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"Peak response of non-linear oscillators under stationary white noise"

by

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Peak response of non-linear oscillators under stationary white noise

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Abstract

The use of the Advanced Censored Closure (ACC) technique, recently proposed by the authors for predicting the peak response of linear structures vibrating under random processes, is extended to the case of non-linear oscillators driven by stationary white noise. The proposed approach requires the knowledge of mean upcrossing rate and spectral bandwidth of the response process, which in this paper are estimated through the Stochastic Averaging method. Numerical applications to oscillators with non-linear stiffness and damping are included, and the results are compared with those given by Monte Carlo Simulation and by other approximate formulations available in the literature.

Keywords: Censored Closure; Computational Stochastic Mechanics; First Passage Time; Gumbel Distribution; Poisson Approach; Random Vibration; Reliability Analysis; Stochastic Averaging.

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1. Introduction

The stochastic analysis of structural and mechanical systems subjected to dynamic actions of a random nature has become very popular in the last decades, given that in a number of engineering situations deterministic approaches are quite unsatisfactory.

When the dynamic excitation is modelled as a Gaussian process, and the system exhibits a linear behaviour, the response is Gaussian too. In this case, then, the knowledge of mean value and standard deviation fully defines the response from a probabilistic point of view. In many circumstances, however, due to a non-linear behaviour of the system, the response may significantly deviate from the Gaussianity, and higher-order statistics are then required. Unfortunately, these are available in exact form just for a restricted class of simple systems: therefore, several approximate methods, with a different degree of complexity and accuracy, have been proposed. Perhaps, the most popular approaches are the methods based on Gaussian and non-Gaussian closure schemes and on approximate solutions of the Fokker-Planck-Kolmogorov (FPK) equation, which are well codified in the literature (e.g. [1, 2, 3, 4, 5]). Of course, different approaches are also available (e.g. the methods based on the maximum entropy principle and on the dissipation energy balancing) and, among these, the Stochastic Averaging (SA) method [6, 7] is applied in this paper in order to estimate the mean upcrossing rate and the Power Spectral Density (PSD) of the response of a Single-Degree-of-Freedom (SDoF) oscillator with non-linear restoring force under white noise input.

It is well known that the mere probabilistic characterization of the response process is not sufficient in a reliability analysis. In fact, under the assumption that a vibrating system fails as soon as the response firstly exits a given safe domain, the statistics of the first passage time have to be estimated, starting from the knowledge of the statistics of the response to the random excitation. This is recognized to be one of the most complicated problem in Computational Stochastic Mechanics, and no exact solutions have been derived, even in the simplest case of SDoF linear oscillators under stationary white noise; hence, a number of approximate formulations are available in the literature.

Among these, the most popular one is the so-called “Poisson approach” (e.g. [3]), in which the response upcrossings of a deterministic threshold are assumed to be statistically independent events. This classical approach, however, proves to be too conservative when the response process is narrowband, and/or when the threshold is not high enough. In these situations, in fact, consecutive upcrossings of the response process cannot be realistically considered as independent events, as they tend to occur in clumps, whose mean size depends on the spectral bandwidth of the response. The latter, then, has to be somehow accounted for in order to improve the results.

The Gaussian Censored Closure (GCC) technique proposed by Senthilnathan and Lutes [8] reveals the same bounds, since also in this case the clumping tendency of the response upcrossings is neglected. With the purpose of overcoming this drawback, Muscolino and Palmeri [9, 10] recently introduced an expedient "censorship factor," which can be directly related to the spectral bandwidth of the response process; the use of the Gumbel model as "uncensored" PDF for the peak response, instead of the Gaussian one, further improves the results. Effectiveness, accuracy and computational advantages of this technique have been proved in the reliability analysis of linear structures, also in the general case of Multi-Degree-of-Freedom systems subjected to coloured noises [11].

Aim of this paper is to extend the use of the proposed technique, termed Advanced Censored Closure (ACC), to non-linear SDoF oscillators under stationary white noise. The results herein presented complement those included in Ref. [12], in which only the case of non-linear damping is coped with. At the best knowledge of the authors, these are the first "consistent" applications of a censored closure technique in the reliability analysis of non-linear dynamical systems. The only examples found in the literature, in fact, are the pioneering papers by Suzuki and Minai [13, 14] in which, however, the response of elastoplastic structures is assumed to be Gaussian.

The proposed ACC technique is amply illustrated by numerical examples, which demonstrate the superiority with respect to Poisson approach and GCC technique, especially when the response process of the non-linear oscillator is narrowband.

2. Response analysis

Let us considerer the random vibration of a non-linear SDoF oscillator driven by a zero-mean stationary white noise W_t:

$$m\,\ddot{X}_t + f\left(X_t, \dot{X}_t\right) = W_t \tag{1}$$

where X_t is the random process that describes the motion, $t \geq 0$ is the generic time instant, m is the inertia, $f\left(x,\dot{x}\right)$ is the non-linear restoring force, which depends on the instantaneous values of displacement, $X_t = x$, and velocity, $\dot{X}_t = \dot{x}$, and the over-dot denotes the time derivate. For the simplicity purpose, the restoring force is assumed to be symmetric with respect to the origin of the phase plane $\{x,\dot{x}\}$, i.e. $f\left(x,\dot{x}\right) = f\left(-x,-\dot{x}\right)$. As a consequence, in our analyses the mean value of the response process X_t is zero.

From a probabilistic point of view, the state variables of the system, X_t and $\dot{X}_t$, are characterized in stationary conditions by the knowledge of the time-independent joint Probability Density Function (PDF), $p_{X\dot{X}}\left(x,\dot{x}\right)$. Given that $f\left(x,\dot{x}\right)$ is a non-linear function, $p_{X\dot{X}}\left(x,\dot{x}\right)$ is non-Gaussian, and as strong is the non-linearity in the reaction force, as largely the PDF of the response deviates from the Gaussianity. These situations, when the exact solution is not available, the PDF of the response can be estimated via a number of approximate methods known to the literature.

Among others, the Stochastic Averaging (SA) method is widely adopted, being versatile and quite straightforward [6]. The method, herein applied in the form recently presented in Ref. [7], operates under the assumption that the motion is pseudo-harmonic, that is:

$$X_t = A_t \cos\left[\omega_{\text{eff}}(A_t)\,t + \Phi_t\right]$$
$$\dot{X}_t = -A_t \sin\left[\omega_{\text{eff}}(A_t)\,t + \Phi_t\right]$$

in which the amplitude A_t and the phase Φ_t constitute a 2-variate random process “slowly” varying with respect to the time t, and $\omega_{\text{eff}}(a)$ is a deterministic function that describes the “effective” value of the amplitude-dependent circular frequency of vibration:

$$\omega_{\text{eff}}(a) = \sqrt{\frac{k_{\text{eff}}(a)}{m}}$$

For a given value of the amplitude, $A_t = a$, the method furnishes the effective stiffness, $k_{\text{eff}}(a)$, and the effective damping coefficient, $c_{\text{eff}}(a)$, as solution of the implicit equations:

$$k_{\text{eff}}(a) = \frac{1}{\pi\, a} I_{\text{c}}(a)$$

$$c_{\text{eff}}(a) = -\frac{1}{\pi\, \omega_{\text{eff}}(a)\, a} I_{\text{s}}(a)$$

where $I_{c}(a)$ and $I_{s}(a)$ are the integral functions associated with the in-phase and out-of-phase components of the restoring force, respectively:

$$I_{c}(a) = \int_{0}^{2\pi} f\left[a\cos(\theta), -a\,\omega_{\mathrm{eff}}(a)\sin(\theta)\right]\cos(\theta)\,\mathrm{d}\theta$$

$$I_{s}(a) = \int_{0}^{2\pi} f\left[a\cos(\theta), -a\,\omega_{\mathrm{eff}}(a)\sin(\theta)\right]\sin(\theta)\,\mathrm{d}\theta$$

