha}^{|\alpha|}(\mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \mathcal{IC}(C^\alpha), \sigma_{\beta,\alpha}^! \mathcal{IC}(\mathcal{Q}^{\alpha+\beta}))$.

The latter space equals $\mathrm{Ext}_{\mathcal{Q}^{\alpha+\beta}}^{|\alpha|}((\sigma_{\beta,\alpha})_* (\mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \mathcal{IC}(C^\alpha)), \mathcal{IC}(\mathcal{Q}^{\alpha+\beta})) = \mathrm{Ext}_{\mathcal{Q}^{\alpha+\beta}}^{|\alpha|}(\mathcal{IC}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}), \mathcal{IC}(\mathcal{Q}^{\alpha+\beta}))$.

Finally, we define $\mathfrak{t}_{\alpha,\beta}$ as the composition of above maps.

4.5.2. Second, we construct the *costabilization map*
$$\mathfrak{x}_{\beta,\alpha} : \ H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta)) \longrightarrow H^{\bullet-|\alpha|}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}, \mathcal{IC}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}))$$
To this end we note that exactly as in 2.6.1, we have $\mathcal{IC}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}) = (\sigma_{\beta,\alpha})_* \mathcal{IC}(\mathcal{Q}^\beta \times C^\alpha) = (\sigma_{\beta,\alpha})_*(\mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \underline{\mathbb{C}}[|\alpha|])$. Let $[C^\alpha] \in H^{-|\alpha|}(C^\alpha, \underline{\mathbb{C}}[|\alpha|])$ denote the fundamental class of C^α.

Now, for $h \in H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ we define $\mathfrak{x}_{\beta,\alpha}(h)$ as $h \otimes [C^\alpha] \in H^{\bullet-|\alpha|}(\mathcal{Q}^\beta \times C^\alpha, \mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \underline{\mathbb{C}}[|\alpha|]) = H^{\bullet-|\alpha|}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}, (\sigma_{\beta,\alpha})_*(\mathcal{IC}(\mathcal{Q}^\beta) \boxtimes \underline{\mathbb{C}}[|\alpha|])) = H^{\bullet-|\alpha|}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}, \mathcal{IC}(\partial_\alpha \mathcal{Q}^{\alpha+\beta}))$.

4.6. Definition. Let $a \in \mathcal{A}_\alpha$, $h \in H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$. We define the action $a(h) \in H^\bullet(\mathcal{Q}^{\alpha+\beta}, \mathcal{IC}(\mathcal{Q}^{\alpha+\beta}))$ as the action of $\mathfrak{t}_{\alpha,\beta}(a)$ on the global cohomology applied to $\mathfrak{x}_{\beta,\alpha}(h)$.

Let us stress that the action of $\mathcal{A}_\alpha$ *preserves cohomological degrees.*

4.6.1. For $a \in \mathcal{A}_\alpha$, $b \in \mathcal{A}_\beta$, $h \in H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ we have $a(b(h)) = a \cdot b(h)$. The proof is entirely similar to the proof of associativity of the multiplication in $\mathcal{A}$.

4.7. For $\beta \in \mathbb{N}[I]$, $i \in I$ both $H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ and $H^\bullet(\mathcal{Q}^{\beta+i}, \mathcal{IC}(\mathcal{Q}^{\beta+i}))$ are Poincaré selfdual. We define the map
$$f_i : \ H^\bullet(\mathcal{Q}^{\beta+i}, \mathcal{IC}(\mathcal{Q}^{\beta+i})) \longrightarrow H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$$
as the dual of the map
$$e_i : \ H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta)) \longrightarrow H^\bullet(\mathcal{Q}^{\beta+i}, \mathcal{IC}(\mathcal{Q}^{\beta+i}))$$
(see 4.4.1).

It follows from 4.4 that the maps f_i, $i \in I$, satisfy the Serre relations of $\mathfrak{n}_-^L$.

4.7.1. Let us sketch a more explicit construction of the operators f_i. Let σ denote the closed embedding $\partial_i \mathcal{Q}^{\beta+i} \hookrightarrow \mathcal{Q}^{\beta+i}$, and let j denote the embedding of the complementary open subset U. We have an exact sequence of perverse sheaves
$$0 \to \sigma^* j_{!*} \mathcal{IC}(U)[-1] \to j_! \mathcal{IC}(U) \to \mathcal{IC}(\mathcal{Q}^{\beta+i}) \to 0$$
defining an element $\mathfrak{f}_i'$ in $\mathrm{Ext}^1(\mathcal{IC}(\mathcal{Q}^{\beta+i}), \sigma^* j_{!*} \mathcal{IC}(U)[-1]$. The weight considerations as in 2.7.1 show that $\sigma^* j_{!*} \mathcal{IC}(U)[-1]$ canonically surjects to $\mathcal{IC}(\partial_i \mathcal{Q}^{\beta+i})$, and thus $\mathfrak{f}_i'$ gives rise to $\mathfrak{f}_i \in \mathrm{Ext}^1(\mathcal{IC}(\mathcal{Q}^{\beta+i}), \mathcal{IC}(\partial_i \mathcal{Q}^{\beta+i}))$. The action of $\mathfrak{f}_i$ on the global cohomology defines the same named map $\mathfrak{f}_i : \ H^\bullet(\mathcal{Q}^{\beta+i}, \mathcal{IC}(\mathcal{Q}^{\beta+i})) \longrightarrow H^{\bullet+1}(\partial_i \mathcal{Q}^{\beta+i}, \mathcal{IC}(\partial_i \mathcal{Q}^{\beta+i}))$. Now recall that $H^\bullet(\partial_i \mathcal{Q}^{\beta+i}, \mathcal{IC}(\partial_i \mathcal{Q}^{\beta+i})) = H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta)) \otimes H^\bullet(C, \underline{\mathbb{C}}[1])$. So the projection of $H^\bullet(C, \underline{\mathbb{C}}[1]) = \mathbb{C}[1] \oplus \mathbb{C}[-1]$ to $\mathbb{C}[-1]$ defines the projection $p : \ H^{\bullet+1}(\partial_i \mathcal{Q}^{\beta+i}, \mathcal{IC}(\partial_i \mathcal{Q}^{\beta+i})) \longrightarrow H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$.

Finally, for $h \in H^\bullet(\mathcal{Q}^{\beta+i}, \mathcal{IC}(\mathcal{Q}^{\beta+i}))$ we have $f_i(h) = p\mathfrak{f}_i(h) \in H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$.

4.7.2. We leave to the reader the absolutely similar elementary construction of the operators e_i. We only mention that the corresponding local element $\mathfrak{e}_i \in \mathrm{Ext}^1_{\mathcal{Q}^{\beta+i}}(\mathcal{IC}(\partial_i \mathcal{Q}^{\beta+i}), \mathcal{IC}(\mathcal{Q}^{\beta+i}))$ comes from the star extension of the constant sheaf on an open subset of $\mathcal{Q}^{\beta+i}$.

4.8. Proposition. If $i, j \in I$, $i \neq j$, then $e_i f_j = f_j e_i : \ H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta)) \longrightarrow H^\bullet(\mathcal{Q}^{\beta+i-j}, \mathcal{IC}(\mathcal{Q}^{\beta+i-j}))$.

Proof. Using the elementary constructions 4.7.2, 4.7.1 of e_i and f_j we reduce the Proposition to the local calculation in a smooth open subset U of $\mathcal{Q}^{\beta+i}$. This local calculation is nothing else than the following fact. Let $D = D_1 \cup D_2$ be a divisor with normal crossings in U, consisting of two smooth irreducible components. If we shriek extend the constant sheaf on $U - D$ through D_1 and then star extend it through D_2 we get the same result if we first star extend the constant sheaf on $U - D$ through D_2, and then shriek extend it through D_1. $\square$

4.9. For $i \in I$ we define the endomorphism h_i of $H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ as the scalar multiplication by $\langle\beta + 2\check\rho, i'\rangle$ where $i' \in X$ is the simple root.

Proposition. $e_i f_i - f_i e_i = h_i : H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta)) \longrightarrow H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$.

Proof. Let $U \subset \mathcal{Q}^{\beta+i}$ be an open subset such that $W := U \cap \partial_i \mathcal{Q}^{\beta+i}$ consists of quasimaps of defect *exactly* i. Then $W = \overset{\circ}{\mathcal{Q}}{}^\beta \times C$ is a smooth divisor in U. Let $T_W U$ be the normal bundle to W in U. This is a line bundle on W. For $\phi \in \overset{\circ}{\mathcal{Q}}{}^\beta$ we can restrict $T_W U$ to $C = \phi \times C \subset \overset{\circ}{\mathcal{Q}}{}^\beta \times C$, and the degree of the restriction does not depend on a choice of ϕ. We denote this degree by $d_{\beta,i}$.

Using the elementary construction of e_i, f_i we see that the commutator $[e_i, f_i]$ acts as the scalar multiplication by $d_{\beta,i}$.

The equality $d_{\beta,i} = \langle\beta + 2\check\rho, i'\rangle$ is proved in [FK] 4.4. Let us make a few comments on this proof. As stated, it works in the case $\mathbf{G} = SL_n$, moreover, it uses the Laumon resolution $\mathcal{Q}_L^\alpha \longrightarrow \mathcal{Q}^\alpha$. In fact, the proof uses not the whole space $\mathcal{Q}_L^\alpha$ but the open subspace $\mathcal{U}_\alpha^L \subset \mathcal{Q}_L^\alpha$ (see *loc. cit.* 4.2.1). This open subset projects isomorphically into $\mathcal{Q}^\alpha$. More precisely, $\mathcal{U}_\alpha^L \subset \mathcal{Q}^\alpha$ consists of all quasimaps with defect *at most a simple coroot*. Now one can see that the calculations in $\mathcal{U}_\alpha^L$ carried out in §4 of *loc. cit.* do not use any specifics of SL_n and carry over to the case of arbitrary $\mathbf{G}$. This completes the proof of the Proposition. $\square$

4.10. Combining the results of 4.4, 4.7, 4.8, 4.9 we see that e_i, f_i, h_i, $i \in I$, generate the action of the Langlands dual Lie algebra $\mathfrak{g}^L$ on $\oplus_{\alpha\in\mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha))$. Since $\mathfrak{g}^L$ is a *simple* Lie algebra, we obtain an *embedding* $\mathfrak{g}^L \hookrightarrow \mathrm{End}\, (\oplus_{\alpha\in\mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha)))$. The image of $\mathfrak{n}_+^L \subset \mathfrak{g}^L$ under this embedding coincides with the image of the Lie algebra $\mathfrak{a} \subset \mathcal{A}$ (see 4.3) in $\mathrm{End}(\oplus_{\alpha\in\mathbb{N}[I]} H^\bullet(\mathcal{Q}^\alpha, \mathcal{IC}(\mathcal{Q}^\alpha)))$. We deduce that $\mathfrak{a} \cong \mathfrak{n}_+^L$. Hence we obtain the embedding of the enveloping algebras $U(\mathfrak{n}_+^L) \cong U(\mathfrak{a}) \hookrightarrow \mathcal{A}$. Comparing the graded dimensions we see that this embedding is in fact an isomorphism. In particular, $\mathrm{Prim}(\mathcal{A}) = \mathfrak{a} \cong \mathfrak{n}_+^L$.

We have proved the following Theorem.

Theorem. The bialgebra $\mathcal{A}$ is isomorphic to the universal enveloping algebra $U(\mathfrak{n}_+^L)$.

5. Closures of Schubert strata and combinatorics of alcoves

5.1. We recall some combinatorics from [L1] and [L5]. Let E be an $\mathbb{R}$-vector space $Y \otimes_\mathbb{Z} \mathbb{R}$. We equip it with a scalar product $(|)$ extending it by linearity from the basis I : $(i|j) \overset{\mathrm{def}}{=} d\frac{i\cdot j}{d_i d_j}$. Here $i \cdot j$ is a part of Cartan datum (see [L4], 1.1), $d_i \overset{\mathrm{def}}{=} \frac{i\cdot i}{2}$, and $d \overset{\mathrm{def}}{=} \max_{i\in I} d_i$.

The Weyl group $\mathcal{W}_f$ acts on E by orthogonal reflections; it is generated by the reflections $s_i(y) \overset{\mathrm{def}}{=} y - \langle y, i'\rangle i$ (notations of *loc. cit.*, 2.2; we have extended the pairing $\langle, \rangle : Y \times X \longrightarrow \mathbb{Z}$ by linearity to $\langle, \rangle : E \times X \longrightarrow \mathbb{R}$.) We shall regard $\mathcal{W}_f$ as acting on E *on the right*. The set $I\mathcal{W}_f$ is the set $\check{\mathcal{R}}$ of *coroots*, and the set $I\mathcal{W}_f \cap \mathbb{N}[I]$ is the set $\check{\mathcal{R}}^+$ of *positive coroots*. Recall that $\check\rho = \frac{1}{2}\sum_{\check\theta\in\check{\mathcal{R}}^+} \check\theta \in E$.

We consider the following collection

$$\mathfrak{F} \overset{\mathrm{def}}{=} \{H_{\theta,n},\ \theta \in \mathcal{R},\ n \in \mathbb{Z}\}$$

of affine hyperplanes in E:

$$H_{\theta,n} = \{y \in E | \langle y, \theta\rangle = n\}$$

Each $H \in \mathfrak{F}$ defines an orthogonal reflection $y \mapsto y\sigma_H$ in E with fixed point set H. Let Ω be the group of affine motions generated by the σ_H ($H \in \mathfrak{F}$). We shall regard Ω as acting on the right on E.

The connected components of $E - \cup_{H \in \mathfrak{F}}$ are called *alcoves*. The group Ω acts simply transitively on the set $\mathfrak{A}$ of alcoves. We shall denote by A_0^- the following alcove:

$$A_0^- \stackrel{\text{def}}{=} \{y \in E | \langle y, i' \rangle < 0 \; \forall i \in I; \; \langle y, \theta_0 \rangle > -1\}$$

where $\theta_0 \in \mathcal{R}^+$ is the highest root. Then Ω is generated by the reflections in the walls of A_0^-. The subgroup of Ω generated by the reflections in the walls $H_{i',0}$ is just the Weyl group $\mathcal{W}_f$.

Note that in [L5], §§1,8 the group Ω is called the *affine Weyl group* and denoted by W (not to be confused with the *dual affine Weyl group* $W^\sharp$). We will follow the notations of [L1] instead. In particular, we will call $\mathcal{W}$ the group defined as in *loc. cit.*, 1.1. It acts on $\mathfrak{A}$ simply transitively *on the left*, commuting with the action of Ω. As a Coxeter group it is canonically isomorphic to the affine Weyl group Ω. In the notations of *loc. cit.* the subgroup $\mathcal{W}_f \subset \Omega$ is nothing else then Ω_0, and $w_0 = \omega_0$.

Recall that the set S of Coxeter generators of $\mathcal{W}$ can be represented as the faces (walls) of A_0^-. The generator corresponding to the wall $H_{i',0}$, $i \in I$, will be denoted s_i, and the generator corresponding to the wall $H_{\theta_0,-1}$ will be denoted by s_0.

It is easy to see that the intersection of Ω with the group of translations of E is (the group of translations by vectors in) Y. Thus we obtain a normal subgroup $Y \subset \Omega$. It is known that Ω is a semidirect product of $\mathcal{W}_f$ and Y (see [L5], 1.5). In particular, each element $\omega \in \Omega$ can be uniquely written in the form $\omega = w\chi$, $w \in \mathcal{W}_f$, $\chi \in Y$. Combining this observation with the action of Ω on $\mathfrak{A}$ we obtain a bijection

$$\mathcal{W}_f \times Y \stackrel{\sim}{\longrightarrow} \mathfrak{A} : \; (w, \chi) \mapsto A_0^- w\chi$$

We will use this bijection freely, so we will write (w, χ) for an alcove often.

We will denote by $\beta_0 \in \check{\mathcal{R}}^+$ the coroot dual to the highest root θ_0, and by $s_{\beta_0} \in \mathcal{W}_f$ the reflection in $\mathcal{W}_f$ taking $y \in E$ to $y - \langle y, \theta_0 \rangle \beta_0$.

5.1.1. *Lemma.* For any $i \in I$ we have $s_i(w, \chi) = (s_i w, \chi)$, and $s_0(w, \chi) = (s_{\beta_0} w, \chi - \beta_0 w)$.

Proof. Clear. □

5.1.2. Recall that for a pair of alcoves $A, B \in \mathfrak{A}$ the *distance* $d(A, B)$ was defined in [L1], 1.4. For $\chi = \sum_{i \in I} a_i i \in Y$ we define $|\chi| \in \mathbb{Z}$ as follows:

$$|\chi| \stackrel{\text{def}}{=} \sum_{i \in I} a_i$$

For $w \in \mathcal{W}_f$ let $l(w)$ denote the usual length function:

$$l(w) = \sharp(\check{\mathcal{R}}^+ w \cap -\check{\mathcal{R}}^+) = \dim \mathbf{X}_w$$

Lemma. Let $B = (w, \chi)$. Then $d(A_0^-, B) = 2|\chi| + l(w)$.

Proof. For $A = (w_1, \chi_1), B = (w_2, \chi_2)$ let us define $d'(A, B) \stackrel{\text{def}}{=} 2|\chi_2| - 2|\chi_1| + l(w_2) - l(w_1)$. We want to prove $d'(A, B) = d(A, B)$. To this end it suffices to check the properties [L1](1.4.1),(1.4.2) for d' instead of d. This in turn follows easily from the Lemma 5.1.1. □

5.1.3. Recall a few properties of the partial order $\leq$ on $\mathfrak{A}$ introduced in [L1], 1.5.

$$(w_1, \chi) \leq (w_2, \chi) \Leftrightarrow w_1 \leq w_2, \tag{3}$$

where $w_1 \leq w_2$ stands for the usual Bruhat order on $\mathcal{W}_f$.

$$(w_1, \chi_1) \leq (w_2, \chi_2) \Leftrightarrow (w_1, \chi_1 + \chi) \leq (w_2, \chi_2 + \chi) \ \forall \chi \in Y, \tag{4}$$

$$(w, 0) \leq (w s_{\check\theta}, \check\theta) \ \forall \check\theta \in \check{\mathcal{R}}^+, \tag{5}$$

where $s_{\check\theta} \in \mathcal{W}_f$ is the reflection in $\mathcal{W}_f$ taking $y \in E$ to $y - \langle y, \theta\rangle\check\theta$, θ being a root dual to the coroot $\check\theta$.

$$s_0(w, \chi) \leq (w, \chi) \Leftrightarrow \beta_0 w > 0. \tag{6}$$

The proofs are easy if not contained explicitly in [L1]. For the converse we have the following

Lemma. Let $\preceq$ be the minimal partial order on $\mathfrak{A}$ enjoying the properties (3), (4), (5). Then $A \preceq B \Leftrightarrow A \leq B$.

