Cliques in Graphs

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. All external results used have been properly attributed.

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Abstract

The main focus of this thesis is to evaluate $k_r(n,\delta)$, the minimal number of r-cliques in graphs with n vertices and minimum degree δ. A fundamental result in Graph Theory states that a triangle-free graph of order n has at most $n^2/4$ edges. Hence, a triangle-free graph has minimum degree at most $n/2$, so if $k_3(n,\delta) = 0$ then $\delta \leq n/2$. For $n/2 \leq \delta \leq 4n/5$, I have evaluated $k_r(n,\delta)$ and determined the structures of the extremal graphs. For $\delta \geq 4n/5$, I give a conjecture on $k_r(n,\delta)$, as well as the structures of these extremal graphs. Moreover, I have proved various partial results that support this conjecture.

Let $k_r^{reg}(n,\delta)$ be the analogous version of $k_r(n,\delta)$ for regular graphs. Notice that there exist n and δ such that $k_r(n,\delta) = 0$ but $k_r^{reg}(n,\delta) > 0$. For example, a theorem of Andrásfai, Erdős and Sós states that any triangle-free graph of order n with minimum degree greater than $2n/5$ must be bipartite. Hence $k_3(n, \lfloor n/2 \rfloor) = 0$ but $k_3^{reg}(n, \lfloor n/2 \rfloor) > 0$ for n odd. I have evaluated the exact value $k_3^{reg}(n,\delta)$ for δ between $2n/5 + 12\sqrt{n}/5$ and $n/2$ and determined the structure of these extremal graphs.

At the end of the thesis, I investigate a question in Ramsey Theory. The Ramsey number $R_k(G)$ of a graph G is the minimum number N, such that any edge colouring of K_N with k colours contains a monochromatic copy of G. The constrained Ramsey number $f(G,T)$ of two graphs G and T is the minimum number N such that any edge colouring of K_N with any number of colours contains a monochromatic copy of G or a rainbow copy of T. It turns out that these two quantities are closely related when T is a matching. Namely, for almost all graphs G, $f(G, tK_2) = R_{t-1}(G)$ for $t \geq 2$.

Contents

Chapter 1

Introduction

1.1 Cliques in graphs - a classical problem

Let G be a graph of order n and size e unless stated otherwise. For a given graph H, a graph G is *H-free* if it does not contain a subgraph isomorphic to H. One of the trademark problems in Extremal Graph Theory is to determine the maximum number of edges in a graph without any triangles. Mantel [34] first proved that a triangle-free graph of order n has at most $n^2/4$ edges. Moreover, he showed that a complete bipartite graph with partition sizes $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ is the only triangle-free graph of order n with $\lfloor n^2/4 \rfloor$ edges. For a graph H, we denote by $ex(n, H)$ the maximum number of edges in a H-free graph of order n. Thus, using this notation, $ex(n, K_3) = \lfloor n^2/4 \rfloor$. The *$r$-partite Turán graph of order n*, $T_r(n)$, is a complete r-partite graph of order n with vertex classes of sizes either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Turán [45] proved that if G is K_{r+1}-free, then $e(G) \leq e(T_r)$ with equality if and only if $G = T_r(n)$. Thus, $ex(n, K_{r+1}) = e(T_r)$.

If we denote by $\mathcal{K}_r(G)$ the set of K_r in G and its size $k_r(G) = |\mathcal{K}_r(G)|$, then Turán's Theorem states that if $k_{r+1}(G) = 0$, then $e(G) \leq e(T_r(n))$. Given $e(G) > e(T_r(n))$, we would like to know what values of $k_r(G)$ are possible. This naturally leads us to the problem of which values of $k_r(G)$ are possible if $e(G) > e(T_r(n))$, a problem which many people have investigated.

Erdős [15] proved that $k_r(G)$ is at most $\binom{a}{r} + \binom{b}{r-1}$, where $a > b \geq 0$ are the unique integers such that $e = \binom{a}{2} + b$. It should be noted that the above statement is independent of n. The graph obtained by joining a new vertex to b vertices of a complete graph of order a has precisely $\binom{a}{2} + b$ edges and $\binom{a}{r} + \binom{b}{r-1}$ r-cliques (complete graphs of order r). Now, this problem of bounding $k_r(G)$ from above for a given e is in fact a special case of the Kruskal-Katona Theorem [31, 30], which is stated below, taking $s = 2$ and $t = r$.

Theorem 1.1.1 (Kruskal-Katona Theorem [31, 30]). *Let s and t be two positive integers with $s < t$. Let $\mathcal{A}$ be a family of s-sets of size $|\mathcal{A}| = \binom{n_s}{s} + \binom{n_{s-1}}{s-1} + \cdots + \binom{n_j}{j}$, where $n_s > n_{s-1} > \cdots > n_j \geq j \geq 1$. Let $\mathcal{B}$ be the union of all t-element supersets of the sets in $\mathcal{A}$. Then, $|\mathcal{B}| \leq \binom{n_s}{t} + \binom{n_{s-1}}{t-1} + \cdots + \binom{n_j}{t-s+j}$.*

Let $f_r(n, e)$ be the minimal $k_r(G)$ for graphs G of order n and size e. Determining $f_r(n, e)$ seems to be difficult. First, we give the structure of a family of graphs that are conjectured to achieve $f_r(n, e)$. Let p be an integer such that $(1 - 1/p)n^2/2 < e \leq (1 - 1/(p + 1))n^2/2$. Note that a simple calculation shows that $e(T_p(n)) \leq (1 - 1/p)n^2/2$. We consider a $(p + 1)$-partite graph with vertex classes $V_1, \ldots, V_{p+1}$ with the following properties. Each vertex class $V_1, \ldots, V_p$ has size at least $|V_{p+1}|$, and $|V_1| \geq \cdots \geq |V_p| \geq |V_1| - 1$. The graph induced by V_i and V_j is a complete bipartite graph with partition V_i and V_j for all $1 \leq i < j \leq p+1$ unless $i = p$ and $j = p+1$. The graph induced by V_p and V_{p+1} is a bipartite graph with partition V_p and V_{p+1} and more than $|V_p||V_{p+1}| - |V_p|$ edges. Since there is a choice of missing edges between V_p and V_{p+1}, the above construction defines a family of graphs. Each such graph achieves the conjectured $f_r(n, e)$. Note that these graphs are independent of r, so is $f_r(n, e)$ (provided that $e > e(T_r(n))$). Also, the structure of these graphs depends on the edge density $e/\binom{n}{2}$. Hence, one of the main parameters of $f_r(n, e)$ may be assumed to be the edge density $e/\binom{n}{2}$. Hence, we sometimes say that e is in the interval between c_1 and c_2, meaning that $c_1n^2/2 < e \leq c_2n^2/2$ for $0 \leq c_1 \leq c_2 \leq 1$.

Erdős [17] was the first to study $f_3(n, e)$ for $e \leq n^2/4 + o(1)$. Lovász and Simonovits [33] proceeded to evaluate $f_3(n, e)$ for $n^2/4 \leq e \leq n^2/4 + n/2$. Bollobás [5] then gave the first concave, actually linear, bound for each interval between $1 - 1/p$ and $1 - 1/(p + 1)$ for positive integers p. Moreover, the lower bound is sharp at the boundary points of each interval, i.e. $e = (1 - 1/p)n^2/2$ for each integer $p \geq r$. Fisher [22] evaluated $f_3(n, e)$ asymptotically for the interval between 1/2 and 2/3, but it was not until nearly twenty years later that a dramatic breakthrough of Razborov [40] proved the asymptotic result of $f_3(n,e)$ for a general e. The proof of this used the concept of flag algebra developed in [39]. Unfortunately, it seemed difficult to generalise Razborov's proof even for $f_4(n,e)$. Nikiforov [35] later gave a simple and elegant proof of the asymptotic result of both $f_3(n,e)$ and $f_4(n,e)$ for general e. However, bounds for $f_r(n,e)$ when $r \geq 5$ have not yet been determined. We recommend Chapter 6 in [6] for a survey of other results related to complete subgraphs.

1.2 Bounding the minimum degree

In this thesis, we are interested in a variant of the classical problem described in the last section, where instead of considering the number of edges we consider the minimum degree. In this case, $k_r(n,\delta)$ is defined to be the minimum value of $k_r(G)$ for graphs G of order n with minimum degree δ. In addition, $k_r^{reg}(n,\delta)$ is defined to be the minimum value of $k_r(G)$ for δ-regular graphs G of order n. It can be seen, by Turán's Theorem, that if $k_r(n,\delta) = 0$ then $\delta \leq (1-1/r)n$. It should be noted that there exist n and δ such that $k_r(n,\delta) = 0$, but $k_r^{reg}(n,\delta) > 0$. For example, if $r = 3$, n odd and $2n/5 < \delta n < 2$, then $k_3(n,\delta) = 0$ by removing edges from $T_2(n)$ until we have minimum degree δ. However, a theorem of Andrásfai, Erdős and Sós [3] states that every triangle-free graph of order n with minimal degree greater than $2n/5$ is bipartite. Since no regular graphs with an odd number of vertices can be bipartite, $k_3^{reg}(n,\delta) > 0$ for n odd and $2n/5 < \delta < n/2$, whilst $k_3(n,\delta) = 0$. In Chapter 2, we determine the exact value of $k_3^{reg}(n,\delta)$ for n odd and δ in the interval between $2n/5 + 12\sqrt{n}/5$ and $n/2$.

Theorem 1.2.1. *For every odd integer $n \geq 10^7$ and even integer δ with $2n/5 + 12\sqrt{n}/5 \leq \delta \leq n/2$, $k_3^{reg}(n,\delta) = \delta(3\delta - n - 1)/4$ holds.*

In addition, we identify the structure of the extremal graphs for $k_3^{reg}(n,\delta)$. The construction given in Chapter 2 implies that $k_3^{reg} = O(n^2)$ for $n/3 \leq \delta \leq n/2$. Therefore, the contribution from the parity of n should be $o(1)$ in terms of density of triangles.

We have also studied $k_3(n, \delta)$ for $\delta > n/2$. Thus, in Chapter 3, we have given what we conjecture to be the exact structure of extremal graphs for $k_r(n, \delta)$ subject to certain conditions on n and δ. These extremal graphs for $k_r(n, \delta)$ share many similarities with the ones for $f_r(n, e)$. For example, the extremal graphs are independent of r (provided $\delta > (1 - 1/r)n$). Our main result proves a sharp lower bound on $k_r(n, \delta)$ for $n/2 < \delta \leq 4n/5$.

Theorem 1.2.2. *Let n and δ be integers with $n/2 < \delta \leq 4n/5$. Let $\beta = 1 - \delta/n$. Then for integers $r \geq 3$,*

$$k_r(n, \delta) \geq g_r(\beta)n^r,$$

where the function $g_r(\beta)$ is explicitly defined in Section 3.1. Moreover, for $3 \leq r \leq \beta^{-1} + 1$ equality holds if and only if (n, β) is feasible, and the extremal graphs are members of $\mathcal{G}(n, \beta)$, where feasible and $\mathcal{G}(n, \beta)$ are also defined in Section 3.1.

The proof of the above theorem is spread over Chapters 3 to 6. In Section 3.3, we give an elegant proof for the case $n/2 < \delta \leq 2n/3$ which then forms the framework for proving Theorem 1.2.2 for $2n/3 \leq \delta \leq 4n/5$. In Chapter 4, we prove that Theorem 1.2.2 is true for K_{p+2}-free graphs, where $p = 3$ if $2n/3 < \delta \leq 3n/4$, and $p = 4$ if $3n/4 < \delta \leq 4n/5$. In fact, we prove a sharp lower bound on $k_r(G)$ for K_{p+2}-free graphs G of order n and minimum degree δ, where $n/2 < \delta < n$ and $p = \lceil (1 - \delta/n)^{-1} \rceil - 1$.

Theorem 1.2.3. *Let n and δ be integers with $n/2 < \delta < n$. Let $\beta = 1 - \delta/n$ and $p = \lceil \beta^{-1} \rceil - 1$. Then for integers $r \geq 3$,*

$$k_r(n, \delta; K_{p+2}\text{-free}) \geq g_r(\beta)n^r$$

holds, where $k_r(n, \delta; K_{p+2}\text{-free})$ is the minimum value of $k_r(G)$ for K_{p+2}-free graphs G of order n with minimum degree δ. Moreover, for $3 \leq r \leq p + 1$ equality holds if and only if (n, β) is feasible, and the extremal graphs are members of $\mathcal{G}(n, \beta)$.

Various technical difficulties arise when we prove Theorem 1.2.2 for the case $2n/3 < \delta \leq 4n/5$ for graphs G containing cliques of sizes $p + 2$. This is because the existence of $(p + 2)$-cliques in G introduces error terms into our estimates for $k_r(G)$ for $3 \leq r \leq p+1$, a situation which we discuss in Section 5.2 when $2n/3 < \delta \leq 3n/4$ and G contains a 5-clique. Chapter 5 and Chapter 6 are dedicated respectively to the cases for $2n/3 < \delta \leq 3n/4$ and $3n/4 < \delta \leq 4n/5$. We study the case $\delta > 4n/5$ in Chapter 7, where we evaluate $k_r(n, \delta)$ exactly for $\delta/n \in I$, where I is a union of infinitely many non-empty intervals between 0 and 1. The following theorem is proved in Section 7.1.

Theorem 1.2.4. *For positive integers p, there exists $1/(p+1) < \beta_p \leq 1/p$ such that for all $1/(p+1) \leq \beta < \beta_p$ and integers n, $\delta = (1-\beta)n$ and r,*

$$k_r(n, \delta) \geq g_r(\beta) n^r$$

holds. Moreover, for $3 \leq r \leq p+1$ equality holds if and only if (n, β) is feasible, and the extremal graphs are members of $\mathcal{G}(n, \beta)$.

1.3 Constrained Ramsey theory

For an edge colouring of a graph G, we say that G is *monochromatic* if and only if all its edge colours are the same. The *Ramsey number* $R(s,t)$ is the minimum number N such that for any edge colouring of K_N with two colours, say red and blue, there exists a red monochromatic copy of K_s or a blue monochromatic copy of K_t. It is easy to see that $R(2, s) = s$ and $R(s,t) = R(t,s)$. Also, it can be easily shown that $R(3,3) = 6$. The existence of $R(s,t)$ was first proved by Ramsey [38] and rediscovered by Erdős and Szekeres [16]. Only a few of the numbers of $R(s,t)$ are known precisely. Erdős and Szekeres [16] showed that $R(s,s) \leq \binom{2s-2}{s-1}$. This bound was later improved by Rödl [24] and Thomason [42]. The best known upper bound was proved by Conlon [12], that is, $R(s,s) \leq s^{-c\frac{\log s}{\log \log s}} \binom{2s-2}{s-1}$, whilst the best known lower bound for $R(s,s)$ is of order $2^{s/2}$, which is obtained by a simple probabilistic argument.

The Ramsey number $R(s_1, \dots, s_k)$ is defined analogously to be the minimum number N such that for any edge colouring of K_N with colours $c_1, \dots, c_k$, there exists a monochromatic copy of K_{s_i} of colour c_i for some i. If $s_i = s$ for all i, then we simply write $R_k(s)$. If an edge colouring of K_N uses infinitely many colours, then it is possible to avoid monochromatic K_s for $s \geq 3$. Nevertheless, there exists a well-structured edge coloured complete subgraph in K_N. For example, there may exist a complete subgraph that is *rainbow* (i.e every edge has a distinct colour). If we let the vertices of G be $v_1, \ldots, v_n$, then a *lexicographically coloured* (or *colexicographically coloured*) G is an edge colouring such that the edge $v_i v_j$ has colour c_i for $i < j$ (or $i > j$ respectively) with c_i distinct. It can be observed that a lexicographically coloured finite graph becomes colexicographically coloured if the ordering on the vertex set is reversed, and vice versa. Erdős and Rado [18] proved that for any edge colouring of K_N with any number of colours, there exists a complete subgraph with one of the above colourings. This is known as the Canonical Ramsey Theorem.

Theorem 1.3.1 (Canonical Ramsey Theorem [18]). *For every positive integer s, there exists an integer $N(s) > 0$ with the following property. For each integer $N \geq N(s)$ and every edge colouring of K_N, there exists a K_s in K_N such that it is either monochromatic, rainbow, lexicographically coloured or colexicographically coloured.*

The Ramsey number $R(G, H)$ of two graphs, G and H, is the minimum number N such that for any 2-edge colouring of K_N with colours red and blue say, there exists a red monochromatic G or a blue monochromatic H. Hence, $R(K_s, K_t) = R(s, t)$. For graphs $G_1, \ldots, G_k$, we define $R(G_1, \ldots, G_k)$ analogously and write $R_k(G)$ if $G_i = G$ for all i.

Until now we have considered only monochromatic subgraphs. However, the canonical Ramsey Theorem states that there exists a monochromatic, rainbow, lexicographically coloured or colexicographically coloured subgraph in any edge colouring of K_N for N sufficiently large. It should be noted that both lexicographical and colexicographical colourings depend on the initial ordering of the vertex set, so they are not preserved under vertex relabelling. Hence, we shall focus our attention on study monochromatic and rainbow subgraphs.

The *constrained Ramsey number* $f(S, T)$ of two graphs, S and T, is the minimum number N such that for any edge colouring of K_N with any number of colours, there exists a monochromatic copy of S or a rainbow copy of T. In the literature, this is sometime called the *rainbow Ramsey number* or the *monochromatic-rainbow Ramsey number*. If follows easily from the canonical

Ramsey Theorem (see Proposition 9.1.2) that $f(S,T)$ exists if and only if S is a star of T is acyclic. An obvious lower bound for $f(S,T)$ is $R_{t-1}(S,T)$, where $t = e(T)$. This is because there exists an edge colouring of $K_{R_{t-1}(S)-1}$ using $t-1$ colours, which does not contain a monochromatic copy of S by the definition of $R_{t-1}(S)$. Since this edge colouring uses fewer than t colours, there is no rainbow T.

Various people have investigated the exact values of $f(S,T)$. Alon, Jiang, Miller and Pritikin [2] studied the case when $S = K_{1,s}$. The number $f(K_{1,s},T)$ is closely related to an m-good colouring. An *m-good colouring* is an edge colouring such that any vertex is incident with at most m edges of the same colour. Thus, $f(K_{1,s},T)$ is the minimum number N such that any $(s-1)$-good colouring of K_N contains a rainbow T. On the other hand, $f(S,K_{1,t})$ coincides with the local $(t-1)$-Ramsey number of a graph S. The *local $(t-1)$-Ramsey number* of a graph S, first introduced by Gyárfás, Lehel, Schelp and Tuza [26], is the analogue of the Ramsey number of S restricted to edge colourings such that each vertex is incident with at most $t-1$ edges of different colours. Jamison, Jiang and Ling [28] studied $f(S,T)$ when S and T are both trees. Wagner [46], Loh and Sudakov [32] investigated further the important case when S is a tree and T is a path.

In Chapter 9, we study $f(S,tK_2)$, where tK_2 is a set of vertex disjoint edges of size t.

Theorem 1.3.2. *Suppose S is a graph of order at least 5 and $R_{k+1}(S) \geq R_k(S)+3$ for all positive integers k. Then, $f(S,tK_2) = R_{t-1}(S)$ for all integers $t \geq 2$.*

We also identify the graphs S that do not satisfy the hypothesis of the theorem above in Proposition 9.3.1. If S has no isolated vertices, then S is bipartite and one of its vertex classes has size at most 3.

Chapter 2

Triangles in regular graphs with density below a half

2.1 Introduction

Mantel [34] proved that a triangle-free graph of order n has at most $n^2/4$ edges. Hence, a triangle-free regular graph of order n has degree $\delta \leq n/2$. If n is even, it is easy to construct $n/2$-regular bipartite graphs of order n. Recall that $k_3^{reg}(n, \delta)$ is the minimum number of triangles in δ-regular graphs of order n. Therefore, for n even, $k_3^{reg}(n, \delta) = 0$ if and only if $\delta \leq n/2$. For n odd, we investigate the largest possible δ such that $k_3^{reg}(n, \delta) = 0$. By considering a blow-up of a cycle of length 5, δ can be as large as $2n/5$ when n is a multiple of 5. In fact, δ is at most $2n/5$ by a theorem of Andrásfai, Erdős and Sós [3]. We state the special case of their theorem for $r = 3$ below.

Theorem 2.1.1 (Andrásfai, Erdős and Sós [3]). *Any triangle-free graph of order n with minimum degree at least $2n/5$ must be bipartite.*

Therefore, if n is odd and $k_3^{reg}(n, \delta) = 0$, then $\delta \leq 2n/5$.

In this chapter, we evaluate $k_3^{reg}(n, \delta)$ for n odd and almost all δ between $2n/5$ and $n/2$.

Theorem 2.1.2. *Let $k \geq 2^{20}$ and $l \geq 0$ be integers with $k \geq 2l + 3\sqrt{30l} + 137$.*

Suppose G is a $2k$-regular graph with order $4k + 2l + 1$. Then,

$$k_3(G) \geq k(k - l - 1)$$

holds. Furthermore equality holds if and only if $G \in \mathcal{G}^{reg}(k, l)$ (defined in Section 2.2).

By rephrasing the above theorem in terms of n and δ, we prove Theorem 1.2.1.

Proof of Theorem 1.2.1. Let $n = 4k + 2l + 1$ and $\delta = 2k$. Since $\delta \geq 2n/5$, we have $k \geq 2^{20}$. Note that $l = (n - 2\delta - 1)/2 \leq n/10$. By the hypothesis, we have $2k \geq 2(4k + 2l + 1)/5 + 12\sqrt{10l}/5$, that is $k \geq 2l + 6\sqrt{10l} + 1$. If $l \geq 2^{12}$, $6\sqrt{10l} + 1 > 3\sqrt{30l} + 137$. If $l \leq 2^{12}$, $2l + 3\sqrt{30l} + 137 < 2^{14} < k$. Hence, k and l satisfies the hypothesis of Theorem 2.1.2. Therefore, $k_3^{reg}(n, \delta) = k(k - l - 1) = \delta(3\delta - n - 1)/4$. □

In the next section, we define the family $\mathcal{G}(n, \beta)$ of extremal graphs as stated in Theorem 2.1.2. In Section 2.3, we investigate the sum of the squares of the degrees of a graph, which turns out to be an important tool for our proof. Finally, we prove Theorem 2.1.2 in Section 2.4.

2.2 Structure of the extremal graphs

First, we look at the case $n = 4k+1$ and $\delta = \lfloor n/2 \rfloor = 2k$. Our task is to construct a $2k$-regular graph of order $4k + 1$ by adding a new vertex to a triangle-free $2k$-regular graph of order $4k$. The only triangle-free $2k$-regular graph of order $4k$ is a complete bipartite graph with vertex classes X and Y with $|X| = |Y| = 2k$. We remove a matching M of size k and join all vertices that are incident with M to a new vertex z. Let G be the resulting graph. Figure 2.2 is G for $k = 2$. Clearly, G is $2k$-regular and every triangle contains the vertex z. Hence, $k_3(G)$, the number of triangles in G, is $k(k-1)$. This is precisely the bound in Theorem 2.1.2. Next, we extend this construction for $\delta < n/2$ as follows.

Let k and l be integers with $k > l \geq 0$. Consider a complete bipartite graph $K_{2k+l,2k+l}$ with vertex sets $\{x_1, \ldots, x_{2k+l}\}$ and $\{y_1, \ldots, y_{2k+l}\}$. First, we remove an $(l+1)$-factor from the graph induced by the vertex set $\{x_i, y_i : 1 \leq i \leq$

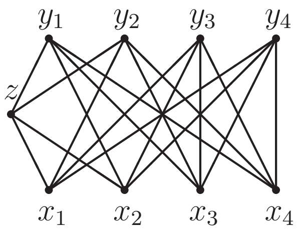

Figure 2.1: G for $k = 2$

$k\}$, and an l-factor from the graph induced by the vertex set $\{x_i, y_i : k < i \leq 2k + l\}$. Join $x_1, \ldots, x_k, y_1, \ldots, y_k$ to a new vertex z. We denote by $\mathcal{G}^{reg}(k, l)$ the family of graphs which can be obtained by the above construction. For $G \in \mathcal{G}^{reg}(k, l)$, G is $2k$-regular with $4k + 2l + 1$ vertices. Each triangle in G contains the vertex z, so there are $k(k - l - 1)$ triangles in G. Note that $\mathcal{G}^{reg}(k, l)$ is family of extremal graphs stated in Theorem 2.1.2.

Also, $|\mathcal{G}^{reg}(k, l)| = 1$ for $l = 0$ or $l = k - 1$. The case $l = 0$ corresponds to the case when $\delta = \lfloor n/2 \rfloor$. On the other hand, for $l = k - 1$, $\mathcal{G}^{reg}(k, l)$ is a family of $n/3 + o(n)$-regular graphs.

2.3 Sum of squared degrees

For a general graph H, we write $\phi(H)$ for the sum of the squares of the degrees of the vertices of H of order n. Clearly, by the Cauchy-Schwarz inequality, $\phi(H)$ is at least $4e(H)^2/n$. However, for given n and e, determining $\max\{\phi(H) : |H| = n$ and $e(H) = e\}$ is non-trivial. Let a and b be the unique non-negative integers such that $e = \binom{a}{2} + b$ with $0 \leq b < a$. The *quasi-complete* graph C_n^e with e edges of order n is the graph with vertex set $v_1, \ldots, v_n$ and $E(C_n^e) = \{v_i v_j : i < j \leq a\} \cup \{v_{a+1} v_i : i \leq b\}$. The *quasi-star* S_n^e is the complement graph of $C_n^{e'}$ where $e' = \binom{n}{2} - e$. Let $S(n, e) = \phi(S_n^e)$ and $C(n, e) = \phi(C_n^e)$. Ahlswede and Katona [1] proved that the maximum of $\phi(H)$ over all graphs H with n vertices and e edges is either $S(n, e)$ or $C(n, e)$. This result was rediscovered by Olpp [37]. We paraphrase their theorem below.