In stationary conditions, the Rayleigh-like approximate PDF of the amplitude can be evaluated once the functions $\omega_{\mathrm{eff}}(a)$ and $c_{\mathrm{eff}}(a)$ are known:

$$p_{A}(a) = \frac{1}{N_{A}} \frac{m\,\omega_{\mathrm{eff}}(a)\,a}{\sqrt{\pi S_{0}}} \exp\left[-\frac{m\,\Pi_{A}(a)}{\pi S_{0}}\right] \tag{2}$$

where S_{0} is the level of the uniform PSD of the white noise input, and N_{A} is just a normalization constant, which can be computed by satisfying the axiomatic condition:

$$\int_{0}^{+\infty} p_{A}(a)\,\mathrm{d}a = 1$$

and where the function $\Pi_{A}(a)$ is given by:

$$\Pi_{A}(a) = \int a\,c_{\mathrm{eff}}(a)\,\omega_{\mathrm{eff}}^{2}(a)\,\mathrm{d}a \tag{3}$$

Notice that the value of N_{A} in Eq. (2) depends on the arbitrary constant of integration arising from the indefinite integral of Eq. (3).

The PDF of Eq. (2) can be conveniently used for determining the joint PDF of X_{t} and $\dot{X}_{t}$ in the approximate form:

$$p_{X\dot{X}}(x,\dot{x}) = \frac{1}{2\pi\sqrt{\overline{\omega}_{\mathrm{eff}}^{2}\,x^{2} + \dot{x}^{2}}}\, p_{A}\left(\sqrt{x^{2} + \dot{x}^{2}/\overline{\omega}_{\mathrm{eff}}^{2}}\right)$$

where $\overline{\omega}_{\mathrm{eff}}$ is the expected value of $\omega_{\mathrm{eff}}(a)$:

$$\overline{\omega}_{\mathrm{eff}} = \mathrm{E}\langle \omega_{\mathrm{eff}}(A_{t})\rangle = \int_{0}^{+\infty} \omega_{\mathrm{eff}}(a)\,p_{A}(a)\,\mathrm{d}a$$

$\mathrm{E}\langle\cdot\rangle$ being the expectation operator.

Finally, the PSD of response process X_t can be estimated in the form:

$$S_X(\omega) = \frac{1}{2\pi} \int_0^{+\infty} \frac{\omega_{\text{eff}}^2(a)\, c_{\text{eff}}(a)\, a^2}{\left[\omega^2 - \omega_{\text{eff}}^2(a)\right]^2 + \left[c_{\text{eff}}(a)\,\omega\right]^2 / m}\; p_A(a)\,\mathrm{d}a \tag{4}$$

and the associated spectral moments [15] are given by:

$$\lambda_{i,X} = 2 \int_0^{+\infty} \omega^i\, S_X(\omega)\,\mathrm{d}\omega \quad , \quad i = 0,1,2,\cdots$$

which allow measuring the spectral bandwidth of X_t through the dimensionless parameter (e.g. [3]):

$$q_X = \sqrt{1 - \frac{\lambda_{1,X}^2}{\lambda_{0,X}\,\lambda_{2,X}}} \tag{5}$$

which is bounded in the interval $[0,1]$: that is, as large is q_X, as large is the spectral bandwidth of the response process.

3. Reliability analysis

Let the first passage time, $T_1(b) \geq 0$, be the random variable that describes the time instant at which the response process X_t firstly upcrosses the (Double) D-barrier of level $b > 0$, which defines the symmetric safe domain $[-b,b]$. The first passage time, then, satisfies the mathematical conditions:

$$\left|X_t\right| \leq b \quad \forall t \in [0, T_1(b)]$$

$$\left|X_{T_1(b)}\right| = b$$

$$X_{T_1(b)}\, \dot{X}_{T_1(b)} > 0$$

Let the peak response, $Y_t \geq 0$, be the non-stationary random process that describes the largest absolute value of the response over the time interval $[0,t]$. The peak response, then, can be so defined:

$$Y_t = \max_{0 \leq \tau \leq t} \left\{ \left| X_\tau \right| \right\} \tag{6}$$

and the samples of Y_t are monotonic non-decreasing function of the time t.

One can easily prove that $T_1(b)$ and Y_t are complementary random variables (e.g. [11]), as the Cumulative Distribution Functions (CDFs) of these quantities sum up to one:

$$F_{T_1(b)}(t) + F_Y(b;t) = 1$$

Interestingly, when the safe domain $[-b,b]$ is assumed for the response process X_t, the CDF of the first passage time gives the probability of failure:

$$P_{\mathrm{f}}(t) = F_{T_1(b)}(t) = \Pr\left\langle T_1(b) \le t \right\rangle$$

while the CDF of the peak response gives the reliability, that is the probability of success:

$$R(t) = 1 - P_{\mathrm{f}}(t) = F_Y(b;t) = \Pr\left\langle Y_t \le b \right\rangle$$

where the symbol $\Pr\left\langle \cdot \right\rangle$ denotes the probability associated with the event into angle brackets.

3.1 Poisson approach

In the reliability analysis of dynamical systems excited by random noises no exact solutions have been derived, even in the simplest case of the stationary vibration of a linear oscillator under white noise. The simplest approximate formulation known to the literature is the so-called Poisson approach (e.g. [3]), in which the spectral bandwidth of the response process is neglected. When applied to the system under consideration, the method gives the reliability as:

$$R(t) = \left[2\, F_X(b) - 1 \right] \exp\left[-2\, \nu_X^+(b)\, t \right]$$

where $\nu_X^+(b)$ is the time-independent mean upcrossing rate of the level b by the response process X_t:

$$\nu_X^+(b) = \int_0^{+\infty} \dot{x}\, p_{X\dot{X}}(b, \dot{x})\, \mathrm{d}\dot{x} \tag{7}$$

and $F_X(x)$ is the CDF of X_t. These quantities are directly furnished by the response analysis.

Unfortunately, although very simple, the Poisson approach proves to be excessively conservative when the response process X_t is narrowband, e.g. because the system is lightly damped, and/or when the level b is not high enough with respect to the standard deviation of the response, σ_X. In these circumstances, in fact, consecutive upcrossings of the selected D-barrier are far to be independent events, and exhibit the tendency to occur in clumps, whose mean size increases as the spectral bandwidth of X_t decreases (e.g. [16]). What is also important to note is that, even in the case of broadband response process, the mean upcrossing rate of Eq. (7) has to be effectively computed, otherwise the reliability given by the Poisson approach may be heavily inaccurate. As a consequence, the popular Stochastic Linearization (SL) method (see Appendix) should be avoided in the reliability analysis of non-linear systems, as the accuracy in predicting the response statistics may be inadequate in practical circumstances.

3.2 Gaussian Censored Closure

The Gaussian Censored Closure (GCC) proposed by Senthilnathan and Lutes [8] suffers the same limitations as the Poisson approach, since also in this case the spectral bandwidth of the response process is not accounted for. The method has been originally applied in the reliability analysis of linear oscillators, but the extension to non-linear oscillators is quite straightforward.