Proof. Easy. $\square$

5.2. Recall the notations of [FM] 8.4, 9.1. The Schubert stratification of $\mathcal{Z}_\chi^\alpha$ reads as follows:

$$\mathcal{Z}_\chi^\alpha = \bigsqcup_{w \in \mathcal{W}_f}^{0 \leq \beta \leq \alpha} \overset{\bullet}{\mathcal{Z}}_{w, \chi - \alpha + \beta}^\beta$$

or, after a change of variable $\eta = \chi - \alpha + \beta$,

$$\mathcal{Z}_\chi^\alpha = \bigsqcup_{w \in \mathcal{W}_f}^{\chi - \alpha \leq \eta \leq \chi} \overset{\bullet}{\mathcal{Z}}_{w, \eta}^{\eta - \chi + \alpha}$$

We want to describe the closure of a stratum $\overset{\bullet}{\mathcal{Z}}_{w, \eta}^{\eta - \chi + \alpha}$.

Theorem. Fix a pair of alcoves $A = (w, \eta), B = (y, \xi)$. For any $\chi \geq \eta, \xi$ and $\alpha \in \mathbb{N}[I]$ sufficiently dominant (such that $\eta - \chi + \alpha \geq 10\check\rho \leq \xi - \chi + \alpha$) we have

a) $\dim \overset{\bullet}{\mathcal{Z}}_{y, \xi}^{\xi - \chi + \alpha} - \dim \overset{\bullet}{\mathcal{Z}}_{w, \eta}^{\eta - \chi + \alpha} = d(A, B)$;

b) $\overset{\bullet}{\mathcal{Z}}_{w, \eta}^{\eta - \chi + \alpha}$ lies in the closure of $\overset{\bullet}{\mathcal{Z}}_{y, \xi}^{\xi - \chi + \alpha}$ iff $A \leq B$.

Proof. a) follows immediately comparing the Lemmas 5.1.2 and [FM] 8.5.2.

The proof of b) occupies the rest of this section.

5.3. Let $\theta \in \mathcal{R}^+$ be a positive root. Let $\mathfrak{g}_\theta$ denote the corresponding $\mathfrak{sl}_2$ Lie subalgebra in $\mathfrak{g}$, and let $\mathbf{G}_\theta$ denote the corresponding SL_2-subgroup in $\mathbf{G}$. Take any $w \in \mathcal{W}_f$ and let $y = s_\theta w$. We will view w, y as $\mathbf{H}$-fixed points in $\mathbf{X}$. The $\mathbf{G}_\theta$-orbit $\mathbf{G}_\theta w = \mathbf{G}_\theta y$ is a smooth rational curve of degree $\check\theta$ in $\mathbf{X}$ (here $\check\theta$ stands for the dual coroot of the root θ). We will view it as a closed subset of $\mathbf{X}$ rather than a parametrized curve. Let us denote this curve by $L_{w,y}$. It is fixed by the $\mathbf{H}$ action on $\mathbf{X}$ (as a subset, not pointwise).

5.3.1. *Lemma.* Let $L \subset \mathbf{X}$ be an irreducible rational curve fixed by the Cartan action. Then $L = L_{w,y}$ for some $w, y = s_\theta w \in \mathcal{W}_f$.

Proof. Since L is a fixed curve it must contain a fixed point $w \in \mathcal{W}_f$. Let $f : \hat{L} \to L \subset \mathbf{X}$ be the normalization of L. The $\mathbf{H}$-action on L extends to the $\mathbf{H}$-action on $\hat{L}$. We choose an $\mathbf{H}$-fixed point in $f^{-1}(w)$ and preserve the name w for this point. Let t be a formal coordinate at $w \in \hat{L}$. The map f gives rise to the homomorphism $f^* : \hat{\mathcal{O}}_{\mathbf{X},w} \to \mathbb{C}[[t]] = \hat{\mathcal{O}}_{\hat{L},w}$ of X-graded rings from the completion of the local ring $\mathcal{O}_{\mathbf{X},w}$ to the ring of formal power series in t. We have $\hat{\mathcal{O}}_{\mathbf{X},w} = \mathbb{C}[[T_w^* \mathbf{X}]]$ — the ring of formal power series on the tangent space of $\mathbf{X}$ at the point w, and the grading is induced by the grading of the cotangent space

$$T_w^* \mathbf{X} = \bigoplus_{\vartheta \in w\mathcal{R}^+} \mathbb{C}x_\vartheta,$$

where $x_\vartheta \in \mathfrak{g}^*$ is an $\mathbf{H}$-eigenvector with the eigenvalue ϑ. Since the roots in $w\mathcal{R}^+$ are pairwise linearly independent, it follows that there exists a root $\theta \in w\mathcal{R}^+$ and a positive integer n such that the map f^* is given by

$$f^*(x_\vartheta) = \begin{cases} t^n, & \text{if } \vartheta = \theta \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the formal jet of the map $f : \hat{L} \to \mathbf{X}$ coincides with the formal jet of the map $\varphi_{w,y}^n : \mathbb{P}^1 \to \mathbf{X}$ given by the composition of the n-fold covering $\mathbb{P}^1 \to L_{w,y}$ ramified over the points w and y, and of the embedding $L_{w,y} \to \mathbf{X}$. Now irreducibility of L implies $L = L_{w,y}$. $\square$

5.3.2. *Remark.* The fixed curves $L_{w,y}$ and $L_{w',y'}$ intersect nontrivially iff $w = w'$ or $w = y'$ or $y = w'$ or $y = y'$. In effect, a point of intersection of two fixed curves has to be a fixed point.

5.4. Definition. Let $f : \mathcal{C} \to \mathbf{X}$ be a stable map (see [Ko]) from a genus 0 curve $\mathcal{C}$ into the flag variety $\mathbf{X}$. Let $w, y \in \mathcal{W}_f$. If $f(\mathcal{C}) \cap \mathbf{X}_w \neq \emptyset$ and $f(\mathcal{C}) \cap \overline{\mathbf{X}}_y \neq \emptyset$ we will say that the pair (w, y) is $(f, \mathcal{C})$-connected. If $\deg \mathcal{C} = \gamma$ we will say also that (w, y) is γ-connected.

5.4.1. *Lemma.* A pair (w, y) is γ-connected iff there exists a collection $(\check{\theta}_1, \ldots, \check{\theta}_k)$ of positive coroots $\check{\theta}_r \in \check{\mathcal{R}}^+$ such that $\check{\theta}_1 + \ldots + \check{\theta}_k \leq \gamma$ and $s_{\theta_k} \ldots s_{\theta_1} w \leq y$.

Proof. Suppose the pair (w, y) is $(f, \mathcal{C})$-connected, where $f : \mathcal{C} \to \mathbf{X}$ is a stable map of degree γ. Then we have $f(\mathcal{C}) \cap \mathbf{X}_w \neq \emptyset$. Acting on f by an element of the Borel subgroup we can assume that $w \in f(\mathcal{C})$. Acting by the Cartan subgroup $\mathbf{H}$ we can degenerate $(f, \mathcal{C})$ into an $\mathbf{H}$-fixed stable map $f' : \mathcal{C}' \to \mathbf{X}$. Since $w \in f(\mathcal{C})$ and w is an $\mathbf{H}$-fixed point, we will have $w \in f'(\mathcal{C}')$. Similarly, since $f(\mathcal{C}) \cap \overline{\mathbf{X}}_y \neq \emptyset$ and $\overline{\mathbf{X}}_y$ is a closed $\mathbf{H}$-invariant subspace, we will have $f'(\mathcal{C}') \cap \overline{\mathbf{X}}_y \neq \emptyset$, hence $f'(\mathcal{C}') \ni y'$ for some $y' \in \mathcal{W}_f$ such that $y' \leq y$. The image of an $\mathbf{H}$-fixed stable map is a connected union of $\mathbf{H}$-fixed curves, hence there exists a sequence $L_1, \ldots, L_k$ of $\mathbf{H}$-fixed curves such that $w \in L_1$, $y' \in L_k$ and $L_r \cap L_{r+1} \neq \emptyset$. We can assume that $L_r = L_{w_r, w_{r+1}}$ $(r = 1, \ldots, k)$, where $w_1 = w$, $w_{r+1} = s_{\theta_r} w_r$ and $w_{k+1} = y'$. Since

$$\gamma = \deg f = \deg f' \geq \deg L_1 + \ldots + \deg L_k = \check{\theta}_1 + \ldots + \check{\theta}_k$$

the Lemma follows. $\square$

5.4.2. *Corollary.* If $w \not\leq y$ and (w, y) is γ-connected then

$$l(w) \leq l(y) + 2|\gamma| - 1$$

and equality holds only if $\gamma = \check{\theta}$, $l(s_\theta) = 2|\check{\theta}| - 1$ and $s_\theta w = y$ for some $\check{\theta} \in \check{\mathcal{R}}^+$.

Proof. By the Lemma 5.4.1 we have

$$l(y) \geq l(s_{\theta_k} \dots s_{\theta_1} w) \geq l(w) - l(s_{\theta_1}) - \dots - l(s_{\theta_k}) \geq$$
$$\geq l(w) - (2|\check{\theta}_1| - 1) - \dots - (2|\check{\theta}_k| - 1) \geq l(w) - 2|\gamma| + k \geq l(w) - 2|\gamma| + 1.$$

The Corollary follows. $\square$

5.5. Proposition. A Schubert stratum $\overset{\bullet}{\mathcal{Z}}_{w,\eta}^{\eta-\chi+\alpha}$ lies in the closure of $\overset{\bullet}{\mathcal{Z}}_{y,\xi}^{\xi-\chi+\alpha}$ iff $\eta \leq \xi$ and the pair (w,y) is $(\xi - \eta)$-connected.

Proof. We start with the "only if" part. Both strata lie in the closed subspace $\mathcal{Z}_\xi^{\xi-\chi+\alpha} \subset \mathcal{Z}_\chi^\alpha$, hence we can consider them as subspaces in $\mathcal{Z}_\xi^{\xi-\chi+\alpha}$.

Let $\mathcal{Q}_K^{\xi-\chi+\alpha} = \overline{M}_{0,0}(\mathbb{P}^1 \times \mathbf{X}, (1, \xi - \chi + \alpha))$ be the Kontsevich space of stable maps from the genus zero curves without marked points to $\mathbb{P}^1 \times \mathbf{X}$ of bidegree $(1, \xi - \chi + \alpha)$ (see [Ko]). The map $\pi : \mathcal{Q}_K^{\xi-\chi+\alpha} \longrightarrow \mathcal{Q}^{\xi-\chi+\alpha}$ is constructed in the Appendix. For a stable map $f : \mathcal{C} \longrightarrow \mathbb{P}^1 \times \mathbf{X}$ in $\mathcal{Q}_K^{\xi-\chi+\alpha}$ we denote by $f' : \mathcal{C} \longrightarrow \mathbb{P}^1$ (resp. $f'' : \mathcal{C} \longrightarrow \mathbf{X}$) its composition with the first (resp. second) projection.

Let $\mathcal{K}_\xi^{\xi-\chi+\alpha} \subset \mathcal{Q}_K^{\xi-\chi+\alpha}$ be the preimage of the subspace $\mathcal{Z}_\xi^{\xi-\chi+\alpha} \subset \mathcal{Q}^{\xi-\chi+\alpha}$ under the map $\pi : \mathcal{Q}_K^{\xi-\chi+\alpha} \to \mathcal{Q}^{\xi-\chi+\alpha}$. It is a locally closed subspace consisting of all stable maps $f : \mathcal{C} \to \mathbb{P}^1 \times \mathbf{X}$ such that $f'^{-1}(\infty)$ is a point (i.e. $\mathcal{C}$ has no vertical component over $\infty \in \mathbb{P}^1$), and $f''(f'^{-1}(\infty)) = w_0 \in \mathbf{X}$. Consider the preimage of the Schubert stratification of $\mathcal{Z}_\xi^{\xi-\chi+\alpha}$:

$$\mathcal{K}_\xi^{\xi-\chi+\alpha} = \bigsqcup_{w \in \mathcal{W}_f}^{\chi-\alpha \leq \eta \leq \chi} \overset{\bullet}{\mathcal{K}}_{w,\eta}^{\eta-\chi+\alpha}.$$

Here $\overset{\bullet}{\mathcal{K}}_{w,\eta}^{\eta-\chi+\alpha}$ is just the subspace of all stable maps $f : \mathcal{C} \to \mathbb{P}^1 \times \mathbf{X}$ such that

$$\deg \mathcal{C}_1 = \xi - \eta \quad \text{and} \quad f''(P) \in \mathbf{X}_w,$$

where $\mathcal{C}_1 = f'^{-1}(0)$ is the vertical component of $\mathcal{C}$ over the point $0 \in \mathbb{P}^1$, and $\mathcal{C}_0$ is the main component of $\mathcal{C}$ (the one which projects isomorphically onto $\mathbb{P}^1$ under f'); the point P is the intersection of these components: $P = \mathcal{C}_0 \cap \mathcal{C}_1$; and as always, $f''(f'^{-1}(\infty)) = w_0$. It follows that $\overset{\bullet}{\mathcal{Z}}_{w,\eta}^{\eta-\chi+\alpha}$ lies in the closure of $\overset{\bullet}{\mathcal{Z}}_{y,\xi}^{\xi-\chi+\alpha}$ iff the map

$$\pi : \overset{\bullet}{\mathcal{K}}_{w,\eta}^{\eta-\chi+\alpha} \bigcap \overline{\overset{\bullet}{\mathcal{K}}_{y,\xi}^{\xi-\chi+\alpha}} \to \overset{\bullet}{\mathcal{Z}}_{w,\eta}^{\eta-\chi+\alpha}$$

is surjective. In particular, the above intersection must be non-empty.

Consider the subspace $\mathcal{K}_{y,\xi}^{\xi-\chi+\alpha} \subset \mathcal{K}_\xi^{\xi-\chi+\alpha}$ formed by the stable maps $f : \mathcal{C} \to \mathbb{P}^1 \times \mathbf{X}$ such that $f(\mathcal{C}) \cap (\{0\} \times \overline{\mathbf{X}}_y) \neq \emptyset$. In other words, $\mathcal{K}_{y,\xi}^{\xi-\chi+\alpha} \subset \mathcal{K}_\xi^{\xi-\chi+\alpha}$ is formed by the stable maps such that $f''(\mathcal{C}_1) \cap \overline{\mathbf{X}}_y \neq \emptyset$. The subspace $\mathcal{K}_{y,\xi}^{\xi-\chi+\alpha}$ is obviously closed and irreducible. It contains $\overset{\bullet}{\mathcal{K}}_{y,\xi}^{\xi-\chi+\alpha}$ as an open subspace, hence

$$\mathcal{K}_{y,\xi}^{\xi-\chi+\alpha} = \overline{\overset{\bullet}{\mathcal{K}}_{y,\xi}^{\xi-\chi+\alpha}}.$$

The intersection $\overset{\bullet}{\mathcal{K}}_{w,\eta}^{\eta-\chi+\alpha} \bigcap \overline{\overset{\bullet}{\mathcal{K}}_{y,\xi}^{\xi-\chi+\alpha}}$ consists of all stable maps such that

$$\deg \mathcal{C}_1 = \xi - \eta, \qquad f''(P) \in \mathbf{X}_w \quad \text{and} \quad f''(\mathcal{C}_1) \cap \overline{\mathbf{X}}_y \neq \emptyset.$$

Therefore the intersection is non-empty only if the pair (w,y) is $(\xi - \eta)$-connected (recall that $P \in \mathcal{C}_1$). This completes the proof of the "only if" part.

On the other hand, if (w,y) is $(\xi-\eta)$-connected, then for any stable map $(f,\mathcal{C})$ we can replace the component $\mathcal{C}_1$ by a stable curve of degree $(\xi-\eta)$, connecting the point $f''(P) \in \mathbf{X}_w$ with $\overline{\mathbf{X}}_y$. Such replacement will not affect $\pi(f,\mathcal{C})$, but the new curve will lie in the intersection $\overset{\bullet}{\mathcal{K}}{}^{\eta-\chi+\alpha}_{w,\eta} \bigcap \overset{\bullet}{\mathcal{K}}{}^{\xi-\chi+\alpha}_{y,\xi}$, hence the above intersection maps surjectively onto $\overset{\bullet}{\mathcal{Z}}{}^{\eta-\chi+\alpha}_{w,\eta}$ and the Proposition follows. $\square$

5.6. Now we can prove the Theorem 5.2.

We define the *adjacency order* $\preceq$ on $\mathfrak{A}$ as follows. We say that $A = (w,\eta) \preceq (y,\xi) = B$ if for any $\chi \geq \eta, \xi$ and sufficiently dominant $\alpha \in \mathbb{N}[I]$ the stratum $\overset{\bullet}{\mathcal{Z}}{}^{\eta-\chi+\alpha}_{w,\eta}$ lies in the closure of $\overset{\bullet}{\mathcal{Z}}{}^{\xi-\chi+\alpha}_{y,\xi}$.

First we check that $\preceq$ satisfies the relations (3–5) of 5.1.3. To this end let us rephrase these properties via criterion 5.5. The property (3) means that (w_1, w_2) is 0-connected iff $w_1 \leq w_2$. The property (4) expresses the fact that the adjacency of strata depends on difference of degrees only, which is evident from the criterion 5.5. The property (5) means that the pair $(w, s_\theta w)$ is $\check{\theta}$-connected. This is indeed so since the curve $L_{w, s_\theta w}$ has degree $\check{\theta}$ and connects w with $s_\theta w$.

Now the adjacency order $\preceq$ is clearly generated by the adjacencies in codimension 1. So it remains to check that the only adjacencies in codimension 1 are the ones of type (5) or type (3) with $l(w_1) = l(w_2) - 1$.