Theorem 2.3.1 (Ahlswede and Katona [1], Olpp [37]). *Let H be a graph with n vertices and e edges. Then*

$$\phi(H) \le \begin{cases} S(n,e) & \text{if } 0 \le e < \binom{n}{2}/2 - n/2 \\ \max\{S(n,e), C(n,e)\} & \text{if } \binom{n}{2}/2 - n/2 \le e \le \binom{n}{2}/2 + n/2 \\ C(n,e) & \text{if } \binom{n}{2}/2 + n/2 < e \le \binom{n}{2} \end{cases}$$

holds. Moreover, for $e < \binom{n}{2}/2 - n/2$ or $e > \binom{n}{2}/2 + n/2$, equality holds if and only if H is S_n^e or C_n^e respectively.

However, both $S(n,e)$ and $C(n,e)$ are difficult to express in terms of e and n. Clark, Entringer and Székely [10], de Caen [14] and Nikiforov [36] gave simpler but weaker upper bounds on $\phi(H)$. Das [13] bounds $\phi(H)$ from above with an extra constraint that the degrees of H are bounded.

However, for our purpose, we need to determine the exact maximum of $\phi(H)$ when the degrees are bounded from above and e is small. Formally speaking, we are going to determine the maximum of $\phi(H)$ over all graphs H with e edges and maximal degree $\Delta(H)$, where e and $\Delta(H)$ are specified in the lemma below. Here, we do not specify n, the number of vertices of H, as it turns out that the maximum value of $\phi(H)$ is independent of n.

Lemma 2.3.2. *Let r be a positive integer greater than 4500, and H be a graph with $\Delta(H) \le r$ and $e(H) = \beta r$ for $1 \le \beta < 6/5$. Then*

$$\phi(H) \le S(r+1, \beta r) = (\beta^2 - 2\beta + 2)r^2 + (5\beta - 4)r.$$

Furthermore, equality holds if and only if H is $S_{r+1}^{\beta r}$ with isolated vertices.

Proof. Suppose H is a graph achieving the maximum value of $\phi(H)$, so $\phi(H) \ge S(r+1, \beta r)$. First, we would like to determine the number of vertices with maximum degree Δ in H. Suppose that $\Delta \le (e(H)+1)/2$. Let W be the set of vertices $v \in V(H)$ with $d(v) > e(H)/16$. Clearly, $|W| \le 32$ and $\sum_{w \in W} d(w) \le$

$e(H) + \binom{|W|}{2} \leq e(H) + 496 \leq 10e(H)/9$. This means that we have

$$\begin{aligned}\phi(H) &= \sum_{w \in W} d(w)^2 + \sum_{v \notin W} d(v)^2 \\ &\leq \frac{1}{2}(e(H)+1)\sum_{w \in W} d(w) + \frac{1}{16}e(H)\sum_{v \notin W} d(v) \\ &= \frac{1}{2}(e(H)+1)\sum_{w \in W} d(w) + \frac{1}{16}e(H)(2e(H) - \sum_{w \in W} d(w)) \\ &\leq \frac{5}{9}e(H)^2 + \frac{1}{8}e(H)^2 = \frac{49}{72}e(H)^2 < r^2,\end{aligned}$$

contradicting the maximality of $\phi(H)$. Hence, $\Delta \geq e(H)/2 + 1$ and there exists a unique vertex u with $d(u) = \Delta$.

Next, we would like to identify the isolated vertices of H. Suppose vw is an edge of H with $v \notin N(u) \cup \{u\}$. If $\Delta < r$, the graph $H' = H - vw + uv$ has

$$\begin{aligned}\phi(H') - \phi(H) &= (d(u)+1)^2 + (d(w)-1)^2 - (d(u)^2 + d(w)^2) \\ &= 2(d(u) - d(w) + 1) > 0.\end{aligned}$$

This contradicts the maximality of $\phi(H)$, so we may assume $\Delta = r$. Since $d(w) < r$, there exists $x \in N(u) \backslash N(w)$. If $d(v) \geq d(x)$, then

$$\phi(H - ux + uv) - \phi(H) = 2(d(v) - d(x) + 1) > 0.$$

Otherwise,

$$\phi(H - vw + xw) - \phi(H) = 2(d(x) - d(v) + 1) > 0.$$

Hence, $d(v) \geq 1$ if and only if $v \in N(u) \cup \{u\}$. This means that the set of isolated vertices of H is precisely the vertex set $V(H) \backslash (N(u) \cup \{u\})$. From now on, we may assume that H contains no isolated vertex.

We now claim that Δ is in fact equal to r. Suppose the contrary. Since $\Delta < e(H)$, there exists an edge lying entirely in the neighbourhood of u. Let

w_1w_2 be such an edge with $d(w_1) + d(w_2)$ minimal. Note that

$$d(w_1) + d(w_2) \le e(H) - (d(u) - 2) + 1 = e(H) - \Delta + 3 \le \Delta + 1$$

as $\Delta \ge e(H)/2 + 1$. Then the graph $\widetilde{H}$ which is obtained by removing the edge w_1w_2 and joining u to a new vertex w_3, has

$$\phi(\widetilde{H}) = \phi(H) + 2(\Delta - d_H(w_1) - d_H(w_2) + 2) > \phi(H),$$

a contradiction and proves the claim. Hence, $V(H) = r + 1$ and $e(H) = \beta r$. This is a special case of Theorem 2.3.1, so $\phi(H) = S(r + 1, \beta r)$. Furthermore, this is achieved if and only if $H = S^{\beta r}_{r+1}$. □

Clearly the bound on $r \ge 4500$ is just an artifact of the proof. This bound could be reduced by a more careful counting argument. In addition, the argument still holds for $\beta \le 2$ and r sufficiently large. Furthermore, we believe for a graph H with degree at most $r \ge 2$ and $e \le \binom{r}{2}/2 - r/2$ edges, the extremal graph which achieves the maximal $\phi(H)$, is S^e_{r+1} with isolated vertices.

2.4 Proof of Theorem 2.1.2

Let G be a $2k$-regular graph with order $4k + 2l + 1$ and $k_3(G)$ minimal. By the construction of $\mathcal{G}^{reg}(k, l)$, $k_3(G) \le k(k-l-1)$. Let X and Y be a vertex partition of $V(G)$ with $e(X, Y)$ maximal. Without loss of generality $|X| = 2k + l + p + 1$ and $|Y| = 2k+l-p$ for some non-negative integer p. We further define $e(X) = \beta k$. Since $2k|X| = \sum_{x\in X} d(x) = e(X, Y)+2e(X) \le 2k|Y|+2e(X)$, we have $\beta \ge 2p+1$. For any vertex v, $d_X(v)$ denotes the number of neighbours of v in X, i.e. $d_X(v) = |N(v)\cap X|$, and $d_Y(v)$ is defined similarly. By the maximality of $e(X, Y)$, $d_X(x) \le k$ for all $x \in X$ and $d_Y(y) \le k$ for all $y \in Y$. For an edge v_1v_2, $d(v_1v_2)$ denotes the number of common neighbours of v_1 and v_2. Hence, $d(v_1v_2)$ is the number of triangles containing the edge v_1v_2. Similarly, we define $d_Y(v_1v_2)$ to be the number of common neighbours of v_1 and v_2 in Y. For $x_1x_2 \in E(X)$,

$$d_Y(x_1x_2) \ge d_Y(x_1) + d_Y(x_2) - |Y| = 2k + p - l - (d_X(x_1) + d_X(x_2)). \tag{2.1}$$

Let t be number of triangles with two vertices in X and one vertex in Y. Summing (2.1) over all edges in X, we get

$$\begin{aligned} t = \sum_{x_1x_2 \in E(X)} d_Y(x_1,x_2) &\geq \sum_{x_1x_2 \in E(X)} 2k+p-l-(d_X(x_1)+d_X(x_2)) \\ &= e(X)(2k+p-l)-\phi(G[X]) \\ &= \beta k(2k+p-l)-\phi(G[X]). \end{aligned}$$

But $t \leq k_3(G) \leq k(k-l-1)$ and so $k(k-l-1) \geq \beta k(2k+p-l)-\phi(G[X])$. Since $p \geq 0$, this implies

$$k(k-l-1) \geq \beta k(2k-l)-\phi(G[X]). \tag{2.2}$$

Moreover since $\beta \geq 1$ and $l \leq k/2$ this gives

$$2\phi(G[X]) \geq (3\beta-1)k^2. \tag{2.3}$$

In order to apply Lemma 2.3.2 to $\phi(G[X])$, we need to bound β from above.

We call the edge e *heavy*, if e is contained in more than $2(k-l-1)/3$ triangles. We write $d(e)$ to denote the number of triangles containing e. Equivalently, an edge e is heavy if and only if $d(e) > 2(k-l-1)/3$. Let T be a triangle with edges e_1, e_2 and e_3. For $i=0,1,2,3$, let n_i denote the number of vertices in G with exactly i neighbours in T. Clearly, $\sum n_i = n$, $\sum in_i = 6k$ and $n_2+3n_3 = \sum d(e_i)$ by counting vertices, edges and triangles respectively. Thus, $\sum d(e_i) \geq n_2+2n_3 \geq \sum in_i - \sum n_i = 2k-2l-1$, so one of e_1,e_2,e_3 is heavy. This means that every triangle contains at least one heavy edge. Hence, G is triangle-free if we remove all the heavy edges. Let h be the number of heavy edges in G. Notice that $h \leq 9k/2$, because $2h(k-l-1)/3 \leq \sum d(e) = 3k_3(G) \leq 3k(k-l-1)$. Let W be the set of vertices incident with at least $3\sqrt{3k/5}$ heavy edges. Then $|W| \leq 2h/(3\sqrt{3k/5}) \leq \sqrt{15k}$.

Let G' be the subgraph formed by removing all the heavy edges and the vertex set W. Note that $|G'| = n-|W| = 4k+2l+1-|W|$ and $\delta(G') > 2k-|W|-3\sqrt{3k/5}$. We claim that $\delta(G') > 2|G'|/5$, i.e. $2k-|W|-3\sqrt{3k/5} > 2(4k+2l+1-|W|)/5$. This is equivalent to $2(k-(2l+1)) > 3(\sqrt{15k}+|W|)$.

Setting $|W| = \sqrt{15k}$, this is true provided $(k-(2l+1))^2 \geq 135k$ as $k > 2l+1$. Expanding and rearranging the inequality we have

$$k^2 - (4l+137)k + (2l+1)^2 > 0,$$

which is true as $k > 2l + 3\sqrt{30l} + 137 > 2l + (3\sqrt{120l+2085} + 137)/2$ by the hypothesis of the theorem. Thus, the claim holds and so $\delta(G') > 2|G'|/5$. Moreover, G' is triangle-free and so is bipartite by Theorem 2.1.1. Now, G' has at least

$$e(G) - 2k|W| - h \geq e(G) - 2k\sqrt{15k} - 9k/2 \geq e(G) - 8k\sqrt{k}$$

edges. Since $e(X) + e(Y)$ is minimal, it follows that $e(X) \leq 8k\sqrt{k}$, in other words $\beta \leq 8\sqrt{k}$.

Now let U be the set of vertices $x \in X$ with $d_X(x) \geq \alpha\sqrt{\beta k}$, where $\alpha \leq \sqrt{k/\beta}$ is some number to be decided later. Then, $|U| \leq 2\beta k/\alpha\sqrt{\beta k} = 2\sqrt{\beta k}/\alpha$. Also, $\sum_{u\in U} d_X(u) \leq e(X) + \binom{|U|}{2} \leq (1+2/\alpha^2)\beta k$. Hence,

$$\begin{aligned}
\phi(G[X]) &= \sum_{u\in U} d_X(u)^2 + \sum_{v\in X\setminus U} d_X(v)^2 \\
&\leq k\sum_{u\in U} d_X(u) + \alpha\sqrt{\beta k}\sum_{v\in X\setminus U} d_X(v) \\
&\leq (1+2/\alpha^2)\beta k^2 + \alpha(1-2/\alpha^2)(\beta k)^{3/2} \\
&\leq (1+2/\alpha^2)\beta k^2 + \alpha(\beta k)^{3/2}.
\end{aligned}$$

The penultimate inequality is true as $\alpha < \sqrt{k/\beta}$. Set $\alpha = (16k/\beta)^{1/6}$, so $\phi(G[X]) \leq \beta k^2 + 3(\beta^4 k^5/2)^{1/3}$. By (2.3), we know that

$$\beta - 1 - 3(4\beta^4/k)^{1/3}$$

is negative. Now we claim that $\beta < 6/5$. It is enough to show that the above equation is strictly positive when $\beta \in [6/5, 8\sqrt{k}]$. In fact, we only need to check the cases when $\beta = 6/5$ and $\beta = 8\sqrt{k}$ by convexity. If $\beta = 6/5$, we are done as $k \geq 2^{20}$. If $\beta = 8\sqrt{k}$, $\beta - 1 - 3(4\beta^4/k)^{1/3}$ is an increasing function in k

for $k \geq 2^{20}$. It is strictly positive when $k = 2^{20}$. Hence, this proves the claim, so $\beta < 6/5$.

Recall that $\beta \geq 2p + 1$, so we have $p = 0$. By Lemma 2.3.2, $\phi(G[X]) \leq (\beta^2 - 2\beta + 2)k^2 + (5\beta - 4)k$, Notice that

$$\begin{aligned}\beta k(2k - l) - \phi(G[X]) &\geq \beta k(2k - l) - \Big((\beta^2 - 2\beta + 2)k^2 + (5\beta - 4)k\Big) \\ &= k(k - l - 1) + (\beta - 1)(3 - \beta)k^2 - (\beta - 1)(l + 5)k \\ &\geq k(k - l - 1) + (\beta - 1)(2 - \beta)k^2 \\ &\geq k(k - l - 1).\end{aligned}$$

Thus equality holds in (2.2). For equality to hold, we need $\beta = 1$, so $e(X) = k$, and by Lemma 2.3.2 $G[X]$ is a star $K_{1,k}$. Let z be the centre of $K_{1,k}$, $X_z = X \cap N(z)$ and $Y_z = Y \cap N(z)$. By (2.1), each $x \in X_z$ has at least $k - l - 1$ neighbours in Y_z. In fact, each has exactly $k-l-1$ neighbours in Y_z as $k(k-l-1) = |k_3| \geq \sum_{x \in X_z} |N(x) \cap Y_z| \geq |X_z|(k - l - 1) = k(k - l - 1)$. By regularity of G, $|N(x) \cap Y \backslash Y_z| = k + l + 1 = |Y \backslash Y_z|$. Thus, $G[X_z, Y \backslash Y_z]$ is a complete bipartite graph. Similarly, $G[Y_z, X \backslash X_z]$ is complete bipartite by considering $X' = Y \cup z$ and $Y' = X \backslash z$. Once again, by the regularity of the graph, $G[X \backslash (X_z \cup \{z\}), Y \backslash Y_z]$ is $(k - l)$-regular. Hence, G is a member of $\mathcal{G}^{reg}(k, l)$. Thus completes the proof of Theorem 2.1.2. □

2.5 Remark

In summary, the main idea of the proof is to first consider the bipartition of the vertex set into X and Y with $e(X, Y)$ maximal. Then, we estimate the number of triangles with exactly two vertices in X using a bound on $\phi(G[X])$ (Lemma 2.3.2). Observe that the lower bound on k in the assumption of Theorem 2.1.2 is also an artifact of the proof. This bound on k can be significantly improved if we improve the argument of bounding β from $\sqrt{k}$ to a constant term. Another way to improve the bound on k would be to prove Lemma 2.3.2 for $\beta \leq 8\sqrt{k}$.

We are now interested in $k_3^{reg}(n, \delta)$ for n odd and $\delta \leq 2n/5$. Also, we would like to identify the structures of graphs that achieve $k_3^{reg}(n, \delta)$. Clearly, $\mathcal{G}^{reg}(k, l)$

is a suitable candidate for the extremal graphs for $k_3^{reg}(n, \delta)$. Other possible candidates are graphs that are obtained by modifying triangle-free δ-regular graphs of order $n-1$. Hence, we would like to know the structures of the triangle-free regular graphs with degree $\delta \leq 2n/5$. Fortunately, triangle-free graphs with minimum degree δ have been well studied. We now give a brief overview.

A *homomorphism* f from a graph G to a graph H maps $V(G)$ to $V(H)$ such that $f(v)f(u) \in E(H)$ if $vu \in E(G)$. We say that G is *homomorphic* to H, if there exists a homomorphism from G to H. Therefore, with this notation, Theorem 2.1.1 states that any triangle-free graph G of order n with minimum degree at least $2n/5$ is homomorphic to an edge. Häggkvist [27] extended this result and showed that if the minimum degree is at least $3n/8$, then G is homomorphic to a C_5. Jin [29] classified the structures of triangle-free graphs for $\delta(G) \geq 10n/29$. Later, Chen, Jin and Koh [9], and Brandt [7] identified the structures of graphs which do not contain a Grötzsch graph, or a modified Petersen Graph (with either an edge deletion or contraction) respectively. It is worth pointing out that such structural results for triangle-free graphs are only valid for $\delta(G) \geq n/3$. For example, Hajnal (see [19]) used Kneser graphs to show that there exist many triangle-free graphs with minimum degree $(1/3-\epsilon)n$ and arbitrarily large chromatic number. Therefore, it is possible to evaluate $k_3(n, \delta)$ as well as identify the extremal graphs for $n/3 \leq \delta \leq n/2$. Also, it might be true that $k_3^{reg}(n, \delta) = 0$ for all n and $\delta < n/3$.

From the theorem of Andrásfai, Erdős and Sós [3], we know that for $r \geq 4$, there also exists n and $\delta < (1-1/(r-1))n$ such that $k_r^{reg}(n, \delta) > 0$, but $k_r(n, \delta) = 0$. It is easy to see that such n is not divisible by $r-1$. The construction of $G^{reg}(k, l)$ can be easily extended for the case $(r-1)|\delta$ and $n \equiv 1 \pmod{r-1}$. Moreover, we believe that $k_r^{reg}(n, \delta)$ is of order n^{r-1}.

Chapter 3

Cliques in graphs with bounded minimum degree

Recall that $k_r(G)$ is the number of r-cliques in a graph G. Define $k_r(n, \delta)$ to be the minimum number of r-cliques in graphs of order n with minimum degree δ. In the next section, we give an explicit construction of a family of graphs $\mathcal{G}(n, \beta)$, which we believe are the extremal graphs for $k_r(n, (1-\beta)n)$. In Section 3.3, we are going to prove that the conjectured value of $k_3(n, \delta)$ is correct for $n/2 < \delta \leq 2n/3$.

Theorem 3. *Let $1/3 \leq \beta < 1/2$. Suppose G is a graph of order n with minimum degree $(1 - \beta)n$. Then*

$$k_3(G) \geq (1 - 2\beta)\beta n k_2(G) \geq g_3(\beta)n^3.$$

Furthermore, equality holds if and only if (n, β) is feasible and G is a member of $\mathcal{G}(n, \beta)$.

The definitions of the term *feasible* and the family $\mathcal{G}(n, \beta)$ are given in Section 3.1 below.

3.1 Conjectured extremal graphs and $k_r(n, \delta)$

Recall that $f_r(n, e)$ is the minimum number of r-cliques in graphs of order n and e edges. In Chapter 1, we observed that the structure of the conjectured extremal graphs for $f_r(n,e)$ depends on p, where $(1-1/p)n^2/2 < e \leq (1-1/(p+1))n^2/2$. We believe that a similar result holds for $k_r(n,\delta)$ as well. Let $\delta = (1-\beta)n$ with $0 < \beta \leq 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Hence, β and βn are assumed to be a rational and an integer respectively. Note that p is defined so that by Turán's Theorem $k_r(n,(1-\beta)n) > 0$ for all n (such that βn is an integer) if and only if $r \leq p+1$. Since the case $\beta = 1$ implies the trivial case $\delta = 0$, we may assume that $0 < \beta < 1$. Furthermore, we consider the cases $1/(p+1) \leq \beta < 1/p$ separately for positive integers p. Hence, the condition $p = 2$ is equivalent to $1/3 \leq \beta < 1/2$, that is, $n/2 < \delta \leq 2n/3$.

Now, we give an upper bound on $k_r(n,\delta)$ by construction. Let n and $(1-\beta)n$ be positive integers not both odd with $0 < \beta < 1$. Let $p = \lceil \beta^{-1} \rceil - 1$. Consider a graph $G = (V,E)$ with the following properties. There is a partition of V into $V_0, V_1, \dots, V_{p-1}$ with $|V_0| = (1-(p-1)\beta)n$ and $|V_i| = \beta n$ for $1 \leq i \leq p-1$. For $0 \leq i < j \leq p-1$, $G[V_i, V_j]$ is a complete bipartite graph. For $1 \leq j \leq p-1$, $G[V_j]$ is empty and $G[V_0]$ is a $(1-p\beta)n$-regular graph such that the number of triangles in $G[V_0]$ is minimal over all $(1-p\beta)n$-regular graphs of order $(1-(p-1)\beta)n$. We define $\mathcal{G}(n,\beta)$ to be the family of graphs, which are obtainable by the above construction. Observe that $\mathcal{G}(n,\beta)$ is only defined if n and $(1-\beta)n$ are not both odd. Thus, whenever we mention $\mathcal{G}(n,\beta)$, we automatically assume that n and $(1-\beta)n$ are not both odd.

We say (n,β) is *feasible* if $G[V_0]$ is triangle-free. Note that $G[V_0]$ is regular of degree $(1-p\beta)n \leq (1-(p-1)\beta)n/2 = |V_0|/2$. Thus, if $|V_0|$ is even, then $G[V_0]$ is triangle-free. Therefore, for a given β, there exist infinitely many choices of n such that (n,β) is a feasible pair. If (n,β) is not a feasible pair, then $|V_0|$ is odd. Notice that $k_3(G[V_0]) \leq k_3(G') \leq n^2/16$, where $G' \in \mathcal{G}^{reg}(k,l)$ (see Section 2.2) $|V_0| = 4k+2l+1$ and $(1-p\beta)n \leq 2k$. Thus, $k_3(G[V_0]) = O(n^2)$.

By our construction, it is easy to see that every $G \in \mathcal{G}(n,\beta)$ is $(1-\beta)n$-regular. In particular, for positive integers $r \geq 3$, the number of r-cliques in G is exactly

$$k_r(G) = g_r(\beta)n^r + \binom{p-1}{r-3}(1-p\beta)^{r-3}n^{r-3}k_3(G[V_0]), \tag{3.1}$$

where,

$$\begin{aligned} g_r(\beta) =& \binom{p-1}{r}\beta^r + \binom{p-1}{r-1}(1-(p-1)\beta)\beta^{r-1} \\ &+ \frac{1}{2}\binom{p-1}{r-2}(1-p\beta)(1-(p-1)\beta)\beta^{r-2} \end{aligned}$$

with $\binom{x}{y}$ defined to be 0 if $x < y$ or $y < 0$. Since $k_3(G[V_0]) = O(n^2)$, the term with $k_3(G[V_0])$ in (3.1) is of order at most n^{r-1}. Therefore, this term only contributes $o(1)$ in terms of density of r-cliques. In fact, most of the time, we consider the case when (n, β) is feasible, i.e. $k_3(G[V_0]) = 0$. In Chapter 8 (Section 8.2), we discuss the case when (n, β) is not feasible including the case when neither n nor $(1-\beta)n$ is even.

If $\beta = 1/(p+1)$, then $(p+1)|n$ and $\mathcal{G}(n, 1/(p+1)) = \{T_{p+1}(n)\}$. Also, $k_r(T_{p+1}(n)) = g_r(n, 1/(p+1))n^r \geq k_r(n, (1-1/(p+1))n)$. Bollobás [5] proved that if $e = (1-1/(p+1))n^2/2$ and $(p+1)|n$, then $f_r(n, e) = k_r(T_{p+1}(n))$. Moreover, $T_{p+1}(n)$ is the only graph of order n with e edges and $f_r(n, e)$ r-cliques. Hence,

$$k_r(T_{p+1}(n)) \geq k_r\left(n, \left(1 - \frac{1}{p+1}\right)n\right) \geq f_r\left(n, \left(1 - \frac{1}{p+1}\right)\frac{n^2}{2}\right) = k_r(T_{p+1}(n)),$$

so $k_r(n, (1-1/(p+1))n) = k_r(T_{p+1}(n)) = g_r(n, 1/(p+1))n^r$. Moreover, $\mathcal{G}(n, 1/(p+1))$ is the extremal family.

We conjecture that if (n, β) is feasible then $\mathcal{G}(n, \beta)$ is the extremal family for $k_r(n, (1-\beta)n)$.

Conjecture 3.1.1. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let n and βn be positive integers. Then*

$$k_r(n, (1-\beta)n) \geq g_r(\beta)n^r$$

for positive integers r. Moreover, for $3 \leq r \leq p+1$ equality holds if and only if (n, β) is feasible and the extremal graphs are members of $\mathcal{G}(n, \beta)$.

By our previous observation, the conjecture is true for the following three cases: $p = 1$, $r > p+1$ and $\beta = 1/(p+1)$. Hence, we consider the situation when $3 \leq r \leq p+1$ and $\beta \notin \{0, 1, 1/2, 1/3, \dots\}$. In Section 3.3, we prove Theorem 3, so Conjecture 3.1.1 is true for $p = 2$, that is $n/2 < \delta \le 2n/3$. Chapters 4-7 are devoted to proving the conjecture for $p \ge 3$.