The basic idea is to operate a convenient censorship in the joint PDF of the random processes X_t, $\dot{X}_t$ and Y_t, $p_{X\dot{X}Y}(x,\dot{x},y;t)$, with the aim of eliminating the probability associated with the impossible event that the absolute value of the response, $|X_t|$, overcomes the peak response, Y_t, in a given time instant, that is:

$$\Pr\left\langle |X_t| > Y_t \right\rangle = 0 \tag{8}$$

In the formulation by Senthilnathan and Lutes [8] this censorship is obtained with the help of a Monte Carlo Simulation (MCS), which is used in each time step in order to evaluate the r.h.s. of the differential equations ruling the statistical moments of the peak response. The latter can be written in the compact form [11]:

$$\dot{m}_{i,Y}(t) = i\,\mathrm{E}\left\langle Y_t^{i-1}\, g\left(X_t, \dot{X}_t, Y_t\right)\right\rangle \quad , \quad i = 1, 2, \cdots \tag{9}$$

where $m_{i,Y}(t) = \mathrm{E}\left\langle Y_t^i \right\rangle$ is the i-th statistical moment of Y_t, and $g\left(x,\dot{x},y\right)$ is the highly non-linear function so defined:

$$g\left(x,\dot{x},y\right) = |\dot{x}|\,\bar{U}\left(x\,\dot{x}\right)\bar{U}\left(|x| - y\right) = \begin{cases} |\dot{x}| & , \quad \mathrm{sign}(x) = \mathrm{sign}(\dot{x}) \quad \text{and} \quad |x| \geq y \\ 0 & , \quad \text{otherwise} \end{cases}$$

$\bar{U}(\cdot)$ being the unit step function continuous from the right:

$$\bar{U}(x)=\begin{cases}0 & , \quad x<0\\ 1 & , \quad x\geq 0\end{cases}$$

Since the joint PDF $p_{X\dot{X}Y}(x,\dot{x},y;t)$ is a priori unknown, at a generic time instant t the expectation in the r.h.s. of Eq. (9) cannot be evaluated by the definition:

$$\mathrm{E}\left\langle Y_t^{i-1}\, g\left(X_t,\dot{X}_t,Y_t\right)\right\rangle=\int_0^{+\infty} y^{i-1}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g\left(x,\dot{x},y\right) p_{X\dot{X}Y}(x,\dot{x},y;t)\,\mathrm{d}x\,\mathrm{d}\dot{x}\right]\mathrm{d}y \quad , \quad i=1,2,\cdots \tag{10}$$

In Ref. [8], then, a numerical scheme is proposed, that requires: (i) the generation of a number n_s of the samples $\left\{X_t^{(j)},\dot{X}_t^{(j)},Y_t^{(j)}\right\}$, $j=1,2,\cdots,n_\mathrm{s}$, under the assumption that $p_{X\dot{X}Y}(x,\dot{x},y;t)$ is jointly Gaussian; (ii) the satisfaction of Eq. (8) through the substitution of the j-th sample $Y_t^{(j)}$ with the value $\left|X_t^{(j)}\right|$ when the generation gives $Y_t^{(j)}<\left|X_t^{(j)}\right|$; (iii) the approximate evaluation of the expectation in the r.h.s. of Eq. (9) as:

$$\mathrm{E}\left\langle Y_t^{i-1}\, g\left(X_t,\dot{X}_t,Y_t\right)\right\rangle\cong\sum_{j=1}^{n_\mathrm{s}} Y_t^{(j)i-1}\, g\left(X_t^{(j)},\dot{X}_t^{(j)},Y_t^{(j)}\right) \quad , \quad i=1,2 \tag{11}$$

Notice that, because the joint PDF $p_{X\dot{X}Y}(x,\dot{x},y;t)$ is assumed to be Gaussian, Eq. (11) has to be evaluated only for the first two statistical moments, $i=1,2$. Interestingly, since the peak response has possible values only in the range $[0,+\infty)$, while a Gaussian process has non-zero probability in the entire real axis $(-\infty,+\infty)$, in the paper by Senthilnathan and Lutes [8] two non-linear transformations of Y_t are also considered, and the consistent definitions of the non-linear function $g\left(x,\dot{x},y\right)$ are derived. Unfortunately, these transformations do not improve substantially the accuracy of the GCC technique.

4. Advanced Censored Closure

Despite the simplicity of the GCC technique proposed by Senthilnathan and Lutes [8], two main flaws may discourage its practical application: (i) the results do not depend on the spectral bandwidth of the response process, and then the clump tendency of the response upcrossings is not accounted for; (ii) the assumption that the joint PDF $p_{X\dot{X}Y}(x,\dot{x},y;t)$ is Gaussian may lead to an unacceptable degree of inaccuracy, especially in the case of non-linear oscillators. With the aim of overcoming these two drawbacks, a novel technique, termed Advanced Censored Closure (ACC), has been proposed in the recent papers by Muscolino and Palmeri [9, 10, 11, 12]. In this approach the spectral bandwidth of the response X_t is taken into account through the so-called “censorship factor,” unknown before to the literature, and the marginal PDF of the peak response Y_t is obtained by manipulating the Gumbel distribution.

About the first flaw, it would be stressed that even the exact knowledge of the joint PDF $p_{X\dot{X}}(x,\dot{x};t)$ at a given time instant t (this is the case, for instance, of linear dynamical systems driven by Gaussian processes) does not include the bandwidth effects, which on the contrary could be appreciated in the time domain through the auto-correlation function of the response process. Moreover, about the second flaw, although it could be formally cured with appropriate non-linear transformations, the use of the Gumbel model, also known as the “first asymptote of extremes,” seems to be preferable from a theoretical point of view. This model, in fact, proves to be the asymptotic distribution of the largest value, Y, of an exponentially-distributed random variable, X (e.g. [17]), and this is precisely the case of the approximate description of the response process X_t given by the SA method. On the contrary, the other two asymptotes known to the literature are inappropriate in our case: the Fréchet model, in fact, can be used when the random variable X is described by a Cauchy-like distribution, while the Webull model requires that the distribution of the random variable X is bounded in a finite interval.

In Ref. [9] it is demonstrated that, without loss of generality, Eq. (9) can be conveniently posed in the form:

$$\dot{m}_{i,Y}(t) = 2\,i\,\chi(t)\int_0^{+\infty} \nu_X^+(b)\,b^{i-1}\,\Phi_Y(b;t)\,\mathrm{d}b \quad , \quad i=1,2 \tag{12}$$

where $\Phi_Y(b;t)$ is the so-called “uncensored” CDF of the peak response, and $\chi(t)$ is the censorship factor, which is bounded in the interval $[0,1]$. Let us emphasize that Eq. (12) has been simply derived by assuming a convenient expression for the joint PDF $p_{X\dot{X}Y}(x,\dot{x},y;t)$ in Eq. (10), where the only approximation is that the values of the random processes $\dot{X}_t$ and Y_t, at a given time instant t, are statistically independent. As demonstrated through numerical simulations, the actual effects of this assumption are negligible in practical applications, and then Eq. (10) and Eq. (12) can be thought to be equivalent (more details can be found in Ref. [9]).

Instead of the Gaussian model considered in the GCC technique [8], the use of Gumbel model is suggested in the ACC technique [9, 10, 11, 12] for the uncensored CDF of the peak response:

$$\Phi_Y(b;t) = \exp\left\{-\exp\left[-\frac{b-\eta_Y(t)}{\kappa_Y(t)}\right]\right\} \tag{13}$$

The latter depends on the two parameters $\eta_Y(t)$ and $\kappa_Y(t)$, which account for the position and the spread of the probability mass, respectively:

$$\eta_Y(t) = \mu_Y(t) - 0.5772\,\kappa_Y(t)$$
$$\kappa_Y(t) = 0.7797\,\sigma_Y(t)$$

$\mu_Y(t)$ and $\sigma_Y(t)$ being the mean value and the standard deviation of the peak response, given by:

$$\mu_Y(t) = m_{1,Y}(t)$$
$$\sigma_Y(t) = \sqrt{m_{2,Y}(t) - \mu_Y(t)^2}$$

Of course, more accurate results can be obtained by using more sophisticated models for the uncensored CDF $\Phi_Y(b;t)$, e.g. based on the truncated type-A and type-C Gram-Charlier series expansions (e.g. [18]). In this case, however, the computational effort may excessively increases as higher-order statistics are required, while in the case of the Gumbel model only the first two statistical moments are needed.