Assume that $\overset{\bullet}{\mathcal{Z}}{}^{\eta-\chi+\alpha}_{w,\eta}$ lies in the closure of $\overset{\bullet}{\mathcal{Z}}{}^{\xi-\chi+\alpha}_{y,\xi}$ and has codimension 1. Since

$$\dim \overset{\bullet}{\mathcal{Z}}{}^{\eta-\chi+\alpha}_{w,\eta} = 2|\eta - \chi + \alpha| + l(w) - \dim \mathbf{X}, \quad \dim \overset{\bullet}{\mathcal{Z}}{}^{\xi-\chi+\alpha}_{y,\xi} = 2|\xi - \chi + \alpha| + l(y) - \dim \mathbf{X}$$

it follows that $l(w) = l(y) + 2|\xi - \eta| - 1$. On the other hand, by the criterion 5.5 the pair (w,y) is $(\xi-\eta)$-connected. If $\xi = \eta$ then $l(w) = l(y) - 1$ and we are in the situation of type (3). If $\xi > \eta$ then $w \not\leq y$ and by the Corollary 5.4.2 it follows that $\xi - \eta = \check{\theta}$ and $y = s_\theta w$ for some $\theta \in \mathcal{R}^+$, i.e. we are in the situation of type (5).

The Theorem follows. $\square$

5.7. Recall that $\overset{\bullet}{\mathcal{Q}}{}^\gamma \subset \mathcal{Q}^\gamma$ is the open subset formed by all the quasimaps without defect at $0 \in C$ (i.e. defined at $0 \in C$, see [FM] 8.1). For $w \in \mathcal{W}_f$ we define the locally closed subset $\overset{\bullet}{\mathcal{Q}}{}^\gamma_w \subset \overset{\bullet}{\mathcal{Q}}{}^\gamma$ formed by all the quasimaps ϕ such that $\phi(0) \in \mathbf{X}_w \subset \mathbf{X}$ (cf. *loc. cit.* 8.4). Its closure will be denoted by $\mathcal{Q}^\gamma_w$; it coincides with the closure of the fine Schubert stratum $\overset{\circ}{\mathcal{Q}}{}^\gamma_w$ defined in *loc. cit.* 8.4.1.

Corollary of the Proof. Let $\gamma \leq \beta \in \mathbb{N}[I]$, $w, y \in \mathcal{W}_f$. Then $\mathcal{Q}^\beta_y \supset \mathcal{Q}^\gamma_w$ iff $(y, \beta) =: B \geq A := (w, \gamma)$. Also, $\dim \mathcal{Q}^\beta_y - \dim \mathcal{Q}^\gamma_w = d(A, B)$. $\square$

5.8. We will denote the closure of a fine Schubert stratum $\overset{\circ}{\mathcal{Z}}{}^\beta_y$ by $\mathcal{Z}^\beta_y$ (note that it was denoted by $\overline{\mathcal{Z}}{}^\beta_y$ in [FM]. We hope this will cause no confusion.) Similarly, the closure of $\overset{\bullet}{\mathcal{Z}}{}^\beta_{y,\xi}$ (see *loc. cit.* 9.4) will be denoted by $\mathcal{Z}^\beta_{y,\xi}$.

We promote "alcovic" notations. Suppose we are in the situation of the above theorem, i.e. $A = (w,\eta) \leq (y,\xi) = B$; $\chi \geq \eta, \xi$, and $\alpha \in \mathbb{N}[I]$ is sufficiently dominant. We introduce new variables $\gamma = \eta - \chi + \alpha$ and $\beta = \xi - \chi + \alpha$ (note that both γ and β necessarily lie in $\mathbb{N}[I]$). Then the above Theorem claims that $\overset{\bullet}{\mathcal{Z}}{}^\gamma_{w,\eta}$ lies in the closure $\mathcal{Z}^\beta_{y,\xi}$ of $\overset{\bullet}{\mathcal{Z}}{}^\beta_{y,\xi}$. We will denote this closure by $\mathcal{Z}^\beta_B$; thus we have the closed embedding $\mathcal{Z}^\gamma_A \hookrightarrow \mathcal{Z}^\beta_B$.

With our new notations, it is only natural to rename the snop $\mathcal{L}(w,\eta)$ (see *loc. cit.* 9.4) into $\mathcal{L}(A)$, and $\mathcal{M}(w,\eta)$ into $\mathcal{M}(A)$, and $\mathcal{DM}(w,\eta)$ into $\mathcal{DM}(A)$.

5.8.1. Recall (see *loc. cit.* 9.3) that we call $\alpha \in \mathbb{N}[I]$ sufficiently dominant, and write $\alpha \gg 0$, iff $\alpha \geq 10\check{\rho}$. For such α we defined the open subvariety $\ddot{\mathcal{Z}}^\alpha \subset \mathcal{Z}^\alpha$. For $\beta \leq \alpha$ the closed subvariety $\mathcal{Z}_y^\beta \cap \ddot{\mathcal{Z}}^\alpha \subset \ddot{\mathcal{Z}}^\alpha$ is nonempty iff $\beta \gg 0$. In this case it will be denoted by the same symbol $\mathcal{Z}_y^\beta$; we hope this will cause no confusion. Again, for $0 \ll \gamma \leq \beta$ we have $\mathcal{Z}_y^\beta \supset \mathcal{Z}_w^\gamma$ iff $(y,\beta) =: B \geq A := (w,\gamma)$. Also, $\dim \mathcal{Z}_y^\beta - \dim \mathcal{Z}_w^\gamma = d(A,B)$.

6. Mixed snops and convolution

6.1. In this section we compute the stalks of irreducible snops. First we prove that they carry a natural Tate Hodge structure. We start with preliminary results about affine Grassmannian. Recall the stratification of the affine Grassmannian $\mathcal{G}$ by the Iwahori orbits $\mathcal{G}_{w,\eta}$ described e.g. in [FM] 10.4. The orbit closure $\overline{\mathcal{G}}_{w,\eta}$ is partitioned into its intersections with semiinfinite orbits: $\overline{\mathcal{G}}_{w,\eta} = \sqcup_{\alpha \in Y}(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha)$. The irreducible perverse sheaf $\mathcal{IC}(\overline{\mathcal{G}}_{w,\eta})$ has the mixed Hodge module counterpart $\mathbf{IC}(\overline{\mathcal{G}}_{w,\eta})$.

Proposition. a) $H_c^\bullet(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha, \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$ is a pure Hodge complex of weight $\dim \overline{\mathcal{G}}_{w,\eta}$. It is a direct sum of Tate Hodge structures;

b) $H_c^\bullet(\overline{\mathcal{G}}_{w,\eta} \cap \overline{T}_\alpha, \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$ is a pure Hodge complex of weight $\dim \overline{\mathcal{G}}_{w,\eta}$. It is a direct sum of Tate Hodge structures.

Proof. a) Let us denote the locally closed embedding of $\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha$ (resp. $\overline{\mathcal{G}}_{w,\eta} \cap S_\alpha$) into $\overline{\mathcal{G}}_{w,\eta}$ by i_T (resp. i_S). One can construct a natural isomorphism $H^\bullet(\overline{\mathcal{G}}_{w,\eta} \cap S_\alpha, i_S^! \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta})) \xrightarrow{\sim} H_c^\bullet(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha, i_T^* \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$ (see [MV], Opposite parabolic restrictions). On the other hand, $\mathbf{IC}(\overline{\mathcal{G}}_{w,\eta})$ is a pure Hodge module of weight $\dim \overline{\mathcal{G}}_{w,\eta}$, while $H_c^\bullet(?)$ and i_T^* decrease weights, and $H^\bullet(?)$ and $i_S^!$ increase weights.

It remains to prove that $H_c^\bullet(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha, \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$ is a direct sum of Tate Hodge structures. To this end consider the Cousin spectral sequence associated with the partition into locally closed subsets $\overline{\mathcal{G}}_{w,\eta} = \sqcup_{\alpha \in Y}(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha)$. We have $E_2^{p,q} \Longrightarrow H^{p+q}(\overline{\mathcal{G}}_{w,\eta}, \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$, and $E_2^{p,q} = \oplus H_c^{p+q}(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha, i_T^* \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$; the sum is taken over α such that the codimension of $\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha$ in $\overline{\mathcal{G}}_{w,\eta}$ equals q. For the weight reasons the differentials in this spectral sequence vanish, and it collapses at the second term. Hence $H^\bullet(\overline{\mathcal{G}}_{w,\eta}, \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta})) \cong \oplus_{\alpha \in Y} H_c^\bullet(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha, i_T^* \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$. On the other hand, the LHS is well known to be a direct sum of Tate Hodge structures.

So a) is proved. Now b) follows by the application of Cousin spectral sequence associated with the partition into the locally closed subsets $\overline{\mathcal{G}}_{w,\eta} \cap \overline{T}_\alpha = \sqcup_{\beta \geq \alpha}(\overline{\mathcal{G}}_{w,\eta} \cap T_\beta)$. $\square$

6.1.1. *Remark.* Let $g \in \mathbf{G}$ lie in the normalizer $N(\mathbf{H})$ of the Cartan subgroup $\mathbf{H}$, and let T_α^g denote $g(T_\alpha)$: the action of g on the semiinfinite orbit in $\mathcal{G}$. The same argument as above proves that $H_c^\bullet(\overline{\mathcal{G}}_{w,\eta} \cap T_\alpha^g, \mathbf{IC}(\overline{\mathcal{G}}_{w,\eta}))$ is a pure Hodge complex of weight $\dim \overline{\mathcal{G}}_{w,\eta}$; it is a direct sum of Tate Hodge structures.

6.2. According to the Theorem 3.3.2, the simple stalk $\mathfrak{s}_\alpha^* \iota_\alpha^* \mathbf{IC}^\alpha = \mathbf{IC}_{\{\{\alpha\}\}}^0$ is pure of weight $2|\alpha|$, and it is a direct sum of Tate Hodge structures. By factorization, the Hodge module $\mathbf{IC}^\alpha$ on $\mathcal{Z}^\alpha$ is pointwise pure, and all its stalks are direct sums of Tate Hodge structures. Hence the Hodge module $\mathbf{IC}(\mathcal{Q}^\alpha)$ on $\mathcal{Q}^\alpha$ is pointwise pure, and all its stalks are direct sums of Tate Hodge structures. In what follows, such Hodge modules will be called *pointwise pure Tate Hodge modules.*

Recall the proalgebraic variety $\mathfrak{Q}^\alpha$ introduced in [FM] 10.6. The theory of mixed Hodge modules on such varieties is developed in [KT1]. In particular, we will be interested in the irreducible Hodge module

$\mathbf{IC}(\mathfrak{Q}^\alpha)$, cf. [FM] 10.7.3. The same argument as in *loc. cit.* 12.7 proves that $\mathbf{IC}(\mathfrak{Q}^\alpha)$ is a pointwise pure Tate Hodge module.

For the future reference, let us collect the above facts into a Theorem.

Theorem. The Hodge modules $\mathbf{IC}^\alpha, \mathbf{IC}(\mathcal{Q}^\alpha), \mathbf{IC}(\mathfrak{Q}^\alpha)$ are the pointwise pure Tate Hodge modules.

6.3. Theorem. $\mathbf{IC}(\mathcal{Q}_w^\alpha)$ (see 5.7) is a pointwise pure Tate Hodge module.

Proof. The usual argument with factorization reduces the proof to the study of the stalk of $\mathbf{IC}(\mathcal{Q}_w^\alpha)$ at the fine Schubert stratum $\overset{\circ}{\mathcal{Q}}{}_y^0 \times (\mathbb{P}^1 - 0)_\Gamma^\beta \times 0^{\alpha-\beta} \subset \mathcal{Q}^\alpha$, $\beta \leq \alpha$, $y \in \mathcal{W}_f$ (see *loc. cit.* 8.4.1). Moreover, by the same factorization argument, this stalk will not change if we add $\gamma \in \mathbb{N}[I]$ to both α and β. We will choose γ sufficiently dominant to ensure that $\alpha + \gamma \in Y^+$. So we may and will suppose that α is dominant. Furthermore, we can choose any point in the stratum $\overset{\circ}{\mathcal{Q}}{}_y^0 \times (\mathbb{P}^1 - 0)_\Gamma^\beta \times 0^{\alpha-\beta}$, and we will choose a point $\phi = (\mathfrak{L}_\lambda)_{\lambda \in X^+} \in \overset{\circ}{\mathcal{Q}}{}_y^\beta$ where $\mathfrak{L}_\lambda = \mathcal{V}_\lambda^{\mathbf{N}_+^y}(\langle \beta - \alpha, \lambda \rangle \cdot 0 - \langle D, \lambda \rangle)$, and $\mathbf{N}_+^y$ is the nilpotent subgroup conjugated to $\mathbf{N}_+$ by any representative $\dot{y}$ of y in the normalizer of $\mathbf{H}$ (see *loc. cit.* 3.3); $D \in (\mathbb{P}^1 - 0)_\Gamma^\beta$.

Recall the setup of the Theorem 12.12 of *loc. cit.* We consider the Hodge module $\mathcal{F} = \mathbf{IC}(\overline{\mathcal{G}}_{w,\alpha})$ on $\overline{\mathcal{G}}_\alpha$ (recall that we assumed $\alpha \in Y^+$), and the convolution $\mathbf{c}_\mathcal{Q}^0(\mathcal{F}) = \mathbf{q}_*(\mathbf{IC}(\mathfrak{Q}^0) \otimes \mathbf{p}^* \mathcal{F})[-\dim \underline{\mathfrak{M}}^\alpha]$ which is a mixed Hodge module on $\mathcal{Q}^\alpha$ smooth along the fine Schubert stratification. By the Proposition 12.3b) of *loc. cit.* $\mathbf{c}_\mathcal{Q}^0(\mathbf{IC}(\overline{\mathcal{G}}_{w,\alpha})) = \mathbf{q}_* \mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0)$. By the Decomposition Theorem, $\mathbf{q}_* \mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0)$ is a direct sum of irreducible Hodge modules on $\mathcal{Q}^\alpha$. By 10.4.2 of *loc. cit.* $\mathbf{q}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0) = \mathcal{Q}_w^\alpha$, and hence $\mathbf{q}_* \mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0)$ contains a direct summand $\mathbf{IC}(\mathcal{Q}_w^\alpha)$. Thus it suffices to prove that $\mathbf{c}_\mathcal{Q}^0(\mathcal{F})$ is a pointwise pure Tate Hodge module.

More precisely, we are interested in the stalk of $\mathbf{c}_\mathcal{Q}^0(\mathcal{F})$ at the point $\phi \in \overset{\circ}{\mathcal{Q}}{}_y^\beta$. As in the proof of 12.9.1 of *loc. cit.*, we have $\mathbf{q}^{-1}(\phi) = \overline{\mathcal{G}}_{w,\alpha} \cap \overline{T}_{\beta-\alpha}^y = \sqcup_{\gamma \geq 0}(\overline{\mathcal{G}}_{w,\alpha} \cap T_{\beta-\alpha+\gamma}^y)$ where $T_{\beta-\alpha+\gamma}^y$ is $\dot{y}(T_{\beta-\alpha+\gamma})$: the action of $\dot{y} \in \mathbf{G}$ on the semiinfinite orbit in $\mathcal{G}$. According to the Lemma 13.1 of *loc. cit.*, we have $\mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0)|_{\overline{\mathcal{G}}_{w,\alpha} \cap T_{\beta-\alpha+\gamma}^y} = \mathbf{IC}(\overline{\mathcal{G}}_{w,\alpha})|_{\overline{\mathcal{G}}_{w,\alpha} \cap T_{\beta-\alpha+\gamma}^y} \otimes \mathbf{IC}_\Gamma^\gamma$. By 6.1.1, $H_c^\bullet(\overline{\mathcal{G}}_{w,\alpha} \cap T_{\beta-\alpha+\gamma}^y, \mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0))$ is a pure direct sum of Tate Hodge structures, and by 6.2, $\mathbf{IC}_\Gamma^\gamma$ is also a pure direct sum of Tate Hodge structures; thus $H_c^\bullet(\overline{\mathcal{G}}_{w,\alpha} \cap T_{\beta-\alpha+\gamma}^y, \mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0) \otimes \mathbf{IC}_\Gamma^\gamma)$ is also a pure direct sum of Tate Hodge structures.

Hence the Cousin spectral sequence associated to the partition of $\mathbf{q}^{-1}(\phi) = \overline{\mathcal{G}}_{w,\alpha} \cap \overline{T}_{\beta-\alpha}^y$ into the locally closed subsets $\overline{\mathcal{G}}_{w,\alpha} \cap T_{\beta-\alpha+\gamma}^y$ collapses at the second term for the weight reasons. We deduce that $\mathbf{c}_\mathcal{Q}^0(\mathcal{F})_{(\phi)} = H^\bullet(\mathbf{q}^{-1}(\phi), \mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0)) \cong \oplus_{\gamma \geq 0} H_c^\bullet(\overline{\mathcal{G}}_{w,\alpha} \cap T_{\beta-\alpha+\gamma}^y, \mathbf{IC}(\mathcal{G}\mathcal{Q}_{w,\alpha}^0) \otimes \mathbf{IC}_\Gamma^\gamma)$ is also a pure direct sum of Tate Hodge structures. This completes the proof of the Theorem. $\square$

6.4. The subsections 6.4–6.8 are quite parallel to §§10–12 of [FM].

6.4.1. Let G be the usual affine flag manifold $\mathbf{G}((z))/\mathbf{I}$. It is the ind-scheme representing the functor of isomorphism classes of triples $(\mathcal{T}, \tau, \mathfrak{t})$ where $\mathcal{T}$ is a $\mathbf{G}$-torsor on $\mathbb{P}^1$, and τ is its section (trivialization) defined off 0, while $\mathfrak{t}$ is its $\mathbf{B}$-reduction at 0. It is equipped with a natural action of proalgebraic Iwahori group $\mathbf{I}$, and the orbits of this action are naturally numbered by the affine Weyl group $\mathcal{W}$: $\mathsf{G} = \sqcup_{x \in \mathcal{W}} \mathsf{G}_x$. An orbit G_y lies in the closure $\overline{\mathsf{G}}_x$ iff $y \leq x$ in the usual Bruhat order on $\mathcal{W}$. The orbit closure $\overline{\mathsf{G}}_x$ has a natural structure of projective variety. The natural projection $\mathrm{pr}: \mathsf{G} \longrightarrow \mathcal{G}$ is a fibration with the fiber $\mathbf{X}$. All these facts are very well known, see e.g. [KT2].