3.2 Key observation

We look at the structure of the extremal graphs $\mathcal{G}(n, \beta)$. Recall that $G[V_i, V_j]$ is complete bipartite for $i \neq j$ and $G[V_0]$ is $(1 - p\beta)$-regular. Figure 3.2 illusrates the structure of $\mathcal{G}(n, \beta)$ for $1/5 \le \beta < 1/4$, that is $p = 4$. It is natural to see that there are three types of edges e according to the number of vertices of e in V_0. However, if we consider $d(e)$ the number of triangles containing e, then there are only two types. To be precise

$$d(e) = \begin{cases} (1-2\beta)n & \text{if } |V(T) \cap V_0| = 0, 1 \text{ and} \\ (p-1)\beta n & \text{if } |V(T) \cap V_0| = 2, \end{cases} \tag{3.2}$$

for $e \in E(G)$ and $p = \lceil \beta^{-1} \rceil - 1$. This simple observation plays an important role.

3.3 Proof of Theorem 3

In this section, $1/3 \le \beta < 1/2$ and $p = 2$, so $n/2 < \delta \le 2n/3$. Note that the only non-trivial case of $k_r(n, \delta)$ is when $r = 3$, that is evaluating the minimum number of triangles. Let G be a graph of order n with $n/2 < \delta(G) = (1 - \beta)n \le 2n/3$.

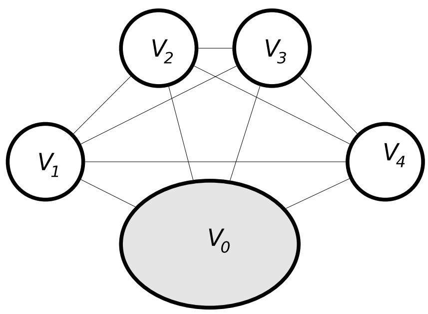

Figure 3.1: a typical member of $\mathcal{G}(n, \beta)$ for $p = 4$

Our aim is to show that

$$k_3(G) \geq (1-2\beta)\beta n k_2(G) \geq g_3(\beta)n^3.$$

Since G has at least $(1-\beta)n^2/2$ edges, the second inequality is true.

Recall that for an edge e, $d(e)$ is the number of triangles containing e. We write $D(e) = d(e)/n$. Clearly, $\sum D(e) = 3k_3(G)/n$ with sum over $E(G)$, where $k_r(G)$ is the number of r-cliques in G. In addition, $D(e) \geq 1-2\beta$ for each edge e, because each vertex in G misses at most βn vertices. Since $\beta < 1/2$, $D(e) > 0$. Thus, every edge is contained in a triangle. Let T be a triangle in G. Similarly, define $d(T)$ to be the number of 4-cliques containing T and write $D(T) = d(T)/n$. We claim that

$$\sum_{e \in E(T)} D(e) \geq 2 - 3\beta + D(T). \tag{3.3}$$

Let n_i be the number vertices in G with exactly i neighbours in T for $i = 0, 1, 2, 3$. Clearly, $n = n_0 + n_1 + n_2 + n_3$. By counting the number of edges incident with T, we obtain

$$3(1-\beta)n \leq \sum_{v \in V(T)} d(v) = 3n_3 + 2n_2 + n_1 \leq 2n_3 + n_2 + n. \tag{3.4}$$

On the other hand, $n_3 = d(T)$ and $n_2 + 3n_3 = \sum_{e \in E(G)} d(e)$. Hence, (3.3) holds. Notice that equality holds in (3.3) only if $d(v) = (1-\beta)n$ for all $v \in T$.

Next, by summing (3.3) over all triangles T in G, we obtain

$$n \sum_{e \in E(G)} D(e)^2 = \sum_{T} \sum_{e \in E(T)} D(e) \geq (2-3\beta)k_3 + 4k_4/n \geq (2-3\beta)k_3. \tag{3.5}$$

We would like to bound $\sum D(e)^2$ above in terms of $\sum D(e)$, which is equal to $3k_3(G)/n$. It is well known (or by Proposition 3.3.1 stated below) that $\sum D(e)^2$ is maximal if $D(e)$ takes only two values.

Proposition 3.3.1. *Let $\mathcal{A}$ be a finite set. Suppose $f, g : \mathcal{A} \to \mathbb{R}$ with $f(a) \leq M$*

and $g(a) \geq m$ for all $a \in \mathcal{A}$. Then

$$\sum_{a\in\mathcal{A}} f(a)g(a) \leq m \sum_{a\in\mathcal{A}} f(a) + M \sum_{a\in\mathcal{A}} g(a) - mM|\mathcal{A}|,$$

with equality if and only if for each $a \in \mathcal{A}$, $f(a) = M$ or $g(a) = m$.

Proof. Observe that $\sum_{a\in\mathcal{A}}(M - f(a))(g(a) - m) \geq 0$. □

From (3.2), for $G \in \mathcal{G}(n, \beta)$ with (n, β) feasible, either $D(e) = 1 - 2\beta$ or $D(e) = \beta$ for each edge e. Suppose first that $D(e) \leq \beta$ for all edges e. Recall that $D(e) \geq 1 - 2\beta$ for all edge e. By Proposition 3.3.1 taking $\mathcal{A} = E(G)$, $f = g = D$, $m = 1 - 2\beta$ and $M = \beta$, we have

$$\sum_{e\in E(G)} D(e)^2 \leq (1-\beta) \sum_{e\in E(G)} D(e) - (1-2\beta)\beta k_2 = 3(1-\beta)k_3/n - (1-2\beta)\beta k_2$$

as $k_2 = e(G)$. Substitute the above inequality into (3.5) and rearrange; we get

$$k_3(G) \geq (1-2\beta)\beta k_2(G)n \geq (1-\beta)(1-2\beta)\beta n^3/2 = g_3(\beta)n^3$$

as required.

Thus, we may assume that there exists an edge e with $D(e) > \beta$. For an edge e, define $D_-(e) = \min\{D(e), \beta\}$ and $D_+(e) = D(e) - D_-(e)$. Notice that $\sum D(e)^2 = \sum D(e)(D_+(e) + D_-(e)) = \sum D(e)D_+(e) + \sum D(e)D_-(e)$. By the definition, $D_-(e)$ is at most β. Thus, we can bound $\sum D(e)D_-(e)$ using Proposition 3.3.1 taking $\mathcal{A} = E(G)$, $f = D_-$, $g = D$, $m = 1 - 2\beta$ and $M = \beta$. Hence,

$$\begin{aligned} n \sum_{e\in E(G)} D(e)D_-(e) &\leq (1-2\beta)n \sum_{e\in E(G)} D_-(e) + \beta n \sum_{e\in E(G)} D(e) - (1-2\beta)\beta n k_2 \\ &\leq (1-\beta)n \sum_{e\in E(G)} D(e) - (1-2\beta)\beta n k_2 \qquad (3.6) \end{aligned}$$

$$n \sum_{e\in E(G)} D(e)D_-(e) \leq 3(1-\beta)k_3 - (1-2\beta)\beta n k_2. \qquad (3.7)$$

However, there is no non-trivial upper bound on $\sum D(e)D_+(e)$. Fortunately, we can avoid the term $\sum D(e)D_+(e)$ by the following trick. We claim that

$$\sum_{e\in E(T)} D_-(e) \geq 2-3\beta \tag{3.8}$$

for every triangle T. If $D(e) = D_-(e)$ for each edge e in T, then (3.8) holds by (3.3). Otherwise, there exists $e_0 \in E(T)$ such that $D(e_0) \neq D_-(e_0)$. This means that $D_-(e_0) = \beta$, so $\sum D_-(e) \geq \beta + 2(1-2\beta) = 2-3\beta$. Hence, (3.8) holds for every triangle T.

Next, we sum (3.8) over all triangles. We obtain an inequality very similar to (3.5) but with the left hand side equal to $n\sum_{e\in E(G)} D(e)D_-(e)$. After substitution of (3.7) and rearrangement, we have $k_3(G) \geq (1-2\beta)\beta k_2(G)n \geq g_3(\beta)n^3$. Thus, we have proved the inequality in Theorem 3.

Now suppose equality holds, i.e. $k_3 = (1-2\beta)\beta k_2 n$. This means that equality holds in (3.6), so (since $\beta < 1/2$) $D(e) = D_-(e)$ for all $e \in E(G)$. Because equality holds in (3.8), $\sum D(e) = 2-3\beta$. Hence, $D(T) = 0$ for every triangle T by (3.3). In particular, by the remark following (3.3), G is $(1-\beta)n$-regular, because every vertex lies in a triangle as $D(e) > 0$ for all edges e. Moreover, G is K_4-free as $D(T) = 0$ for every triangle T. Since equality holds in Proposition 3.3.1, either $D(e) = 1-2\beta$ or $D(e) = \beta$ for each edge e. Recall that equality holds for (3.3), so every triangle T contains exactly one edge e_1 with $D(e_1) = \beta$ and two edges, e_2 and e_3, with $D(e_2) = D(e_3) = 1-\beta$. Pick an edge e with $D(e) = \beta$ and let W be the set of common neighbours of the end vertices of e, so $|W| = \beta n$. Clearly W is an independent set, otherwise G contains a K_4. For each $w \in W$, $d(w) = (1-\beta)n$ implies $N(w) = V(G)\backslash W$. Therefore, $G[V(G)\backslash W]$ is $(1-2\beta)n$-regular. If there is a triangle T in $G[V(G)\backslash W]$, then $T \cup w$ forms a K_4 for $w \in W$. This contradicts the assumption that G is K_4-free, so $G[V(G)\backslash W]$ is triangle-free. Hence, G is a member of $\mathcal{G}(n,\beta)$ and (n,β) is feasible. Therefore, we have now proved Conjecture 3.1.1 for the case $p = 2$.

Chapter 4

Degrees of cliques and K_{p+2}-free graphs

We define the degree for a general clique in the coming section. Also, we study some of its basic properties. By mimicking the proof of Theorem 3, we show that the conjecture is true for all K_{p+2}-free graphs in Section 4.2

Theorem 4. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Suppose G is a K_{p+2}-free graph of order n with minimum degree $(1-\beta)n$. Then*

$$\frac{k_s(G)}{g_s(\beta)n^s} \geq \frac{k_t(G)}{g_t(\beta)n^t} \tag{4.1}$$

holds for $2 \leq t < s \leq p+1$. Moreover, the following three statements are equivalent:

(i) Equality holds for some $2 \leq t < s \leq p+1$.

(ii) Equality holds for all $2 \leq t < s \leq p+1$.

(iii) The pair (n, β) is feasible and G is a member of $\mathcal{G}(n, \beta)$.

Notice that the theorem implies Theorem 1.2.3, that is, Conjecture 3.1.1 for K_{p+2}-free graphs and all $\beta \in (0, 1)$, because $k_2(G) \geq (1-\beta)n^2/2 = g_2(\beta)$.

4.1 Degree of a clique

Let $G \in \mathcal{G}(n,\beta)$ with (n,β) feasible. Let T be a t-clique in G. It is easy to see that there are three types of cliques according to $|T \cap V_0|$. However, if we consider $d(T)$, the number of $(t+1)$-cliques containing T, then there are only two types. To be precise

$$d(T) = \begin{cases} (1-t\beta)n & \text{if } |V(T) \cap V_0| = 0, 1 \text{ and} \\ (p+1-t)\beta n & \text{if } |V(T) \cap V_0| = 2, \end{cases}$$

for $T \in \mathcal{K}_t(G)$, $2 \le t \le p+1$ and $p = \lceil \beta^{-1} \rceil - 1$. This observation helps us to define the correct notion of degree for a general clique.

For a graph G and a vertex set $U \subset V(G)$, we write $\mathcal{K}_t(U)$ to be the set of t-cliques in $G[U]$. Let $k_t(U) = |\mathcal{K}_t(U)|$. If $U = V(G)$, we simply write $\mathcal{K}_t$ and k_t. Define the *degree* $d(T)$ of a t-clique T to be the number of $(t+1)$-cliques containing T. In other words, $d(T) = |\{S \in \mathcal{K}_{t+1} : T \subset S\}|$. If $t = 1$, then $d(v)$ coincides with the ordinary definition of the degree for a vertex v. If $t = 2$, then $d(uv)$ is the number of common neighours of the end vertices of the edge uv, that is the codegree of u and v. The number $d(uv)$ is known as the *book number* in the literature. Clearly, $n\sum_{T\in\mathcal{K}_t} d(T) = (t+1)k_{t+1}$ for $t \ge 1$. For convenience, we write $D(T)$ to denote $d(T)/n$.

Next, define the functions D_+ and D_- as follows. For a graph G with $\delta(G) = (1-\beta)n$ and $1 \le t \le p+1$ (where $p = \lceil \beta^{-1} \rceil - 1$),

$$\begin{aligned} D_-(T) &= \min\{D(T), (p+1-t)\beta\}, \text{ and} \\ D_+(T) &= D(T) - D_-(T) = \max\{0, D(T) - (p+1-t)\beta\} \end{aligned}$$

for $T \in \mathcal{K}_t$. We say that a clique T is *heavy* if $D_+(T) > 0$. Let $\mathcal{K}_t^+(U)$ denote the set of heavy t-cliques in $G[U]$ and $k_t^+(U) = |\mathcal{K}_t^+(U)|$. A graph G is *heavy-free* if G does not contain any heavy cliques. Note that both D_- and D_+ are functions depending on β i.e. $\delta(G)$. Hence, the notation of a heavy clique is defined only if the base graph is known. Now, we study some basic properties of $D(T)$ and $D_-(T)$.

Lemma 4.1.1. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Suppose G is a graph of* *order n with minimum degree $(1-\beta)n$. Suppose $S \in \mathcal{K}_s$ and $T \in \mathcal{K}_t(S)$ for $1 \leq t < s$. Then*

(i) $D(S) \geq 1 - s\beta$,

(ii) $D(S) \geq D(T) - (s-t)\beta$,

(iii) $D_+(T) \leq D_+(S) \leq D_+(T) + (s-t)\beta$ *for* $s \leq p+1$,

(iv) if $D_+(S) > 0$ *and* $s \leq p+1$, *then* $D_-(S) = (p-s+1)\beta$.

Moreover, G is K_{p+2}-free if and only if G is heavy-free.

Proof. For each $v \in S$, there are at most βn vertices not joined to v. Hence, $D(S) \geq 1 - s\beta$, so (i) is true. Similarly, consider the vertices in $S \backslash T$, so (ii) is also true. If $s \leq p+1$ and $D_+(T) > 0$, then by (ii) we have

$$\begin{aligned} D_+(S) + (p-s+1)\beta \geq & D(S) \\ \geq & D(T) - (s-t)\beta \\ = & D_+(T) + (p-t+1)\beta - (s-t)\beta, \end{aligned}$$

so the left inequality of (iii) is true. Since $D(S) \leq D(T)$, the right inequality of (iii) is also true by the definition of $D_+(S)$ and $D_+(T)$. Finally, (iv) is a straightforward consequence of the definition of $D_+(S)$. Notice that $D(U) = D_+(U)$ for $U \in \mathcal{K}_{p+1}$. Hence, by (iii), G is K_{p+2}-free if and only if G is heavy-free. □

Notice that (iii) implies that if a t-subclique T of a clique S is heavy, so is S. However, the converse is false even if all t-subcliques are not heavy. For example, consider an isolated clique of size $(p-t+1)\beta n + t$. Then, $D(T) = (p-t+1)\beta$ for every t-sunclique T. However, for an s-clique S with $t < s < p+1$, $D(S) = (p-t+1)\beta - (s-t)/n$, so S is heavy provided n is not too small.

In the next lemma, we prove a generalisation of (3.3).

Lemma 4.1.2. *Let $0 < \beta < 1$. Let s and t be integers with $2 \leq t < s$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$\sum_{T \in \mathcal{K}_t(S)} D(T) \geq (1-\beta)s\binom{s-2}{t-1} - (t-1)\binom{s-1}{t} + \binom{s-2}{t-2}D(S)$$

for $S \in \mathcal{K}_s$. Moreover, if equality holds, then $d(v) = (1-\beta)n$ for all $v \in S$.

Proof. Let n_i be the number of vertices with exactly i neighbours in S. The following three equations :

$$\sum_i n_i = n, \tag{4.2}$$

$$\sum_i i n_i = \sum_{v \in V(S)} d(v) \geq s(1-\beta)n, \tag{4.3}$$

$$\sum_i \binom{i}{t} n_i = \sum_{T \in \mathcal{K}_t(S)} D(T)n, \tag{4.4}$$

follow from a count of the number of vertices, edges and $(t+1)$-cliques respectively. Next, by considering $(t-1)\binom{s-1}{t}(4.2) - \binom{s-2}{t-1}(4.3) +$ (4.4), we have

$$\sum_{T \in \mathcal{K}_t(S)} D(T)n \geq \left((1-\beta)s\binom{s-2}{t-1} - (t-1)\binom{s-1}{t}\right)n + \sum_{0 \leq i \leq s} x_i n_i,$$

where $x_i = \binom{i}{t} + (t-1)\binom{s-1}{t} - i\binom{s-2}{t-1}$. Notice that $x_i = x_{i+1} + \binom{s-2}{t-1} - \binom{i}{t-1} \geq x_{i+1}$ for $0 \leq i \leq s-2$. For $i = s-1$, we have

$$\begin{aligned} x_{s-1} &= \binom{s-1}{t} + (t-1)\binom{s-1}{t} - (s-1)\binom{s-2}{t-1} \\ &= t\binom{s-1}{t} - (s-1)\binom{s-2}{t-1} = 0. \end{aligned}$$

For $i = s$, $n_s = D(S)n$ and

$$\begin{aligned}
x_s =&\binom{s}{t} + (t-1)\binom{s-1}{t} - s\binom{s-2}{t-1}\\
=&t\binom{s-1}{t} + \binom{s-1}{t-1} - s\binom{s-2}{t-1}\\
=&(s-t+1)\binom{s-1}{t-1} - s\binom{s-2}{t-1}\\
=&(s-t+1)\binom{s-2}{t-2} - (t-1)\binom{s-2}{t-1}\\
=&\binom{s-2}{t-2}.
\end{aligned}$$

In particular, if equality holds in the lemma, then equality holds in (4.3). This means that $d(v) = (1-\beta)n$ for all $v \in S$. □

In fact, most of the time, we are only interested in the case when $s = t+1$. Hence, we state the following corollary.

Corollary 4.1.3. *Let $0 < \beta < 1$. Suppose G is a graph or order n with minimum degree $(1-\beta)n$. Then*

$$\sum_{T \in \mathcal{K}_t(S)} D(T) \geq 2 - (t+1)\beta + (t-1)D(S)$$

for $S \in \mathcal{K}_{t+1}$ and integer $t \geq 2$. Moreover, if equality holds, then $d(v) = (1-\beta)n$ for all $v \in S$. □

4.2 K_{p+2}-free graphs

Here, all graphs are assumed to be K_{p+2}-free. Lemma 4.1.1 implies that these graphs are also heavy-free. This means that $D_+(T) = 0$ and $D(T) \leq (p+1-t)\beta$ for all $T \in \mathcal{K}_t$. We would also like to point out that the family of K_{p+2}-free graphs is a natural family to study in its own right. Let G be a graph of order n with minimum degree $(1-\beta)n$. Note that G contains a K_{p+1} by Turán's theorem, because $e(G) = \delta(G)n/2 = (1-\beta)n^2/2 > ex(n, K_{p+1})$ as $1/(p+1) \leq \beta <$

$1/p$. Hence, K_{p+2}-free graphs are those graphs with the minimum possible clique number given $\delta(G)$.

Following the same approach as in proof of Theorem 3, we are going to first sum Corollary 4.1.3 over $S \in \mathcal{K}_{t+1}$ and then apply Proposition 3.3.1. We obtain the following lemma.

Lemma 4.2.1. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Suppose G is a K_{p+2}-free graph of order n with minimum degree $(1-\beta)n$. Then*

$$k_{t+1}(G) \geq \frac{(1-t\beta)(p-t+1)\beta n k_t(G) + (t-1)(t+2)k_{t+2}(G)/n}{t-1+(p-2t+2)(t+1)\beta}$$

for $2 \leq t \leq p$. Moreover, if equality holds, then G is $(1-\beta)n$-regular and, for each $T \in \mathcal{K}_t$, either $D(T) = 1 - t\beta$ or $D(T) = (p-t+1)\beta$.

Proof. Summing Corollary 4.1.3 over $S \in \mathcal{K}_{t+1}$ gives

$$(2-(t+1)\beta)k_{t+1}(G) + (t-1)\sum_{S \in \mathcal{K}_{t+1}} D(S) \leq \sum_{S \in \mathcal{K}_{t+1}} \sum_{T \in \mathcal{K}_t(S)} D(T) = n \sum_{T \in \mathcal{K}_t} D(T)^2. \tag{4.5}$$

Recall that G is heavy-free, so $1 - t\beta \leq D(T) \leq (p-t+1)\beta$, where the lower bound is proved in Lemma 4.1.1 (i). By Proposition 3.3.1 taking $\mathcal{A} = \mathcal{K}_t$, $f = D$, $g = D$, $m = 1 - t\beta$ and $M = (p-t+1)\beta$, the right hand side of (4.5) is at most

$$\begin{aligned}
&(1-t\beta)n \sum_{T \in \mathcal{K}_t(S)} D(T) + (p-t+1)\beta n \sum_{T \in \mathcal{K}_t(S)} D(T) \\
&\quad - (1-t\beta)(p-t+1)\beta n k_t \\
=&(1+(p-2t+1)\beta)n \sum_{T \in \mathcal{K}_t(S)} D(T) - (1-t\beta)(p-t+1)\beta n k_t \\
=&(1+(p-2t+1)\beta)(t+1)k_{t+1} - (1-t\beta)(p-t+1)\beta n k_t.
\end{aligned}$$

Note that $\sum D(S) = (t+2)k_{t+2}/n$. Thus, we prove the inequality in the lemma by rearranging (4.5).

Suppose equality holds in the lemma, so that equality holds in Corollary 4.1.3 for every $(t+1)$-clique S. Since $D(T) \geq (1-t\beta) > 0$ for $T \in \mathcal{K}_t$ and $t = 2, \ldots, p$, every vertex v is in a $(t+1)$-clique. Therefore, G is $(1-\beta)n$-regular by the case of equality in Corollary 4.1.3. In addition, we have equality in Proposition 3.3.1, so for each $T \in \mathcal{K}_t$, either $D(T) = 1 - t\beta$ or $D(T) = (p-t+1)\beta$. □

For a fixed β, Lemma 4.2.1 gives a linear relationship between k_t, k_{t+1} and k_{t+2}. Note that $k_{p+2} = 0$ and $k_2 \geq (1-\beta)n^2/2$. By taking $t = 2, \ldots, p$, we obtain $p-1$ linear relationships with $p-1$ unknowns variables, $k_3, \ldots, k_{p+1}$. Therefore, we can solve this set of equations. To keep our calculations simple, we are going to establish a few relationships between $g_t(\beta)$ and $g_{t+1}(\beta)$ in the next lemma.

Lemma 4.2.2. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let t be an integer with $2 \leq t \leq p$. Then*

$$\begin{aligned}(t+1)g_{t+1}(\beta) =&(1-t\beta)g_t(\beta)\\ &+\frac{1}{2}\binom{p-1}{t-2}((p+1)\beta-1)(1-(p-1)\beta)(1-p\beta)\beta^{t-2}, \quad (4.6)\\ g_{t+1}(\beta) =&\frac{(1-t\beta)(p-t+1)\beta g_t(\beta)+(t-1)(t+2)g_{t+2}(\beta)}{t-1+(t+1)(p-2t+2)\beta}. \quad (4.7)\end{aligned}$$

Moreover

$$\frac{g_p(\beta)}{g_{p+1}(\beta)} = \frac{1}{\beta}\left(1 + \frac{\beta g_{p-1}(\beta')}{(1-\beta)g_p(\beta')}\right), \quad (4.8)$$

where $\beta' = \beta/(1-\beta)$.

Proof. We fix β (and p) and write g_t to denote $g_t(\beta)$. Pick n such that (n,β) is feasible and let $G \in \mathcal{G}(n,\beta)$ with partition classes $V_0, V_1, \ldots, V_{p-1}$ as described in Section 3.1. For $T \in \mathcal{K}_t$, $D(T) = 1 - t\beta$ or $D(T) = (p-t+1)\beta$. Since $D(T) = (p-t+1)\beta$ if and only if $|V(T) \cap V_0| = 2$, there are exactly

$$\frac{1}{2}\binom{p-1}{t-2}(1-(p-1)\beta)(1-p\beta)\beta^{t-2}n^t$$

t-cliques T with $D(T) = (p-t+1)\beta$. We have

$$(t+1)g_{t+1}n^{t+1} = (t+1)k_{t+1} = n\sum_{T\in\mathcal{K}_t} D(T).$$

Hence, (4.6) is true, by expanding the right hand side of the above equation. For $2 \leq s < p$, let f_s and f_{s+1} be (4.6) with $t = s$ and $t = s + 1$ respectively. Then (4.7) follows by considering $(p - s + 1)f_s - (s - 1)\beta f_{s+1}$.