Eq. (12) shows that the rate of change of the statistical moments of the peak response process is proportional to the censorship factor, i.e. as large is $\chi(t)$, as fast the statistical moments $m_{i,Y}(t)$ increase, and as conservative are the results: in particular, when the proposed ACC technique is applied with $\chi(t)=1$ one can prove that the results become consistent with those of the Poisson approach. The accuracy is improved when the censorship factor is estimated as the expected value of the semi-empirical correction term, $\beta(b)$, proposed by Vanmarcke [9] in the reliability analysis of stationary Gaussian processes:

$$\chi(t) = \mathrm{E}\left\langle \beta(Y_t) \right\rangle = \int_0^{+\infty} \beta(y)\, p_Y(y;t)\,\mathrm{d}y \tag{14}$$

where:

$$\beta(b)=\frac{1-\exp\left(-1.253\,q_X^{1.2}\,b/\sigma_X\right)}{1-\exp\left[-0.5\left(b/\sigma_X\right)^2\right]}$$

σ_X being the standard deviation of the response process, and q_X being the bandwidth parameter of Eq. (5). Moreover, in Eq. (14) $p_Y(y;t)$ is the marginal PDF of the peak response at time t, which is given by:

$$p_Y(y;t)=2\left\{\phi_Y(y;t)\left[F_X(y)-0.5\right]+\Phi_Y(y;t)\,p_X(y;t)\right\}\bar{U}(y) \tag{15}$$

where $\phi_Y(y;t)$ is the uncensored PDF of the peak response:

$$\phi_Y(y;t)=\frac{\partial}{\partial y}\Phi_Y(y;t)=\frac{1}{\kappa_Y(t)}\exp\left\{\frac{\eta_Y(t)-y}{\kappa_Y(t)}-\exp\left[\frac{\eta_Y(t)-y}{\kappa_Y(t)}\right]\right\}$$

Interestingly, the PDF of Eq. (15) can be viewed as the combination of two terms: the first one is proportional to the PDF of the response process, $p_X(y;t)$, which gives an essential contribution when the time t is relatively small, i.e. only in the first cycles of the response process; the second one is proportional to the uncensored PDF of the peak response, $\phi_Y(y;t)$, whose contribution prevails when the time t increases (in particular, in stationary conditions $p_Y(y;t)\to\phi_Y(y;t)$ as $t\to+\infty$).

Once Eq. (12), for $i=1,2$, are numerically integrated, the time evolution of the first two statistical moments of the peak response, $m_{1,Y}(t)$ and $m_{2,Y}(t)$, are sufficient in evaluating the approximate PDF of Eq. (15). An effective numerical scheme of solution, based on the so-called "midpoint" method [19], also known as the second-order Runge-Kutta method, can be found in Ref. [11], where the complete flow-chart is displayed and discussed, and where it is also emphasized the computational efficiency of the proposed ACC technique in comparison with the classical approaches.

Finally, one can prove that the results of the GCC (without transformations of the random process Y_t) are recovered when: (i) in Eq. (12) the censorship factor is assumed to be one, $\chi(t)=1$; (ii) the mean upcrossing rate is that one of the "equivalent" linear system given by the Stochastic Linearization (SL) method (Eq. (A.2)); and (iii) the uncensored CDF takes the Gaussian expression:

$$\Phi_Y(y;t) = \frac{1}{2}\left\{1 + \mathrm{erf}\left[\frac{y - \mu_Y(t)}{\sqrt{2}\,\sigma_Y(t)}\right]\right\} \tag{16}$$

$\mathrm{erf}(\cdot)$ being the error function [20]. Notice that the proposed approach allows saving some computational time with respect to the original formulation by Senthilnathan and Lutes [8], given that the MCS in computing Eqs. (11) is avoided.

5. Numerical applications

For the validation purpose, the Advanced Censored Closure (ACC) technique, summarized in the previous section, is implemented in a simple code running on Mathematica [21], with different degrees of complexity:

1. In the first analysis, the censorship factor $\chi(t)$ is given by Eq. (14), the Stochastic Averaging (SA) method is applied in computing the PSD of Eq. (4) and the mean upcrossing rate of Eq. (7), and the Gumbel model is considered for the uncensored distribution of the peak response (Eq. (13)): the label “ACC” is used for this solution.
2. In the second analysis, the only difference with respect to the first one is that the censorship factor takes the constant value $\chi(t) = 1$: the label “Poisson” is used for this solution, given that the value $\chi(t) = 1$ is consistent with the assumption of independent upcrossings by the response process.
3. In the third analysis, the censorship factor is still $\chi(t) = 1$, while the mean upcrossing rate is evaluated by the Stochastic Linearization (SL) method (Eq. (A.2)), and the Gaussian model is considered for the uncensored distribution of the peak response (Eq. (16)): the label “GCC” is used for this solution.

The analyses are conducted on oscillators with non-linear stiffness (sub-section 5.1) and damping (sub-section 5.2), and the results, in terms of the evolutionary mean value and standard deviation of the peak response, are compared with the statistics from Monte Carlo Simulation (MCS), with $n_s = 500$ samples, which is performed with a house code running on MATLAB [22].

5.1 Linear-plus-cubic stiffness

In a first stage, a SDoF oscillator with non-linear stiffness is considered. The non-linear restoring force in Eq. (1) is modelled as the reaction of an elastic spring with linear-plus-cubic stiffness in parallel with a linear viscous dashpot:

$$f(x,\dot{x}) = k_1\, x + k_3\, x^3 + c\,\dot{x}$$

in which $k_1/m = 1\,\mathrm{s}^{-2}$, $k_3/m = 0.1\,\mathrm{cm}^{-2}\,\mathrm{s}^{-2}$ and $c/m = 0.1\mathrm{s}^{-1}$, m being the inertia of the oscillator. The analyses are carried out with three increasing levels of the PSD of the white noise input, $S_0/m^2 = 0.1$, 1.0, $10\,\mathrm{cm}^2\,\mathrm{s}^{-3}$, and the results are displayed in Figs. 1, 2 and 3, respectively. In the MCS the excitation is approximated as a pink noise with circular frequency of cut-off $\omega_\mathrm{c} = 40\,\mathrm{rad/s}$, i.e. as a broadband process with uniform PSD of level S_0 in the interval $[0,\omega_\mathrm{c}]$.

In Fig. 1.a, for the case of "low" level of excitation ($S_0/m^2 = 0.1\,\mathrm{cm}^2\,\mathrm{s}^{-3}$), the mean upcrossing rates of the response process given by the SA (Eq. (7), solid line) and the SL (Eq. (A.2), dot-dashed line) methods are compared, in a logarithmic scale, with that one estimated by MCS (circles). In both cases the agreement is quite satisfactory: only at higher levels of the barrier b, in fact, the SL method overestimates the actual mean upcrossing rate of X_t.

Fig. 1.b shows, in a logarithmic scale, that the approximate PSD of the response process X_t given by the SA method (Eq. (4), solid line) compares very closely with the PSD obtained by MCS (circles). Each sample of the PSD is computed in the MATLAB code by the function $\mathrm{pwelch}(\cdot)$, which implements the so-called Welch method [23].

Fig. 1.c plots the mean value of the peak response, μ_Y, as a function of the time, t. The proposed ACC technique (solid line) is in good agreement with the results of the MCS (circles). The Poisson solution (dashed line) and the GCC technique (dot-dashed line) overestimate the peak response. This is because the response process, X_t, is narrowband: in fact, the equivalent damping ratio given by the SL method is $\zeta_\mathrm{eq} = c_\mathrm{eq}/\left(2\,m\,k_\mathrm{eq}\right) = 0.0396$ (see Appendix), and the bandwidth parameter given by the SA method is $q_X = 0.234$. The accuracy of the ACC technique reduces when the standard deviation of the peak response, σ_Y, is considered (Fig. 1.d). This prediction, however, is conservative, as the probability of failure, P_f, increases with the standard deviation of the peak response. Moreover, it is worth noting that generally the probability of failure is much

more sensitive to the mean value μ_Y, which controls the position of the PDF of the peak response and which is accurately predicted by the proposed ACC technique, than to the standard deviation σ_Y, which controls the spread around μ_Y.