6.4.2. Let M be the affine flag scheme representing the functor of isomorphism classes of $\mathbf{G}$-torsors on $\mathbb{P}^1$ equipped with trivialization in the formal neighbourhood of ∞ and with $\mathbf{B}$-reduction at 0 (see [Ka]

and [KT1]). It is equipped with a natural action of the opposite Iwahori group $\mathbf{I}_- \subset \mathbf{G}[[z^{-1}]]$, and the orbits of this action are naturally numbered by the affine Weyl group $\mathcal{W}$: $\mathsf{M} = \sqcup_{x \in \mathcal{W}} \mathsf{M}_x$. An orbit M_y lies in the closure $\underline{\mathsf{M}}_x$ iff $y \geq x$ in the usual Bruhat order on $\mathcal{W}$. For any $x \in \mathcal{W}$ the union of orbits $\mathsf{M}^x := \sqcup_{\mathcal{W} \ni y \leq x} \mathsf{M}_y$ forms an open subscheme of M. This subscheme is a projective limit of schemes of finite type, all the maps in projective system being fibrations with affine fibers. Moreover, M^x is equipped with a free action of a prounipotent group $\mathbf{G}^x$ (a congruence subgroup in $\mathbf{G}[[z^{-1}]]$) such that the quotient $\underline{\mathsf{M}}^x$ is a smooth scheme of finite type.

The natural projection pr : $\mathsf{M} \longrightarrow \mathfrak{M}$ (see [FM] 10.6) forgetting a $\mathbf{B}$-reduction at 0 is a fibration with the fiber $\mathbf{X}$.

Restricting a trivialization of a $\mathbf{G}$-torsor from $\mathbb{P}^1 - 0$ to the formal neighbourhood of ∞ we obtain the closed embedding $\mathbf{i}$: $\mathsf{G} \longrightarrow \mathsf{M}$. The intersection of M_x and G_y is nonempty iff $y \leq x$, and then it is transversal. Thus, $\overline{\mathsf{G}}_x \subset \mathsf{M}^x$. According to [KT2], the composition $\overline{\mathsf{G}}_x \hookrightarrow \mathsf{M}^x \longrightarrow \underline{\mathsf{M}}^x$ is a closed embedding.

6.4.3. Following 10.6.3 of [FM] we define for *arbitrary* $\alpha \in Y$ the scheme $\overset{\circ}{\mathsf{Q}}{}^\alpha$ (resp. Q^α) representing the functor of isomorphism classes of triples $(\mathcal{T}, \mathfrak{t}, (\mathfrak{L}_\lambda)_{\lambda \in X^+})$ where $\mathcal{T}$ is a $\mathbf{G}$-torsor trivialized in the formal neighbourhood of $\infty \in \mathbb{P}^1$, and $\mathfrak{t}$ is its $\mathbf{B}$-reduction at 0, while $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda^{\mathcal{T}}$, $\lambda \in X^+$, is a collection of line subbundles (resp. invertible subsheaves) of degree $\langle -\alpha, \lambda \rangle$ satisfying the Plücker conditions (cf. *loc. cit.*). The evident projection $\overset{\circ}{\mathsf{Q}}{}^\alpha \longrightarrow \mathsf{M}$ (resp. $\mathsf{Q}^\alpha \longrightarrow \mathsf{M}$) will be denoted by $\overset{\circ}{\mathbf{p}}$ (resp. $\mathbf{p}$). The open embedding $\overset{\circ}{\mathsf{Q}}{}^\alpha \hookrightarrow \mathsf{Q}^\alpha$ will be denoted by $\mathbf{j}$. Clearly, $\mathbf{p}$ is projective, and $\overset{\circ}{\mathbf{p}} = \mathbf{p} \circ \mathbf{j}$.

The free action of prounipotent group $\mathbf{G}^x$ on M^x lifts to the free action of $\mathbf{G}^x$ on the open subscheme $\mathbf{p}^{-1}(\mathsf{M}^x) \subset \mathsf{Q}^\alpha$. The quotient is a scheme of finite type $\underline{\mathsf{Q}}^{\alpha,x}$ equipped with the projective morphism $\mathbf{p}$ to $\underline{\mathsf{M}}^x$. There exists a $\mathbf{I}_-$-invariant stratification $\mathfrak{S}$ of Q^α such that $\mathbf{p}$ is stratified with respect to $\mathfrak{S}$ and the stratification $\mathsf{M} = \sqcup_{x \in \mathcal{W}} \mathsf{M}_x$. One can define perverse sheaves and mixed Hodge modules on Q^α smooth along $\mathfrak{S}$ following the lines of [KT1]. In particular, we have the irreducible Hodge module $\mathbf{IC}(\mathsf{Q}^\alpha)$.

6.4.4. Following 8.4.1 of [FM] we introduce the *fine stratification* of Q^α according to defects of invertible subsheaves:

$$\mathsf{Q}^\alpha = \bigsqcup_{\Gamma \in \mathfrak{P}(\beta-\gamma)}^{\alpha \geq \beta \geq \gamma} \overset{\circ}{\mathsf{Q}}{}^\gamma \times (C-0)_\Gamma^{\beta-\gamma} \times 0^{\alpha-\beta}$$

and the *fine Schubert stratification* of Q^α:

$$\mathsf{Q}^\alpha = \bigsqcup_{w \in \mathcal{W}_f, \ \Gamma \in \mathfrak{P}(\beta-\gamma)}^{\alpha \geq \beta \geq \gamma} \overset{\circ}{\mathsf{Q}}{}_w^\gamma \times (C-0)_\Gamma^{\beta-\gamma} \times 0^{\alpha-\beta}$$

where $\overset{\circ}{\mathsf{Q}}{}_w^\gamma \subset \overset{\circ}{\mathsf{Q}}{}^\gamma$ is defined as follows. A collection of line subbundles $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda^{\mathcal{T}}$ defines a collection of lines $L_\lambda \subset (\mathcal{V}_\lambda^{\mathcal{T}})_{(0)}$ in the fiber over $0 \in C$. This collection of lines defines a $\mathbf{B}$-reduction of $\mathcal{T}$ at $0 \in C$, and if the relative position of this reduction and $\mathfrak{t}$ is $w \in \mathcal{W}_f$, we say that the triple $(\mathcal{T}, \mathfrak{t}, (\mathfrak{L}_\lambda)_{\lambda \in X^+})$ lies in $\overset{\circ}{\mathsf{Q}}{}_w^\gamma$.

The closure of $\overset{\circ}{\mathsf{Q}}{}_w^\gamma$ is denoted by $\mathsf{Q}_w^\gamma \subset \mathsf{Q}^\gamma$. Also, for an alcove $A = (w, \gamma)$ we will often write $\overset{\circ}{\mathsf{Q}}_A$ for $\overset{\circ}{\mathsf{Q}}{}_w^\gamma$, and Q_A for Q_w^γ. An open subset of $\overset{\circ}{\mathsf{Q}}{}_w^\gamma$ formed by the triples $(\mathcal{T}, \mathfrak{t}, (\mathfrak{L}_\lambda)_{\lambda \in X^+})$ such that the collection $(\mathfrak{L}_\lambda)_{\lambda \in X^+}$ has no defect at $0 \in C$ will be denoted by $\overset{\bullet}{\mathsf{Q}}{}_w^\gamma$.

The *refined stratification* of Q^α is a refinement of the fine Schubert stratification: it additionally subdivides $\overset{\circ}{\mathsf{Q}}{}^\gamma_w$ into the strata $\overset{\circ}{\mathsf{Q}}{}^{\gamma,x}_w := \mathbf{p}^{-1}(\mathsf{M}_x)$. The map $\mathbf{p} : \mathsf{Q}^\alpha \longrightarrow \mathsf{M}$ is clearly stratified with respect to the refined stratification of Q^α and the stratification $\mathsf{M} = \sqcup_{x \in \mathcal{W}} \mathsf{M}_x$.

6.4.5. For $\beta \geq \alpha$ we have a closed embedding $\mathsf{Q}^\alpha \hookrightarrow \mathsf{Q}^\beta$, sending a triple $(\mathcal{T}, \tau, (\mathfrak{L}_\lambda)_{\lambda \in X^+})$ to $(\mathcal{T}, \tau, (fL_\lambda(\langle \beta - \alpha, \lambda \rangle \cdot 0))_{\lambda \in X^+})$. These embeddings form an inductive system, and we will denote its union by Q. A *perverse sheaf* or a *mixed Hodge module on Q supported on* Q^α is just the same as a perverse sheaf or a mixed Hodge module on Q^α. Let $\aleph(\mathsf{Q})$ be the additive category formed by the (finite) direct sums of irreducible Hodge modules $\mathbf{IC}(\mathsf{Q}^\gamma_w)$ and their Tate twists on Q. Let $\mho(\mathsf{Q})$ be the additive category formed by the direct sums of Goresky-MacPherson sheaves $\mathcal{IC}(\mathsf{Q}^\gamma_w)$ and their shifts in the derived category. The subcategories of sheaves supported on $\mathsf{Q}^\alpha \subset \mathsf{Q}$ will be denoted by $\aleph(\mathsf{Q}^\alpha)$ and $\mho(\mathsf{Q}^\alpha)$.

6.4.6. *Remark.* The category of perverse sheaves on Q of finite length, with all the irreducible constituents of the form $\mathcal{IC}(\mathsf{Q}^\gamma_w)$, is equivalent to the category $\mathcal{PS}$ defined in [FM]. This is the good working definition of $\mathcal{PS}$ promised in [FM] instead of the ugly provisional definition given there.

6.4.7. Similarly, let $\aleph(\mathsf{G})$ be the additive category formed by the (finite) direct sums of irreducible Hodge modules $\mathbf{IC}(\overline{\mathsf{G}}_x)$ and their Tate twists on G. Let $\mho(\mathsf{G})$ be the additive category formed by the direct sums of Goresky-MacPherson sheaves $\mathcal{IC}(\overline{\mathsf{G}}_x)$ and their shifts in the derived category. The subcategories of sheaves supported on $\overline{\mathsf{G}}_x \subset \mathsf{G}$ will be denoted by $\aleph(\overline{\mathsf{G}}_x)$ and $\mho(\overline{\mathsf{G}}_x)$.
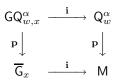
For $\alpha \in \mathbb{N}[I]$ let $\aleph(\mathcal{Q}^\alpha)$ be the additive category formed by the (finite) direct sums of irreducible Hodge modules $\mathbf{IC}(\mathcal{Q}^\gamma_w)$, $\gamma \leq \alpha$, and their Tate twists on $\mathcal{Q}^\alpha$. Let $\mho(\mathcal{Q}^\alpha)$ be the additive category formed by the direct sums of Goresky-MacPherson sheaves $\mathcal{IC}(\mathcal{Q}^\gamma_w)$, $\gamma \leq \alpha$, and their shifts in the derived category.

For $\alpha \in \mathbb{N}[I]$ sufficiently dominant let $\aleph(\ddot{\mathcal{Z}}^\alpha)$ be the additive category formed by the (finite) direct sums of irreducible Hodge modules $\mathbf{IC}(\mathcal{Z}^\gamma_w)$, $0 \ll \gamma \leq \alpha$, and their Tate twists on $\ddot{\mathcal{Z}}^\alpha$ (see 5.8.1). Let $\mho(\ddot{\mathcal{Z}}^\alpha)$ be the additive category formed by the direct sums of Goresky-MacPherson sheaves $\mathcal{IC}(\mathcal{Z}^\gamma_w)$, $0 \ll \gamma \leq \alpha$, and their shifts in the derived category.

Finally, let $\aleph(\mathcal{PS})$ be the additive category formed by the collections $\mathfrak{H}^\alpha_\chi \in \aleph(\ddot{\mathcal{Z}}^\alpha_\chi)$ together with factorization isomorphisms as in [FM] 9.3. Let $\mho(\mathcal{PS})$ be the additive category formed by the direct sums of irreducible snops $\mathcal{L}(A) \in \mathcal{PS}$ (see 5.8) and their shifts in the derived category.

6.5. For $\mathfrak{F}, \mathfrak{G} \in \mho(\mathsf{G})$ their convolution $\mathfrak{F} \star \mathfrak{G} \in \mho(\mathsf{G})$ was studied e.g. in [KT2]. We will define the convolution $\mho(\mathsf{G}) \times \mho(\mathsf{Q}) \longrightarrow \mho(\mathsf{Q})$. Its construction occupies the subsections 6.5–6.8.

6.5.1. Let $x \in \mathcal{W}$, $w \in \mathcal{W}_f$, $\alpha \in Y$. We define the *convolution diagram* $\mathsf{GQ}^\alpha_{w,x}$ as the cartesian product of $\overline{\mathsf{G}}_x$ and Q^α_w over M. Thus we have the following cartesian diagram (cf. [FM] 12.2):

$$
\begin{array}{ccc}
\mathsf{GQ}^\alpha_{w,x} & \overset{\mathbf{i}}{\longrightarrow} & \mathsf{Q}^\alpha_w \\
{\scriptstyle \mathbf{p}} \downarrow & & {\scriptstyle \mathbf{p}} \downarrow \\
\overline{\mathsf{G}}_x & \overset{\mathbf{i}}{\longrightarrow} & \mathsf{M}
\end{array}
$$

The same argument as in *loc. cit.* 12.2 proves that $\mathbf{IC}(\mathsf{Q}^\alpha_w) \otimes \mathbf{p}^* \mathbf{i}_* \mathbf{IC}(\overline{\mathsf{G}}_x)[-\dim \underline{\mathsf{M}}^x] \cong \mathbf{IC}(\mathsf{GQ}^\alpha_{w,x})$, and moreover, for any $\mathfrak{F} \in \aleph(\overline{\mathsf{G}}_x)$ the complex $\mathbf{IC}(\mathsf{Q}^\alpha_w) \otimes \mathbf{p}^* \mathbf{i}_* \mathfrak{F}[-\dim \underline{\mathsf{M}}^x]$ is in fact a semisimple Hodge module (living in cohomological degree 0). Furthermore, the Verdier duality $\mathcal{D}$ takes $\mathbf{IC}(\mathsf{Q}^\alpha_w) \otimes \mathbf{p}^* \mathbf{i}_* \mathfrak{F}[-\dim \underline{\mathsf{M}}^x]$ to $\mathbf{IC}(\mathsf{Q}^\alpha_w) \otimes \mathbf{p}^* \mathbf{i}_* \mathcal{D} \mathfrak{F}[-\dim \underline{\mathsf{M}}^x]$ (see *loc. cit.* 12.1.b).

6.5.2. It is well known that $\mathcal{W}_f \backslash \mathcal{W} / \mathcal{W}_f = Y^+$. Let us denote the double coset of $x \in \mathcal{W}$ by $\eta \in Y^+$. Then the image $\mathrm{pr}(\overline{\mathsf{G}}_x)$ in the affine Grassmannian $\mathcal{G}$ lies in $\overline{\mathcal{G}}_\eta$. Suppose $\eta + \alpha \in \mathbb{N}[I]$. Comparing the above definition of $\mathsf{GQ}^\alpha_{w,x}$ with *loc. cit.* 12.2 we obtain the natural projection $\mathrm{pr} : \mathsf{GQ}^\alpha_{w,x} \longrightarrow \mathcal{GQ}^\alpha_\eta$. Recall the map $\mathbf{q} : \mathcal{GQ}^\alpha_\eta \longrightarrow \mathcal{Q}^{\eta+\alpha}$ defined in *loc. cit.* 11.2.

For $\mathfrak{F} \in \mho(\overline{\mathcal{G}}_x)$ we define the *convolution*

$$\mathfrak{F} \star \mathcal{IC}(\mathsf{Q}^\alpha_w) := \mathbf{q}_* \mathrm{pr}_* (\mathbf{IC}(\mathsf{Q}^\alpha_w) \otimes \mathbf{p}^* \mathbf{i}_* \mathfrak{F}[-\dim \underline{\mathsf{M}}^x])$$

By the decomposition theorem $\mathfrak{F} \star \mathcal{IC}(\mathsf{Q}^\alpha_w) \in \mho(\mathcal{Q}^{\eta+\alpha})$. By additivity, the convolution extends to the functor

$$\star : \ \mho(\overline{\mathsf{G}}_x) \times \mho(\mathsf{Q}^\alpha) \longrightarrow \mho(\mathcal{Q}^{\eta+\alpha})$$

By the last sentence of 6.5.1 this functor commutes with Verdier duality:

$$\mathcal{D}\mathfrak{F} \star \mathcal{D}\mathfrak{H} \xrightarrow{\ \sim\ } \mathcal{D}(\mathfrak{F} \star \mathfrak{H})$$

6.6. Let $\alpha \in Y$, $w \in \mathcal{W}_f$. Suppose a fine Schubert stratum $\overset{\circ}{\mathsf{Q}}{}^\gamma_w \times (C - 0)^{\beta-\gamma}_\Gamma \times 0^{\alpha-\beta}$ lies in Q^α_y (see 6.4.4). The stalk of $\mathbf{IC}(\mathsf{Q}^\alpha_y)$ at (any point ϕ in) the stratum $\overset{\circ}{\mathsf{Q}}{}^\gamma_w \times (C - 0)^{\beta-\gamma}_\Gamma \times 0^{\alpha-\beta}$ will be denoted by $\mathbf{IC}(\mathsf{Q}^\alpha_y)_{\beta,\gamma,\Gamma,w}$. Let $\xi \in Y^+$ be dominant enough so that $\xi + \gamma \in \mathbb{N}[I]$ (hence $\xi + \alpha \in \mathbb{N}[I]$). Then we may consider the stalk $\mathbf{IC}(\mathcal{Q}^{\xi+\alpha}_y)_{\xi+\beta,\xi+\gamma,\Gamma,w}$ of $\mathbf{IC}(\mathcal{Q}^{\xi+\alpha}_y)$ at (any point in) the stratum $\overset{\circ}{\mathcal{Q}}{}^{\xi+\gamma}_w \times (C - 0)^{\beta-\gamma}_\Gamma \times 0^{\alpha-\beta} \subset \mathcal{Q}^{\xi+\alpha}$.