Now let $G' = G \backslash V_{p-1}$. Notice that G' is $(1 - 2\beta)n$-regular with $(1 - \beta)n$ vertices. We observe that G' is a member of $\mathcal{G}(n', \beta')$, where $n' = (1 - \beta)n$. Observe that $\lceil \beta'^{-1} \rceil - 1 = p - 1$, so $1/p \leq \beta' < 1/(p - 1)$. Recall that $k_t(G) = g_t(\beta)n^t$ for all $2 \leq t \leq p$, so $k_{p+1}(G)g_p(\beta) = k_p(G)g_{p+1}(\beta)n$. Similarly, $k_p(G')g_{p-1}(\beta') = k_{p-1}(G')g_p(\beta')n$. By considering $\mathcal{K}_p(G)$ and $\mathcal{K}_{p+1}(G)$, we obtain the following two equations :

$$k_{p+1}(G) = \beta n k_p(G'), \tag{4.9}$$

$$\begin{aligned} k_p(G) &= \beta n k_{p-1}(G') + k_p(G') = \beta n \frac{g_{p-1}(\beta')k_p(G')}{n' g_p(\beta')} + k_p(G') \\ &= \left(1 + \frac{\beta g_{p-1}(\beta')}{(1 - \beta)g_p(\beta')}\right) k_p(G'). \end{aligned} \tag{4.10}$$

By substituting (4.9) and (4.10) into $k_p(G)n/k_{p+1}(G) = g_p(\beta)/g_{p+1}(\beta)$, we obtain (4.8). The proof is complete. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. Fix β and write g_t to denote $g_t(\beta)$. First, we are going to prove (4.1). In fact, it is sufficient to prove the case when $s = t + 1$. We proceed by induction on t from above. For $t = p$, Lemma 4.2.1 gives

$$(p - 1 - (p - 2)(p + 1)\beta)k_{p+1} \geq (1 - p\beta)\beta n k_p.$$

Since $g_{p+2} = 0$, (4.7) implies $k_{p+1}/g_{p+1}n^{p+1} \geq k_p/g_p n^p$. Hence, (4.1) is true for $t = p$. For $t < p$, Lemma 4.2.1 shows that

$$\begin{aligned} &(t - 1 + (t + 1)(p - 2t + 2)\beta)k_{t+1} \\ &\geq (1 - t\beta)(p + 1 - t)\beta n k_t + (t - 1)(t + 2)k_{t+2}/n \end{aligned}$$

and by the induction hypothesis,

$$\geq (1-t\beta)(p+1-t)\beta n k_t + (t-1)(t+2) g_{t+2} k_{t+1}/g_{t+1}. \tag{4.11}$$

Thus, (4.1) follows from (4.7).

It is clear that (iii) implies both (i) and (ii) by the construction of $\mathcal{G}(n,\beta)$ and the feasibility of (n,β). Suppose (i) holds, so equality holds in (4.1) for $t = t_0$ and $s = s_0$ with $t_0 < s_0$. We claim that equality must also hold for $t = p$ and $s = p+1$. Suppose the claim is false and equality holds for $t = t_0$ and $s = s_0$, where s_0 is maximal. Since equality holds for $t = t_0$, by (4.1), equality holds for $t = t_0, \ldots, s_0 - 1$ with $s = s_0$. We may assume that $t = s_0 - 1$ and $s_0 \neq p+1$ and $k_{s_0+1}/g_{s_0+1} n > k_{s_0}/g_{s_0}$. However, this would imply a strictly inequality in (4.11) contradicting the fact that equality holds for $s = s_0$ and $t = s_0 - 1$. Thus, the proof of the claim is complete, that is, if (i) holds then equality holds in (4.1) for $t = p$ and $s = p+1$.

Therefore, in order to prove that (i) implies (iii), it is sufficient to show that if $k_{p+1}/g_{p+1} n^{p+1} = k_p/g_p n^p$, then (n,β) is feasible and G is a member of $\mathcal{G}(n,\beta)$. We proceed by induction on p. It is true for $p = 2$ by Theorem 3, so we may assume $p \geq 3$. Since equality holds in (4.1), we have equality in Lemma 4.2.1 and Corollary 4.1.3. Thus, G is $(1-\beta)n$-regular. In addition, for each $T \in \mathcal{K}_p$, either $D(T) = 1 - p\beta$ or $D(T) = \beta$. Moreover, Corollary 4.1.3 implies that $\sum_{T \in \mathcal{K}_p(S)} D(T) \geq 2 - (p+1)\beta$ for $S \in \mathcal{K}_{p+1}$. Thus, there exists $T \in \mathcal{K}_p(S)$ with $D(T) = \beta$. Pick $T \in \mathcal{K}_p$ with $D(T) = \beta$ and let $W = \bigcap\{N(v) : v \in V(S)\}$, so $|W| = \beta$. Since G is K_{p+2}-free, W is a set of independent vertices. For each $w \in W$, $d(w) = (1-\beta)n$, so $N(w) = V(G) \backslash W$. Thus, the graph $G' = G[V(G)\backslash W]$ is $(1-\beta')n'$-regular, where $n' = (1-\beta)n$, $\beta' n' = (1-2\beta)n$ and $\beta' = \beta/(1-\beta)$. Note that $\lceil \beta'^{-1} \rceil - 1 = p - 1$. Since G is K_{p+2}-free, G' is K_{p+1}-free. Also, $k_{p+1}(G) = \beta n k_p(G')$ and

$$\begin{aligned} k_p(G) =& \beta n k_{p-1}(G') + k_p(G') \overset{\text{by (4.1)}}{\leq} \beta \frac{g_{p-1}(\beta') k_p(G')}{g_p(\beta')} + k_p(G') \\ =& \left(1 + \beta \frac{g_{p-1}(\beta')}{(1-\beta) g_p(\beta')}\right) k_p(G') \overset{\text{by (4.8)}}{=} \frac{g_p(\beta)\beta}{g_{p+1}(\beta)} k_p(G'). \end{aligned} \tag{4.12}$$

Hence,

$$g_p(\beta)\beta n k_p(G') = g_p(\beta)k_{p+1}(G) \overset{\text{by (4.1)}}{=} g_{p+1}(\beta)nk_p(G) \overset{\text{by (4.12)}}{\leq} g_p(\beta)\beta n k_p(G').$$

Therefore, we have $k_p(G')/g_p(\beta')n'^p = k_{p-1}(G')/g_{p-1}(\beta')n'^{p-1}$. By the induction hypothesis, $G' \in \mathcal{G}(n', \beta')$, which implies $G \in \mathcal{G}(n, \beta)$. This completes the proof of the theorem. □

Chapter 5

Heavy cliques

In the previous chapter, we have proved that Conjecture 3.1.1 holds for K_{p+2}-free graphs. By Lemma 4.1.1, K_{p+2}-free graphs do not contain any heavy cliques. Hence, in order to prove Conjecture 3.1.1 completely, it remains to tackle heavy cliques. In the proof of Theorem 3, we apply D_- and (3.8) to handle heavy cliques. In the following section, we look at the natural generalisation of (3.8). Unfortunately, this natural generalisation is not sufficient to prove the Conjecture 3.1.1 even for $p = 3$. By a detailed analysis of heavy cliques, we prove that Conjecture 3.1.1 is true for $p = 3$.

Theorem 5. *Let $1/4 \leq \beta < 1/3$. Let s and t be integers with $2 \leq t < s \leq 4$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$\frac{k_s(G)}{g_s(\beta)n^s} \geq \frac{k_t(G)}{g_t(\beta)n^t}.$$

Moreover, the following three statements are equivalent:

(i) Equality holds for some $2 \leq t < s \leq 4$.

(ii) Equality holds for all $2 \leq t < s \leq 4$.

(iii) The pair (n, β) is feasible and G is a member of $\mathcal{G}(n, \beta)$.

5.1 Basic inequalities for heavy cliques

When $p = 2$ (see Section 3.3), we tackled the heavy cliques by considering (3.8), that is, $\sum_{e \in E(T)} D_-(e) \geq 2 - 3\beta$ for $T \in \mathcal{K}_3$. The contributions of the heavy cliques are completely removed when we sum (3.8) over all triangles T. Therefore, we need to generalise (3.8) for a general clique S and $p \geq 3$. Clearly the left hand side of the desired inequality would be $\sum D_-(T)$ with sum over subcliques T in S. Notice that for $p = 2$, D_- is the zero function on triangles. Hence, (3.8) is equivalent to $\sum D_-(e) \geq 2 - 3\beta + D_-(T)$. Thus, a natural generalisation of (3.8) is to replace the function D with D_- in Lemma 4.1.2.

Lemma 5.1.1. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let s and t be integers with $2 \leq t < s \leq p+1$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$\sum_{T \in \mathcal{K}_t(S)} D_-(T) \geq (1-\beta)s\binom{s-2}{t-1} - (t-1)\binom{s-1}{t} + \binom{s-2}{t-2} D_-(S).$$

for $S \in \mathcal{K}_s$.

Proof. Since $D_+(S) \geq D_+(T)$ for every $T \in \mathcal{K}_t(S)$ by Lemma 4.1.1 (*iii*), there is nothing to prove by Lemma 4.1.2 if there are at most $\binom{s-2}{t-2}$ heavy t-cliques in S. Now suppose there are more than $\binom{s-2}{t-2}$ heavy t-cliques in S. In particular, S contains a heavy t-clique, so S is itself heavy with $D_-(S) = (p+1-s)\beta$ by Lemma 4.1.1 (*iv*). Thus, the right hand side of the inequality is $\binom{s}{t}(1 - t\beta) + \binom{s-2}{t-2}((p+1)\beta - 1)$. By Lemma 4.1.1 (*i*) we have that $D_-(T) \geq (1 - t\beta)$ for $T \in \mathcal{K}_t(S)$. Furthermore, by Lemma 4.1.1 (*iv*) $D_-(T) = (p - t + 1)\beta$ if T is heavy, so summing $D_-(T)$ over $T \in \mathcal{K}_t(S)$ gives

$$\begin{aligned}\sum_{T \in \mathcal{K}_t(S)} D_-(T) &\geq k_t^+(S)(p-t+1)\beta + \left(\binom{s}{t} - k_t^+(S)\right)(1-t\beta) \\ &= \binom{s}{t}(1-t\beta) + k_t^+(S)((p+1)\beta - 1).\end{aligned}$$

This completes the proof of the lemma. □

The following notation will help us to keep the calculations simple. For $2 \leq$

$t \le p$, we define the function $\widetilde{D} : \mathcal{K}_{t+1} \to \mathbb{R}$ such that

$$\widetilde{D}(S) = \sum_{T \in \mathcal{K}_t(S)} D_-(T) - \Big(2 - (t+1)\beta + (t-1)D_-(S)\Big)$$

for $S \in \mathcal{K}_{t+1}$. Hence, Lemma 5.1.1 has the following corollary for $s = t+1$.

Corollary 5.1.2. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let t be integer with $2 \le t \le p$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then $\widetilde{D}(S) \ge 0$ for $S \in \mathcal{K}_{t+1}$.* □

Next, we sum $\widetilde{D}(S)$ over $S \in \mathcal{K}_{t+1}$ and bound it from above. The proof is an application of Proposition 3.3.1, which is similar to the proof of Lemma 4.2.1.

Lemma 5.1.3. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let t be an integer with $2 \le t \le p$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$\sum_{S \in \mathcal{K}_{t+1}} \widetilde{D}(S) \le \big(t - 1 + (p - 2t + 2)(t+1)\beta\big)k_{t+1} + (t-1) \sum_{S \in \mathcal{K}_{t+1}} D_+(S)$$
$$-(1 - t\beta)(p - t + 1)\beta n k_t - (t-1)(t+2)\frac{k_{t+2}}{n} - (1 - t\beta)n \sum_{T \in \mathcal{K}_t} D_+(T).$$

Moreover, equality holds if and only if for each $T \in \mathcal{K}_t$, either $D_-(T) = 1 - t\beta$ or $D_-(T) = (p - t + 1)\beta$.

Proof. Notice that the sum $\widetilde{D}(S)$ over $S \in \mathcal{K}_{t+1}$ is equal to

$$\sum_{S \in \mathcal{K}_{t+1}} \sum_{T \in \mathcal{K}_t(S)} D_-(T) - (2 - (t+1)\beta)k_{t+1} - (t-1) \sum_{S \in \mathcal{K}_{t+1}} D_-(S). \tag{5.1}$$

We look at each term separately. Recall that $D = D_- + D_+$, so

$$\sum D_-(S) = \sum D(S) - \sum D_+(S) = (t+2)\frac{k_{t+2}}{n} - \sum D_+(S).$$

By interchanging the order of summations, $\sum \sum D_-(T) = n \sum D_-(T) D(T)$ with sum over $T \in \mathcal{K}_t$. Hence, by Proposition 3.3.1 taking $\mathcal{A} = \mathcal{K}_t$, $f = D_-$, $g = D$,

$m = 1 - t\beta$ and $M = (p - t + 1)\beta$, we obtain

$$\begin{aligned}
&n \sum_{T \in \mathcal{K}_t} D_-(T) D(T) \\
\leq &(1 - t\beta) n \sum_{T \in \mathcal{K}_t} D_-(T) + (p - t + 1)\beta n \sum_{T \in \mathcal{K}_t} D(T) - (1 - t\beta)(p - t + 1)\beta n k_t \\
= &(1 + (p - 2t + 1)\beta) n \sum_{T \in \mathcal{K}_t} D(T) - (1 - t\beta) n \sum_{T \in \mathcal{K}_t} D_+(T) \\
&- (1 - t\beta)(p - t + 1)\beta n k_t \\
= &(1 + (p - 2t + 1)\beta)(t + 1) k_{t+1} - (1 - t\beta) n \sum_{T \in \mathcal{K}_t} D_+(T) \\
&- (1 - t\beta)(p - t + 1)\beta n k_t.
\end{aligned}$$

Hence, substituting these identities back into (5.1), we obtain the desired inequality in the lemma.

By Proposition 3.3.1, equality holds if and only if for each $T \in \mathcal{K}_t$, either $D(T) = 1 - t\beta$ or $D_-(T) = (p - t + 1)\beta$. □

By Corollary 5.1.2, $\sum_{S \in \mathcal{K}_{t+1}} \widetilde{D}(S) \geq 0$. Together with Lemma 5.1.3, we obtain that for $2 \leq t \leq p + 1$,

$$\begin{aligned}
&\big(t - 1 + (p - 2t + 2)(t + 1)\beta\big) k_{t+1} + (t - 1) \sum_{S \in \mathcal{K}_{t+1}} D_+(S) \geq \\
&(1 - t\beta)(p - t + 1)\beta n k_t + (t - 1)(t + 2)\frac{k_{t+2}}{n} + (1 - t\beta) n \sum_{T \in \mathcal{K}_t} D_+(T). \qquad (5.2)
\end{aligned}$$

In fact, the above inequality implies Lemma 4.2.1 as D_+ is the zero function for K_{p+2}-free graphs. Following the proof of Theorem 4, our next task is to solve for $k_3, \ldots, k_{p+1}$. In the next section, we look at the case when $p = 3$. However, we are going to see that (5.2) is not sufficient to prove Conjecture 3.1.1.

5.2 $k_r(n,\delta)$ for $2n/3 < \delta \leq 3n/4$

From now on, p is assumed to be 3, so $1/4 \leq \beta < 1/3$ (that is $2n/3 < \delta \leq 3n/4$). Explicitly, (5.2) becomes

$$(1+3\beta)k_3 + \sum_{T\in\mathcal{K}_3} D_+(T) \geq 2(1-2\beta)\beta nk_2 + 4k_4/n + (1-2\beta)n\sum_{e\in\mathcal{K}_2} D_+(e), \tag{5.3}$$

$$(2-4\beta)k_4 + 2\sum_{S\in\mathcal{K}_4} D_+(S) \geq (1-3\beta)\beta nk_3 + 10k_5/n + (1-3\beta)n\sum_{T\in\mathcal{K}_3} D_+(T), \tag{5.4}$$

for $t = 2$ and $t = 3$ respectively. Since D_- is a zero function on 4-cliques, $\sum_{S\in\mathcal{K}_4} D_+(S) = \sum_{S\in\mathcal{K}_4} D(S) = \sum_{S\in\mathcal{K}_4} D(S) = 5k_5/n$. Hence, the terms with k_5 and $\sum D_+(S)$ cancel in (5.4). Since $(1-2\beta) > 0$, we may ignore the term with $\sum D_+(e)$ in (5.3). Substituting (5.4) into (5.3), we get

$$\begin{aligned}(1+3\beta)k_3 + \sum_{T\in\mathcal{K}_3} D_+(T) \geq{}& 2(1-2\beta)\beta nk_2 \\ &+ \frac{4}{(2-4\beta)n}\left((1-3\beta)\beta nk_3 + (1-3\beta)n\sum_{T\in\mathcal{K}_3} D_+(T)\right) \\ (1-\beta)k_3 \geq{}& 2(1-2\beta)^2\beta nk_2 - (4\beta-1)\sum_{T\in\mathcal{K}_3} D_+(T). \end{aligned} \tag{5.5}$$

Recall that $g_2(\beta) = (1-\beta)/2$ and $g_3(\beta) = (1-2\beta)^2\beta$. If $(4\beta-1) \geq 0$, then (5.5) would imply

$$k_3(G) \geq 2(1-2\beta)^2\beta nk_2(G)/(1-\beta) \geq (1-2\beta)^2\beta n^3 = g_3(\beta)n^3,$$

which prove Conjecture 3.1.1 for $1/4 \leq \beta < 1/3$ and $r = 3$. However, $(4\beta-1) \geq 0$ only if $\beta = 1/4$. Thus, the natural extension of (3.8) is not sufficient even for the case $p = 3$.

Therefore, we are going to strengthen both (5.3) and (5.4). The methods used to strengthen (5.3) and (5.4) are different. Thus, we will look at them separately.

5.2.1 Strengthening (5.3)

Recall that (5.3) is a consequence of Corollary 5.1.2 and Lemma 5.1.3 for $t = 2$. Therefore, a strengthening of Corollary 5.1.2 would lead to a strengthening of (5.3). For $t = 2$ and $p = 3$, Corollary 5.1.2 explicitly states that

$$\sum_{e \in \mathcal{K}_2(T)} D_-(e) \geq 2 - 3\beta + D_-(T) \tag{5.6}$$

for $T \in \mathcal{K}_3$. Even though we already know that the above inequality is not sufficient to prove the Conjecture 3.1.1 for $p = 3$, they are indeed the best possible by considering $\mathcal{G}(n, \beta)$. Therefore, a strengthening of Corollary 5.1.2 must involve D_+.

Note that the coefficient of $\sum D_+(e)$ term on the right hand side of (5.3) is $(1 - 2\beta) > 0$. Next, observe that

$$\sum_{e \in \mathcal{K}_2} D_+(e)n = \sum_{T \in \mathcal{K}_3} \sum_{e \in \mathcal{K}_2(T)} D_+(e)/D(e).$$

Hence, in order to exploit the $\sum D_+(e)$ term, we are going to prove

$$\widetilde{D}(T) + (1 - 2\beta) \sum_{e \in \mathcal{K}_2(T)} D_+(e)/D(e) \geq c(4\beta - 1)D_+(T)/(1 - 2\beta),$$

for some constant $c > 0$. Notice that for a heavy edge e, $D(e) = D_+(e) + 2\beta$. Thus, $D_+(e)/D(e)$ is equivalent to $D_+(e)/(D_+(e) + 2\beta)$ for all edges e.

Suppose we have proved that the above inequality is true for $c = 1$. Hence, we obtain a strengthening of (5.3). It turns out that if we substitute this strengthening of (5.3) into (5.4), we would obtain (5.5) without the $\sum D_+(T)$ terms on the right hand side. Thus, Conjecture 3.1.1 is true for $p = 3$ without needing to strengthen (5.4). Unfortunately, we are only able to prove it for the case $c = 1 - 2/(29 - 75\beta) < 1$.

Lemma 5.2.1. *Let $1/4 \leq \beta < 1/3$. Suppose G is a graph order n with minimum* *degree* $(1-\beta)n$. *Let* $T \in \mathcal{K}_3$. *Then*

$$\widetilde{D}(T) + (1-2\beta)\sum_{e\in\mathcal{K}_2(T)} \frac{D_+(e)}{D_+(e)+\beta} \ge c\frac{(4\beta-1)D_+(T)}{1-2\beta}. \tag{5.7}$$

with $c = 1 - 2/(29-75\beta)$. *Moreover, equality holds only if* T *is not heavy and* $d(v) = (1-\beta)n$ *for all* $v \in T$.

Proof. Corollary 5.1.2 gives $\widetilde{D}(T) \ge 0$, so we may assume that T is heavy. In addition, Corollary 4.1.3 implies that

$$\widetilde{D}(T) + \sum_{e\in\mathcal{K}_2(T)} D_+(e) \ge D_+(T). \tag{5.8}$$

Since $(4\beta-1)/(1-2\beta) < 1$, we may further assume that T contains at least one heavy edge, or else (5.7) holds as (5.8) becomes $\widetilde{D}(T) \ge D_+(T)$. Let $e_0 \in \mathcal{K}_2(T)$ with $D(e_0) = \max\{D(e) : e \in \mathcal{K}_2(T)\}$. By substituting (5.8) into (5.7), it is sufficient to show that the function

$$f = \left(1 - \frac{1-2\beta}{D_+(e_0)+2\beta}\right)\widetilde{D}(T) - \left(c\frac{(4\beta-1)}{1-2\beta} - \frac{1-2\beta}{D_+(e_0)+2\beta}\right)D_+(T)$$

is non-negative.

First consider the case when $D_+(T) \le 1-3\beta$. Lemma 4.1.1 (*ii*) implies $D_+(e_0) \le D_+(T) \le 1-3\beta$. Hence,

$$\begin{aligned}\frac{1-2\beta}{D_+(e_0)+2\beta} - c\frac{(4\beta-1)}{1-2\beta} \ge& \frac{1-2\beta}{1-\beta} - c\frac{(4\beta-1)}{1-2\beta} \\ =& \frac{(1-3\beta)(56-233\beta+200\beta^2)}{(1-\beta)(1-2\beta)(29-75\beta)} > 0.\end{aligned}$$

Also, $1-2\beta \le 2\beta$. Therefore, $f > 0$ by considering the coefficients of $\widetilde{D}(T)$ and $D(T)$.

Hence, we may assume $D_+(T) > 1-3\beta$. Since T is heavy, $D_-(T) = \beta$. Therefore, by the definition of $\widetilde{D}$, we have

$$\widetilde{D}(T) = \sum D_-(e) - 2(1-\beta). \tag{5.9}$$

We split into different cases separately depending on the number of heavy edges in T.

Suppose all edges are heavy. Thus, $\widetilde{D}(T) = 2(4\beta - 1)$ by (5.9), because $D_-(e) = 2\beta$ for all edges e in T. Clearly $D_+(T) = D(T) - \beta \leq 1 - \beta$. Hence, (5.7) is true as

$$\widetilde{D}(T) = 2(4\beta - 1) \geq (4\beta - 1)(1 - \beta)/(1 - 2\beta) \geq (4\beta - 1)D_+(T)/(1 - 2\beta).$$

Thus, there exists an edge in T that is not heavy. We now show that $D_+(T) \leq \beta$. If not, Lemma 4.1.1 (*iii*) implies that $D_+(e) \geq D_+(T) - \beta > 0$ for all edges e in T, which is a contradiction. Since $D_+(T) \leq \beta$,

$$D(e_0) = D_-(e_0) + D_+(e_0) \leq 2\beta + D_+(T) \leq 3\beta.$$

Suppose T contains one or two heavy edges. We are going to show that in both cases

$$\widetilde{D}(T) \geq 2(D_+(T) - (1 - 3\beta)).$$

First assume that there is exactly one heavy edge in T. Let e_1 and e_2 be the two non-heavy edges in T. Note that $D_-(e_i) = D(e_i) \geq D(T) = D_+(T) + \beta > 1 - 2\beta$ for $i = 1, 2$. Thus, (5.9) and Lemma 4.1.1 imply that $\widetilde{D}(T) \geq 2(D_+(T) - (1 - 3\beta))$. Assume that T contains two heavy edges. Let e_1 be the non-heavy edge in T. Similarly, we have $D_-(e_1) \geq D_+(T) + \beta > 1 - 2\beta$. Recall that $D_+(T) \leq \beta$, so (5.9) and Lemma 4.1.1 imply

$$\begin{aligned}\widetilde{D}(T) \geq& (4\beta + D_+(T) - (1 - 3\beta)) \\ =& 4\beta - 1 + D_+(T) - (1 - 3\beta) \geq 2(D_+(T) - (1 - 3\beta)).\end{aligned}$$

Since $\widetilde{D}(T) \geq 2(D_+(T) - (1 - 3\beta))$, in proving (5.7), it is enough to show that

$$\begin{aligned}D(e_0)f \geq& 2(D_+(e_0) + 4\beta - 1)(D_+(T) - (1 - 3\beta)) \\ &- \left(c\frac{(4\beta - 1)}{1 - 2\beta}(D_+(e_0) + 2\beta) - (1 - 2\beta)\right) D_+(T) \qquad (5.10)\end{aligned}$$

is non-negative for $0 < D_+(e_0) \leq D_+(T)$ and $1 - 3\beta \leq D_+(T) \leq \beta$. Notice that for a fixed $D_+(T)$ it is enough to check the boundary points of $D_+(e_0)$. For $D_+(e_0) = 0$, we have

$$\begin{aligned}
D(e_0)f \geq& 2(4\beta-1)(D_+(T)-(1-3\beta)) + \frac{(1-3\beta)(29-50\beta-100\beta^2)}{(29-75\beta)(1-2\beta)}D_+(T)\\
=&\frac{1500\beta^3-1314\beta^2+361\beta-29}{(1-2\beta)(29-75\beta)}D_+(T) - 2(4\beta-1)(1-3\beta)\\
\geq&\frac{1500\beta^3-1314\beta^2+361\beta-29}{(1-2\beta)(29-75\beta)}(1-3\beta) - 2(4\beta-1)(1-3\beta)\\
=&\frac{(1-3\beta)^2(29-50\beta-100\beta^2)}{(1-2\beta)(29-75\beta)} > 0.
\end{aligned}$$

For $D_+(e_0) = D_+(T)$,

$$\begin{aligned}
D(e_0)f \geq& 2(D_+(T)+4\beta-1)(D_+(T)-(1-3\beta))\\
&-\left(\left(1-\frac{2}{29-75\beta}\right)\frac{(4\beta-1)}{1-2\beta}(2\beta+D_+(T)) - (1-2\beta)\right)D_+(T).
\end{aligned}$$

Notice that this is a quadratic function in $D_+(T)$. Moreover, the coefficients of $D_+(T)^2$ and $D_+(T)$ are $(600\beta^2-449\beta+85)/(29-75\beta)(1-2\beta)$ and $3(4\beta-1)(200\beta^2-151\beta+29)/(1-2\beta)(29-75\beta)$ respectively. More importantly, they are both positive. Thus, it enough to check for $D_+(T) = 1-3\beta$.