The same results are displayed in Figs. 2 and 3 for the levels of the excitation "medium" ($S_0/m^2 = 1.0\ \mathrm{cm}^2\,\mathrm{s}^{-3}$) and "high" ($S_0/m^2 = 10\ \mathrm{cm}^2\,\mathrm{s}^{-3}$), respectively. In these cases the deviation of the response process from the Gaussianity increases: this is confirmed by Figs. 2.a and 3.a, in which the mean upcrossing rate given by the SL method (dot-dashed lines), operating under the assumption of Gaussianity, deviates from the results of the MCS (circles), while the SA method (solid lines) is still in good agreement. On the contrary, some non-negligible differences emerge in the evaluation of the PSD of the response process via the SA method (solid lines). The most relevant portion of these differences is shadowed in Figs. 2.b and 3.b, which prove that in the cases of medium and high levels of excitation the SA method (solid lines) overestimates energy content and spectral bandwidth of the response process of the non-linear oscillator under consideration. As a consequence, the accuracy of the proposed ACC technique decreases (see Fig. 2.c and 3.c), although the advantages of this approach (solid lines) with respect to Poisson solution (dashed lines) and GCC technique (dot-dashed lines) persist. Interestingly, the differences among these three approaches tend to decrease when the level of excitation increases, although the bandwidth parameter is almost constant, being $q_X = 0.251$ in the second case ($S_0/m^2 = 1.0\ \mathrm{cm}^2\,\mathrm{s}^{-3}$) and $q_X = 0.257$ in the third case ($S_0/m^2 = 10\ \mathrm{cm}^2\,\mathrm{s}^{-3}$).

5.2 Linear-plus-cubic damping

In a second stage, the same analyses as in the previous sub-section are carried out on a SDoF oscillator in which the non-linear restoring force is modelled as the reaction of a linear elastic spring in parallel with a linear-plus-cubic viscous dashpot:

$$f(x,\dot{x}) = k\,x + c_1\,\dot{x} + c_3\,\dot{x}^3$$

in which $k/m = 1\,\mathrm{s}^{-2}$, $c_1/m = 0.01\mathrm{s}^{-1}$, and $c_3/m = 0.001\mathrm{cm}^{-2}\,\mathrm{s}$, m being the inertia of the system. The analyses are performed with the same levels of the white noise input considered in the previous sub-section, i.e. $S_0/m^2 = 0.1$, 1.0, $10\,\text{cm}^2\,\text{s}^{-3}$, and the results are displayed in Figs. 4, 5 and 6, respectively.

Figs. 4.a, 5.a and 6.a demonstrate the ability of the SA method (solid line) in predicting the actual mean upcrossing rate of the response process, estimated by MCS (circles), even when the level of the excitation is high. Figs. 4.b, 5.b and 6.b demonstrate the same accuracy in estimating the PSD of the response process. Notice that, as opposite to the previous case of non-linearity in the stiffness, the bandwidth parameter increases with the level of the input, from $q_X = 0.135$ when the excitation is low ($S_0/m^2 = 0.1\,\text{cm}^2\,\text{s}^{-3}$), to $q_X = 0.369$ when the excitation is high ($S_0/m^2 = 10\,\text{cm}^2\,\text{s}^{-3}$); correspondingly, the equivalent damping ratio given by the SL method is $\zeta_{\text{eq}} = 0.0181$ when the excitation is low, $\zeta_{\text{eq}} = 0.156$ when the excitation is high.

Figs. 4.c, 5.c and 6.c show that the accuracy of the proposed ACC technique (solid line) in predicting the mean value of the peak response is substantially independent of the spectral bandwidth of the response process. The Poisson approach (dashed line), on the contrary, overestimates the results of the MCS (circles) when the response is more narrowband (Fig. 4.c), while the GCC technique gives an acceptable estimate only when the response is more broadband (Fig. 6.c). Finally, also in the case of non-linear damping the proposed ACC results to be conservative in terms of standard deviation of the peak response, and the discrepancy with respect to the MCS tends to decrease as the spectral bandwidth of the response process increases (Figs. 4.d, 5.d and 6.d).

6. Conclusions

In the framework of the reliability analysis of dynamical system excited by random processes, the Advanced Censored Closure (ACC) technique has been extended in this paper to the stationary vibration of non-linear SDoF oscillators under white noise.

The proposed approach enables the statistics of the non-stationary peak response process to be accurately predicted with a moderate computational effort, once mean upcrossing rate and spectral bandwidth of the response process are known. The Stochastic Averaging method is applied in computing these quantities.

The numerical applications to oscillators with non-linear stiffness and damping demonstrate that the ACC technique is very versatile, and that the results are in good agreement with those given by Monte Carlo Simulation even when the level of the input increases, and then the response process deviates from the Gaussianity. The superiority of the proposed approach with respect to the classical Poisson approach, as well as to the more simple Gaussian Censored Closure, is also stressed.

Appendix. Stochastic Linearization

The Stochastic Linearization (SL) method, widely adopted in practical applications because of its simplicity, substitutes the non-linear restoring force in Eq. (1) with the linear expression:

$$f_{\mathrm{eq}}\left(x,\dot{x}\right)=k_{\mathrm{eq}}\,x+c_{\mathrm{eq}}\,\dot{x}$$

where the "equivalent" values of stiffness and damping, k_{eq} and c_{eq}, are evaluated under the assumption that the PDF of the response is Gaussian [1, 24]. These values depend on the approximate variances $\tilde{\sigma}_X^2$ and $\tilde{\sigma}_{\dot{X}}^2$ of X_t and $\dot{X}_t$:

$$\begin{aligned} k_{\mathrm{eq}} &= \frac{\tilde{E}\left[\,f(X_t,\dot{X}_t)\,X_t\,\right]}{\tilde{\sigma}_X^2} \\ c_{\mathrm{eq}} &= \frac{\tilde{E}\left[\,f(X_t,\dot{X}_t)\,\dot{X}_t\,\right]}{\tilde{\sigma}_{\dot{X}}^2} \end{aligned} \tag{A.1}$$

where the symbol $\tilde{E}\langle\cdot\rangle$ stands for the "Gaussian" expectation operator. In stationary conditions, then, one obtains:

$$\tilde{E}\left\langle\,f(X_t,\dot{X}_t)\,X_t\right\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,\dot{x})\,x\,\tilde{p}_X(x)\,\tilde{p}_{\dot{X}}(\dot{x})\,\mathrm{d}x\,\mathrm{d}\dot{x}$$

$$\tilde{E}\left\langle\,f(X_t,\dot{X}_t)\,\dot{X}_t\right\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,\dot{x})\,\dot{x}\,\tilde{p}_X(x)\,\tilde{p}_{\dot{X}}(\dot{x})\,\mathrm{d}x\,\mathrm{d}\dot{x}$$

$\tilde{p}_X(x)$ and $\tilde{p}_{\dot{X}}(\dot{x})$ being the approximate Gaussian PDFs of X_t and $\dot{X}_t$:

$$\tilde{p}_X(x) = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}_X} \exp\left[-\frac{x^2}{2\,\tilde{\sigma}_X^2} \right]$$

$$\tilde{p}_{\dot{X}}(\dot{x}) = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}_{\dot{X}}} \exp\left[-\frac{\dot{x}^2}{2\,\tilde{\sigma}_{\dot{X}}^2} \right]$$