Theorem. $\mathbf{IC}(\mathsf{Q}^\alpha_y)_{\beta,\gamma,\Gamma,w}$ is isomorphic, up to a shift, to $\mathbf{IC}(\mathcal{Q}^{\xi+\alpha}_y)_{\xi+\beta,\xi+\gamma,\Gamma,w}$.

Proof. The argument is parallel to the one in [FM] 12.6, 12.7. Let us consider another copy $\mathcal{G}^1$ of affine Grassmannian: the ind-scheme representing the functor of isomorphism classes of pairs $(\mathcal{T}, \tau')$ where $\mathcal{T}$ is a $\mathbf{G}$-torsor on C, and τ' is its section (trivialization) *defined off* $1 \in C$. Restricting a trivialization to the formal neighbourhood of $\infty \in C$ we obtain the closed embedding $\mathbf{i}^1 : \mathcal{G}^1 \hookrightarrow \mathfrak{M}$. Furthermore, since τ' identifies the fiber of $\mathcal{T}$ over $0 \in C$ with $\mathbf{G}$, the set of $\mathbf{B}$-reductions of $\mathcal{T}$ at 0 is canonically isomorphic to $\mathbf{X}$. Choosing the reduction $\mathfrak{t} = e = \mathbf{B} \in \mathbf{X}$ we obtain the same named closed embedding $\mathbf{i}^1 : \mathcal{G}^1 \hookrightarrow \mathsf{M}$.

We consider the cartesian product $\mathcal{G}^1\mathsf{Q}^\alpha_{y,\eta}$ of $\overline{\mathcal{G}}_\eta$ and Q^α_y over M. Thus we have the following cartesian diagram (cf. 6.5.1):

$$
\begin{array}{ccc}
\mathcal{G}^1\mathsf{Q}^\alpha_{y,\eta} & \xrightarrow{\ \mathbf{i}^1\ } & \mathsf{Q}^\alpha_y \\
{\scriptstyle \mathbf{p}}\big\downarrow & & {\scriptstyle \mathbf{p}}\big\downarrow \\
\overline{\mathcal{G}}^1_\eta & \xrightarrow{\ \mathbf{i}^1\ } & \mathsf{M}
\end{array}
$$

The same argument as in 6.5.1 proves that up to a shift $\mathbf{IC}(\mathsf{Q}^\alpha_y) \otimes \mathbf{p}^* \mathbf{i}^1_* \mathbf{IC}(\overline{\mathcal{G}}^1_\eta) \cong \mathbf{IC}(\mathcal{G}^1\mathsf{Q}^\alpha_{y,\eta})$.

Recall the convolution diagram $\mathcal{GQ}^\alpha_\eta$ (see [FM] 11.1, 12.2). The point $0 \in C$ played a special role in the definition of this diagram. Let us replace 0 by 1 in the definition of $\mathcal{GQ}^\alpha_\eta$. Let us call the result $\mathcal{G}^1\mathcal{Q}^\alpha_\eta$. One defines the map $\mathbf{q}^1 : \mathcal{G}^1\mathcal{Q}^\alpha_\eta \longrightarrow \mathcal{Q}^{\eta+\alpha}$ as in *loc. cit.* 11.2. Then the Proposition 12.6 of *loc. cit.* states that the restriction of $\mathbf{q}^1$ to $\overset{\bullet}{\mathcal{Q}}{}^{1,\eta+\alpha}$ is an isomorphism, where $\overset{\bullet}{\mathcal{Q}}{}^{1,\eta+\alpha} \subset \mathcal{Q}^{\eta+\alpha}$ is an open subset formed by all the quasimaps defined at $1 \in C$ (i.e. without defect at 1). A moment of reflection shows that there is no difference between the definitions of $\mathcal{G}^1\mathsf{Q}^\alpha_{w_0,\eta}$ and $\mathcal{G}^1\mathcal{Q}^\alpha_\eta$. Identifying them and imbedding $\mathcal{G}^1\mathsf{Q}^\alpha_{y,\eta}$ into $\mathcal{G}^1\mathsf{Q}^\alpha_{w_0,\eta}$ we obtain the map $\mathbf{q}^1 : \mathcal{G}^1\mathsf{Q}^\alpha_{y,\eta} \longrightarrow \mathcal{Q}^{\eta+\alpha}$ which is a closed embedding over $\overset{\bullet}{\mathcal{Q}}{}^{1,\eta+\alpha}$. Clearly, the image of this embedding coincides with $\mathcal{Q}^{1,\eta+\alpha}_y := \mathcal{Q}^{\eta+\alpha}_y \cap \overset{\bullet}{\mathcal{Q}}{}^{1,\eta+\alpha}$. Evidently, $\mathcal{Q}^{1,\eta+\alpha}_y$ is open in $\mathcal{Q}^{\eta+\alpha}_y$.

Now let us consider the stalk of $\mathbf{IC}(\mathsf{Q}_y^\alpha)$ at a point $\phi = (\mathcal{T}, \mathfrak{t}, (\mathfrak{L}_\lambda)_{\lambda \in X^+})$ in the stratum $\overset{\circ}{\mathsf{Q}}{}_w^\gamma \times (C - 0)_\Gamma^{\beta - \gamma} \times 0^{\alpha - \beta}$. Suppose that the isomorphism class of **G**-torsor $\mathcal{T}$ equals $\eta \in Y^+$, i.e. $\mathcal{T} \in \mathfrak{M}_\eta$. The stalk in question does not depend on a choice of $\mathcal{T} \in \mathfrak{M}_\eta$ and the defect $D \in (C - 0)_\Gamma^{\beta - \gamma}$. In particular, we may (and will) suppose that $\mathcal{T} \in \mathbf{i}^1(\mathcal{G}_\eta^1)$, and $D \in (C - 0 - 1)^{\beta - \gamma}$. We have seen that up to a shift $\mathbf{IC}(\mathsf{Q}_y^\alpha)_\phi \otimes \mathbf{p}^* \mathbf{i}_*^1 \mathbf{IC}(\overline{\mathcal{G}}_\eta^1)_\mathcal{T} \cong \mathbf{IC}(\mathcal{G}^1 \mathsf{Q}_{y,\eta}^\alpha)_\phi$. But the stalk of $\mathbf{IC}(\overline{\mathcal{G}}_\eta^1)$ at any point $\mathcal{T} \in \mathcal{G}_\eta^1$ is isomorphic, up to a shift, to the trivial Tate Hodge structure $\mathbb{Q}(0)$. We deduce that up to a shift $\mathbf{IC}(\mathsf{Q}_y^\alpha)_\phi \cong \mathbf{IC}(\mathcal{G}^1 \mathsf{Q}_{y,\eta}^\alpha)_\phi$.

On the other hand, the latter stalk is isomorphic to the stalk of $\mathbf{IC}(\mathcal{Q}_y^{1,\eta + \alpha})$ at the point $\mathbf{q}^1(\phi)$. This point lies in $\overset{\circ}{\mathcal{Q}}{}_w^{\eta + \gamma} \times (C - 0)_\Gamma^{\beta - \gamma} \times 0^{\alpha - \beta}$.

Thus we have proved that up to a shift $\mathbf{IC}(\mathsf{Q}_y^\alpha)_{\beta, \gamma, \Gamma, w}$ is isomorphic to $\mathbf{IC}(\mathcal{Q}_y^{\eta + \alpha})_{\eta + \beta, \eta + \gamma, \Gamma, w}$.

The standard use of factorization shows that the latter stalk is independent of the shift $\eta \mapsto \xi$. This completes the proof of the Theorem. $\square$

6.6.1. *Corollary.* The Hodge module $\mathbf{IC}(\mathsf{Q}_y^\alpha)$ is pointwise pure Tate. $\square$

6.6.2. *Corollary.* Let $\beta, \gamma \in Y$, $w, y \in \mathcal{W}_f$. Then $\mathsf{Q}_y^\beta \supset \mathsf{Q}_w^\gamma$ iff $(y, \beta) =: B \geq A := (w, \gamma)$.

Proof. $\mathsf{Q}_y^\beta \supset \mathsf{Q}_w^\gamma$ iff the stalk of $\mathbf{IC}(\mathsf{Q}_y^\beta)$ at the generic point of Q_w^γ does not vanish. Also, $\mathcal{Q}_y^\beta \supset \mathcal{Q}_w^\gamma$ iff the stalk of $\mathbf{IC}(\mathcal{Q}_y^\beta)$ at the generic point of $\mathcal{Q}_w^\gamma$ does not vanish. Now recall that the order on alcoves is translation invariant, and apply 5.7. $\square$

6.7. Before we finish the construction of convolution $\star : \mho(\mathsf{G}) \times \mho(\mathsf{Q}) \longrightarrow \mho(\mathsf{Q})$ we have to define the numerous functors between the categories $\aleph$ and $\mho$ of 6.4.5 and 6.4.7.

First of all, we choose an additive equivalence $\mathfrak{E}_\mathsf{Q}^{\mathcal{PS}} : \aleph(\mathsf{Q}) \longrightarrow \aleph(\mathcal{PS})$ sending $\mathbf{IC}(\mathsf{Q}_w^\eta)$, $\eta \in Y$, $w \in \mathcal{W}_f$, to the collection $\mathbf{IC}(\mathcal{Z}_{w,\eta}^\alpha) \in \aleph(\ddot{\mathcal{Z}}_\eta^\alpha)$ (see 5.8.1 and [FM] 9.3). Let $\mathfrak{E}_{\mathcal{PS}}^\mathsf{Q} : \aleph(\mathcal{PS}) \longrightarrow \aleph(\mathsf{Q})$ be an inverse equivalence.

We preserve the name $\mathfrak{E}_\mathsf{Q}^{\mathcal{PS}}$ for an additive equivalence $\mho(\mathsf{Q}) \longrightarrow \mho(\mathcal{PS})$ sending $\mathcal{IC}(\mathsf{Q}_w^\eta)$, $\eta \in Y$, $w \in \mathcal{W}_f$, to $\mathcal{L}(w, \eta) \in \mho(\mathcal{PS})$ and commuting with shifts. Let $\mathfrak{E}_{\mathcal{PS}}^\mathsf{Q} : \mho(\mathcal{PS}) \longrightarrow \mho(\mathsf{Q})$ be an inverse equivalence.

For $\alpha \in \mathbb{N}[I]$ let $\aleph_{\gg 0}(\mathcal{Q}^\alpha) \subset \aleph(\mathcal{Q}^\alpha)$ be the direct summand subcategory formed by the direct sums of irreducible Hodge modules $\mathbf{IC}(\mathcal{Q}_w^\gamma)$, $0 \ll \gamma \leq \alpha$, and their Tate twists. Let $\mho_{\gg 0}(\mathcal{Q}^\alpha) \subset \mho(\mathcal{Q}^\alpha)$ be the direct summand subcategory formed by the direct sums of Goresky-MacPherson sheaves $\mathcal{IC}(\mathcal{Q}_w^\gamma)$, $0 \ll \gamma \leq \alpha$, and their shifts in derived category. Let $\mathfrak{E}_{\mathcal{Q}^\alpha}^{\gg 0} : \aleph(\mathcal{Q}^\alpha) \longrightarrow \aleph_{\gg 0}(\mathcal{Q}^\alpha)$, $\mho(\mathcal{Q}^\alpha) \longrightarrow \mho_{\gg 0}(\mathcal{Q}^\alpha)$ denote the projections to the direct summands, and let $\mathfrak{E}_{\gg 0}^{\mathcal{Q}^\alpha} : \aleph_{\gg 0}(\mathcal{Q}^\alpha) \longrightarrow \aleph(\mathcal{Q}^\alpha)$, $\mho_{\gg 0}(\mathcal{Q}^\alpha) \longrightarrow \mho(\mathcal{Q}^\alpha)$ denote the embeddings of direct summands.

Let $\ss_\alpha$ denote the locally closed embedding $\ddot{\mathcal{Z}}^\alpha \hookrightarrow \mathcal{Q}^\alpha$. One can easily see that for $\gamma \gg 0$ we have $\ss_\alpha^* \mathbf{IC}(\mathcal{Q}_w^\gamma)[-\dim \mathbf{X}] = \mathbf{IC}(\mathcal{Z}_w^\gamma)$. Hence the functor $\ss_\alpha^*[-\dim \mathbf{X}]$ defines an equivalence $\mathfrak{E}_{\mathcal{Q}^\alpha}^{\ddot{\mathcal{Z}}^\alpha} : \aleph_{\gg 0}(\mathcal{Q}^\alpha) \longrightarrow \aleph(\ddot{\mathcal{Z}}^\alpha)$, $\mho_{\gg 0}(\mathcal{Q}^\alpha) \longrightarrow \mho(\ddot{\mathcal{Z}}^\alpha)$. Let $\mathfrak{E}_{\ddot{\mathcal{Z}}^\alpha}^{\mathcal{Q}^\alpha} : \aleph(\ddot{\mathcal{Z}}^\alpha) \longrightarrow \aleph_{\gg 0}(\mathcal{Q}^\alpha)$, $\mho(\ddot{\mathcal{Z}}^\alpha) \longrightarrow \mho_{\gg 0}(\mathcal{Q}^\alpha)$ be an inverse equivalence.

Finally, for $\xi \in Y$ let $\xi_* : \aleph(\mathsf{Q}) \longrightarrow \aleph(\mathsf{Q})$ (resp. $\mho(\mathsf{Q}) \longrightarrow \mho(\mathsf{Q})$) be an additive equivalence taking $\mathbf{IC}(\mathsf{Q}_w^\alpha)$ to $\mathbf{IC}(\mathsf{Q}_w^{\xi + \alpha})$ (resp. $\mathcal{IC}(\mathsf{Q}_w^\alpha)$ to $\mathcal{IC}(\mathsf{Q}_w^{\xi + \alpha})$ and commuting with shifts).

6.8. Finally, we are in a position to define the convolution $\star : \mho(\mathsf{G}) \times \mho(\mathsf{Q}) \longrightarrow \mho(\mathsf{Q})$. To this end recall the convolution $\mho(\overline{\mathsf{G}}_x) \times \mho(\mathcal{Q}^\alpha) \longrightarrow \mho(\mathcal{Q}^{\eta + \alpha})$ defined in 6.5. To stress the dependence on $x \in \mathcal{W}$ and $\alpha \in Y$ we will call this convolution $\star_x^\alpha$ (recall that $\eta \in Y^+ = \mathcal{W}_f \backslash \mathcal{W} / \mathcal{W}_f$ is just the double coset of x).

Let $\mathfrak{F} \in \mho(\overline{\mathsf{G}}_x)$, $\mathfrak{H} \in \mho(\mathsf{Q}^\alpha)$. The collection $\mathfrak{E}_{\mathcal{Q}^\beta}^{\ddot{\mathcal{Z}}^\beta} \circ \mathfrak{E}_{\mathcal{Q}^\beta}^{\gg 0}(\mathfrak{F} \star_x^{\beta - \eta}((\beta - \alpha - \eta)_* \mathfrak{H})) \in \mho(\ddot{\mathcal{Z}}_{\alpha + \eta}^\beta)$, $\beta \in \mathbb{N}[I]$, defines an object in $\mho(\mathcal{PS})$. Applying $\mathfrak{E}_{\mathcal{PS}}^{\mathsf{Q}}$ to this object we obtain the desired convolution $\mathfrak{F} \star_x^\alpha \mathfrak{H} \in \mho(\mathsf{Q})$. Note that if $y \leq x$, $\gamma \leq \alpha$, and $\mathfrak{F}$ is actually supported on $\overline{\mathsf{G}}_y$, while $\mathfrak{H}$ is actually supported on Q^γ, then we have the canonical isomorphism $\mathfrak{F} \star_y^\gamma \mathfrak{H} \equiv \mathfrak{F} \star_x^\alpha \mathfrak{H}$. Hence the convolutions $\star_x^\alpha : \mho(\overline{\mathsf{G}}_x) \times \mho(\mathsf{Q}^\alpha) \longrightarrow \mho(\mathsf{Q})$ extends to the convolution $\star : \mho(\mathsf{G}) \times \mho(\mathsf{Q}) \longrightarrow \mho(\mathsf{Q})$.

As we have seen in 6.5, this convolution commutes with Verdier duality.

Moreover, for $\mathfrak{F}, \mathfrak{G} \in \mho(\mathsf{G})$, $\mathfrak{H} \in \mho(\mathsf{Q})$ one can easily construct a canonical isomorphism $\mathfrak{F} \star (\mathfrak{G} \star \mathfrak{H}) \cong (\mathfrak{F} \star \mathfrak{G}) \star \mathfrak{H}$.

Finally, composing this convolution with the mutually inverse equivalences $\mathfrak{E}_{\mathsf{Q}}^{\mathcal{PS}} : \mho(\mathsf{Q}) \rightleftharpoons \mho(\mathcal{PS}) : \mathfrak{E}_{\mathcal{PS}}^{\mathsf{Q}}$ we obtain the same named convolution functor $\star : \mho(\mathsf{G}) \times \mho(\mathcal{PS}) \longrightarrow \mho(\mathcal{PS})$. It is also associative with respect to convolution in $\mho(\mathsf{G})$, and commutes with Verdier duality.

6.9. Let $K(\mathsf{Q})$ (resp. $K(\mathcal{PS})$) denote the K-group of the category $\mho(\mathsf{Q})$ (resp. $\mho(\mathcal{PS})$): it is generated by the isomorphism classes of objects in $\mho(\mathsf{Q})$ with relations $[K_1 \oplus K_2] = [K_1] + [K_2]$. Both $K(\mathsf{Q})$ and $K(\mathcal{PS})$ are equipped with the structure of $\mathbb{Z}[v, v^{-1}]$-modules: $-v$ acts as the shift $[-1]$.

$K(\mathcal{PS})$ is a free $\mathbb{Z}[v, v^{-1}]$-module with a basis $[\mathcal{L}(A)]$, $A \in \mathfrak{A}$ (see 5.8). Similarly, $K(\mathsf{Q})$ is a free $\mathbb{Z}[v, v^{-1}]$-module with a basis $[\mathcal{L}'(A)]$, $A \in \mathfrak{A}$ where for $A = (w, \eta)$ we denote by $\mathcal{L}'(A)$ the Goresky-MacPherson sheaf $\mathcal{IC}(\mathsf{Q}_w^\eta) = \mathcal{IC}(\mathsf{Q}_A)$ (see 6.4.4).