For $D(T)_+ = 1-3\beta$, we have

$$D(e_0)f \geq \frac{(1-3\beta)^2(200\beta^2-233\beta+56)}{(29-75\beta)(1-2\beta)} > 0.$$

Hence, we have proved (5.7).

It is easy to check that if equality holds in (5.7) then $D_+(T) = 0$. Thus, for all edges e in T, $D_+(e) = 0$ by Lemma 4.1.1. Furthermore, equality holds in (5.8), so equality holds in Corollary 4.1.3 as $D_+(T) = 0 = D_+(e)$. Hence, $d(v) = (1-\beta)n$ for $v \in S$. This completes the proof of the lemma. □

Now we sum (5.7) over all $T \in \mathcal{K}_3$. After rearrangement, we have

$$\sum_{T\in\mathcal{K}_3}\widetilde{D}(T) \geq \left(1-\frac{2}{29-75\beta}\right)\frac{(4\beta-1)}{1-2\beta}\sum_{T\in\mathcal{K}_3}D_+(T) - (1-2\beta)n\sum_{e\in\mathcal{K}_2}D_+(e).$$

The left hand side is bounded above by Lemma 5.1.3 with $t = 2$. Thus, we obtain a strengthening of (5.3).

Corollary 5.2.2. *Let $1/4 \leq \beta < 1/3$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$(1+3\beta)k_3 + \frac{2}{1-2\beta}\left(1-3\beta+\frac{4\beta-1}{29-75\beta}\right)\sum_{T\in\mathcal{K}_3} D_+(T) \geq 2(1-2\beta)\beta n k_2 + 4\frac{k_4}{n}$$

holds. Moreover, if equality holds, then G is $(1-\beta)n$-regular and for each edge e, either we have $D(e) = 1-2\beta$ or $D(e) = 2\beta$. □

As mentioned before, if Lemma 5.2.1 holds for $c = 1$, then Conjecture 3.1.1 is true for $p = 3$ without needing to strengthen (5.4). However, for $\epsilon > 0$, it is easy to construct graphs which contain a triangle T with $D_+(T) = 1 - 3\beta + \epsilon$ and edge degrees $1-2\beta+\epsilon$, $1-2\beta+\epsilon$ and $2\beta+\epsilon$ respectively. For $c = 1$, (5.10) becomes

$$D(e_0)f \geq -(4\beta^2 + 2\beta^2 - 1)(1-3\beta)/(1-2\beta) + o(\epsilon).$$

Hence, for ϵ sufficiently small, the right hand is greater than zero only if $\beta \leq (\sqrt{5}-1)/4 \approx 0.309 < 2/3$. It can be checked that for $1/4 \leq \beta \leq (\sqrt{5}-1)/4$, the proof of Lemma 5.2.1 also holds with $c = 1$. Thus, we can prove Theorem 5 for $1/4 < \beta < (\sqrt{5}-1)/4$.

Surprisingly, it is easier to prove Conjecture 3.1.1 for $\delta = (3/4-\epsilon)n$ than for $\delta = (2/3+\epsilon)n$ for small $\epsilon > 0$. To see this, suppose that we can prove Conjecture 3.1.1 for $p = 3$ providing for each $T \in \mathcal{K}_3$, we can approximate $\sum_{e\in\mathcal{K}_2(T)} D_-(e)$ accurately, say within an absolute error. By definition of D_-, $1-2\beta \leq D_-(e) \leq 2\beta$ for all edges e. Notice that the interval is of size $4\beta-1$, which tends to zero as β tends to $1/4$. Thus, approximating $\sum_{e\in\mathcal{K}_2(T)} D_-(e)$ accurately for $\beta = 1/3 - \epsilon$ is likely to be more difficult than for $\beta = 1/4+\epsilon$, where $\epsilon > 0$ sufficiently small.

5.2.2 Strengthening (5.4)

Next, we are going to strengthen (5.4). Note that by mimicking the proof of Lemma 5.2.1, we could obtain a strengthening of Corollary 5.1.2 for $t = 3$. It would lead to a strengthening of (5.4). However, it is still not sufficient to prove the Conjecture 3.1.1 when β is close to 1/3.

Fortunately, we are able to prove the following strengthening of (5.4). The proof requires a detailed analysis of $\mathcal{K}_5$, so it is postponed to Section 5.3.

Theorem 5.2.3. *Let $1/4 \leq \beta < 1/3$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$(2-4\beta)k_4 \geq (1-3\beta)\beta n k_3 + \left(1 - 3\beta + \frac{4\beta - 1}{29 - 75\beta}\right) n \sum_{T \in \mathcal{K}_3} D_+(T). \tag{5.11}$$

Moreover, if equality holds, then (n,β) is feasible and $G \in \mathcal{G}(n,\beta)$.

5.2.3 Proof of Conjecture 3.1.1 for $1/4 \leq \beta < 1/3$

By using the two strengthened versions of (5.3) and (5.4), that is, Corollary 5.2.2 and Theorem 5.2.3, we prove Theorem 5.

Proof of Theorem 5. Note that in proving the inequality in Theorem 5, it is sufficient to prove the case when $s = t+1$. Recall that $p = 3$ as $1/4 \leq \beta < 1/3$, so

$$g_2(\beta) = (1-\beta)/2,\ g_3(\beta) = (1-2\beta)^2\beta \text{ and } g_4 = (1-2\beta)(1-3\beta)\beta^2/2.$$

Theorem 5.2.3 states that $(2-4\beta)k_4 \geq (1-3\beta)\beta n k_3$. This implies $k_4/g_4(\beta)n^4 \geq k_3/g_3(\beta)n^3$ by (4.7) with $t = 3$. Hence, the theorem is true for $t = 3$.

For $t = 2$, by substituting Corollary 5.2.2 into Theorem 5.2.3, we obtain

$$\begin{aligned}(1+3\beta)k_3 + \frac{2}{1-2\beta}\left(1 - 3\beta + \frac{4\beta-1}{29-75\beta}\right)\sum_{T\in\mathcal{K}_3} D_+(T) \geq 2(1-2\beta)\beta n k_2 \\ + \frac{4}{(2-4\beta)n}\left((1-3\beta)\beta n k_3 + \left(1 - 3\beta + \frac{4\beta-1}{29-75\beta}\right) n \sum D_+(T)\right).\end{aligned}$$

Observe that the $\sum D_+(T)$ terms on both sides cancel. Hence, after rearrangement, we have $(1-\beta)k_3 \geq 2(1-2\beta)^2\beta n k_2$. Thus, $k_3/g_3(\beta)n^4 \geq k_2/g_2(\beta)n^3$.

This is clear that (iii) implies (i) and (ii) by the construction of $\mathcal{G}(n,\beta)$ and the feasibility of (n,β). Suppose (i) holds, so equality holds for $t=t_0$ and $s=s_0$ with $t_0<s_0$. Since equality holds for $t_0<s_0$, equality holds for $t=t_0,\dots,s_0-1$ and $s=s_0$. Hence, we may assume that $s=t+1$. In both cases, we have equality in Theorem 5.2.3. Hence, (n,β) is feasible and $G\in\mathcal{G}(n,\beta)$. □

5.3 Proof of Theorem 5.2.3

In this section, T, S and U always denote a 3-clique, 4-clique and 5-clique respectively. First, we establish some basic facts almost T, S and U, which we use in the proof. Observe that $D_-(S)=0$ for $S\in\mathcal{K}_4$, so $D_+(S)=D(S)$. Recall that $\widetilde{D}(S)=\sum_{T\in\mathcal{K}_3(S)}D_-(T)-(2-4\beta)$. Let $T_1,\dots,T_4$ be triangles in S with $D(T_i)\leq D(T_{i+1})$ for $1\leq i\leq 3$. Since $D_-(T)\leq\beta$, we have

$$\widetilde{D}(S)=\begin{cases}2(4\beta-1) & \text{if } k_3^+(S)=4,\\ 4\beta-1+(D(T_1)-(1-3\beta)) & \text{if } k_3^+(S)=3,\\ D(T_1)+D(T_2)-2(1-3\beta) & \text{if } k_3^+(S)=2,\end{cases} \tag{5.12}$$

where $k_3^+(S)$ is the number of heavy triangles in S. Also recall that $D(T)\geq 1-3\beta$ by Lemma 4.1.1 (i). We will often make reference to these formulae throughout this section.

Our first aim is to prove a result corresponding to Lemma 5.2.1 for the sum of the degrees of triangles in a 4-clique. Define the function $\eta:\mathcal{K}_4\to\mathbb{R}$ to be

$$\eta(S)=\widetilde{D}(S)-\frac{4\beta-1}{29-75\beta}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+\beta}$$

for $S\in\mathcal{K}_4$. Recall that for a heavy triangle T, $D(T)=D_+(T)+\beta$. Thus, only heavy 3-cliques in S contribute to $\sum D_+(T)/(D_+(T)+\beta)$. Ideally, we would like $\eta(S)\geq 0$ for all $S\in\mathcal{K}_4$. This would imply $\sum_{S\in\mathcal{K}_4}\eta(S)\geq 0$. Therefore, we would have

$$
\begin{aligned}
0 \leq \sum_{S\in\mathcal{K}_4} \eta(S) = &\sum_{S\in\mathcal{K}_4} \widetilde{D}(S) - \frac{4\beta-1}{29-75\beta} n \sum_{T\in\mathcal{K}_3} D_+(T) \\
\leq &(2-4\beta)k_4 - (1-3\beta)\beta n k_3 - \left(1-3\beta+\frac{4\beta-1}{29-75\beta}\right) n \sum_{T\in\mathcal{K}_3} D_+(T) \\
&+ 2\sum_{S\in\mathcal{K}_4} D_+(S) - 10k_5/n,
\end{aligned}
$$

where the last inequality is due to Lemma 5.1.3 with $t = 3$. Observe that $\sum_{S\in\mathcal{K}_4} D_+(S) = \sum_{S\in\mathcal{K}_4} D(S) = 5k_5/n$, so the terms with $\sum D_+(S)$ and k_5/n cancel. Rearranging the inequality, we obtain the inequality in Theorem 5.2.3. Actually, it is enough to show that $\sum_{S\in\mathcal{K}_4} \eta(S) \geq 0$.

Suppose $\sum_{S\in\mathcal{K}_4} \eta(S) < 0$. Then, there exists a 4-clique S with $\eta(S) < 0$. Such a 4-clique is called *bad*, otherwise it is called *good*. The sets of bad and good 4-cliques are denoted by $\mathcal{K}_4^{bad}$ and $\mathcal{K}_4^{good}$ respectively. In the next claim, we identify the structure of a bad 4-clique.

Claim 5.3.1. *Suppose S is a bad 4-clique. Let*

$$\Delta = (1-3\beta)(1+\epsilon) \text{ and } \epsilon = (4\beta-1)/(150\beta^2-137\beta+30).$$

Then, the following hold

(i) S contains exactly one heavy edge and two heavy triangles,

(ii) $0 < D(S) < \Delta$,

(iii) $D(T) + D(T') < 2\Delta$, where T and T' are the two non-heavy triangles in S.

Proof. Let $T_1, \ldots, T_4$ be triangles in S with $D(T_i) \leq D(T_{i+1})$ for $1 \leq i \leq 3$. We may assume that $D_+(T_4) > 0$, otherwise S is good by Corollary 5.1.2 as $\eta(S) = \widetilde{D}(S) \geq 0$. Hence, S is also heavy by Lemma 4.1.1. We separate cases by the number of heavy triangles in S.

First, suppose all triangles are heavy. Hence, $\widetilde{D}(S) = 2(4\beta - 1)$ by (5.12).

Clearly, $D_+(T_i) \le 1-\beta$ for $1 \le i \le 4$, so

$$\begin{aligned}\eta(S) \ge& 2(4\beta-1) - \frac{4\beta-1}{29-75\beta}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+\beta}\\ \ge& 2(4\beta-1)\left(1-\frac{2(1-\beta)}{29-75\beta}\right) = \frac{2(4\beta-1)(27-73\beta)}{29-75\beta} \ge 0.\end{aligned}$$

This contradicts the assumption that S is bad. Thus, not all triangles in S are heavy. By Lemma 4.1.1 (*iii*), we see that $D_+(T) \ge D_+(S) - \beta = D(S) - \beta$ for $T \in \mathcal{K}_3(S)$, so $0 < D(S) \le \beta$ or else all $T \in \mathcal{K}_3(S)$ are heavy. Also, $D_+(T) \le D_+(S) = D(S) \le \beta$.

Suppose all but one triangles are heavy, so $\widetilde{D}(S) \ge 4\beta - 1$ by (5.12). Hence,

$$\begin{aligned}\eta(S) \ge& 4\beta-1 - \frac{4\beta-1}{29-75\beta}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+\beta} \ge (4\beta-1)\left(1-\frac{3}{29-75\beta}\frac{D(S)}{D(S)+\beta}\right)\\ \ge& (4\beta-1)\left(1-\frac{3}{2(29-75\beta)}\right) = \frac{5(4\beta-1)(11-30\beta)}{2(29-75\beta)} \ge 0.\end{aligned}$$

Suppose there is only one heavy triangle, T_4, in S. Corollary 4.1.3 implies that $\widetilde{D}(S) + D_+(T_4) \ge 2D_+(S) = 2D(S)$. Note that $D_+(T_4) \le D_+(S) = D(S)$, so $\widetilde{D}(S) \ge D(S)$. Thus,

$$\begin{aligned}\eta(S) \ge& D(S) - \frac{4\beta-1}{29-75\beta}\frac{D_+(T_4)}{D_+(T_4)+\beta} \ge D(S) - \frac{4\beta-1}{29-75\beta}\frac{D(S)}{D(S)+\beta}\\ =& \left(1-\frac{4\beta-1}{(29-75\beta)(D(S)+\beta)}\right)D(S) \ge \left(1-\frac{4\beta-1}{(29-75\beta)\beta}\right)D(S) > 0.\end{aligned}$$

Hence, S has exactly two heavy triangles, namely T_3 and T_4. If $D(S) \ge \Delta$, then

$$\begin{aligned}\eta(S) =& D(T_1) + D(T_2) - 2(1-3\beta) - \frac{4\beta-1}{29-75\beta}\left(\frac{D_+(T_3)}{D_+(T_3)+\beta} + \frac{D_+(T_4)}{D_+(T_4)+\beta}\right)\\ \ge& 2(D(S) - (1-3\beta)) - \frac{2(4\beta-1)}{29-75\beta}\frac{D(S)}{D(S)+\beta}\\ >& 2(1-3\beta)\epsilon - \frac{2(4\beta-1)}{29-75\beta}\frac{\Delta}{\Delta+\beta} \ge 2(1-3\beta)\epsilon - \frac{2(4\beta-1)\Delta}{(29-75\beta)(1-2\beta)} = 0.\end{aligned}$$

Thus, $D(S) < \Delta$. If $D(T_1) + D(T_2) \geq 2\Delta$, then $\widetilde{D}(S) \geq 2(\Delta - (1-3\beta)) = 2(1-3\beta)\epsilon$ by (5.12). Moreover, since $D_+(T_i) \leq D(S) < \Delta$ for $i = 3, 4$,

$$\eta(S) > 2(1-3\beta)\epsilon - \frac{2(4\beta-1)}{29-75\beta}\frac{\Delta}{\Delta+\beta} \geq 2(1-3\beta)\epsilon - \frac{2(4\beta-1)\Delta}{(29-75\beta)(1-2\beta)} = 0.$$

Thus, (iii) is true.

We have shown that S contains two heavy triangles. Therefore, to prove (i), it is sufficient to prove that S contains exactly one heavy edge. A triangle containing a heavy edge is heavy by Lemma 4.1.1 (iii). Since S contains two heavy triangle, there is at most one heavy edge in S. It is enough to show that if S does not contain any heavy edge and $D(S) < \Delta$, then S is good, which is a contradiction. Assume that S contains no heavy edge. Let $e_i = T_i \cap T_4$ be an edge of T_4 for $i = 1, 2, 3$. We claim that $\widetilde{D}(S) \geq D_+(T_4)$. By Corollary 4.1.3 taking $S = T_4$ and $t = 2$, we obtain

$$\begin{aligned} D(e_1) + D(e_2) + D(e_3) \geq & 2 - 3\beta + D(T_4) \\ D(e_1) + D(e_2) \geq & 2 - 4\beta + D_+(T_4). \end{aligned}$$

as $D(e_3) \leq 2\beta$ and $D_-(T_4) = \beta$. By Lemma 4.1.1 (ii), we get

$$D(T_1) + D(T_2) \geq D(e_1) + D(e_2) - 2\beta \geq 2(1-3\beta) + D_+(T_4).$$

Hence, $\widetilde{D}(S) \geq D_+(T_4)$ by (5.12). Therefore,

$$\begin{aligned} \eta(S) \geq & D_+(T_4) - \frac{4\beta-1}{29-75\beta} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T)+\beta} \\ \geq & \left(1 - \frac{2(4\beta-1)}{(29-75\beta)(D_+(T_4)+\beta)}\right) D_+(T_4) \\ \geq & \left(1 - \frac{2(4\beta-1)}{(29-75\beta)\beta}\right) D_+(T_4) > 0 \end{aligned}$$

and so S is good, a contradiction. This completes the proof of the claim. □

Since a bad 4-clique S must be heavy, that is, $D(S) > 0$, it is contained in some 5-clique. A 5-clique is called *bad* if it contains at least one bad 4-clique. We denote $\mathcal{K}_5^{bad}$ to be the set of bad 5-cliques.

Notice that

$$2\sum_{T\in\mathcal{K}_3(U)} D(T) = \sum_{S\in\mathcal{K}_4(U)}\sum_{T\in\mathcal{K}_3(S)} D(T) \tag{5.13}$$

for a 5-clique U. For a 4-clique S, Corollary 4.1.3 states that

$$\sum_{T\in\mathcal{K}_3(S)} D(T) \geq 2-4\beta+2D(S) \geq 2-4\beta.$$

Therefore, this implies (5.13) is at least $5(2-4\beta)$. On the other hand, Lemma 4.1.2 taking $s=5$ and $t=3$ implies that

$$\sum_{T\in\mathcal{K}_3(U)} D(T) \geq 7-15\beta+3D(U) \geq 7-15\beta.$$

Hence, (5.13) is at least $2(7-15\beta) > 5(2-4\beta)$ as $1/4 \leq \beta < 1/3$. This observation suggests that summing $D(T)$ over $T\in\mathcal{K}_3(U)$ would give a better lower bound than summing $D(T)$ over $T\in\mathcal{K}_3(S)$ followed by summing over $S\in\mathcal{K}_4(U)$.

Recall that a 4-clique S lies in $D(S)n$ bad 5-cliques U. We define $\widetilde{\eta}(S)$ to be $\eta(S)/D(S)$ for $S\in\mathcal{K}_4$ with $D(S)>0$. Clearly for a fixed $S\in\mathcal{K}_4$ with $D(S)>0$, summing $\widetilde{\eta}(S)$ over $U\in\mathcal{K}_5$, which contains S, is exactly equal to $\eta(S)n$. Thus,

$$n\sum_{S\in\mathcal{K}_4}\eta(S) = \sum_{U\in\mathcal{K}_5}\sum_{S\in\mathcal{K}_4(U)}\widetilde{\eta}(S) + n\sum_{S\in\mathcal{K}_4:D(S)=0}\eta(S).$$

Recall that our aim is to show that $\sum_{S\in\mathcal{K}_4}\eta(S)\geq 0$. Since $D(S)=0$ implies that S is good, we have $\eta(S)\geq 0$. Hence, it is enough to show that $\sum_{S\in\mathcal{K}_4(U)}\widetilde{\eta}(S)\geq 0$ for each bad 5-clique U.

Now, we give a lower bound on $\widetilde{\eta}(S)$ for bad 4-cliques S. By Claim 5.3.1,

$$\eta(S) \geq -\frac{4\beta-1}{29-75\beta}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+\beta} \geq -\frac{2(4\beta-1)}{29-75\beta}\frac{D(S)}{D(S)+\beta}.$$

Hence,

$$\widetilde{\eta}(S) \geq -\frac{2(4\beta-1)}{(29-75\beta)(D(S)+\beta)} > -\frac{2(4\beta-1)}{(29-75\beta)\beta}. \tag{5.14}$$

Next, we are going to bound $D(S)$ above for $S \in \mathcal{K}_4(U)\backslash\mathcal{K}_4^{bad}$ and $U \in \mathcal{K}_5^{bad}$. Let $S^b \in \mathcal{K}_4^{bad}(U)$. Observe that $S \cap S^b$ is a 3-clique. Then, by Lemma 4.1.1 and Claim 5.3.1, we have

$$D(S) \leq D(S \cap S^b) = D_+(S \cap S^b) + \beta \leq D(S^b) + \beta < \Delta + \beta. \tag{5.15}$$

Recall that a bad 4-clique S contains a heavy edge by Claim 5.3.1 and hence so does a bad 5-clique U. We split $\mathcal{K}_5^{bad}$ into subcases depending on the number of heavy edges in U. The next claim studies the relationship between the number of heavy edges and bad 4-cliques in a bad 5-clique U.

Claim 5.3.2. *Suppose $U \in \mathcal{K}_5^{bad}$ with $h \geq 2$ heavy edges and b bad 4-cliques. Then $b \leq 2h/(h-1) = 2 + 2/(h-1)$. Moreover, if there exist two heavy edges sharing a common vertex, $b \leq 3$.*

Proof. Define H to be the graph induced by the heavy edges in U. Write u_S for the vertex in U not in $S \in \mathcal{K}_4(U)$. This defines a bijection between $V(U)$ and $\mathcal{K}_4(U)$. If S is bad, u_S is adjacent to all but one heavy edges by Claim 5.3.1. By summing the degrees of H, $2h = \sum_{S \in \mathcal{K}_4(U)} d(u_S) \geq b(h-1)$. Thus, $b \leq 2h/(h-1)$.

If there exist two heavy edges sharing a common vertex in H, then every bad 4-clique must miss one of the vertices of these two heavy edges. Hence, $b \leq 3$. □

Suppose $U \in \mathcal{K}_5^{bad}$ has at least two heavy edges, say e and e'. In the following two claims, we study the cases whether e and e' intersect or not respectively.

Claim 5.3.3. *Suppose U is a bad 5-clique and there exist two heavy edges e and e' in U sharing a common vertex. Then $\sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S) > 0$.*

Proof. Suppose U contains b bad 4-cliques. Clearly $b \leq 3$ by Claim 5.3.2. Notice that $\sum_{S \in \mathcal{K}_4^{bad}(U)} \widetilde{\eta}(S) > -b\gamma \geq -3\gamma$ by (5.14), where $\gamma = 2(4\beta-1)/(29-75\beta)\beta$. Observe that there are exactly two heavy 4-cliques containing both e and e'.

Therefore, it is enough to show that $\eta(S) \geq 3D(S)\gamma/2$ for each $S \in \mathcal{K}_4(U)$ containing both e and e'.

Since S contains two heavy edges, namely e and e', it contains at least three heavy triangles. Let S' be a 4-clique in U distinct to S. Then, for $T = S \cap S'$, $D_+(T) \leq \min\{D_+(S), D_+(S')\} \leq D_+(S') = D(S')$ by Lemma 4.1.1 (*iii*). In particular, if S' is bad, then $D_+(T) \leq D(S') < \Delta$ by Claim 5.3.1.

Suppose S contains exactly three heavy triangles. Since not all triangles in S are heavy, $D_+(S) \leq \beta$ by Lemma 4.1.1 (*iii*). Also, $\widetilde{D}(S) \geq 4\beta - 1$ by (5.12). Therefore, $\eta(S)$ is at least

$$\begin{aligned}
&4\beta - 1 - \frac{4\beta-1}{29-75\beta}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+\beta} \geq 4\beta - 1 - \frac{4\beta-1}{29-75\beta}\sum_{S'\in\mathcal{K}_4(U)\backslash S}\frac{D(S')}{D(S')+\beta}\\
>&4\beta - 1 - \frac{4\beta-1}{29-75\beta}\left(\frac{b\Delta}{\Delta+\beta} + \frac{3-b}{2}\right) \geq 4\beta - 1 - \frac{4\beta-1}{29-75\beta}\left(\frac{\Delta}{\Delta+\beta}+1\right)\\
\geq& 3\beta\gamma/2 \geq 3D(S)\gamma/2
\end{aligned}$$

as required.

Now suppose all triangles in S are heavy. Notice that $D(S) < \Delta + \beta$ by (5.15) and $\widetilde{D}(S) = 2(4\beta - 1)$ by (5.12). Hence,

$$\begin{aligned}
\eta(S) =& 2(4\beta-1) - \frac{4\beta-1}{29-75\beta}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+\beta}\\
>&2(4\beta-1)\left(1 - \frac{2(\Delta+\beta)}{(29-75\beta)(\Delta+2\beta)}\right)\\
>&3(\Delta+\beta)\gamma/2 > 3D(S)\gamma/2
\end{aligned}$$

as required. The proof of the claim is completed. □

Claim 5.3.4. *Suppose U is a bad 5-clique and there exist two vertex disjoint heavy edges e and e' in U. Then $\sum_{S\in\mathcal{K}_4(U)} \widetilde{\eta}(S) > 0$.*

Proof. By the previous claim, we may assume that U has exactly two heavy edges. Otherwise, there exist two heavy edges sharing a vertex. Hence, U has $b \leq 4$ bad 4-cliques by Claim 5.3.2. Clearly, $\sum_{S\in\mathcal{K}_4^{bad}(U)} \widetilde{\eta}(S) > -b\gamma$ by (5.14), where as before $\gamma = 2(4\beta - 1)/(29 - 75\beta)\beta$. Also, there is exactly one heavy 4-clique S

containing both e and e'. Therefore, it is sufficient to prove that $\eta(S) \geq bD(S)\gamma$.