Eqs. (A.1), then, require the knowledge of $\tilde{\sigma}_X^2$ and $\tilde{\sigma}_{\dot{X}}^2$, which can be evaluated as solution of the implicit equations:

$$\begin{cases} \tilde{E}\left\langle f(X_t, \dot{X}_t)\, X_t \right\rangle = m^2\, \tilde{\sigma}_{\dot{X}}^2 \\ \tilde{E}\left\langle f(X_t, \dot{X}_t)\, \dot{X}_t \right\rangle = \dfrac{\pi\, S_0}{m^2} \end{cases}$$

Finally, the corresponding mean upcrossing rate of the level b by the response process takes the expression:

$$\tilde{\nu}_X^+(b) = \frac{1}{2\pi} \frac{\tilde{\sigma}_{\dot{X}}}{\tilde{\sigma}_X} \exp\left[-\frac{b^2}{2\,\tilde{\sigma}_X^2} \right] \qquad \text{(A.2)}$$

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List of Figures

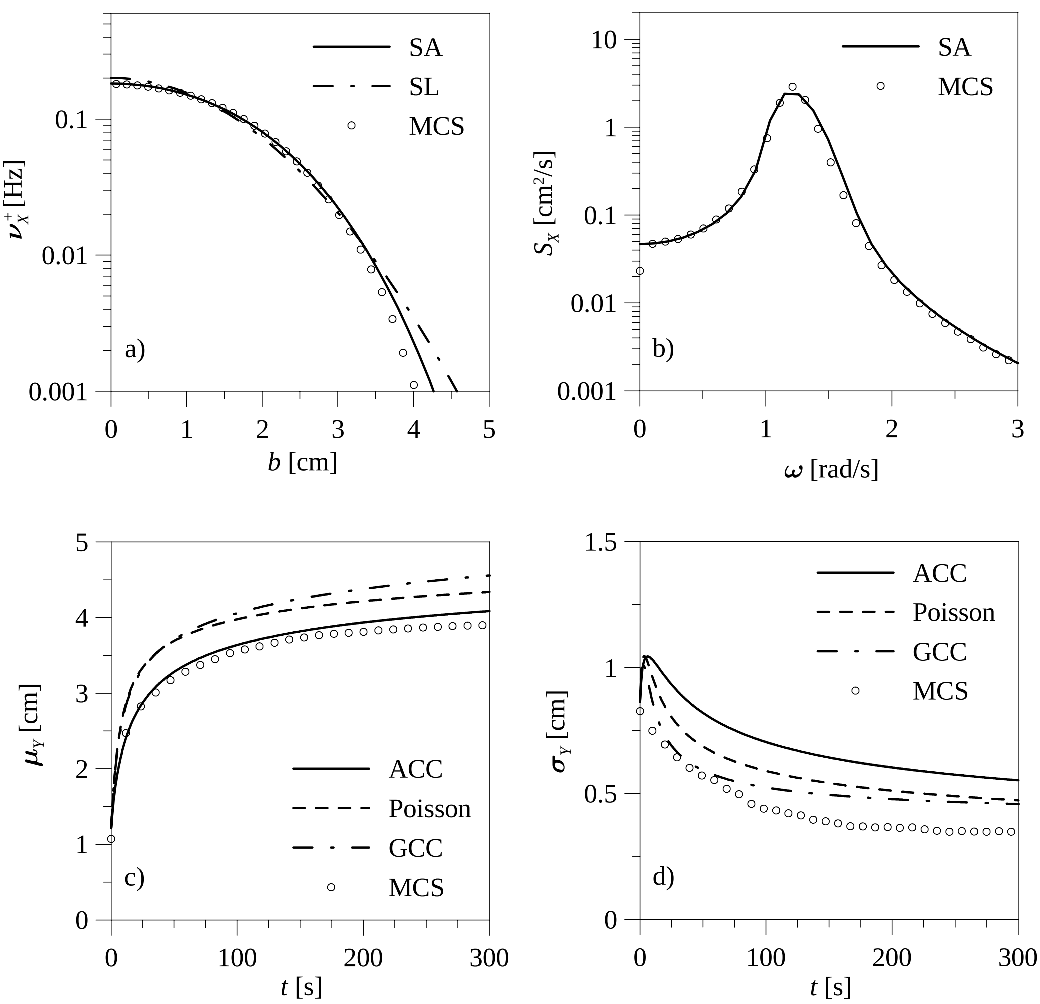

Figure 1. Mean upcrossing rate (a) and Power Spectral Density (b) of the stationary response of a SDoF oscillator with linear-plus-cubic stiffness under white noise of "low" intensity ($S_0/m^2 = 0.1\,\mathrm{cm}^2\,\mathrm{s}^{-3}$). Evolutionary mean value (c) and standard deviation (d) of the peak response.

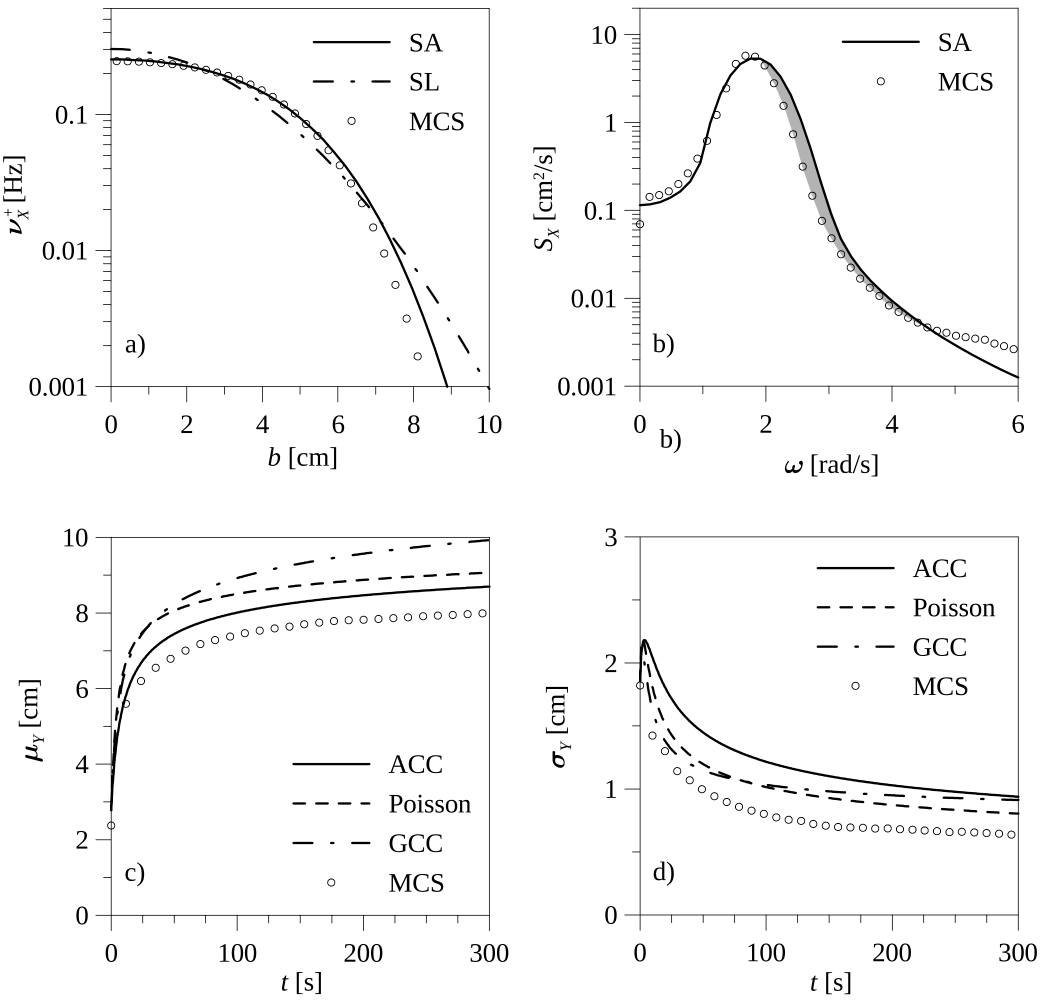

Figure 2. Mean upcrossing rate (a) and Power Spectral Density (b) of the stationary response of a SDoF oscillator with linear-plus-cubic stiffness under white noise of "medium" intensity ($S_0/m^2 = 1.0\,cm^2\,s^{-3}$). Evolutionary mean value (c) and standard deviation (d) of the peak response.