We denote by $\hat{K}(\mathcal{PS}) \supset K(\mathcal{PS})$ (resp. $\hat{K}(\mathsf{Q}) \supset K(\mathsf{Q})$) the completed module formed by the possibly infinite sums $\sum_{A \in \mathfrak{A}} a_A[\mathcal{L}(A)]$ (resp. $\sum_{A \in \mathfrak{A}} a_A[\mathcal{L}'(A)]$) $a_A \in \mathbb{Z}[v, v^{-1}]$ such that $\exists B \in \mathfrak{A}$ with the property $(a_A \neq 0 \Rightarrow A \leq B)$.

We introduce another (topological) basis $\{[\mathcal{M}'(A)], A \in \mathfrak{A}\}$ of $\hat{K}(\mathsf{Q})$. It is characterized by the following property. The stalk at a fine Schubert stratum $\overset{\circ}{\mathsf{Q}}_B \subset \mathsf{Q}_B$ defines the functional $f_B : \hat{K}(\mathsf{Q}) \longrightarrow \mathbb{Z}[v, v^{-1}]$. We require $f_A([\mathcal{M}'(A)]) = f_A([\mathcal{L}'(A)])$ and $f_B([\mathcal{M}'(A)]) = 0$ for $B \neq A$. In a similar way one defines a (topological) basis $\{[\mathcal{M}(A)], A \in \mathfrak{A}\}$ of $\hat{K}(\mathcal{PS})$.

The main goal of this section is a computation of the transition matrix between the bases $[\mathcal{L}(A)]$ and $[\mathcal{M}(A)]$. We will make a free use of the notations and results of the excellent exposition [S]. In particular, the polynomials $\overline{q}_{B,A}$ (generic Kazhdan-Lusztig polynomials) are introduced in *loc. cit.*, Theorems 6.1, 6.4. We will prove the following

Theorem. a) $[\mathcal{L}(A)] = \sum_{B \leq A}(-1)^{d(A,B)}\overline{q}_{B,A}[\mathcal{M}(B)]$;

b) $[\mathcal{L}'(A)] = \sum_{B \leq A}(-1)^{d(A,B)}\overline{q}_{B,A}[\mathcal{M}'(B)]$.

The proof of the Theorem occupies the subsections 6.10–6.13.

6.10. For the time being let us define $\mathfrak{q}_{B,A} \in \mathbb{Z}[v, v^{-1}]$ (resp. $\mathfrak{q}'_{B,A} \in \mathbb{Z}[v, v^{-1}]$) by the requirement $[\mathcal{L}(A)] = \sum_{B \leq A} \mathfrak{q}_{B,A}[\mathcal{M}(B)]$ (resp. $[\mathcal{L}'(A)] = \sum_{B \leq A} \mathfrak{q}'_{B,A}[\mathcal{M}'(B)]$). Thus we have to prove $\mathfrak{q}_{B,A} = \mathfrak{q}'_{B,A} = (-1)^{d(A,B)}\overline{q}_{B,A}$.

By the Corollary 6.6.1 the Hodge module $\mathbf{IC}(\mathsf{Q}_A)$ is pointwise pure Tate. This allows us to reconstruct the stalk $\mathbf{IC}(\mathsf{Q}_A)_B$ of $\mathbf{IC}(\mathsf{Q}_A)$ at $\overset{\circ}{\mathsf{Q}}_B \subset \mathsf{Q}_A$ from the polynomials $\mathfrak{q}'_{B,A}$. Namely, suppose that the stalk of $\mathbf{IC}(\mathsf{Q}_A)$ at $\overset{\circ}{\mathsf{Q}}_A$ is the Hodge structure $\mathbb{Q}(0)$ living in cohomological degree δ_A (this is the definition of δ_A). Then for an odd integer k we have $H^{\delta_A + k}\mathbf{IC}(\mathsf{Q}_A)_B = 0$. For arbitrary integer k the cohomology $H^{\delta_A + 2k}\mathbf{IC}(\mathsf{Q}_A)_B$ is a sum of a few copies of $\mathbb{Q}(k)$. Finally, $\dim H^{\delta_A + k}\mathbf{IC}(\mathsf{Q}_A)_B$ equals the coefficient

of $(-v)^{-d(B,A)+k}$ in $\mathfrak{q}'_{B,A}$ (for arbitrary $k \in \mathbb{Z}$). The same discussion applies to the stalks $\mathbf{IC}(\mathcal{Z}_A)_B$ of $\mathbf{IC}(\mathcal{Z}_A)$.

But the stalks $\mathbf{IC}(\mathcal{Z}_A)_B$ and $\mathbf{IC}(\mathsf{Q}_A)_B$ are isomorphic. In effect, if $A = (w, \alpha)$, $B = (y, \beta)$, and $\xi \in Y^+$ is big enough, then by the Theorem 6.6 $\mathbf{IC}(\mathsf{Q}_A)_B$ is isomorphic to the stalk of $\mathbf{IC}(\mathcal{Q}_w^{\xi+\alpha})$ at $\overset{\circ}{\mathcal{Q}}_y^{\xi+\beta}$ (up to a shift; but we may disregard shifts: as we have just seen the cohomological degrees contain exactly the same information as the Hodge structures by the pointwise Tate purity). And the latter stalk is isomorphic to $\mathbf{IC}(\mathcal{Z}_A)_B$ since $\mathbf{IC}(\mathcal{Z}_w^{\xi+\alpha}) = \text{ß}^*_{\xi+\alpha}\mathbf{IC}(\mathcal{Q}_w^{\xi+\alpha})[-\dim \mathbf{X}]$ where $\text{ß}_{\xi+\alpha}$ stands for the locally closed embedding of $\overset{..}{\mathcal{Z}}{}^{\xi+\alpha}$ into $\mathcal{Q}^{\xi+\alpha}$.

This implies that the functor $\mathfrak{E}^{\mathsf{Q}}_{\mathcal{PS}} : \mho(\mathcal{PS}) \longrightarrow \mho(\mathsf{Q})$ inducing an isomorphism of $\hat{K}(\mathcal{PS})$ and $\hat{K}(\mathsf{Q})$ sending $[\mathcal{L}(A)]$ to $[\mathcal{L}'(A)]$ *also sends* $[\mathcal{M}(A)]$ *to* $[\mathcal{M}'(A)]$. In other words, $\mathfrak{E}^{\mathsf{Q}}_{\mathcal{PS}}$ preserves stalks. In particular, $\mathfrak{q}_{B,A} = \mathfrak{q}'_{B,A}$ for all $B \leq A$. From now on we will identify $\hat{K}(\mathcal{PS})$ and $\hat{K}(\mathsf{Q})$, we will write $[\mathcal{L}(A)]$ for $[\mathcal{L}'(A)]$, and $[\mathcal{M}(A)]$ for $[\mathcal{M}'(A)]$.

6.10.1. *Remark.* As we have seen, the stalk of $\mathbf{IC}(\mathcal{Q}_w^\alpha)$ at a fine Schubert stratum $\overset{\circ}{\mathcal{Q}}_y^\beta$ can be reconstructed in terms of polynomials $\mathfrak{q}_{B,A}$ for $A = (w, \alpha), B = (y, \beta)$. Since we know the simple stalks $\mathbf{IC}^0_{\{\{\gamma\}\}}$, the factorization property allows us to reconstruct all the other stalks of $\mathbf{IC}(\mathcal{Q}_w^\alpha)$.

6.11. We know the simple stalks $\mathbf{IC}^0_{\{\{\gamma\}\}}$ by the Theorems 3.1, 3.3.2. They coincide with the stalks of $\mathbf{IC}(\mathsf{Q}^\alpha) = \mathbf{IC}(\mathsf{Q}_{w_0}^\alpha)$. Thus we deduce the following formula:

$$[\mathcal{L}(w_0, \alpha)] = \sum_{w \in \mathcal{W}_f}^{\beta \in \mathbb{N}[I]} (-v)^{l(w)-l(w_0)} \mathcal{K}^\beta(v^2)[\mathcal{M}(w, \alpha - \beta)]$$

Comparing with Kato's Theorem (see [Kat] or [S] 6.3) we see that for any $B \in \mathfrak{A}$ and $A = (w_0, \alpha)$ we have $\mathfrak{q}_{B,A} = (-1)^{d(A,B)}\overline{q}_{B,A}$.

6.12. The K-group $K(\mathsf{G}) := K(\mho(\mathsf{G}))$ is also a $\mathbb{Z}[v, v^{-1}]$-module in a natural way ($-v$ acts as a shift $[-1]$). The convolution $\star : \mho(\mathsf{G}) \times \mho(\mathsf{G}) \longrightarrow \mho(\mathsf{G})$ makes $K(\mathsf{G})$ into a $\mathbb{Z}[v, v^{-1}]$-algebra. This is the affine Hecke algebra $\mathcal{H}$ as described e.g. in [S] §2. The basis $\{H_x, \ x \in \mathcal{W}\}$ corresponds to perverse shriek extensions of constant sheaves on orbits G_x. The involution $H \mapsto \overline{H}$ of *loc. cit.* is induced by the Verdier duality $\mathcal{D} : \mho(\mathsf{G}) \longrightarrow \mho(\mathsf{G})$.

Let $s_i \in \mathcal{W}$, $i \in I \sqcup 0$ be a simple reflection (see 5.1). Then $\overline{\mathsf{G}}_{s_i} \subset \mathsf{G}$ is a projective line, to be denoted by $\overline{\mathsf{G}}_i$. The class $[\mathcal{IC}(\overline{\mathsf{G}}_i)] \in \mathcal{H}$ equals $H_{s_i} - v^{-1}$ (we identify H_e with $1 \in \mathcal{H}$). Following W.Soergel, we will denote $[\mathcal{IC}(\overline{\mathsf{G}}_i)]$ by $\tilde{C}_i = \tilde{C}_{s_i}$.

The convolution $\star : \mho(\mathsf{G}) \times \mho(\mathsf{Q}) \longrightarrow \mho(\mathsf{Q})$ gives rise to the structure of $\mathcal{H}$-module on $\hat{K}(\mathsf{Q})$.

Proposition. Let $i \in I \sqcup 0$, $A \in \mathfrak{A}$. If $s_i A > A$ then $\tilde{C}_i[\mathcal{M}(A)] = [\mathcal{M}(s_i A)] - v^{-1}[\mathcal{M}(A)]$, and if $s_i A < A$ then $\tilde{C}_i[\mathcal{M}(A)] = [\mathcal{M}(s_i A)] - v[\mathcal{M}(A)]$.

Proof. As everything is invariant with respect to translations, we may (and will) assume that $A = (w, \alpha)$ for $\alpha \in \mathbb{N}[I]$. Consider the convolution diagram $\mathsf{GQ}_{w,s_i}^\alpha \overset{\mathbf{i}}{\hookrightarrow} \mathsf{Q}_w^\alpha$. We will denote by $\mathsf{G}\overset{\bullet}{\mathsf{Q}}_{w,s_i}^\alpha$ the intersection of $\mathsf{GQ}_{w,s_i}^\alpha$ with $\overset{\bullet}{\mathsf{Q}}_w^\alpha \subset \mathsf{Q}_w^\alpha$ (see 6.4.4). To compute $\tilde{C}_i[\mathcal{M}(A)]$ we have to compute the stalks of $(\mathbf{q} \circ \mathrm{pr})_! \mathcal{IC}(\mathsf{G}\overset{\bullet}{\mathsf{Q}}_{w,s_i}^\alpha)$. Recall that $\mathbf{q} \circ \mathrm{pr}$ maps $\mathsf{GQ}_{w,s_i}^\alpha$ to $\mathcal{Q}^{\eta+\alpha}$ where $\eta \in Y^+ = \mathcal{W}_f \backslash \mathcal{W} / \mathcal{W}_f$ is the double coset of s_i. Note that the double coset of s_0 equals β_0 (see 5.1) — the coroot dual to the highest root. And the double coset of s_i, $i \in I$, equals $0 \in Y^+$. So in any case we have the map $\mathbf{q} \circ \mathrm{pr} : \mathsf{G}\overset{\bullet}{\mathsf{Q}}_{w,s_i}^\alpha \longrightarrow \mathcal{Q}^{\beta_0+\alpha}$.

According to 6.6 the sheaf $\mathcal{IC}(\mathsf{G}\overset{\bullet}{\mathsf{Q}}{}^\alpha_{w,s_i})$ is constant along the fibers of $\mathbf{q} \circ \mathrm{pr}$. So the Proposition follows from the following Claim.

6.12.1. Claim. Let $\phi \in \overset{\bullet}{\mathcal{Q}}{}^\beta_y \subset \mathcal{Q}^{\beta_0+\alpha}$, and let $B = (y, \beta) \in \mathfrak{A}$. Then

a) if $B \neq A, s_i A$, then $(\mathbf{q} \circ \mathrm{pr})^{-1}(\phi) = \emptyset$;

b) if B equals A or $s_i A$, and $s_i A > A$, then $(\mathbf{q} \circ \mathrm{pr})^{-1}(\phi)$ consists of exactly one point;

c) if B equals A or $s_i A$, and $s_i A < A$, then $(\mathbf{q} \circ \mathrm{pr})^{-1}(\phi)$ is isomorphic to the affine line $\mathbb{A}^1$.

6.12.2. Remark. If $i \in I$ then the double coset of s_i equals 0, and the above map $\mathbf{q}$ is just an isomorphism. In this case the convolution boils down to the usual convolution on $\mathbf{X}$. The order relation between $s_i A$ and A also reduces to the usual Bruhat order on $\mathcal{W}_f$ (see 5.1.1 and 5.1.3). So 6.12.1 follows in this case from the standard facts about the geometry of $\mathbf{X}$.

6.12.3. In the general case note first that the (isomorphism type of the) fiber $(\mathbf{q} \circ \mathrm{pr})^{-1}(\phi)$ is independent of $\phi \in \overset{\bullet}{\mathcal{Q}}{}^\beta_y$. The proof is absolutely similar to the one in [FM] 12.6.b). It makes use of the *local convolution diagram* (see *loc. cit.* 11.5). We spare the reader the bulk of notation needed to carry this notion over to our situation.

So we may (and will) choose ϕ lying in the fine Schubert stratum $\overset{\circ}{\mathcal{Q}}{}^0_y \times (C-0)^\beta_\Gamma \times 0^{\alpha+\beta_0-\beta} \subset \overset{\bullet}{\mathcal{Q}}{}^\beta_y$. Moreover, we will assume $\phi = y \times D \times 0^{\alpha+\beta_0-\beta}$ where $D \in (C-0)^\beta_\Gamma$, and $y \in \mathbf{X}_y = \overset{\circ}{\mathcal{Q}}{}^0_y$ is the $\mathbf{H}$-fixed point. We may (and will) view the fiber $(\mathbf{q} \circ \mathrm{pr})^{-1}(\phi)$ as a subset of $\overline{\mathsf{G}}_i$. As such, it is isomorphic to the intersection of $\overline{\mathsf{G}}_i$ with the orbit $T^y_{y,\beta-\alpha-\beta_0}$ of $\mathbf{N}^y_-((z))$ in G passing through the $\mathbf{H}$-fixed point $(y, \beta - \alpha - \beta_0) \in \Omega$ (we use the identification $\mathcal{W}_f \times Y \overset{\sim}{\longrightarrow} \Omega$ of 5.1 along with the fact that the set of $\mathbf{H}$-fixed points of G is naturally identified with Ω), cf. the proof of 12.6.b) in *loc. cit.* Here as always $\mathbf{N}^y_- \subset \mathbf{G}$ denotes the subgroup $\dot{y}\mathbf{N}_-\dot{y}^{-1}$ for a representative $\dot{y} \in N(\mathbf{H})$ of y. Now the Claim follows by the standard Bruhat-Tits theory.

This completes the proof of the Proposition. $\square$

6.13. Comparing the above Proposition with [L1] or [S] 4.1 we see that the $\mathcal{H}$-module $\hat{K}(\mathbf{Q})$ is isomorphic to Lusztig's completed *periodic Hecke module* $\hat{\mathcal{P}}$ (see [S] §6), and this isomorphism φ takes $[\mathcal{M}(A)] \in \hat{K}(\mathbf{Q})$ to a basic element $A \in \hat{\mathcal{P}}$. Moreover, comparing 6.11 with the Theorem 6.4.2 (due to Kato) of *loc. cit.* we see that for $A = (w_0, \alpha)$ we have $\varphi([\mathcal{L}(A)]) = \underline{\tilde{P}}_A$ (notations of *loc. cit.* §6).

The Verdier duality $\mathcal{D}: \mho(\mathbf{Q}) \longrightarrow \mho(\mathbf{Q})$ gives rise to an involution $N \mapsto \mathcal{D}N$ of $\hat{K}(\mathbf{Q})$. As Verdier duality commutes with the convolution, this involution is compatible with the involution $H \mapsto \overline{H}$ of $\mathcal{H}$, that is, $\mathcal{D}(H(N)) = \overline{H}(\mathcal{D}N)$. Certainly, $\mathcal{D}[\mathcal{L}(A)] = [\mathcal{L}(A)]$ for any $A \in \mathfrak{A}$.

Recall the Lusztig's involution $P \mapsto \overline{P}$ of the periodic module $\hat{\mathcal{P}}$ (see e.g. *loc. cit.* 4.3). It is also compatible with the involution on the Hecke algebra, i.e. $\overline{H(P)} = \overline{H}(\overline{P})$. It also preserves the elements $\underline{\tilde{P}}_A$, $A \in \mathfrak{A}$ (see *loc. cit.* 6.4). Since the elements $\underline{\tilde{P}}_A$, $A = (w_0, \alpha) \in \mathfrak{A}$ clearly generate the $\mathcal{H}$-module $\hat{\mathcal{P}}$, we conclude that $\varphi(\mathcal{D}N) = \overline{\varphi(N)}$ for any $N \in \hat{K}(\mathbf{Q})$.