Since S contains two disjoint heavy edges, all triangles in S are heavy by Lemma 4.1.1. Thus, $\widetilde{D}(S) = 2(4\beta - 1)$ by (5.12). Observe that $T = S \cap S'$ is a triangle for $S' \in \mathcal{K}_4(U)\backslash S$. Moreover, $D_+(T) \leq \min\{D_+(S), D_+(S')\} \leq D_+(S') = D(S')$ by Lemma 4.1.1 (*iii*). Hence

$$\begin{aligned}\eta(S) \geq & 2(4\beta - 1) - \frac{4\beta - 1}{29 - 75\beta} \sum_{S' \in \mathcal{K}_4(U)\backslash S} \frac{D(S')}{D(S') + \beta} \\ > & (4\beta - 1)\left(2 - \frac{1}{29 - 75\beta}\left(\frac{b\Delta}{\Delta + \beta} + \frac{(4-b)(\Delta + \beta)}{\Delta + 2\beta}\right)\right).\end{aligned}$$

Therefore, $\eta(S) - bD(S)\gamma$ is at least

$$\begin{aligned}&(4\beta - 1)\left(2 - \frac{1}{29 - 75\beta}\left(\frac{b\Delta}{\Delta + \beta} + \frac{(4-b)(\Delta + \beta)}{\Delta + 2\beta}\right)\right) - b(\Delta + \beta)\gamma \\ \geq & (4\beta - 1)\left(2 - \frac{4\Delta}{(29 - 75\beta)(\Delta + \beta)}\right) - 4(\Delta + \beta)\gamma > 0.\end{aligned}$$

This completes the proof. □

Recall that a bad 5-clique contains at least one heavy edge. Thus, we are left with the case $U \in \mathcal{K}_5^{bad}$ containing exactly one heavy edge.

Claim 5.3.5. *Suppose U is a bad 5-clique and there is exactly one heavy edge e in U. Then, U contains at most two bad 4-cliques. Moreover, $\sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S) > 0$.*

Proof. Let $u_1, \ldots, u_5$ be the vertices of U and u_4u_5 be the heavy edge. Write S_i and η_i to be $U - u_i$ and $\eta(S_i)$ respectively for $1 \leq i \leq 5$. Similarly write $T_{i,j}$ for $U - u_i - u_j$ for $1 \leq i < j \leq 5$. Recall that a bad 4-clique contains a heavy edge by Claim 5.3.1. Hence, S_i is a bad 4-clique only if $i \leq 3$. Without loss of generality, $S_1, \ldots, S_b$ are the bad 4-cliques in U.

Since S_3 contains a heavy edge, it contains at least 2 heavy triangles. First suppose that all triangles in S_3 are heavy. Thus, S_3 is not bad, so $b \leq 2$ and it is enough to show that $\eta_3 \geq 2\gamma D(S_3)$, where as before $\gamma = 2(4\beta - 1)/(29 - 75\beta)\beta$.

Notice that $D(S_3) < \Delta + \beta$ by (5.15) and $\widetilde{D}(S_3) = 2(4\beta - 1)$ by (5.12). Hence,

$$\begin{aligned}
\eta_3 =& 2(4\beta - 1) - \frac{4\beta - 1}{29 - 75\beta} \sum_{T \in \mathcal{K}_3(S_3)} \frac{D_+(T)}{D_+(T) + \beta} \\
\geq& 2(4\beta - 1)\left(1 - \frac{2D(S_3)}{(29 - 75\beta)(D(S_3) + \beta)}\right) \\
>& 2(4\beta - 1)\left(1 - \frac{2(\Delta + \beta)}{(29 - 75\beta)(\Delta + 2\beta)}\right) \geq 2(\Delta + \beta)\gamma > 2D(S_3)\gamma.
\end{aligned}$$

Second, suppose that S_3 contains exactly three heavy triangles. Once again, it is enough to show that $\eta_3 \geq 2\gamma D(S_3)$. If $D(S_3) \leq 1 - 3\beta$, then $\widetilde{D}(S_3) \geq 4\beta - 1$ by (5.12). Hence

$$\begin{aligned}
\eta_3 \geq& \widetilde{D}(S_3) - \frac{4\beta - 1}{29 - 75\beta} \sum_{T \in \mathcal{K}_3(S_3)} \frac{D_+(T)}{D_+(T) + \beta} \\
\geq& (4\beta - 1)\left(1 - \frac{3D(S_3)}{(29 - 75\beta)(D(S_3) + \beta)}\right) \\
>& (4\beta - 1)\left(1 - \frac{3(1 - 3\beta)}{(29 - 75\beta)(1 - 2\beta)}\right) \geq 2(1 - 3\beta)\gamma \geq 2D(S_3)\gamma.
\end{aligned}$$

So we may assume $D(S_3) > 1 - 3\beta$. Since not all triangles in S_3 are heavy, $D(S_3) = D_+(S_3) \leq \beta$ by Lemma 4.1.1 (*iii*). Also, $\widetilde{D}(S_3) \geq 7\beta - 2 + D(S_3)$ by (5.12) as $D(T) \geq D(S_3)$. Therefore,

$$\begin{aligned}
\eta_3 \geq& 7\beta - 2 + D(S_3) - \frac{3(4\beta - 1)}{29 - 75\beta} \frac{D(S_3)}{D(S_3) + \beta} \\
\geq& 7\beta - 2 + D(S_3)\left(1 - \frac{3(4\beta - 1)}{(29 - 75\beta)(1 - 2\beta)}\right) \\
\geq& 7\beta - 2 + (1 - 3\beta)\left(1 - \frac{3(4\beta - 1)}{(29 - 75\beta)(1 - 2\beta)}\right) \geq 2\beta\gamma \geq 2D(S_3)\gamma.
\end{aligned}$$

Hence, S_3 contains exactly two heavy triangles. By a similar argument, we may assume there are exactly two heavy triangles in S_i for $1 \leq i \leq 3$. Moreover, $D(S_i) < \beta$ for $1 \leq i \leq 3$, otherwise all triangles in S_i are heavy by

Lemma 4.1.1 (*iii*). For $1 \leq i \leq b$,

$$D(T_{i,4}) + D(T_{i,5}) < 2\Delta = 2(1-3\beta)(1+\epsilon)$$

by Claim 5.3.1 (*iii*). For $b < i \leq 3$, $\widetilde{D}(S_i) = D(T_{i,4}) + D(T_{i,5}) - 2(1-3\beta)$ by (5.12). Thus,

$$\begin{aligned}
D(T_{i,4}) + D(T_{i,5}) =& \eta_i + 2(1-3\beta) + \frac{4\beta-1}{29-75\beta} \sum_{T\in\mathcal{K}_3(S_i)} \frac{D_+(T)}{D_+(T)+\beta} \\
\leq& \eta_i + 2(1-3\beta) + \frac{2(4\beta-1)D(S_i)}{(29-75\beta)(D(S_i)+\beta)} \\
\leq& \eta_i + 2(1-3\beta) + \gamma\beta/2.
\end{aligned}$$

After applying Corollary 5.1.2 to S_4 and S_5 taking $t = 3$, and adding the two inequalities together, we obtain

$$\begin{aligned}
2(2-4\beta) \leq& \sum_{1\leq i\leq 3} (D_-(T_{i,4}) + D_-(T_{i,5})) + 2D_-(T_{4,5}) \\
\leq& \sum_{1\leq i\leq 3} (D_-(T_{i,4}) + D_-(T_{i,5})) + 2\beta \\
2(2-5\beta) \leq& \sum_{1\leq i\leq b} (D(T_{i,4}) + D(T_{i,5})) + \sum_{b<i\leq 3} (D(T_{i,4}) + D(T_{i,5})) \\
<& 2b(1-3\beta)(1+\epsilon) + \sum_{b<i\leq 3} \eta_i + (3-b)\left(2(1-3\beta) + \gamma\beta/2\right) \\
2(4\beta-1) <& 2b(1-3\beta)\epsilon + \sum_{b<i\leq 3} \eta_i + (3-b)\gamma\beta/2 \qquad (5.16)
\end{aligned}$$

If $b = 3$, the above inequality becomes $2(4\beta-1) < 6(1-3\beta)\epsilon < 2(4\beta-1)$, which is a contradiction. Thus, $b \leq 2$.

Notice that $\eta_i > -D(S_i)\gamma > -\gamma$ for $1 \leq i \leq b$. Hence, $\sum_{S\in\mathcal{K}_4^{bad}(U)} \widetilde{\eta}(S) > -b\gamma$. Also, recall that $D(S_i) \leq \beta$ for $1 \leq i \leq 3$. It is enough to show that $\sum_{b<i\leq 3} \eta_i \geq b\gamma\beta$. Suppose the contrary, so $\sum_{b<i\leq 3} \eta_i < b\gamma\beta$. Then, (5.16) becomes

$$2(4\beta-1) < 2b(1-3\beta)\epsilon + (3+b)\gamma\beta/2 \leq 4(1-3\beta)\epsilon + 5\gamma\beta/2 < 2(4\beta-1),$$

which is a contradiction. The proof of the claim is complete. □

By combining Claim 5.3.3, Claim 5.3.4 and Claim 5.3.5, we obtain that $\sum_{S\in\mathcal{K}_4(U)}\widetilde{\eta}(S) > 0$ for a bad 5-clique U. Now, we ready to prove the inequality of Theorem 5.2.3. By the remark at the beginning of the section, it is sufficient to show that $\sum\eta(S)\geq 0$ with sum over the 4-cliques S. Recall that by Claim 5.3.1 $\eta(S)\geq 0$ for $S\in\mathcal{K}_4$ with $D(S)=0$ and $\sum_{S\in\mathcal{K}_4(U)}\eta(S)\geq 0$ if U is not a bad 5-clique. Notice that

$$\begin{aligned} n\sum_{S\in\mathcal{K}_4}\eta(S) &= \sum_{U\in\mathcal{K}_5}\sum_{S\in\mathcal{K}_4(U)}\widetilde{\eta}(S) + n\sum_{S\in\mathcal{K}_4:D(S)=0}\eta(S) \\ &\geq \sum_{U\in\mathcal{K}_5^{bad}}\sum_{S\in\mathcal{K}_4(U)}\widetilde{\eta}(S)\geq 0. \end{aligned}$$

Hence, we prove the inequality in Theorem 5.2.3.

Now suppose equality holds in Theorem 5.2.3. Claim 5.3.3, Claim 5.3.4 and Claim 5.3.5 imply that no 4-clique is bad. Furthermore, we must have $\eta(S)=0$ for all $S\in\mathcal{K}_4$. It can be checked that if the definition of a bad 4-clique includes heavy 4-cliques S with $\eta(S)=0$, then all arguments still hold. Thus, we can deduce that G is K_5-free. Hence, G is also K_5-free. By Theorem 4 taking $s=4$ and $t=3$, we obtain that (n,β) is feasible and $G\in\mathcal{G}(n,\beta)$. □

Chapter 6

$k_r(n, \delta)$ for $3n/4 < \delta \leq 4n/5$

In this chapter, we evaluate $k_r(n, \delta)$ for $3n/4 < \delta \leq 4n/5$, which is implied by the theorem below as $k_2(G) \geq (1 - \beta)n^2/2 = g_2(\beta)n^2$.

Theorem 6. *Let $1/5 \leq \beta < 1/4$. Let s and t be integers with $2 \leq t < s \leq 5$. Suppose G is a graph of order n with minimum degree $(1 - \beta)n$. Then,*

$$\frac{k_s(G)}{g_s(\beta)n^s} \geq \frac{k_t(G)}{g_t(\beta)n^t}.$$

Moreover, the following three statements are equivalent:

(i) Equality holds for some $2 \leq t < s \leq 5$.

(ii) Equality holds for all $2 \leq t < s \leq 5$.

(iii) The pair (n, β) is feasible and G is a member of $\mathcal{G}(n, \beta)$.

The general approach of the proof is the same as the proof of Theorem 5, so often lemmas and claims are very similar to ones in Chapter 5. However, additional arguments and refinements are needed in various places. We will point out the similarities and differences before giving the proofs.

We would also like to make the following remark about proving Conjecture 3.1.1 for all p. Our proof of Theorem 5 boils down to proving Corollary 5.2.2

and Theorem 5.2.3. Each statement is an inequality of the form

$$c_1^t \sum_{S\in\mathcal{K}_{t+1}} D_+(S) + c_2^t k_{t+1}(G) \geq c_3^t k_t(G) + c_4^t k_{t+2}(G) + c_5^t \sum_{T\in\mathcal{K}_t} D_+(T), \tag{6.1}$$

for $2 \leq t \leq p(=3)$, where each c_i^t is a positive coefficient depending only on β, p and t. Note that (5.2) is an inequality of this form. However, by considering K_{p+2}-free graphs (see Section 4.2 for the case $p=3$), it is easy to deduce that the coefficients c_2^t, c_3^t, c_4^t in (5.2) are optimal. Thus, the key to proving Conjecture 3.1.1 is obtaining suitable c_1^t and c_5^t. Since t ranges from 2 to p, for a given p, it seems very likely that one may obtain an ad-hoc proof of Conjecture 3.1.1 for $1/(p+1) \leq \beta < 1/p$ by mimicking the proof of Corollary 5.2.2 and Theorem 5.2.3. However, finding a proof that works for all p looks very hard due to the inequalities involved.

6.1 $k_r(n,\delta)$ for $3n/4 < \delta \leq 4n/5$

for $p=4$, (5.2) gives three inequalities, namely for $t=2,3,4$. All three inequalities require strengthening in order to prove Theorem 6. One obvious choice would be to use the method of Section 5.2.1 for $t=2$, and of Section 5.2.2 for $t=4$. We are now left with a choice when $t=3$. It turns out that the method of Section 5.2.1 for $t=3$ is not sufficient to prove Theorem 6 when β is close to 1/4. Thus, we adapt the method of Section 5.2.2, namely the proof of Theorem 5.2.3, to strengthen (5.2) for $t=3$. We state the three strengthened versions of (5.2) for $t=2,3,4$ respectively below. Each lemma improves the coefficients of $\sum D_+(T)$ and $\sum D_+(S)$. These coefficients are not the best possible, but are chosen to be linear in β if possible.

Lemma 6.1.1. *Let $1/5 \leq \beta < 1/4$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$\big(1+6\beta\big)k_3(G) + \left(1 - \frac{14\beta-1}{15\beta}\right)\sum_{S\in\mathcal{K}_3} D_+(S) \geq 3(1-2\beta)\beta n k_2 + \frac{4k_4(G)}{n}.$$

Moreover, if equality holds then G is $(1-\beta)n$-regular, and for each edge e, either *we have* $D(e) = 1 - 2\beta$ *or* $D(e) = 3\beta$.

Lemma 6.1.2. *Let* $1/5 \le \beta < 1/4$. *Suppose* G *is a graph of order* n *with minimum degree* $(1-\beta)n$. *Then*

$$\begin{aligned} &2k_4(G) + \left(2 - \frac{64\beta + 4}{35}\right) \sum_{S \in \mathcal{K}_4} D_+(S) \\ &\ge 2(1-3\beta)\beta n k_3(G) + \left(1 - 3\beta - \frac{13 - 47\beta}{15}\right) n \sum_{T \in \mathcal{K}_3} D_+(T) + \frac{10k_5(G)}{n}. \end{aligned}$$

Lemma 6.1.3. *Let* $1/5 \le \beta < 1/4$. *Suppose* G *is a graph of order* n *with minimum degree* $(1-\beta)n$. *Then*

$$(3 - 10\beta)k_5(G) \ge (1-4\beta)\beta n k_4(G) + \left(1 - 4\beta + \frac{418\beta - 92}{175}\right) n \sum_{T \in \mathcal{K}_4} D_+(T).$$

Moreover, if equality holds, then (n, β) *is feasible and* $G \in \mathcal{G}(n, \beta)$.

Their proofs are postponed to later sections. Here, we give an outline of each proof. Lemma 6.1.1 is proved in Section 6.2, using virtually the same argument as the one in Section 5.2.1. We then skip Lemma 6.1.2 and prove Lemma 6.1.3 instead in Section 6.3. Lemma 6.1.3 can be seen as the natural generalisation of Theorem 5.2.3. Thus, its proof bears many similarities to the proof of Theorem 5.2.3 (Section 5.3). However, some refinements are needed. Finally, in Section 6.4, we prove Lemma 6.1.2. Roughly speaking, the proof proceeds by applying the proof of Theorem 5.2.3 twice.

Assuming these three lemmas, we now prove Theorem 6.

Proof of Theorem 6. Recall that

$$\begin{aligned} g_2(\beta) &= (1-\beta)/2, & g_3(\beta) &= (20\beta^2 - 15\beta + 3)\beta, \\ g_4(\beta) &= (1-3\beta)(3-10\beta)\beta^2/2, \text{ and} & g_5(\beta) &= (1-3\beta)(1-4\beta)\beta^3/2. \end{aligned}$$

First, we are going to show that the inequality in Theorem 6 holds. Observe that it is enough to prove the case $s = t + 1$.

By Lemma 6.1.3, we have

$$\begin{aligned}(3-10\beta)k_5 \geq& (1-4\beta)\beta n k_4 + \left(1 - 4\beta + \frac{418\beta - 92}{175}\right) n \sum_{S\in\mathcal{K}_4} D_+(S)\\ \frac{k_5}{g_5(\beta)n^5} \geq& \frac{k_4}{g_4(\beta)n^4} + \frac{1-4\beta+(418\beta-92)/175}{(3-10\beta)g_5(\beta)n^4}\sum_{S\in\mathcal{K}_4} D_+(S). \qquad (6.2)\end{aligned}$$

Thus, the inequality in Theorem 6 holds for $t = 4$ (and $s = 5$) as the coefficient of $\sum_{S\in\mathcal{K}_4} D_+(T)$ is non-negative.

Next, we substitute (6.2) into Lemma 6.1.2 replacing the k_5 term. We obtain

$$\begin{aligned}&2k_4 + \left(2 - \frac{64\beta+4}{35}\right)\sum_{S\in\mathcal{K}_4} D_+(S)\\ \geq& 2(1-3\beta)\beta n k_3 + \left(1 - 3\beta - \frac{13-47\beta}{15}\right) n \sum_{T\in\mathcal{K}_3} D_+(T)\\ &+ 10\left(\frac{g_5(\beta)k_4}{g_4(\beta)} + \frac{1-4\beta+(418\beta-92)/175}{3-10\beta}\sum_{S\in\mathcal{K}_4} D_+(S)\right)\end{aligned}$$

A simple calculation shows that the coefficient of $\sum_{S\in\mathcal{K}_4} D_+(S)$ on the left hand side is less than the one oo the right, so we can remove the terms involving $\sum_{S\in\mathcal{K}_4} D_+(S)$. After further rearranging, we obtain

$$\frac{k_4}{g_4(\beta)n^4} \geq \frac{k_3}{g_3(\beta)n^3} + \frac{(1-3\beta-(13-47\beta)/15)}{2(1-3\beta)\beta g_3(\beta)n^3}\sum_{T\in\mathcal{K}_3} D_+(T). \qquad (6.3)$$

Therefore, the inequality in Theorem 6 holds for $t = 3$, because once again the coefficient of $\sum_{T\in\mathcal{K}_3}$ is non-negative.

Finally, we substitute (6.3) into Lemma 6.1.1 replacing the k_4 term. We obtain

$$\begin{aligned}&\big(1+6\beta\big)k_3 + \left(1 - \frac{14\beta-1}{15\beta}\right)\sum_{T\in\mathcal{K}_3} D_+(T)\\ \geq& 3(1-2\beta)\beta n k_2 + 4g_4(\beta)\left(\frac{k_3}{g_3(\beta)} + \frac{(1-3\beta-(13-47\beta)/15}{2(1-3\beta)\beta g_3(\beta)}\sum_{T\in\mathcal{K}_3} D_+(T)\right)\end{aligned}$$

Again, a simple calculation shows that the coefficient of $\sum_{T\in\mathcal{K}_3} D_+(T)$ on the right is at least $1-(14\beta-1)/15\beta$, so we can remove the terms involving $\sum_{T\in\mathcal{K}_3} D_+(T)$. After rearrangement, we have $k_3/g_3(\beta)n^3 \geq k_2/g_2(\beta)n^2$. Therefore we have shown that inequality in Theorem 6 holds.

It is clear that (iii) implies (i) and (ii) by the construction of $\mathcal{G}(n,\beta)$ and the feasibility of (n,β). Suppose (i) holds, so equality holds in Theorem 6 for $t = t_0$ and $s = s_0$ with $t_0 < s_0$. We claim that equality must also hold for $t = 4$ and $s = 5$. Suppose the contrary, so $k_5(G)/g_5(\beta) = k_4(G)/g_4(\beta)$. By (6.2), $\sum_{S\in\mathcal{K}_4} D_+(S) > 0$. This implies a strict inequality in (6.3). Moreover, this implies that we have strict inequality in Theorem 6 for $2 \leq t < s \leq 5$, which contradicts (i). Therefore equality holds for $t = 4$ and $s = 5$ and so we have that equality holds in Lemma 6.1.3. Hence, (n,β) is feasible, and $G \in \mathcal{G}(n,\beta)$. □

6.2 Proof of Lemma 6.1.1

The proof follows by mimicking Section 5.2.1. We prove the corresponding version of Lemma 5.2.1 below. First, recall that the definition of $\widetilde{D}(S)$ and the results in Section 5.1 hold for all p.

Lemma 6.2.1. *Let $1/5 \leq \beta < 1/4$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$ and $S \in \mathcal{K}_3(G)$. Then*

$$\widetilde{D}(S) \geq \frac{14\beta - 1}{15\beta} D_+(S) - (1-2\beta) \sum_{T\in\mathcal{K}_2(S)} \frac{D_+(T)}{D_+(T) + 3\beta}. \tag{6.4}$$

Moreover, if equality holds, then S is not heavy, and $d(v) = (1-\beta)n$ for all $v \in S$.

Proof. Suppose the lemma is false and let S be such a 3-clique in G. Corollary 5.1.2 states that $\widetilde{D}(S) \geq 0$, so S is heavy as $(14\beta-1)/15\beta \geq 0$. Also, by Corollary 4.1.3, we have

$$\widetilde{D}(S) + \sum_{T\in\mathcal{K}_2(S)} D_+(T) \geq D_+(S). \tag{6.5}$$

Thus, S contains at least one heavy edge, or else (6.4) holds as (6.5) becomes $\widetilde{D}(S) \geq D_+(S) \geq (14\beta-1)D_+(S)/15\beta$. Let $T_0 \in \mathcal{K}_2(S)$ with $D(T_0) = \max\{D(T) :$

$T \in \mathcal{K}_2(S)\}$. By substituting (6.5) into (6.4), if

$$f = \left(1 - \frac{1-2\beta}{D_+(T_0)+3\beta}\right)\widetilde{D}(S) - \left(\frac{14\beta-1}{15\beta} - \frac{1-2\beta}{D_+(T_0)+3\beta}\right)D_+(S) \tag{6.6}$$

is non-negative, then S satisfies (6.4), which is a contradiction.

Suppose that $D_+(S) \leq 1-4\beta$. Since T_0 is heavy, $D(T_0) > 3\beta \geq 1-2\beta$ by Lemma 4.1.1 (*iv*). Lemma 4.1.1 (*iii*) gives $D(T_0) \leq D(S) + \beta = 3\beta + D_+(S) \leq 1-\beta$. Notice that

$$\frac{14\beta-1}{15\beta} - \frac{1-2\beta}{D(T_0)} \leq \frac{14\beta-1}{15\beta} - \frac{1-2\beta}{1-\beta} = -\frac{(1-4\beta)(1+4\beta)}{15\beta(1-\beta)} \leq 0$$

Therefore, by considering the coefficients of both $\widetilde{D}(S)$ and $D_+(S)$ in (6.6), $f > 0$. Hence, $D_+(S) > 1-4\beta$. We address different cases separately depending on the number of heavy edges in S.

First, suppose all edges are heavy. Then, $\widetilde{D}(S) = 2(5\beta-1)$ by Lemma 4.1.1 (*iv*). Note that $2(5\beta-1) \leq D_+(S)$, otherwise (6.4) holds as $\widetilde{D}(S) = 2(5\beta-1) \geq D_+(S) \geq (14\beta-1)D_+(S)/15\beta$. Clearly, $D(T_0) \leq 1$ and $D_+(S) = D(S) - 2\beta \leq 1-2\beta$. Hence,

$$\begin{aligned} f =& 2(5\beta-1)\left(1-\frac{1-2\beta}{D(T_0)}\right) - \left(\frac{14\beta-1}{15\beta} - \frac{1-2\beta}{D(T_0)}\right)D_+(S) \\ \geq& 4(5\beta-1)\beta - \frac{(1+6\beta)(5\beta-1)}{15\beta}D_+(S) \\ \geq& 4(5\beta-1)\beta - \frac{(1+6\beta)(5\beta-1)(1-2\beta)}{15\beta} = \frac{(5\beta-1)(72\beta^2-4\beta-1)}{15\beta} \geq 0 \end{aligned}$$

Thus, there is an edge in S that is not heavy. This implies $D_+(S) \leq \beta$. If not, Lemma 4.1.1 (*iii*) implies $D_+(T) > 0$ for all $T \in \mathcal{K}_t(S)$, which is a contradiction. Since $D_+(S) \leq \beta$, so $D_+(T) \leq \beta$ for $T \in \mathcal{K}_t(S)$.

Suppose there are either one or two heavy edges in S. We claim that in both cases $\widetilde{D}(S) \geq 2(D_+(S) - (1-4\beta))$. Assume that S contains exactly one heavy edge. Let T_1 and T_2 be the two non-heavy edges in S. By Lemma 4.1.1 (*iii*),

$D_-(T_i) = D(T_i) \geq D(S) = D_+(S)+2\beta > 0$ for $i = 1, 2$. Thus by Lemma 4.1.1 (iv),

$$\widetilde{D}(S) = \sum D_-(T) - (2-\beta) \geq 2(D_+(S) - (1-4\beta)).$$

Next, assume that S contains two heavy edges. Let T_1 be the non-heavy edge in S. By Lemma 4.1.1 (iii), we have $D_-(T_1) \geq D_+(S)+2\beta > 1-2\beta$. Recall that $D_+(S) \leq \beta$. Therefore, $\widetilde{D}(S) \geq 5\beta - 1 + D_+(S) - (1-4\beta) \geq 2(D_+(S) - (1-4\beta))$. This completes the claim, so $\widetilde{D}(S) \geq 2(D_+(S) - (1-4\beta))$.