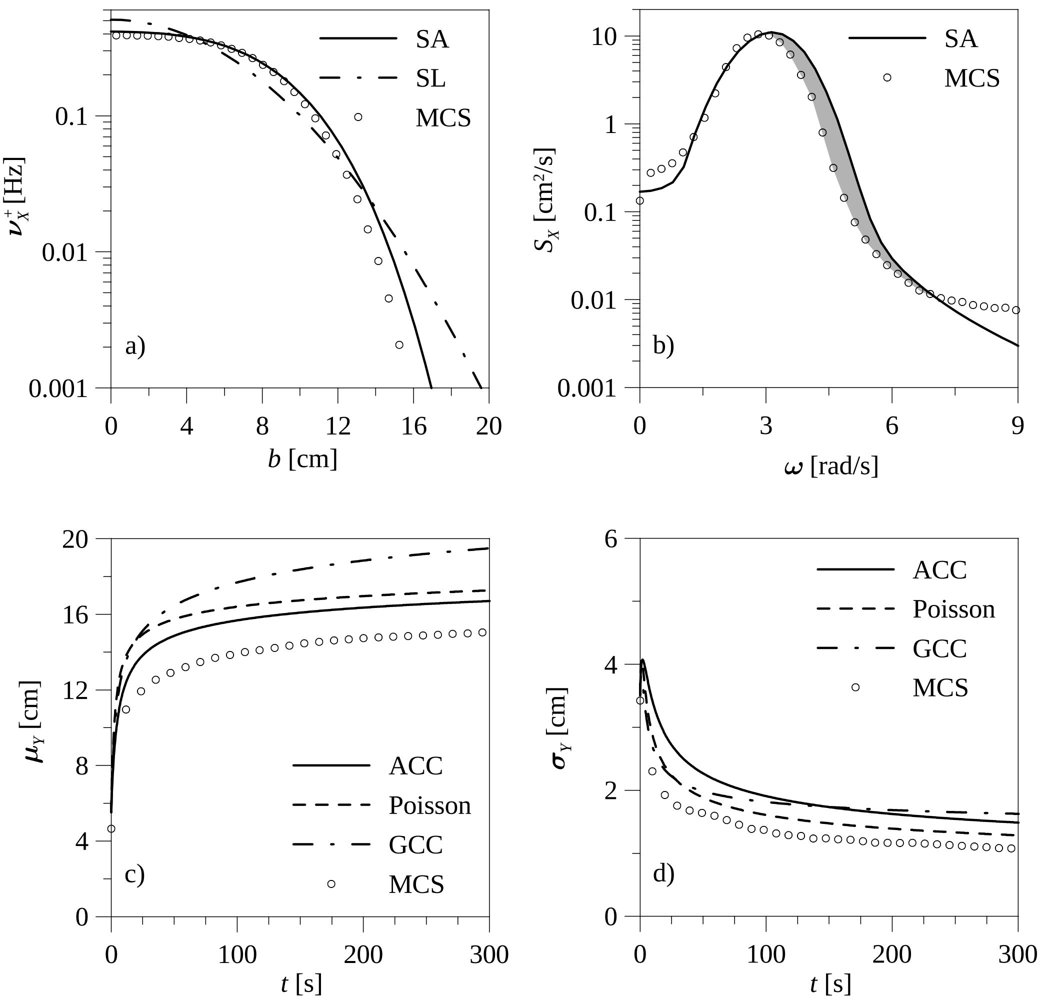

Figure 3. Mean upcrossing rate (a) and Power Spectral Density (b) of the stationary response of a SDoF oscillator with linear-plus-cubic stiffness under white noise of "high" intensity ($S_0/m^2 = 10\,cm^2\,s^{-3}$). Evolutionary mean value (c) and standard deviation (d) of the peak response.

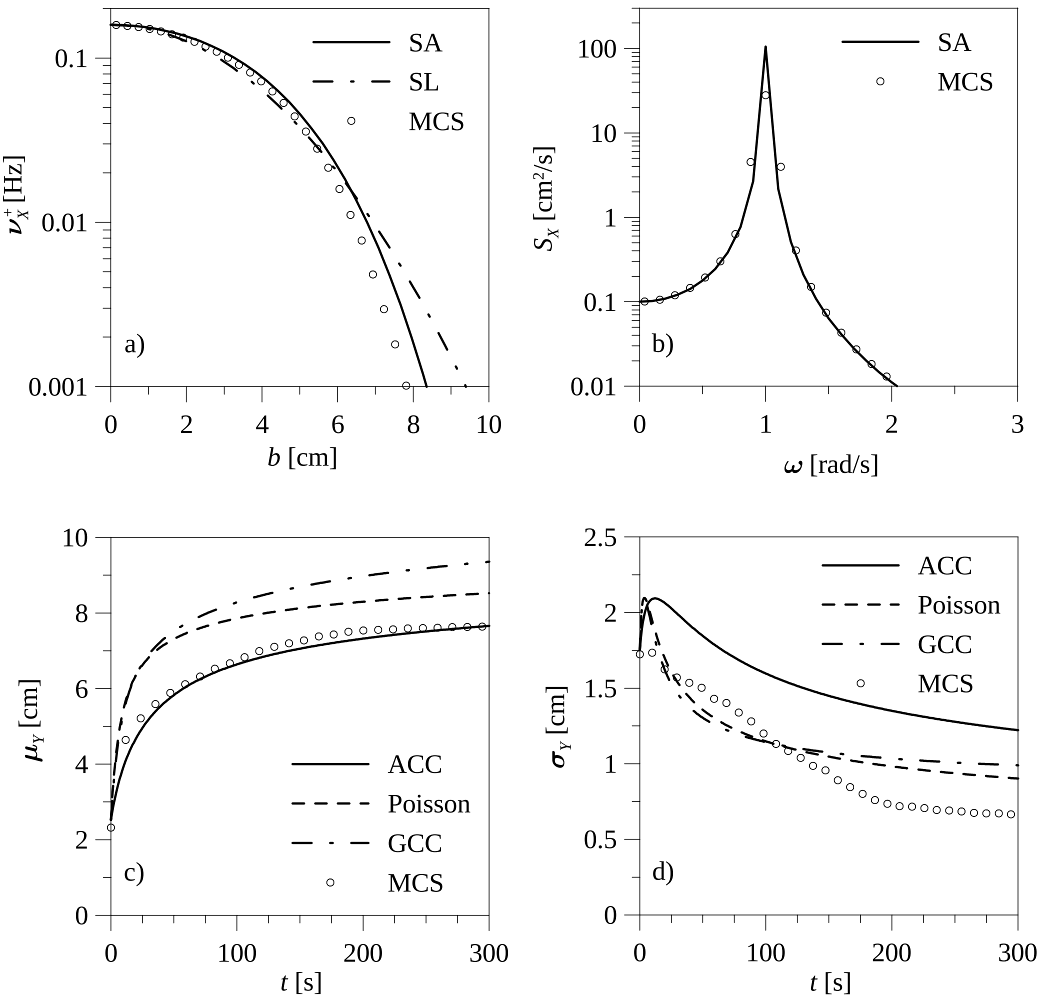

Figure 4. Mean upcrossing rate (a) and Power Spectral Density (b) of the stationary response of a SDoF oscillator with linear-plus-cubic damping under white noise of "low" intensity ($S_0/m^2 = 0.1\,cm^2\,s^{-3}$). Evolutionary mean value (c) and standard deviation (d) of the peak response.