In other words, we may identify $\mathcal{H}$-modules $\hat{\mathcal{P}}$ and $\hat{K}(\mathbf{Q})$ with their involutions and standard bases. Now the Theorem 6.4.2 of *loc. cit.* carries over to $\hat{K}(\mathbf{Q})$ and claims that for any $A \in \mathfrak{A}$ the element $\varphi^{-1}(\underline{\tilde{P}}_A) = \sum_B (-1)^{d(A,B)} \overline{q}_{B,A}[\mathcal{M}(B)]$ is the only $\mathcal{D}$-invariant element of $\hat{K}(\mathbf{Q})$ lying in $[\mathcal{M}(A)] + \sum_{B \leq A} v^{-1}\mathbb{Z}[v^{-1}][\mathcal{M}(B)]$. But $[\mathcal{L}(A)]$ is also $\mathcal{D}$-invariant and it also lies in

$[\mathcal{M}(A)] + \sum_{B \leq A} v^{-1} \mathbb{Z}[v^{-1}][\mathcal{M}(B)]$ by the definition of Goresky-MacPherson extension. This completes the proof of the Theorem 6.9. $\square$

7. TILTING CONJECTURES

7.1. Recall the notations of [FM] 12.12, 12.13. So let $\eta \in Y^+$, $\beta \in Y$, $\eta + \beta \in \mathbb{N}[I]$. For a sheaf $\mathcal{F} \in \mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$ we consider the complex $\mathbf{c}_{\mathcal{Q}}^\beta(\mathcal{F}) = \mathbf{q}_*(\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})[-\dim \underline{\mathfrak{M}}^\eta]$ on $\mathcal{Q}^{\eta+\beta}$. If $\eta + \beta \notin \mathbb{N}[I]$ we set $\mathcal{Q}^{\eta+\beta} = \emptyset$, and $\mathbf{c}_{\mathcal{Q}}^\beta(\mathcal{F}) = 0$. We will define a natural action of $\mathfrak{g}^L$ on $\oplus_{\beta \in Y} H^\bullet(\mathcal{Q}^{\eta+\beta}, \mathbf{c}_{\mathcal{Q}}^\beta(\mathcal{F})) = \oplus_{\beta \in Y} H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$ closely following 4.5–4.9.

7.1.1. For $\beta \in Y$, $\alpha \in \mathbb{N}[I]$ we have the usual *twisting map* $\sigma_{\beta,\alpha} : \mathcal{G}\mathcal{Q}_\eta^\beta \times C^\alpha \longrightarrow \mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}$ (resp. $\mathfrak{Q}^\beta \times C^\alpha \longrightarrow \mathfrak{Q}^{\alpha+\beta}$). We define $\partial_\alpha \mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}$ (resp. $\partial_\alpha \mathfrak{Q}^{\alpha+\beta}$) as the image of this map. We will construct the *stabilization map* $\mathfrak{v}_{\alpha,\beta} : \mathcal{A}_\alpha \longrightarrow \mathrm{Ext}_{\mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}}^{|\alpha|}(\mathcal{IC}(\partial_\alpha \mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F}, \mathcal{IC}(\mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F})$.

We will follow the construction of 4.5.2. First, tensoring with $\mathrm{Id} \in \mathrm{Ext}_{\mathcal{G}\mathcal{Q}_\eta^\beta}^0(\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F}, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$, we obtain the map from $\mathcal{A}_\alpha = \mathrm{Ext}_{C^\alpha}^{|\alpha|}(\mathcal{IC}(C^\alpha), \mathfrak{F}_\alpha)$ to $\mathrm{Ext}_{\mathcal{G}\mathcal{Q}_\eta^\beta \times C^\alpha}^{|\alpha|}(\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \mathcal{IC}(C^\alpha), \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \mathfrak{F}_\alpha)$ (notations of 4.5.2).

As in 4.5.2, there is a canonical map c from $\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \mathfrak{F}_\alpha$ to $\sigma_{\beta,\alpha}^!(\mathcal{IC}(\mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F})$ extending the factorization isomorphism from an open part $\mathring{\mathcal{G}}\mathcal{Q}_\eta^\beta \times \mathbb{A}^\alpha \subset \mathcal{G}\mathcal{Q}_\eta^\beta \times C^\alpha$. Thus c induces the map from $\mathrm{Ext}_{\mathcal{G}\mathcal{Q}_\eta^\beta \times C^\alpha}^{|\alpha|}(\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \mathcal{IC}(C^\alpha), \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \mathfrak{F}_\alpha)$ to $\mathrm{Ext}_{\mathcal{G}\mathcal{Q}_\eta^\beta \times C^\alpha}^{|\alpha|}(\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \mathcal{IC}(C^\alpha), \sigma_{\beta,\alpha}^!(\mathcal{IC}(\mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F}))$.

The latter space equals $\mathrm{Ext}_{\mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}}^{|\alpha|}((\sigma_{\beta,\alpha})_*(\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \mathcal{IC}(C^\alpha)), \mathcal{IC}(\mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F}) = \mathrm{Ext}_{\mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}}^{|\alpha|}(\mathcal{IC}(\partial_\alpha \mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F}, \mathcal{IC}(\mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F})$.

Finally, we define $\mathfrak{v}_{\alpha,\beta}$ as the composition of above maps.

7.1.2. We construct the *costabilization map*

$$\mathfrak{y}_{\beta,\alpha} : H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F}) \longrightarrow H^{\bullet-|\alpha|}(\partial_\alpha \mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}, \mathcal{IC}(\partial_\alpha \mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F})$$

To this end we note that exactly as in 2.6.1, we have $\mathcal{IC}(\partial_\alpha \mathfrak{Q}^{\alpha+\beta}) = (\sigma_{\beta,\alpha})_* \mathcal{IC}(\mathfrak{Q}^\beta \times C^\alpha) = (\sigma_{\beta,\alpha})_*(\mathcal{IC}(\mathfrak{Q}^\beta) \boxtimes \underline{\mathbb{C}}[|\alpha|])$. Let $[C^\alpha] \in H^{-|\alpha|}(C^\alpha, \underline{\mathbb{C}}[|\alpha|])$ denote the fundamental class of C^α.

Now, for $h \in H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$ we define $\mathfrak{y}_{\beta,\alpha}(h)$ as $h \otimes [C^\alpha] \in H^{\bullet-|\alpha|}(\mathcal{G}\mathcal{Q}_\eta^\beta \times C^\alpha, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \underline{\mathbb{C}}[|\alpha|]) = H^{\bullet-|\alpha|}(\partial_\alpha \mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}, (\sigma_{\beta,\alpha})_*(\mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F} \boxtimes \underline{\mathbb{C}}[|\alpha|])) = H^{\bullet-|\alpha|}(\partial_\alpha \mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}, \mathcal{IC}(\partial_\alpha \mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F})$.

7.2. Definition. Let $a \in \mathcal{A}_\alpha$, $h \in H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$. We define the action $a(h) \in H^\bullet(\mathcal{G}\mathcal{Q}_\eta^{\alpha+\beta}, \mathcal{IC}(\mathfrak{Q}^{\alpha+\beta}) \otimes \mathbf{p}^*\mathcal{F})$ as the action of $\mathfrak{v}_{\alpha,\beta}(a)$ on the global cohomology applied to $\mathfrak{y}_{\beta,\alpha}(h)$.

Let us stress that the action of $\mathcal{A}_\alpha$ *preserves cohomological degrees.*

7.2.1. For $a \in \mathcal{A}_\alpha$, $b \in \mathcal{A}_\beta$, $h \in H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$ we have $a(b(h)) = a \cdot b(h)$. The proof is entirely similar to the proof of associativity of the multiplication in $\mathcal{A}$.

7.3. For $\beta \in Y$ the graded space $H^\bullet(\mathcal{GQ}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$ is up to a shift Poincaré dual to $H^\bullet(\mathcal{GQ}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$ (cf. [FM] 12.1b). We define the map

$$f_i : \; H^\bullet(\mathcal{GQ}_\eta^{\beta+i}, \mathcal{IC}(\mathfrak{Q}^{\beta+i}) \otimes \mathbf{p}^*\mathcal{F}) \longrightarrow H^\bullet(\mathcal{GQ}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$$

as the dual of the map

$$e_i : \; H^\bullet(\mathcal{GQ}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F}) \longrightarrow H^\bullet(\mathcal{GQ}_\eta^{\beta+i}, \mathcal{IC}(\mathfrak{Q}^{\beta+i}) \otimes \mathbf{p}^*\mathcal{F})$$

(see 4.4.1).

It follows from 4.4 that the maps f_i, $i \in I$, satisfy the Serre relations of $\mathfrak{n}_-^L$.

7.3.1. We safely leave to the reader an elementary construction of the operators f_i, e_i absolutely similar to the one in 4.7.1, 4.7.2.

7.4. For $i \in I$ we define the endomorphism h_i of $H^\bullet(\mathcal{GQ}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$ as the scalar multiplication by $\langle \beta + 2\check\rho, i' \rangle$ where $i' \in X$ is the simple root. Exactly as in 4.8, 4.9, one proves that the operators e_i, f_i, h_i, $i \in I$, generate the action of the Langlands dual Lie algebra $\mathfrak{g}^L$ on $\oplus_{\beta \in Y} H^\bullet(\mathcal{GQ}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F}) = \oplus_{\beta \in Y} H^\bullet(\mathcal{Q}^{\eta+\beta}, \mathbf{c}_\mathcal{Q}^\beta(\mathcal{F}))$.

In the particular case $\eta = 0$, $\mathcal{F}$ is the irreducible skyscraper sheaf, we have $\mathbf{c}_\mathcal{Q}^\beta(\mathcal{F}) = \mathcal{IC}(\mathcal{Q}^\beta)$, and we return to the $\mathfrak{g}^L$-action on $\oplus_{\beta \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ defined in 4.10.

We will formulate a few conjectures concerning this action, following [FK] §6.

7.5. Let $\mathcal{N}^L \subset \mathfrak{g}^L$ be the nilpotent cone. The Lie algebra $\mathfrak{g}^L$ acts on the cohomology $H_{\mathfrak{n}_-^L}^{\dim \mathbf{X}}(\mathcal{N}^L, \mathcal{O})$ of the structure sheaf of $\mathcal{N}^L$ with supports in $\mathfrak{n}_-^L$. The character of this module is well known to be $\frac{|\mathcal{W}_f| e^{2\check\rho}}{\prod_{\check\theta \in \check{\mathcal{R}}^+}(1 - e^{\check\theta})^2}$. Thus it coincides with the character of $\mathfrak{g}^L$-module $\oplus_{\beta \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$.

Conjecture. $\mathfrak{g}^L$-modules $\oplus_{\beta \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ and $H_{\mathfrak{n}_-^L}^{\dim \mathbf{X}}(\mathcal{N}^L, \mathcal{O})$ are isomorphic.

As explained in [FK] §6 the module $H_{\mathfrak{n}_-^L}^{\dim \mathbf{X}}(\mathcal{N}^L, \mathcal{O})$ is *tilting*, and to prove the Conjecture it suffices to check that $\oplus_{\beta \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ is tilting as well. To this end it is enough to check that the action of the algebra $\mathcal{A} = U(\mathfrak{n}_+^L)$ on $\oplus_{\beta \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathcal{Q}^\beta))$ is free.

7.6. For $\eta \in Y^+$ the irreducible $\mathfrak{g}^L$-module with the highest weight η is denoted by W_η. The Theorem 13.2 of [FM] implies that the character of $\mathfrak{g}^L$-module $\oplus_{\beta \in Y} H^\bullet(\mathcal{Q}^\beta, \mathbf{c}_\mathcal{Q}^\beta(\mathcal{IC}(\overline{\mathcal{G}}_\eta)))$ equals $\mathrm{char}(W_\eta) \times \frac{|\mathcal{W}_f| e^{2\check\rho}}{\prod_{\check\theta \in \check{\mathcal{R}}^+}(1 - e^{\check\theta})^2}$.

Conjecture. There is an isomorphism of $\mathfrak{g}^L$-modules $\oplus_{\beta \in Y} H^\bullet(\mathcal{Q}^\beta, \mathbf{c}_\mathcal{Q}^\beta(\mathcal{IC}(\overline{\mathcal{G}}_\eta)))$ and $W_\eta \otimes H_{\mathfrak{n}_-^L}^{\dim \mathbf{X}}(\mathcal{N}^L, \mathcal{O})$.

This Conjecture is again equivalent to the statement that the $\mathfrak{g}^L$-module $\oplus_{\beta \in Y} H^\bullet(\mathcal{Q}^\beta, \mathbf{c}_\mathcal{Q}^\beta(\mathcal{IC}(\overline{\mathcal{G}}_\eta)))$ is tilting. To check this it suffices to prove that the action of $\mathcal{A} = U(\mathfrak{n}_+^L)$ on $\oplus_{\beta \in Y} H^\bullet(\mathcal{Q}^\beta, \mathbf{c}_\mathcal{Q}^\beta(\mathcal{IC}(\overline{\mathcal{G}}_\eta)))$ is free.

7.7. Let $\mathcal{F}$ be a $\mathbf{G}[[z]]$-equivariant perverse sheaf on the affine Grassmannian $\mathcal{G}$. According to [MV], $H^\bullet(\mathcal{G}, \mathcal{F})$ is equipped with a canonical $\mathfrak{g}^L$-action. While the action of $\mathfrak{h}^L \subset \mathfrak{g}^L$ is defined pretty explicitly in geometric terms in [MV], the actions of $\mathfrak{n}_-^L, \mathfrak{n}_+^L \subset \mathfrak{g}^L$ were not constructed as explicitly so far. Motivated by the Conjecture 7.6 above we propose the following conjectural constructions of the $\mathfrak{n}_\pm^L$-actions on $H^\bullet(\mathcal{G}, \mathcal{F})$.

7.7.1. Let $\mathcal{Q}^{\eta+\beta} = \sqcup_{0 \leq \gamma \leq \eta+\beta} \overset{\bullet}{\mathcal{Q}}{}^\gamma$ be the partition of $\mathcal{Q}^{\eta+\beta}$ according to the defect at $0 \in C$. Let $\mathcal{Q}^{\eta+\beta} = \cup_{0 \leq \gamma \leq \eta+\beta} \mathcal{Q}^\gamma$ be the corresponding filtration by closed subspaces. In particular, we have $\mathbf{X} = \overset{\bullet}{\mathcal{Q}}{}^0 = \mathcal{Q}^0 \subset \mathcal{Q}^{\eta+\beta}$. The perverse sheaf $\mathbf{c}_\mathcal{Q}^\beta(\mathcal{F})$ is isomorphic to a direct sum of a few copies of the sheaves $\mathcal{IC}(\mathcal{Q}^\gamma)$. More precisely, the multiplicity of $\mathcal{IC}(\mathcal{Q}^\gamma)$ in $\mathbf{c}_\mathcal{Q}^\beta(\mathcal{F})$ equals $H_c^{2|\beta-\gamma|}(T_{\gamma-\beta}, \mathcal{F})$ (see the Theorem 13.2 of [FM]). In particular, $H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) \otimes \mathcal{IC}(\mathcal{Q}^0) = H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) \otimes \underline{\mathbb{C}}_\mathbf{X}[n]$ is canonically a direct summand of $\mathbf{c}_\mathcal{Q}^\beta(\mathcal{F})$. Hence $H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) = H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) \otimes H^{-n}(\mathbf{X}, \underline{\mathbb{C}}[n])$ is canonically a direct summand of $H^\bullet(\mathcal{Q}^{\eta+\beta}, \mathbf{c}_\mathcal{Q}^\beta(\mathcal{F})) = H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$.

We conjecture that $\oplus_{\beta \in Y} H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) = \oplus_{\beta \in Y} H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) \otimes H^{-n}(\mathbf{X}, \underline{\mathbb{C}}[n])$ is an $\mathcal{A}^{opp} = U(\mathfrak{n}_-^L)$-submodule of $\oplus_{\beta \in Y} H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$ (with respect to the action defined in 7.2), and the resulting action of $U(\mathfrak{n}_-^L)$ on $\oplus_{\beta \in Y} H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) = H^\bullet(\mathcal{G}, \mathcal{F})$ coincides with the action of [MV].

7.7.2. Similarly, $H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) = H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) \otimes H^n(\mathbf{X}, \underline{\mathbb{C}}[n])$ is canonically a direct summand of $H^\bullet(\mathcal{Q}^{\eta+\beta}, \mathbf{c}_\mathcal{Q}^\beta(\mathcal{F})) = H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F})$. We conjecture that the kernel of the natural projection $\oplus_{\beta \in Y} H^\bullet(\mathcal{G}\mathcal{Q}_\eta^\beta, \mathcal{IC}(\mathfrak{Q}^\beta) \otimes \mathbf{p}^*\mathcal{F}) \longrightarrow \oplus_{\beta \in Y} H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) = \oplus_{\beta \in Y} H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) \otimes H^n(\mathbf{X}, \underline{\mathbb{C}}[n])$ is invariant under the action of $\mathcal{A} = U(\mathfrak{n}_+^L)$ defined in 7.3, and the induced action of $U(\mathfrak{n}_+^L)$ on $\oplus_{\beta \in Y} H_c^{2|\beta|}(T_{-\beta}, \mathcal{F}) = H^\bullet(\mathcal{G}, \mathcal{F})$ coincides with the action of [MV].

7.8. Recall the moduli scheme $\mathfrak{M}$ of $\mathbf{G}$-torsors on $\mathbb{P}^1$ equipped with the formal trivialization at $\infty \in \mathbb{P}^1$. Its stratification according to the isomorphism classes of $\mathbf{G}$-torsors $\mathfrak{M} = \sqcup_{\eta \in Y^+} \mathfrak{M}_\eta$ was described in [FM] 10.6. Let $\overline{\mathfrak{M}}_\eta$ denote the closure of the stratum $\mathfrak{M}_\eta$, and let $\mathcal{IC}(\overline{\mathfrak{M}}_\eta)$ denote the corresponding $\mathcal{IC}$-sheaf. According to the Decomposition Theorem, for $\alpha \in Y$, the direct image $\mathbf{p}_* \mathcal{IC}(\mathfrak{Q}^\xi)$ is a semisimple complex on $\mathfrak{M}$, isomorphic to a direct sum of various $\mathcal{IC}(\overline{\mathfrak{M}}_\eta)$ with shifts and multiplicities. We have $\bigoplus_{\xi \in Y} \mathbf{p}_* \mathcal{IC}(\mathfrak{Q}^\xi) = \bigoplus_{\eta \in Y^+} K_\eta^\bullet \otimes \mathcal{IC}(\overline{\mathfrak{M}}_\eta)$ for (infinite dimensional in general) graded multiplicities spaces K_η.