Recall that our aim is to show that (6.6) is non-negative. By substituting $\widetilde{D}(S) \geq 2(D_+(S) - (1-4\beta))$ into (6.6) and multiplying by $D_+(T_0) + 3\beta)$, it is enough to show that

$$\begin{aligned} D(T_0)f \geq & 2(5\beta-1)\left(D_+(T_0)+5\beta-1\right) \\ & - \left(\frac{14\beta-1}{15\beta}(D_+(T_0)+3\beta) - (1-2\beta)\right) D_+(S) \\ = & 2(5\beta-1)\left(D_+(T_0)+5\beta-1\right) - \left(\frac{(14\beta-1)D_+(T_0)}{15\beta} - \frac{6(1-4\beta)}{5}\right) D_+(S) \end{aligned}$$

is non-negative for $0 < D_+(T_0) \leq D_+(S)$ and $1-4\beta \leq D_+(S) \leq \beta$. By considering the Hessian matrix, all stationary points are saddle points. Thus, it enough to check the above inequality on the boundary of $D_+(T_0)$ and $D_+(S)$. In fact, it is sufficient to check it at the extreme values of $D_+(T_0)$ and $D_+(S)$, because if say $D_-(T_0)$ is fixed, then we take the extremal values of $D_+(S)$, and, if $D_+(T) = D_+(S)$, then the above inequality is a concave function. If $D_+(T_0) = 0$, we have

$$D(T_0)f \geq 2(5\beta-1)^2 + 6(1-4\beta)D_+(S)/5 > 0$$

If $D_+(T_0) = D_+(S) = \beta$,

$$D(T_0)f \geq 2(5\beta-1)\left(6\beta-1\right) - \frac{(86\beta-19)\beta}{15} = \frac{814\beta^2 - 311\beta + 30}{15} > 0.$$

Finally, if $D_+(T_0) = D_+(S) = 1 - 4\beta$, we have

$$D(T_0)f \geq 2(5\beta - 1)\beta + (1 - 4\beta)^2(1 + 4\beta)/15\beta > 0.$$

Hence, (6.4) holds.

It is easy to check thatif equality holds in (6.4) then $D_+(S) = 0$. By Lemma 4.1.1, $D_+(T) = 0$ for $T \in \mathcal{K}_2(S)$. Furthermore, equality holds in (6.5) implies equality also holds in Corollary 4.1.3 as $D_+(T) = 0 = D_+(S)$. Hence, $d(v) = (1 - \beta)n$ for $v \in V(S)$. This completes the proof of the lemma. □

Proof of Lemma 6.1.1. Lemma 6.2.1 gives a lower bound on $\widetilde{D}(S)$ for $S \in \mathcal{K}_3$. Lemma 5.1.3 gives an upper bound on $\sum_{S \in \mathcal{K}_3} \widetilde{D}(S)$. Together, they prove the lemma. □

6.3 Proof of Lemma 6.1.3

As we mentioned earlier, the proof of Lemma 6.1.3 closely follows the proof of Theorem 5.2.3 from Section 5.3. Thus, many claims and statements are similar, but sometimes the proofs require refinements e.g. Claim 6.3.1 and Claim 6.3.6.

Here we give an outline of the proof. In this section, T, S and U will always denote a 4-clique, 5-clique and 6-clique respectively. First, we are going to define a function η on 5-cliques such that $\sum_{S \in \mathcal{K}_5} \eta(S) \geq 0$ would imply the inequality in Lemma 6.1.3. Call a 5-clique S *bad* if $\eta(S) < 0$. Claim 6.3.1 identifies the structure of a bad 5-cliques, namely every bad 5-clique is contained in some 6-clique U. Note that $\sum_{U \in \mathcal{K}_6(U)} \sum_{S \in \mathcal{K}_5(U)} \widetilde{\eta}(S) = n \sum_{S \in \mathcal{K}_5 : D(S) > 0} \eta(S)$, where $\widetilde{\eta}(S) = \eta(S)/D(S)$ for $S \in \mathcal{K}_5$ with $D(S) > 0$. Thus, it is enough to show that $\sum_{S \in \mathcal{K}_5(U)} \widetilde{\eta}(S) \geq 0$ for 6-cliques U containing a bad 5-clique. Claim 6.3.4, Claim 6.3.5 and Claim 6.3.6 verify this inequality.

We recall some basic properties of T, S and U. Observe that $D_-(S) = 0$ for $S \in \mathcal{K}_5$, so $D_+(S) = D(S)$. Thus, $\widetilde{D}(S) = \sum D_-(T) - (2 - 5\beta)$. Let $T_1, \ldots, T_5$ be 4-cliques in S with $D(T_i) \leq D(T_{i+1})$ for $1 \leq i \leq 4$. Since $D_-(T) \leq \beta$ by definition, we have

$$\widetilde{D}(S) = \begin{cases} 2(5\beta - 1) & \text{if } k_3^+(S) = 5, \\ 5\beta - 1 + (D(T_1) - (1 - 4\beta)) & \text{if } k_3^+(S) = 4, \\ D(T_1) + D(T_2) - 2(1 - 4\beta) & \text{if } k_3^+(S) = 3, \end{cases} \tag{6.7}$$

where $k_4^+(S)$ is the number of heavy 4-cliques in S. Notice that $D(T) \geq 1 - 4\beta$ for $T \in \mathcal{K}_4$ by Lemma 4.1.1 (i). Once again, these formulae will be used repeatedly throughout this section. Now, we now give the formal proof.

Proof of Lemma 6.1.3. Define the function $\eta : \mathcal{K}_5 \to \mathbb{R}$ to be

$$\eta(S) = \widetilde{D}(S) - \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta}$$

for $S \in \mathcal{K}_5$. For a heavy 4-clique T, $D(T) = D_+(T) + \beta$ and so only heavy 4-cliques contribute to $\sum_{T \in \mathcal{K}_4(S)} D_+(T)/(D_+(T) + \beta)$. A 5-clique S is called *bad* if $\eta(S) < 0$, otherwise it is called *good*. The sets of bad and good 5-cliques are denoted by $\mathcal{K}_5^{bad}$ and $\mathcal{K}_5^{good}$ respectively. For $S \in \mathcal{K}_5$ with $D(S) > 0$, define $\widetilde{\eta}(S)$ to be $\eta(S)/D(S)$.

If $\sum \eta(S) \geq 0$ with sum over $S \in \mathcal{K}_5$, then

$$\begin{aligned} 0 \leq \sum_{S \in \mathcal{K}_5} \eta(S) &= \sum_{S \in \mathcal{K}_5} \widetilde{D}(S) - \frac{418\beta - 92}{175} \sum_{S \in \mathcal{K}_5} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} \\ &= \sum_{S \in \mathcal{K}_5} \widetilde{D}(S) - \frac{418\beta - 92}{175} n \sum_{T \in \mathcal{K}_4} D_+(T) \\ &\leq (3 - 10\beta)k_5 - (1 - 4\beta)\beta n k_4 - \left(1 - 4\beta + \frac{418\beta - 92}{175}\right) n \sum_{T \in \mathcal{K}_4} D_+(T) \end{aligned}$$

The last inequality is due to Lemma 5.1.3. Thus, we have obtained the inequality in Lemma 6.1.3.

Suppose to the contrary that we have $\sum \eta(S) < 0$, so there exists a bad 5-clique S, i.e. $\eta(S) < 0$. By Corollary 5.1.2, $\widetilde{D}(S) \geq 0$. Thus, by the definition of $\eta(S)$, if $\beta \leq 46/209$, then $\eta(S) \geq 0$ for all $S \in \mathcal{K}_5$. Therefore, for the remainder of this section, we assume that $46/209 \leq \beta \leq 1/4$.

In the next claim, we study the structure of a bad 5-clique S. It turns out that S has similar characteristics to a bad 4-clique from Section 5.3. For example, S has exactly one heavy edge e and every heavy subclique in S contains e. Actually, proving this statement requires a strengthening of Lemma 4.1.2, which we will state in the proof, when it is needed.

Claim 6.3.1. *Suppose S is a bad 5-clique. Let*

$$\Delta = (1 - 4\beta)(1 + \epsilon),\ \epsilon = \frac{3(209\beta - 46)}{313 - 1152\beta} \text{ and } \gamma = \frac{3(418\beta - 92)}{175\beta}.$$

Then, the following holds

(i) S contains exactly one heavy edge e and every heavy subclique in S contains e. In particular, S has exactly three heavy 4-subcliques.

(ii) $0 < D(S) < \Delta$,

(iii) $D(T) + D(T') < 2\Delta$, where T and T' are the two non-heavy 4-cliques in S,

(iv) $\eta(S) \ge -3(418\beta - 92)D(S)/175(D(S) + \beta)$,

(v) $\widetilde{\eta}(S) \ge -3(418\beta - 92)/175(D(S) + \beta) > -\gamma$.

Proof. Let $T_1, \dots, T_5$ be 4-cliques in S with $D(T_i) \le D(T_{i+1})$ for $1 \le i \le 4$. Since S is bad and $\widetilde{D}(S) \ge 0$ by Corollary 5.1.2, $D_+(T_5) > 0$. Thus, S is heavy by Lemma 4.1.1 (*iii*). We separate cases by the number of heavy 4-cliques in S.

First, suppose all 4-cliques are heavy. Hence, $\widetilde{D}(S) = 2(5\beta - 1)$ by (6.7). Clearly, $D_+(T_i) \le 1 - \beta$ for $1 \le i \le 4$, so

$$\begin{aligned}\eta(S) \ge& 2(5\beta - 1) - \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} \\ \ge& 2(5\beta - 1) - \frac{(418\beta - 92)(1 - \beta)}{35} = \frac{418\beta^2 - 160\beta + 22}{35} > 0.\end{aligned}$$

This contradicts the assumption that S is bad. Thus, not all 4-cliques in S are heavy. By Lemma 4.1.1 (*iii*), we see that $D_+(T) \ge D_+(S) - \beta$ for $T \in \mathcal{K}_4(S)$, so $0 < D(S) = D_+(S) \le \beta$ or else all $T \in \mathcal{K}_4(S)$ are heavy. Also, $D_+(T) \le D_+(S) = D(S) \le \beta$.

Suppose all but one of the 4-cliques are heavy, so $\widetilde{D}(S) \geq 5\beta - 1$ by (6.7). Hence,

$$\begin{aligned}\eta(S) \geq & 5\beta - 1 - \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} \geq 5\beta - 1 - \frac{4(418\beta - 92)D(S)}{175(D(S) + \beta)} \\ \geq & 5\beta - 1 - \frac{2(418\beta - 92)}{175} = \frac{39\beta + 9}{175} > 0,\end{aligned}$$

which contradicts S being bad.

Suppose S contains at most two heavy 4-cliques, say $k_4^+(S) = 3 - i$ for $i \in \{1, 2\}$. Recall that $D_+(T) \leq D_+(S) = D(S)$ for $T \in \mathcal{K}_4(S)$, so $\widetilde{D}(S) \geq iD(S)$ by Corollary 4.1.3. Thus, we obtain

$$\begin{aligned}\eta(S) \geq & iD(S) - \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} \\ \geq & iD(S) - \frac{(3 - i)(418\beta - 92)D(S)}{175(D(S) + \beta)} \\ \eta(S) \geq & \left(1 - \frac{2(418\beta - 92)}{175\beta}\right) D(S) = \frac{(184 - 661\beta)D(S)}{175\beta} > 0. \end{aligned} \tag{6.8}$$

Again, this contradicts the fact that S is bad.

Therefore, S has exactly three heavy 4-cliques. This means that T_1 and T_2 are non-heavy. Hence, $\widetilde{D}(S) = D(T_1) + D(T_2) - 2(1 - 4\beta)$ by (6.7). Recall that $D(T_i) \geq D(S)$ for $i = 1, 2$. If $D(S) \geq \Delta \geq 1 - 4\beta$, then

$$\begin{aligned}\eta(S) = & D(T_1) + D(T_2) - 2(1 - 4\beta) - \frac{418\beta - 92}{175} \sum_{3 \leq i \leq 5} \frac{D_+(T_i)}{D_+(T_i) + \beta} \\ \geq & 2(D(S) - (1 - 4\beta)) - \frac{3(418\beta - 92)D(S)}{175(D(S) + \beta)} \\ \geq & 2(D(S) - (1 - 4\beta)) - \frac{3(418\beta - 92)D(S)}{175(1 - 3\beta)} \\ \geq & 2(1 - 4\beta)\epsilon - \frac{3(418\beta - 92)\Delta}{175(1 - 3\beta)} = 0,\end{aligned}$$

so S is good, a contradiction. Thus, (ii) holds. If (iii) is false, then $D(T_1) +$

$D(T_2) \geq 2\Delta$ and so by (6.7) $\widetilde{D}(S) \geq 2(1-4\beta)\epsilon$. Once again, we have

$$\begin{aligned}\eta(S) \geq & 2(1-4\beta)\epsilon - \frac{418\beta - 92}{175} \sum_{3 \leq i \leq 5} \frac{D_+(T_i)}{D_+(T_i)+\beta} \\ \geq & 2(1-4\beta)\epsilon - \frac{3(418\beta-92)D(S)}{175(D(S)+\beta)} \geq 2(1-4\beta)\epsilon - \frac{3(418\beta-92)\Delta}{175(1-3\beta)} = 0.\end{aligned}$$

Thus, (iii) holds. By the definition of $\eta(S)$,

$$\eta(S) \geq -\frac{418\beta-92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T)+\beta} \geq -\frac{3(418\beta-92)D(S)}{175(D(S)+\beta)}.$$

Hence, (iv) and (v) holds.

Therefore, we are left to prove (i). Note that we have already shown that S contains exactly three heavy 4-cliques. Since a 4-clique containing a heavy edge is heavy by Lemma 4.1.1 (iii), there is at most one heavy edge in S. Thus, it suffices to show that S contains an heavy edge. Suppose the contrary, so S does not contain any heavy edge and $D(S) < \Delta$ by (ii). Let the vertices of S be $v_1, \ldots, v_5$ and $T_i = S \backslash v_i$ for $1 \leq i \leq 5$. If $\widetilde{D}(S) \geq D_+(T_5)$, then

$$\begin{aligned}\eta(S) \geq & D_+(T_5) - \frac{418\beta-92}{175} \sum_{3 \leq i \leq 5} \frac{D_+(T_i)}{D_+(T_i)+\beta} \\ \geq & \left(1 - \frac{3(418\beta-92)}{175\beta}\right) D_+(T_5) = \frac{(276-1079\beta)D_+(T_5)}{175\beta} > 0,\end{aligned}$$

which contradicts the fact that S is bad. Therefore, it is enough to show that $\widetilde{D}(S) \geq D_+(T_5)$. By (6.7), it is equivalent to show that

$$D(T_1) + D(T_2) \geq 2(1-4\beta) + D_+(T_5). \tag{6.9}$$

Let W_i be $T_i \cap T_5$ for $1 \leq i \leq 4$. The set of W_i are precisely the 3-cliques in T_5. By Lemma 4.1.1 (ii), $D(T_i) \geq D(W_i) - \beta$ for $1 \leq i \leq 4$. Thus,

$$D(W_1) + D(W_2) \geq 2(1-3\beta) + D_+(T_5) \tag{6.10}$$

implies (6.9). By Corollary 4.1.3 taking $t = 3$ and $S = T_5$, we have

$$D(W_1) + D(W_2) + D_+(W_3) + D_+(W_4) \geq 2(1 - 3\beta) + 2D_+(T_5)$$

as $D_-(W_3),\ D_-(W_4) \leq 2\beta$. We may assume that $D_+(W_3) + D_+(W_4) > D_+(T_5)$, otherwise (6.10) holds, which would imply (6.9) and that S is good, a contradiction. First suppose that $D_+(W_3) = 0$, so $D_+(W_4) > D_+(T_5)$. The edges of W_4 are precisely v_1v_2, v_1v_3 and v_2v_3. Again, by Corollary 4.1.3 taking $t = 2$ and $S = W_4$, we have

$$D(v_1v_2) + D(v_1v_3) + D(v_2v_3) \geq 2 - 3\beta + D_+(W_4) \geq 2 - 3\beta + D_+(T_5),$$

where the last inequality is due to Lemma 4.1.1 (*iii*). Since S has no heavy edge, $D(v_1v_2)$ is at most 3β. Hence, we have $D(v_1v_3) + D(v_2v_3) \geq 2(1 - 2\beta) + D_+(T_5)$. Notice that, by Lemma 4.1.1 (*ii*),

$$D(W_1) + D(W_2) \geq (D(v_2v_3) - \beta) + (D(v_1v_3) - \beta) \geq 2(1 - 3\beta) + D_+(T_5),$$

so (6.10) holds. By similar argument, we may assume that both $D_+(W_3)$ and $D_+(W_4)$ are strictly positive.

We claim that

$$\sum_{e \in E(W_4)} D(e) \geq 1 - 2\beta + D(W_4) + D(v_1) = 1 + D_+(W_4) + D(v_1). \qquad (6.11)$$

Notice that the above statement is a strengthening of Lemma 4.1.2 for $t = 2$, $s = 3$ and $S = W_4$, The proof of (6.11) follows easily from the proof of Lemma 4.1.2 by replacing (4.3) with

$$\sum_i in_i = \sum_{v \in V(S)} d(v) \geq 2(1 - \beta)n + d(v_1).$$

Expanding the left hand side of (6.11) yields

$$D(v_1v_3) + D(v_2v_3) - D(v_1) \geq 1 - D(v_1v_2) + D_+(W_4) \geq 1 - 3\beta + D_+(W_4).$$

By similar argument,

$$D(v_1v_4) + D(v_2v_4) - D(v_2) \geq 1 - 3\beta + D_+(W_3).$$

By the pigeonhole principle, it is easy to see that $D(W_1) = D(v_2v_3v_4) \geq D(v_2v_3) + D(v_2v_4) - D(v_2)$. Similarly, $D(W_2) \geq D(v_1v_3) + D(v_1v_4) - D(v_1)$. Thus,

$$\begin{aligned} D(W_1) + D(W_2) \geq & (D(v_2v_3) + D(v_2v_4) - D(v_2)) + (D(v_1v_3) + D(v_1, v_4) - D(v_1)) \\ \geq & 2(1-3\beta) + D_+(W_3) + D_+(W_4) \geq 2(1-3\beta) + D_+(T_5). \end{aligned}$$

Hence, (6.10) holds, which would lead to a contradiction. The proof of the claim is complete. □

Since a bad 5-clique S must be heavy by Claim 6.3.1, it is contained in some 6-clique U as $D(S) > 0$. A 6-clique is called *bad* if it contains at least one bad 5-clique. We denote the set of bad 6-cliques by $\mathcal{K}_6^{bad}$. Let $S, S^b \in \mathcal{K}_5(U)$ be distinct such that S^b is a bad 5-clique in a bad 6-clique U. Observe that $S \cap S^b$ is a 4-clique. Then, by Lemma 4.1.1 and Claim 6.3.1 (ii), we have

$$D(S) \leq D(S \cap S^b) = D_+(S \cap S^b) + \beta \leq D(S^b) + \beta < \Delta + \beta. \tag{6.12}$$

A bad 5-clique S contains a heavy edge by Claim 6.3.1 and so does a bad 6-clique U. We study U according to the number of heavy edges it contains. The next claim shows the relationship between the number of heavy edges and bad 5-cliques in U. The proof is identical to the proof of Claim 5.3.2. Hence, the proof is omitted.

Claim 6.3.2. *Suppose $U \in \mathcal{K}_6^{bad}$ with $h \geq 2$ heavy edges and b bad 5-cliques. Then $b \leq 2h/(h-1) = 2 + 2/(h-1)$. Moreover, if there exist two heavy edges sharing a common vertex, then $b \leq 3$.* □

Now, our aim is to show that for a bad 6-clique U, $\sum_{S\in\mathcal{K}_5(U)} \widetilde{\eta}(S) > 0$. First of all, for a 5-clique S that contains at least four heavy 4-cliques, we bound $\widetilde{\eta}(S)$ below

Claim 6.3.3. (i) *If $S \in \mathcal{K}_5$ has exactly four heavy 4-cliques, then $\widetilde{\eta}(S) \geq 3\gamma$.*

(ii) If all 4-cliques in $S \in \mathcal{K}_5$ are heavy and $D_+(S) < \Delta + \beta$, then $\widetilde{\eta}(S) \geq \gamma$.

Proof. (i) It is enough to show that $\eta(S) \geq 3\gamma D(S)$. If $D(S) = D_+(S) \leq 1 - 4\beta$, then $\widetilde{D}(S) \geq 5\beta - 1$ by (6.7) and so

$$\begin{aligned}
\eta(S) \geq & 5\beta - 1 - \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} \\
\geq & 5\beta - 1 - \frac{4(418\beta - 92)D(S)}{175(D(S) + \beta)} \geq 5\beta - 1 - \frac{4(418\beta - 92)(1 - 4\beta)}{175(1 - 3\beta)} \\
\geq & 3(1 - 4\beta)\gamma \geq 3\gamma D(S).
\end{aligned}$$

If $D(S) = D_+(S) > 1 - 4\beta$, then $\widetilde{D}(S) \geq 5\beta - 1 + D(S) - (1 - 4\beta)$ by (6.7) and Lemma 4.1.1 (iii). Recall that $D_+(S) \leq \beta$ by Lemma 4.1.1 (iii) as not all 4-cliques in S are heavy. Hence

$$\begin{aligned}
\eta(S) - 3\gamma D(S) \geq & 9\beta - 2 + D(S) - \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} - 3\gamma D(S) \\
\geq & 9\beta - 2 + (1 - 3\gamma)D(S) - \frac{4(418\beta - 92)D(S)}{175(D(S) + \beta)} \\
\geq & 9\beta - 2 + (1 - 3\gamma)(1 - 4\beta) - \frac{2(418\beta - 92)}{175} \\
= & \frac{3(5029\beta^2 - 2355\beta + 276)}{175\beta} > 0
\end{aligned}$$

as required.

(ii) Note that $\widetilde{D}(S) = 2(5\beta - 1)$ by (6.7). Hence,

$$\begin{aligned}
\eta(S) = & 2(5\beta - 1) - \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} \geq 2(5\beta - 1) - \frac{(418\beta - 92)D(S)}{35(D(S) + \beta)} \\
> & 2(5\beta - 1) - \frac{(418\beta - 92)(\Delta + \beta)}{35(\Delta + 2\beta)} \geq (\Delta + \beta)\gamma > D(S)\gamma
\end{aligned}$$

as required. □

The next two claims consider the case when a bad 6-clique U contains at least two heavy edges, e and e'.

Claim 6.3.4. *Suppose U is a bad 6-clique and there exist two heavy edges e and e' in U sharing a common vertex. Then $\sum_{S \in \mathcal{K}_5(U)} \widetilde{\eta}(S) > 0$.*

Proof. Suppose U contains b bad 5-cliques and h heavy edges. Clearly, $b \leq 3$ by Claim 6.3.2. Hence, $\sum_{S' \in \mathcal{K}_5^{bad}(U)} \widetilde{\eta}(S) > -b\gamma \geq -3\gamma$ by Claim 6.3.1 (v). There are exactly three heavy 5-cliques S containing both e and e'. Each such S contains at least four heavy 4-cliques. Also, $D(S) < \Delta + \beta$ by (6.12). Thus, $\widetilde{\eta}(S) > \gamma$ by Claim 6.3.3 (ii). Therefore, $\sum_{S \in \mathcal{K}_5(U)} \widetilde{\eta}(S) > 0$. □

Claim 6.3.5. *Suppose U is a bad 6-clique. Suppose also that there exist two vertex disjoint heavy edges e and e' in U. Then $\sum_{S \in \mathcal{K}_5(U)} \widetilde{\eta}(S) > 0$.*

Proof. By Claim 6.3.4, we may assume that any two heavy edges in U are disjoint and so U has at most three heavy edges. If U has three disjoint heavy edges, then every 5-clique in U contains at least two heavy edges. Therefore, no 5-clique in U is bad by Claim 6.3.1 (i). Thus, the heavy edges in U are precisely e and e'. In addition, U has $b \leq 4$ bad 4-cliques by Claim 6.3.2.

Let S_0 be a bad 4-clique in U with $D(S_0)$ minimal. Note that $\sum_{S \in \mathcal{K}_5^{bad}(U)} \widetilde{\eta}(S) \geq -b\gamma\beta/(D(S_0) + \beta)$ by Claim 6.3.1 (v). Note that there are two heavy 5-cliques S containing both e and e'. Recall that $D(S) \leq D(S_0) + \beta$ by (6.12). Therefore, it is sufficient to prove that

$$\eta(S) > b\gamma\beta/2$$

for each heavy 5-clique S containing both e and e', because the right hand side of the inequality is at least $b\gamma\beta D(S)/2(D(S_0) + \beta)$.