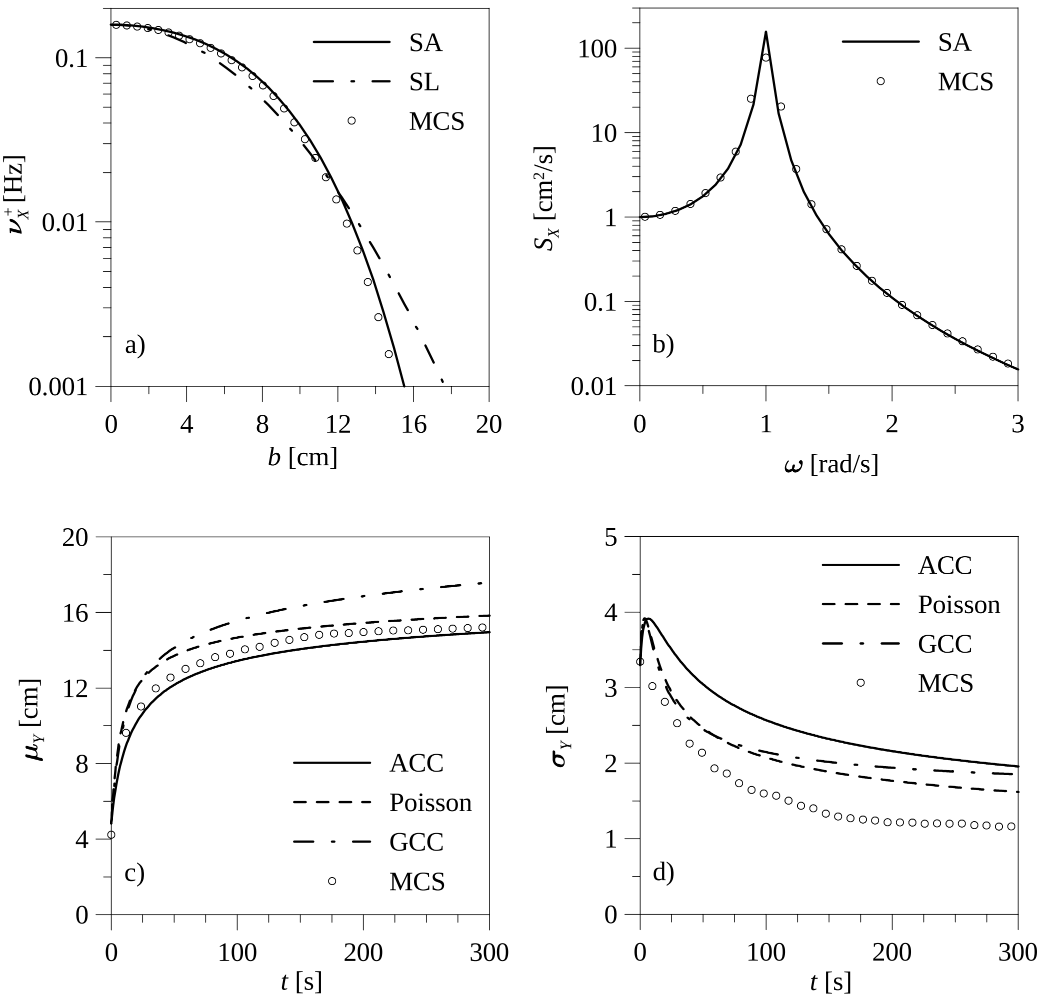

Figure 5. Mean upcrossing rate (a) and Power Spectral Density (b) of the stationary response of a SDoF oscillator with linear-plus-cubic damping under white noise of "medium" intensity ($S_0/m^2 = 1.0\,cm^2\,s^{-3}$). Evolutionary mean value (c) and standard deviation (d) of the peak response.

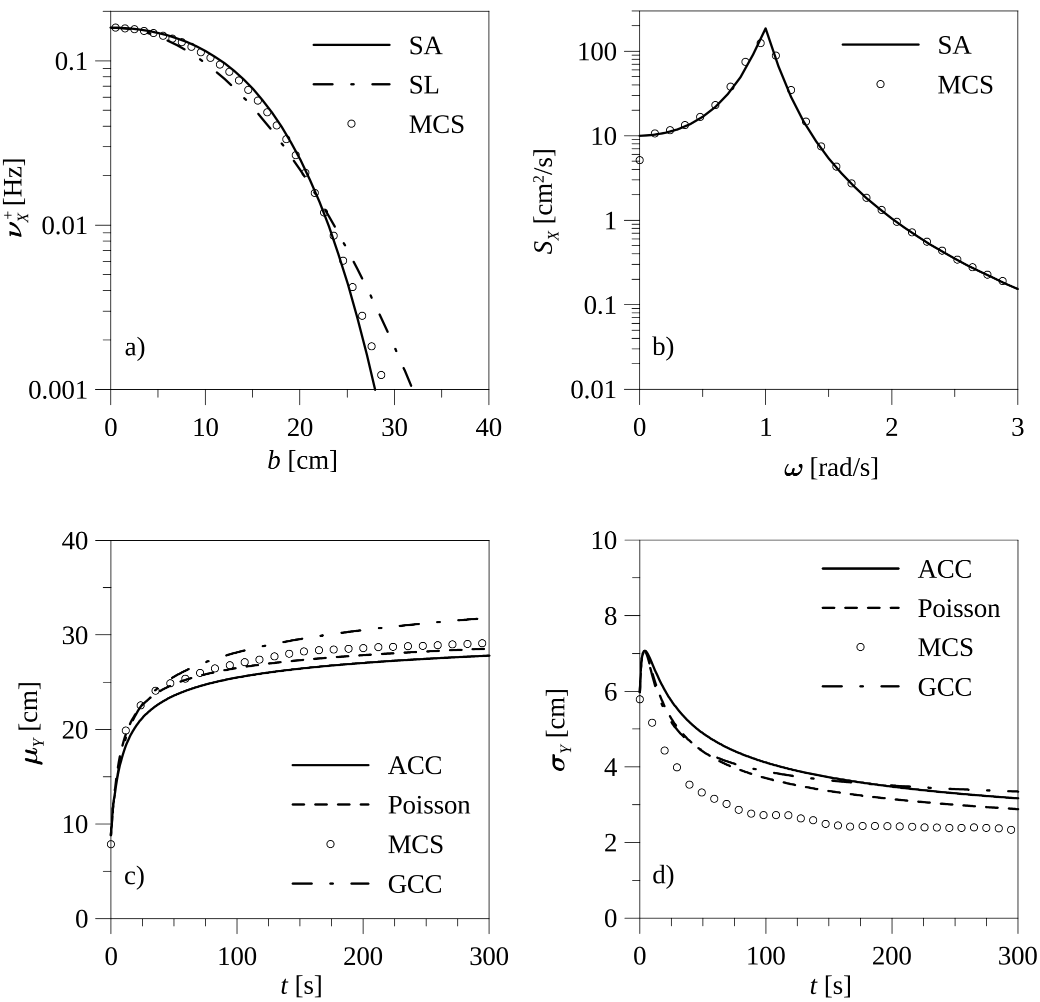

Figure 6. Mean upcrossing rate (a) and Power Spectral Density (b) of the stationary response of a SDoF oscillator with linear-plus-cubic damping under white noise of "high" intensity ($S_0/m^2 = 10\,cm^2\,s^{-3}$). Evolutionary mean value (c) and standard deviation (d) of the peak response.