One can introduce the stalkwise action of $\mathfrak{g}^L$ on $\bigoplus_{\xi \in Y} \mathbf{p}_* \mathcal{IC}(\mathfrak{Q}^\xi)$ entirely similar to 7.4. In other words, one obtains the action of $\mathfrak{g}^L$ on the multiplicities spaces $K_\eta^\bullet$ for any $\eta \in Y^+$. For instance, $K_0^\bullet = \oplus_{\beta \in \mathbb{N}[I]} H^\bullet(\mathcal{Q}^\beta, \mathcal{IC}(\mathfrak{Q}^\beta))$ by definition. The action of $\mathfrak{g}^L$ on $K_0^\bullet$ was described in 7.5. We will extend the conjecture 7.5 to the case of arbitrary $\eta \in Y^+$.

Recall the Lusztig's quantum groups $\mathfrak{u} \subset \mathfrak{U}$, and the representations' category $\mathfrak{C}$, introduced in [FM] 1.3. The highest weights of the regular block $\mathfrak{C}^0$ of the category of $\mathfrak{U}$-modules are contained in the $\mathcal{W}$-orbit $\mathcal{W} \cdot 0 \subset X$ (the strong linkage principle). Let $m_\eta \in \mathcal{W}$ be the shortest representative of the double coset $\mathcal{W}_f \eta \mathcal{W}_f$ in $\mathcal{W}$. Let $T(m_\eta \cdot 0) \in \mathfrak{C}^0$ be the indecomposable *tilting* $\mathfrak{U}$-module with the highest weight $m_\eta \cdot 0$ (see [S]). For example, $T(m_0 \cdot 0)$ is the trivial module. Finally, recall the notion of semiinfinite cohomology $H^{\frac{\infty}{2}+\bullet}(\mathfrak{u}, ?)$ introduced in [A]. It is known that for $T \in \mathfrak{C}^0$ the semiinfinite cohomology $H^{\frac{\infty}{2}+\bullet}(\mathfrak{u}, T)$ carries a natural structure of $\mathfrak{n}_-^L$-integrable $\mathfrak{g}^L$-module.

Conjecture. For $\eta \in Y^+$ there is an isomorphism of graded $\mathfrak{g}^L$-modules $K_\eta^\bullet \simeq H^{\frac{\infty}{2}+\bullet}(\mathfrak{u}, T(m_\eta \cdot 0))$. Under this isomorphism, the weight $\xi \in Y$ part of the RHS corresponds to the multiplicity $_{(\xi)}K_\eta^\bullet \subset K_\eta^\bullet$ of $\mathcal{IC}(\overline{\mathfrak{M}}_\eta)$ in $\mathbf{p}_* \mathcal{IC}(\mathfrak{Q}^\xi)$.

Corollary of the Conjecture. Suppose $\eta - 2\check{\rho} \in Y^+$. Then $K_\eta^\bullet$ is concentrated in degree 0, and is isomorphic to the irreducible $\mathbf{G}^L$-module $W_{\eta - 2\check{\rho}}$.

Remark. S.Arkhipov has recently proved that $H^{\frac{\infty}{2}+\bullet}(\mathfrak{u}, T(0)) \simeq H^{\dim \mathbf{X}}_{\mathfrak{n}_-^L}(\mathcal{N}^L, \mathcal{O})$. Together with the results of [FKM] this establishes the $\eta = 0$ case of the above conjecture.

8. APPENDIX. KONTSEVICH RESOLUTION OF QUASIMAPS' SPACE

8.1. For $\alpha = \sum_{i \in I} a_i i \in \mathbb{N}[I]$ let $\mathcal{Q}_K^\alpha = \overline{M}_{0,0}(\mathbb{P}^1 \times \mathbf{X}), (1, \alpha))$ be the Kontsevich space of stable maps from the genus zero curves without marked points to $\mathbb{P}^1 \times \mathbf{X}$ of bidegree $(1, \alpha)$ (see [Ko]). In this Appendix we construct a regular birational map $\pi : \mathcal{Q}_K^\alpha \longrightarrow \mathcal{Q}^\alpha$.

To this end recall that $\mathbf{X}$ is a closed subvariety in $\prod_{i \in I} \mathbb{P}(V_{\omega_i})$ given by Plücker relations. Drinfeld's space $\mathcal{Q}^\alpha$ is a closed subvariety in $\prod_{i \in I} \mathbb{P} H^0(C, V_{\omega_i} \otimes \mathcal{O}(a_i))$ also given by Plücker relations.

So to construct the desired map it suffices to solve the following problem. For a vector space V one has 2 compactifications of the space M^a of algebraic maps $f : C \longrightarrow \mathbb{P}(V)$ of degree $a \in H_2(\mathbb{P}(V), \mathbb{Z})$. Kontsevich compactification $M_K^a = \overline{M}_{0,0}(C \times \mathbb{P}(V), (1, a))$ is the space of all stable maps from the genus 0 curves into the product $C \times \mathbb{P}(V)$ of bidegree $(1, a)$; the embedding of M^a into M_K^a takes f to its graph Γ_f. Drinfeld compactification M_D^a is the space of invertible subsheaves of degree $-a$ in $V \otimes \mathcal{O}_C$. Twisting by $\mathcal{O}(a)$ we identify M_D^a with $\mathbb{P} H^0(C, V \otimes \mathcal{O}(a))$. The embedding of M^a into M_D^a takes f to the line subbundle $\mathcal{L}_f := p_* q^* \mathcal{O}_{\mathbb{P}(V)}(-1) \subset V \otimes \mathcal{O}_C$ where $C \xleftarrow{p} \Gamma_f \xrightarrow{q} \mathbb{P}(V)$ are the natural projections. We want to prove that the identification $M_K^a \supset M^a = M^a \subset M_D^a$ extends to the regular map $\pi : M_K^a \longrightarrow M_D^a$.

This problem was addressed by A.Givental in [G], "Main Lemma". Unfortunately, the validity of his proof is controversial. In the rest of the Appendix we prove that Id: $M^a \xrightarrow{\sim} M^a$ extends to the regular map $\pi : M_K^a \longrightarrow M_D^a$.

8.2. One of the most important properties of the Drinfeld space M_D^a is the following.

Lemma. Let S be a scheme and $\mathcal{L} \subset V \otimes \mathcal{O}_{C \times S}$ be an invertible subsheaf in the trivial vector bundle $V \otimes \mathcal{O}_{C \times S}$ on $C \times S$ of degree $-a$ over S (i.e. for any $s \in S$ the restriction $\mathcal{L}_s$ of $\mathcal{L}$ to the fiber $C \times s \subset C \times S$ is equal to $\mathcal{O}(-a)$). If the open subset $U = \{(x, s) \in C \times S \mid \mathcal{L}_{(x,s)} \to V \otimes \mathcal{O}_{x,s}$ is embedding$\}$ contains generic points of all fibers of $C \times S$ over S then the map $f : S \to M_D^a$ sending each point $s \in S$ to the subsheaf $\mathcal{L}_s \subset V \otimes \mathcal{O}_{C \times s} \cong C \otimes \mathcal{O}_C$ is regular.

Proof. Let $p : C \times S \to S$ be the natural projection and $L = p_*(\mathcal{L} \otimes \mathcal{O}_C(a))$. It is evident that for any $s \in S$ we have

$$
\begin{array}{ccccc}
H^0(p^{-1}(s), \mathcal{L} \otimes \mathcal{O}_C(a)) & = & H^0(C, \mathcal{O}_C) & = & \mathbb{C}, \\
H^{>0}(p^{-1}(s), \mathcal{L} \otimes \mathcal{O}_C(a)) & = & H^{>0}(C, \mathcal{O}_C) & = & 0
\end{array}
$$

and the map

$$H^0(p^{-1}(s), \mathcal{L} \otimes \mathcal{O}_C(a)) \to H^0(p^{-1}(s), V \otimes \mathcal{O}_C(a))$$

is injection, hence L is a line subbundle in $p_*(V \otimes \mathcal{O}_C(a)) = H^0(C, V \otimes \mathcal{O}_C(a)) \otimes \mathcal{O}_S$. The map f is just the regular map associated with the subbundle L. □

8.3. Recall the statement of Givental's "Main Lemma".

"Main Lemma" a) The map Id: $M^a \xrightarrow{\sim} M^a$ extends to the regular map $\pi : M_K^a \to M_D^a$;

b) Let $(\varphi : \mathcal{C} \to C \times \mathbb{P}(V)) \in M_K^a$ be a stable map and let $\varphi' : \mathcal{C} \to C$, $\varphi'' : \mathcal{C} \to \mathbb{P}(V)$ denote the induced maps. Let $\mathcal{C}_0$ denote the irreducible component of $\mathcal{C}$ such that $\varphi' : \mathcal{C}_0 \to C$ is dominant, and let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be the connected components of $\mathcal{C} \setminus \mathcal{C}_0$. Finally, let $x_1 = \varphi'(\mathcal{C}_1), \ldots, x_m = \varphi'(\mathcal{C}_m)$. Then $\pi(\mathcal{C})$ is the subsheaf with normalisation equal to $\varphi'_*(\varphi''_{|\mathcal{C}_0})^* \mathcal{O}_{\mathbb{P}(V)}(-1)$ and with defect at points $x_1, \ldots x_m$ of degree $\varphi''_*[\mathcal{C}_1], \ldots, \varphi''_*[\mathcal{C}_m]$ respectively.

Proof. a) Consider the space $\widetilde{M}_K^a = \overline{M}_{0,1}(C \times \mathbb{P}(V); (1, a))$ of stable maps with one marked point. There are natural maps $\mathsf{f} : \widetilde{M}_K^a \to M_K^a$ forgetting the marked point, and $\mathrm{ev} : \widetilde{M}_K^a \to C \times \mathbb{P}(V)$ evaluating at the marked point. Let $p : \widetilde{M}_K^a \to C \times M_K^a$ denote the product of the map $p_1 \circ \mathrm{ev} : \widetilde{M}_K^a \xrightarrow{\mathrm{ev}} C \times \mathbb{P}(V) \xrightarrow{p_1} C$ and of the map f. Let $q : \widetilde{M}_K^a \to \mathbb{P}(V)$ denote the map $p_2 \circ \mathrm{ev} : \widetilde{M}_K^a \xrightarrow{\mathrm{ev}} C \times \mathbb{P}(V) \xrightarrow{p_2} \mathbb{P}(V)$.

The map p is an isomorphism over the open subspace $U \subset C \times M_K^a$ of pairs $(x, \mathcal{C})$ such that $x \in \varphi'(\mathcal{C}_0 - \mathcal{C}^{\mathrm{sing}})$. Consider the sheaf $\mathcal{B} = p_* q^* \mathcal{O}_{\mathbb{P}(V)}(-1) \subset p_* q^*(V \otimes \mathcal{O}_{\mathbb{P}(V)}) = V \otimes \mathcal{O}_{C \times M_K^a}$. Its reflexive hull $\mathcal{B}^{**}$ is a reflexive rank 1 sheaf on the smooth orbifold $C \times M_K^a$, hence $\mathcal{B}^{**}$ is an invertible sheaf. Note that since $\mathcal{B}$ is invertible over U, we have $\mathcal{B}^{**}_{|U} = \mathcal{B}_{|U}$. Since U contains $C \times M^a$, the restriction of $\mathcal{B}^{**}$ to the generic fiber of $C \times M_K^a$ over M_K^a is $\mathcal{O}(-a)$, hence the restriction of $\mathcal{B}^{**}$ to any fiber is $\mathcal{O}(-a)$ as well. Over the set U, the embedding $\mathcal{B}^{**}_{|U} \to V \otimes \mathcal{O}_U$ is an embedding of vector bundles (because over U the map p is an isomorphism). On the other hand, U contains generic points of all fibers of $C \times M_K^a$ over M_K^a, therefore we can apply the Lemma 8.2 and the first part of the "Main Lemma" follows.

b) Let $s = (\varphi : \mathcal{C} \to C \times \mathbb{P}(V)) \in M_K^a$ be a stable map, and let $L \subset V \otimes \mathcal{O}_C$ denote a subsheaf $\pi(s) \in M_D^a$; let $L' = \varphi'_*(\varphi''_{|\mathcal{C}_0}{}^*) \mathcal{O}_{\mathbb{P}(V)}(-1)$. The Lemma 8.2 implies $L = \mathcal{B}_s^{**}$. We want to prove that the normalization $\widetilde{L}$ of L equals L'.

Denote $C - \{x_1, \ldots, x_m\}$ by C^0. Then $U \bigcap (C \times s) = C^0 \times s$, hence

$$L_{|C^0 \times s} = \mathcal{B}_{s|C^0 \times s} = (\varphi'_* \varphi''^* \mathcal{O}_{\mathbb{P}(V)}(-1))_{|C^0 \times s} = (\varphi'_*(\varphi''_{|\mathcal{C}_0})^* \mathcal{O}_{\mathbb{P}(V)}(-1))_{|C^0 \times s} = L'_{|C^0 \times s},$$

i.e. L and L' coincide on the open subset of $C \times s$. Since L' is a line subbundle in $V \otimes \mathcal{O}_{C \times s}$ this means that L is a subsheaf in L' and the quotient L'/L is concentrated at the points $x_1, \ldots, x_m$. Therefore $\widetilde{L} = L'$ and the defect of L is concentrated at the points $x_1, \ldots, x_m$.

Note that in the case $m = 1$ the "Main Lemma" follows. In the general case we proceed as follows. Let $c_k = \varphi''_*[\mathcal{C}_k]$, $(k = 1, \ldots, m)$. Let $\mathbf{M}_K^k = \{(\psi : \mathsf{C} \to C \times \mathbb{P}(V))\} \subset M_K^a$ denote the subspace of all stable maps such that a curve C has two irreducible components $\mathsf{C} = \mathsf{C}_0 \cup \mathsf{C}_1$, $\psi_*[\mathsf{C}_0] = (1, a - c_k)$, $\psi_*[\mathsf{C}_1] = (0, c_k)$ and $\psi'(\mathsf{C}_1) = x_k$. Then we have $s \in \overline{\mathbf{M}}_K^1 \bigcap \overline{\mathbf{M}}_K^2 \bigcap \cdots \bigcap \overline{\mathbf{M}}_K^m$.

Let $\mathbf{M}_D^k \subset M_D^a$ denote the subspace of all invertible subsheaves with defect of degree c_k concentrated at the point x_k. According to the case $m = 1$ we have $\pi(\mathbf{M}_K^k) \subset \mathbf{M}_D^k$, hence $\pi(s) \in \overline{\mathbf{M}}_D^1 \bigcap \overline{\mathbf{M}}_D^2 \bigcap \cdots \bigcap \overline{\mathbf{M}}_D^m$. Since the degree of defect at a given point increases under specialization, the degrees of the defect of $\pi(s)$ at the points x_k are greater or equal than c_k, hence they are equal to c_k, and the "Main Lemma" follows. $\square$

References

[A]   S.Arkhipov, Semiinfinite cohomology of quantum groups, *Communications in Mathematical Physics* **188** (1997), 379-405.

[BBD]   A.Beilinson, J.Bernstein, P.Deligne, Faisceaux Pervers, *Astérisque* **100** (1982).

[BL]   J.Bernstein, V.Lunts, Equivariant Sheaves and Functors, *Lecture Notes in Math.* **1578** (1994).

[EM]   S.Evens, I.Mirković, Fourier transform and the Iwahori-Matsumoto involution, *Duke Math. J.* **86**, No. 3 (1997), 435-464.

[FK]   M.Finkelberg, A.Kuznetsov, Global Intersection Cohomology of Quasimaps' spaces, *Int. Math. Res. Notices,* **7** (1997), 301-328.

[FKM]   M.Finkelberg, A.Kuznetsov, I.Mirković, Frobenius realization of $U(\mathfrak{n}^L)$, in preparation.

[FM]   M.Finkelberg, I.Mirković, Semiinfinite Flags. I. Case of global curve $\mathbb{P}^1$, this volume.

[G]   A.Givental, Equivariant Gromov-Witten invariants, *Int. Math. Res. Notices,* **13** (1996), 613-663.

[Ka]   M.Kashiwara, Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebra, *Progress in Math.,* **87** (1991), 407-433.

[Kat]   S.I.Kato, On the Kazhdan-Lusztig polynomials for affine Weyl groups, *Adv. in Math.* **55** (1985), 103-130.

[Ko]   M.Kontsevich, Enumeration of rational curves via torus actions, *Progress in Math.* **129** (1995), 335-368.

[KT1]   M.Kashiwara, T.Tanisaki, Kazhdan-Lusztig conjecture for symmetryzable Kac-Moody Lie algebra II, *Progress in Math.,* **82** (1990), 159-195.

[KT2]   M.Kashiwara, T.Tanisaki, Characters of the negative level highest weight modules for affine Lie algebras, *Duke Math. J.,* **77** (1995), 21-62.

[Ku]   A.Kuznetsov, Laumon's resolution of Drinfeld's compactification is small, *Math. Research Letters,* **4**, No. 2–3 (1997), 349-364.

[L1]   G.Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, *Adv. in Math.* **37**, No. 2 (1980), 121-164.

[L2]   G.Lusztig, Singularities, character formulas, and a q-analogue of weight multiplicities, *Astérisque* **101-102** (1983), 208-229.

[L3]   G.Lusztig, Intersection cohomology methods in representation theory, *ICM90* (1990), 155-174.

[L4]   G.Lusztig, Introduction to quantum groups, Boston, Birkhauser, 1993.

[L5]   G.Lusztig, Monodromic systems on affine flag manifolds, *Proc. R. Soc. Lond. A* **445** (1994), 231-246. Errata, **450** (1995), 731-732.

[MV]   I.Mirković, K.Vilonen, Perverse sheaves on loop Grassmannians and Langlands duality, in preparation, also Preprint alg-geom/9703010.

[S]   W.Soergel, Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln, *Representation Theory, Electronic Journal of AMS*, **1** (1997), 37-68 (engl. 83-114).

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