Since S contains two vertex disjoint heavy edges, all 4-cliques in S are heavy by Lemma 4.1.1. Thus, $\widetilde{D}(S) = 2(5\beta - 1)$ by (6.7). Note that for a 4-clique $T = S \cap S'$ and $S' \in \mathcal{K}_5(U) \backslash S$, $D_+(T) \leq \min\{D_+(S), D_+(S')\} \leq D_+(S') = D(S')$

by Lemma 4.1.1 (*iii*). Therefore,

$$\begin{aligned}\eta(S) =& 2(5\beta-1) - \frac{418\beta-92}{175}\sum_{T\in\mathcal{K}_4(S)}\frac{D_+(T)}{D_+(T)+\beta}\\ \geq& 2(5\beta-1) - \frac{418\beta-92}{175}\sum_{S'\in\mathcal{K}_5(U)\setminus S}\frac{D(S')}{D(S')+\beta}\\ \geq& 2(5\beta-1) - \frac{418\beta-92}{175}\left(\frac{b\Delta}{\Delta+\beta}+\frac{(5-b)(\Delta+\beta)}{\Delta+2\beta}\right),\end{aligned}$$

where the last inequality is due to Claim 6.3.1 (*ii*) and (6.12). Hence,

$$\begin{aligned}\eta(S) - b\gamma\beta/2 \geq& 2(5\beta-1) - \frac{418\beta-92}{175}\left(\frac{b\Delta}{\Delta+\beta}+\frac{(5-b)(\Delta+\beta)}{\Delta+2\beta}\right) - b\gamma\beta/2\\ \geq& 2(5\beta-1) - \frac{418\beta-92}{175}\left(\frac{4\Delta}{\Delta+\beta}+\frac{\Delta+\beta}{\Delta+2\beta}\right) - 2\gamma\beta > 0\end{aligned}$$

as required. The proof is complete. □

Now, we look at the case when $U \in \mathcal{K}_6^{bad}$ contains only one heavy edge.

Claim 6.3.6. *Suppose U is a bad 6-clique containing exactly one heavy edge e. Then, U contains at most three bad 5-cliques. Moreover, $\sum_{S\in\mathcal{K}_5(U)}\widetilde{\eta}(S) > 0$.*

Proof. Let $u_1,\ldots,u_6$ be the vertices of U and u_5u_6 be the heavy edge. Write S_i and η_i to be $U - u_i$ and $\eta(S_i)$ respectively for $1 \leq i \leq 6$. Since a bad 5-clique contains a heavy edge by Claim 6.3.1 (*i*), if S_i is a bad 5-clique, then $i \leq 4$. Without loss of generality, $S_1,\ldots,S_b$ are the bad 5-cliques in U. Similarly, write $T_{i,j}$ to be $U - u_i - u_j$ for $1 \leq i < j \leq 6$.

Since S_i contains the heavy edge e for $1 \leq i \leq 4$, S_i contains at least three heavy 4-cliques. By Claim 6.3.4, $\widetilde{\eta}(S_i) \geq \gamma$ if S_i contains at least four heavy 4-cliques. Recall that $\sum_{S\in\mathcal{K}_5^{bad}(U)}\widetilde{\eta}(S) \geq -b\gamma$ by Claim 6.3.1 (*v*). Thus, we may assume that at most one of $S_1,\ldots,S_4$ contains at least four heavy 4-cliques and that if there is one it is S_4. Otherwise, $\sum_{S\in\mathcal{K}_5(U)}\widetilde{\eta}(S) > 0$.

Suppose S_4 contains exactly four heavy 4-cliques. Claim 6.3.1 (*i*) implies that S_4 is not heavy. Furthermore, $\widetilde{\eta}(S_4) \geq 3\gamma$ by Claim 6.3.3 (*i*). Since there are at most three bad 4-cliques in U, $\sum_{S\in\mathcal{K}_5(U)}\widetilde{\eta}(S) > 0$.

Suppose all 4-cliques in S_4 are heavy, so $T_{4,5}$ and $T_{4,6}$ are heavy. For $1 \leq i \leq 3$, S_i contains exactly three heavy 4-cliques, because each contains the heavy edge u_5u_6. Thus, $T_{i,5}$ and $T_{i,6}$ are not heavy for $1 \leq i \leq 3$. This implies both S_5 and S_6 contains either one or two heavy 4-cliques. By (6.8), it is easy to deduce that

$$\widetilde{\eta}(S_i) \geq 184 - 661\beta/175\beta \geq \gamma/2$$

for $i = 5, 6$. Note that $\sum_{S \in \mathcal{K}_5^{bad}(U)} \widetilde{\eta}(S) \geq -(b-1)\gamma\beta/(D(S_0)+\beta) + \gamma$ by Claim 6.3.1 (v), where S_0 is a bad 5-clique with $D(S_0)$ minimal and $1 \leq b \leq 3$. If $\widetilde{\eta}(S_4) \geq (b-1)\gamma\beta/(D(S_0)+\beta)$, then $\sum_{1 \leq i \leq 6} \widetilde{\eta}(S_i) \geq 0$. Thus, it is sufficient to prove that $\eta(S_4) > (b-1)\gamma\beta$ as $D(S_4) \leq D(S_0) + \beta$ by (6.12). Note that $\widetilde{D}(S_4) = 2(5\beta - 1)$ by (6.7). For $i \in [6]\backslash\{4\}$, $D_+(T_{i,4}) \leq D_+(S_i) = D(S_i)$ by Lemma 4.1.1 (iii). Therefore,

$$\begin{aligned}
\eta(S_4) =& 2(5\beta-1) - \frac{418\beta - 92}{175} \sum_{i \in [6]\backslash\{4\}} \frac{D_+(T_{i,4})}{D_+(T_{i,4}) + \beta} \\
\geq& 2(5\beta-1) - \frac{418\beta - 92}{175} \sum_{i \in [6]\backslash\{4\}} \frac{D(S_i)}{D(S_i) + \beta} \\
\geq& 2(5\beta-1) - \frac{418\beta - 92}{175} \left(\frac{b\Delta}{\Delta+\beta} + \frac{(5-b)(\Delta+\beta)}{\Delta+2\beta} \right).
\end{aligned}$$

The last inequality is due to Claim 6.3.1 (ii) and (6.12). Hence, we have

$$\begin{aligned}
&\eta(S) - (b-1)\gamma\beta \\
\geq& 2(5\beta-1) - \frac{418\beta - 92}{175} \left(\frac{b\Delta}{\Delta+\beta} + \frac{(5-b)(\Delta+\beta)}{\Delta+2\beta} \right) - (b-1)\gamma\beta \\
\geq& 2(5\beta-1) - \frac{418\beta - 92}{175} \left(\frac{3\Delta}{\Delta+\beta} + \frac{2(\Delta+\beta)}{\Delta+2\beta} \right) - 2\gamma\beta > 0
\end{aligned}$$

as required.

Thus, we may assume that there are exactly three heavy 4-cliques in S_i for $1 \leq i \leq 4$ as each S_i contains the heavy edge u_5u_6. Moreover, $D(S_i) \leq \beta$ for $1 \leq i \leq 4$, otherwise all 4-cliques in S_i are heavy by Lemma 4.1.1 (iii). For $1 \leq i \leq b$,

$$D(T_{i,5}) + D(T_{i,6}) < 2\Delta = 2(1-4\beta)(1+\epsilon)$$

by Claim 6.3.1 (iii). For $b < i \leq 4$, $\widetilde{D}(S_i) = D(T_{i,5}) + D(T_{i,6}) - 2(1-4\beta)$ by (6.7). Also $D_+(T_{i,j}) \leq D(S_i)$ for $b < i \leq 4$. Thus,

$$\begin{aligned} D(T_{i,5}) + D(T_{i,6}) =& \eta_i + 2(1-4\beta) + \frac{418\beta - 92}{175} \sum_{T \in \mathcal{K}_4(S)} \frac{D_+(T)}{D_+(T) + \beta} \\ \leq& \eta_i + 2(1-4\beta) + \gamma\beta D(S)/(D(S)+\beta) \\ \leq& \eta_i + 2(1-4\beta) + \gamma\beta/2 \end{aligned}$$

for $b < i \leq 4$. By Corollary 5.1.2 taking $t = 4$ and $S = S_5, S_6$ and adding the two inequalities together, we have

$$\begin{aligned} 2(2-5\beta) \leq& \sum_{1 \leq i \leq 4} (D_-(T_{i,5}) + D_-(T_{i,6})) + 2D_-(T_{5,6}) \\ \leq& \sum_{1 \leq i \leq 4} (D_-(T_{i,5}) + D_-(T_{i,6})) + 2\beta \\ 2(2-6\beta) \leq& \sum_{1 \leq i \leq b} (D(T_{i,5}) + D(T_{i,6})) + \sum_{b < i \leq 4} (D(T_{i,5}) + D(T_{i,6})) \\ 4(5\beta - 1) <& 2b(1-4\beta)\epsilon + \sum_{b < i \leq 4} \eta_i + (4-b)\gamma\beta/2. \end{aligned} \tag{6.13}$$

If $b = 4$, the above inequality becomes $4(5\beta-1) < 8(1-4\beta)\epsilon \leq 4(5\beta-1)$, which is a contradiction. Hence, $b \leq 3$.

Note that $\sum_{S \in \mathcal{K}_5^{bad}(U)} \widetilde{\eta}(S) > -b\gamma$ by Claim 6.3.1 (v). Recall that $D(S_i) \leq \beta$ for $i \leq 4$. Hence, it is enough to show that $\sum_{b<i\leq 4} \eta_i \geq b\beta\gamma$. Suppose the contrary, so that $\sum \eta_i < b\beta\gamma$. Then, (6.13) becomes

$$4(5\beta - 1) < 2b(1-4\beta)\epsilon + (4+b)\gamma\beta/2 \leq 6(1-4\beta)\epsilon + 7\gamma\beta/2 < 4(5\beta - 1),$$

which is a contradiction. The proof of the claim is complete. □

By Claim 6.3.4, Claim 6.3.5 and Claim 6.3.6, we know that $\sum_{S \in \mathcal{K}_5(U)} \widetilde{\eta}(S) \geq 0$ for all $U \in \mathcal{K}_6^{bad}$. By the remark at the beginning of the section, in proving the inequality in Lemma 6.1.3 it is sufficient to show that $\sum_{S \in \mathcal{K}_5} \eta(S) \geq 0$. Recall that if S is bad, i.e. $\eta(S) < 0$, then S is contained in some 6-clique by Claim 6.3.1, so $D(S) > 0$. Thus, we have

$$\begin{aligned} n \sum_{S \in \mathcal{K}_5} \eta(S) =& n \sum_{S \in \mathcal{K}_4 : D(S) > 0} \eta(S) + n \sum_{S \in \mathcal{K}_4 : D(S) = 0} \eta(S) \\ & \geq n \sum_{S \in \mathcal{K}_4 : D(S) > 0} \eta(S) = \sum_{U \in \mathcal{K}_6} \sum_{S \in \mathcal{K}_5(U)} \widetilde{\eta}(S) \geq 0 \end{aligned}$$

Therefore, we have proved the inequality in Lemma 6.1.3.

Now suppose equality holds. By Claim 6.3.6, Claim 6.3.4 and Claim 6.3.5, no 5-clique is bad. Furthermore, we must have $\eta(S) = 0$ for all $S \in \mathcal{K}_5$. It is easy to check that if the definition of a bad 5-clique includes heavy 5-cliques S with $\eta(S) = 0$, then the argument still holds. Thus, we can deduce that G is K_6-free. By Theorem 4 taking $s = 5$ and $t = 4$, we obtain that (n, β) is feasible, and $G \in \mathcal{G}(n, \beta)$.

□

6.4 Proof of Lemma 6.1.2

In this section, we are going to prove Lemma 6.1.2. We are going to mimic the proof of Theorem 5.2.3 (Section 5.3). Here, T, S and U always denote a 3-clique, 4-clique and 5-clique respectively.

We now give an outline of the proof and show that it is not a straightforward generalisation of the proof of Theorem 5.2.3 as we first thought. Again, we define the appropriate function η on 4-cliques such that $\sum_{S \in \mathcal{K}_4} \eta(S) \geq 0$ would imply the inequality in Lemma 6.1.2. Call a 4-clique S *bad* if $\eta(S) < 0$. Claim 6.4.1 identifies the structure of a bad 4-cliques. Our next task is to show that for every 5-clique U containing a bad 4-clique, $\sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S) \geq 0$, where again $\widetilde{\eta}(S) = \eta(S)/D(S)$. However, the above inequality does not always holds as one could construct a 5-clique U such that $\sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S) < 0$. Observe that $\sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S)$ is actually a function on 5-cliques U. For a 5-clique U, define $\zeta(U)$ to be $\sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S)$. It turns out that $\sum_{U \in \mathcal{K}_5} \zeta(U) \geq 0$ would imply $\sum_{S \in \mathcal{K}_4} \eta(S) \geq 0$. Thus, we consider $\zeta(U)$ in the same way we consider $\eta(S)$, and repeat the same argument on $\zeta(U)$. Thus, a 5-clique U is called *bad* if and only if $\zeta(U) < 0$. It is not surprising that a bad 5-clique is also contained in some 6-clique, which we will verify in Claim 6.4.3. Finally, in Claim 6.4.6, we show that $\sum_{U\in\mathcal{K}_5(W)}\widetilde{\zeta}(U)\geq 0$ for 6-cliques W containing a bad 5-clique, where $\widetilde{\zeta}(U)=\zeta(U)/D(U)$ for $U\in\mathcal{K}_5$ with $D(U)>0$. We point out that the main difficulty is to bound $D(U)$ above and below for a bad 5-clique U.

Now, we recall some basic facts about degrees of cliques. For a 4-clique S, $\widetilde{D}(S)=\sum D_-(T)-(2-4\beta+2D_-(S))$. If S is heavy, then $D_-(S)=\beta$ by Lemma 4.1.1 (iv). Let $T_1,\dots,T_4$ be 3-cliques in S with $D(T_i)\leq D(T_{i+1})$ for $1\leq i\leq 3$. Since $1-3\beta\leq D_-(T)\leq 2\beta$, we have

$$\widetilde{D}(S)=\begin{cases}2(5\beta-1) & \text{if } k_3^+(S)=4,\\ 5\beta-1+(D(T_1)-(1-3\beta)) & \text{if } k_3^+(S)=3,\\ D(T_1)+D(T_2)-2(1-3\beta) & \text{if } k_3^+(S)=2,\end{cases}\tag{6.14}$$

where $k_3^+(S)$ is the number of 3-cliques in S. We often refer to these formlae throughout this section.

Proof of Lemma 6.1.2. Let $\eta:\mathcal{K}_4\to\mathbb{R}$ be a function such that

$$\eta(S)=\widetilde{D}(S)+\frac{13-47\beta}{15}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+2\beta}-\frac{64\beta+4}{35}D_+(S)$$

for $S\in\mathcal{K}_4$. Let $\widetilde{\eta}:\mathcal{K}_4\to\mathbb{R}$ be a function such that $\widetilde{\eta}(S)=\eta(S)/D(S)$ for $S\in\mathcal{K}_4$. Note that $D(S)\geq 1-4\beta>0$ for $S\in\mathcal{K}_4$ by Lemma 4.1.1 (i), so $\widetilde{\eta}(S)$ is well defined. A 4-clique S is called *bad* if $\eta(S)<0$.

Recall that for a heavy 3-clique T, $D(T)=D_+(T)+2\beta$. If $\sum_{S\in\mathcal{K}_4}\eta(S)\geq 0$, then

$$\begin{aligned}0\leq&\sum_{S\in\mathcal{K}_4}\eta(S)=\sum_{S\in\mathcal{K}_4}\widetilde{D}(S)+\frac{13-47\beta}{15}\sum_{S\in\mathcal{K}_4}\sum_{T\in\mathcal{K}_3}\frac{D_+(T)}{D_+(T)+2\beta}-\frac{64\beta+4}{35}\sum_{S\in\mathcal{K}_4}D_+(S)\\ =&\sum_{S\in\mathcal{K}_4}\widetilde{D}(S)+\frac{13-47\beta}{15}n\sum_{T\in\mathcal{K}_3}D_+(T)-\frac{64\beta+4}{35}\sum_{S\in\mathcal{K}_4}D_+(S)\\ \leq&2k_4+2\sum_{S\in\mathcal{K}_4}D_+(S)-2(1-3\beta)\beta nk_3-(1-3\beta)n\sum_{T\in\mathcal{K}_3}D_+(T)\\ &+\frac{13-47\beta}{15}n\sum_{T\in\mathcal{K}_3}D_+(T)-\frac{64\beta+4}{35}\sum_{S\in\mathcal{K}_4}D_+(S),\end{aligned}$$

where the last inequality is due to Lemma 5.1.3 with $t = 3$ and $p = 4$. Rearranging the above inequality, we obtain the inequality in Lemma 5.2.3. Thus, our aim is to show that $\sum_{S\in\mathcal{K}_4} \eta(S) \geq 0$.

Next, we define the functions ζ and $\widetilde{\zeta}$ on 5-cliques. Let $\zeta : \mathcal{K}_5 \to \mathbb{R}$ be a function such that $\zeta(U) = \sum_{S\in\mathcal{K}_4(U)} \widetilde{\eta}(S)$ for $U \in \mathcal{K}_5$. Define $\widetilde{\zeta} : \mathcal{K}_5 \to \mathbb{R}$ to be the function such that $\widetilde{\zeta}(U) = \zeta(U)/D(U)$ for $U \in \mathcal{K}_5$ with $D(U) > 0$. Analogously, a 5-clique U is called *bad* if $\zeta(U) < 0$.

Before identifying the structure of a bad 4-clique, we give a lower bound on $\eta(S)$ for $S \in \mathcal{K}_4$. This lower bound will be used instead of the definition whenever we evaluate $\eta(S)$ unless stated otherwise. Recall that

$$\eta(S) = \widetilde{D}(S) + \frac{13-47\beta}{15} \sum_{T\in\mathcal{K}_3(S)} \frac{D_+(T)}{D_+(T)+2\beta} - \frac{64\beta+4}{35} D_+(S)$$

for $S \in \mathcal{K}_4$. By Corollary 5.1.2, $\widetilde{D}(S) \geq 0$ for $S \in \mathcal{K}_4$, so a bad 4-clique S must be heavy. In addition, Corollary 4.1.3 states that for $S \in \mathcal{K}_4$

$$\widetilde{D}(S) + \sum_{T\in\mathcal{K}_3(S)} D_+(T) \geq 2D_+(S). \tag{6.15}$$

Moreover, this implies that a bad 4-clique contains at least one heavy 3-clique or else $\eta(S) \geq 2D_+(S) - (64\beta+4)D_+(S)/35 > 0$. Also, (6.15) gives an upper bound on $\sum_{T\in\mathcal{K}_3(S)} D_+(T)$. Recall that $D_+(T) \leq D_+(S)$ by Lemma 5.1.3 (*iii*). Hence

$$\begin{aligned}
\eta(S) =& \widetilde{D}(S) + \frac{13-47\beta}{15} \sum_{T\in\mathcal{K}_3(S)} \frac{D_+(T)}{D_+(T)+2\beta} - \frac{64\beta+4}{35} D_+(S) \\
\geq& \widetilde{D}(S) + \frac{13-47\beta}{15(D_+(S)+2\beta)} \sum_{T\in\mathcal{K}_3(S)} D_+(T) - \frac{64\beta+4}{35} D_+(S) \\
\overset{(6.15)}{\geq}& \widetilde{D}(S) + \frac{13-47\beta}{15(D_+(S)+2\beta)} \left(2D_+(S) - \widetilde{D}(S)\right) - \frac{64\beta+4}{35} D_+(S) \\
\eta(S) \geq& \left(1 - \frac{13-47\beta}{15(D_+(S)+2\beta)}\right) \widetilde{D}(S) - \left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{15(D_+(S)+2\beta)}\right) D_+(S).
\end{aligned} \tag{6.16}$$

We will refer to this lower bound on $\eta(S)$ throughout this section.

In the next claim, we identify the structure of a bad 4-clique S. The proof is very similar to Claim 5.3.1.

Claim 6.4.1. *Suppose S is a bad 4-clique. Let*

$$\Delta = \frac{32(1-4\beta)(92\beta-13)}{45\beta(31-16\beta)} \text{ and } \gamma = \frac{720\beta^2+1549\beta-416}{720\beta}.$$

Then, the following hold:

(i) *S contains exactly one heavy edge and two heavy 3-cliques,*

(ii) $0 < D(S) < \Delta$,

(iii) $D(T) + D(T') < 2(\Delta+\beta)$, *where T and T' are the two non-heavy 3-cliques in S,*

(iv) $\eta(S) \geq -\gamma D_+(S)$,

(v) $\widetilde{\eta}(S) \geq -\gamma\Delta/(\Delta+\beta)$.

Proof. First, we show that *(ii)* implies both *(iv)* and *(v)*. By *(ii)*, $D_+(S) < \Delta \leq \beta$. Recall that $\widetilde{D}(S) \geq 0$ by Corollary 5.1.2, so (6.16) becomes

$$\begin{aligned}
\eta(S) &\geq \left(1 - \frac{13-47\beta}{15(D_+(S)+2\beta)}\right)\widetilde{D}(S) - \left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{15(D_+(S)+2\beta)}\right)D_+(S) \\
&\geq -\left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{15(D_+(S)+2\beta)}\right)D_+(S) \\
&\geq -\left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{45\beta}\right)D_+(S) = -\gamma D_+(S),
\end{aligned}$$

so *(iv)* holds, and *(v)* easily follows as $D(S) = D_+(S) + \beta < \Delta + \beta$.

Next, we are going to show that *(ii)* holds, and S contains two heavy 3-cliques in S. By the discussion earlier, S must contain a heavy 3-clique and S is heavy. We separate cases by the number of heavy 3-cliques in S.

First, suppose all 3-cliques are heavy. Hence, $\widetilde{D}(S) = 2(5\beta-1)$ by (6.14). Clearly from the definition $D_+(S) = D(S) - \beta \leq 1-\beta$. By (6.16), we have

$$\eta(S) \geq 2(5\beta-1)\left(1 - \frac{13-47\beta}{15(D_+(S)+2\beta)}\right) - \left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{15(D_+(S)+2\beta)}\right)D_+(S).$$

We claim the above inequality is strictly positive, so S is good which is a contradiction. Notice the right hand side of the inequality is a concave function in $D_+(S)$. Hence, it is enough to check the inequality at the boundary points of $D_+(S)$. If $D_+(S) = 0$, then $\eta(S) \geq 0$. If $D_+(S) = 1 - \beta$, then

$$\begin{aligned}\eta(S) \geq & 2(5\beta - 1)\left(1 - \frac{13 - 47\beta}{15\beta}\right) - \left(\frac{64\beta + 4}{35} - \frac{2(13 - 47\beta)}{15\beta}\right)(1 - \beta) \\ = & \frac{240\beta^3 + 11439\beta^2 - 3824\beta + 337}{240(1 + \beta)} > 0.\end{aligned}$$

Thus, there is at least one 3-clique in S that is not heavy. By Lemma 4.1.1 (*iii*), we have $D_+(T) \geq D_+(S) - \beta$ for $T \in \mathcal{K}_3(S)$, so $0 < D_+(S) \leq \beta$ or else all 3-cliques in S are heavy.

Suppose there are exactly three heavy 3-cliques in S, so $\widetilde{D}(S) \geq 5\beta - 1$ by (6.14). Hence, by (6.16), we have

$$\eta(S) \geq (5\beta - 1)\left(1 - \frac{13 - 47\beta}{15(D_+(S) + 2\beta)}\right) - \left(\frac{64\beta + 4}{35} - \frac{2(13 - 47\beta)}{15(D_+(S) + 2\beta)}\right) D_+(S).$$

Again, we are going to show that the inequality is strictly positive. The right hand side of the inequality is a concave function of $D_+(S)$, so it is enough to check at the boundary points. If $D_+(S) = 0$, then the right hand side is positive. If $D_+(S) = \beta$, then

$$\begin{aligned}\eta(S) \geq & (5\beta - 1)\left(1 - \frac{13 - 47\beta}{45\beta}\right) - \left(\frac{64\beta + 4}{35} - \frac{2(13 - 47\beta)}{45\beta}\right)\beta \\ = & \frac{208 - 2096\beta + 5811\beta^2 - 720\beta^3}{720\beta} > 0.\end{aligned}$$

Thus, S contains less than three heavy 3-cliques.

Now suppose, there is only one heavy 3-clique T in S. Since $D_+(T) \leq D_+(S)$

by Lemma 4.1.1 (*iii*), (6.15) implies that $\widetilde{D}(S) \geq D_+(S)$. Thus, (6.16) becomes

$$\begin{aligned}\eta(S) \geq &\left(1 - \frac{13-47\beta}{15(D_+(S)+2\beta)}\right) D_+(S) - \left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{15(D_+(S)+2\beta)}\right) D_+(S)\\ =&\frac{15(15-16\beta)D_+(S)+208-302\beta-480\beta^2}{240(D_+(S)+2\beta)} D_+(S)\\ \geq&\frac{(208-302\beta-480\beta^2)D_+(S)}{240(D_+(S)+2\beta)} > 0,\end{aligned}$$

so S is good, which is a contradiction.

Therefore, S has exactly two heavy 3-cliques. Let T_1 and T_2 be the non-heavy 3-cliques in S. Hence, by (6.14)

$$\begin{aligned}\widetilde{D}(S) =&D(T_1) + D(T_2) - 2(1-3\beta)\\ \geq&2(D(S) - (1-3\beta)) = 2(D_+(S) - (1-4\beta)).\end{aligned}$$

If $D(S) \geq \Delta$, then (6.16) becomes

$$\begin{aligned}\eta(S) \geq&2(D_+(S) - (1-4\beta))\left(1 - \frac{13-47\beta}{15(D_+(S)+2\beta)}\right)\\ &-\left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{15(D_+(S)+2\beta)}\right) D_+(S)\\ \geq&2(D_+(S) - (1-4\beta))\left(1 - \frac{13-47\beta}{45\beta}\right) - \left(\frac{64\beta+4}{35} - \frac{2(13-47\beta)}{45\beta}\right) D_+(S)\\ =&\frac{31-16\beta}{16} D_+(S) - \frac{2(1-4\beta)(92\beta-13)}{45\beta}\\ \geq&\frac{31-16\beta}{16}\Delta - \frac{2(1-4\beta)(92\beta-13)}{45\beta} = 0.\end{aligned}$$

This is a contradiction as S is assumed to be bad. Hence, (*ii*) holds.

Furthermore, if $D(T_1) + D(T_2) \geq 2(\Delta + \beta)$, then by (6.14) $\widetilde{D}(S) \geq 2(\Delta - (1 -$

$4\beta)$). Again, by (6.16) with $D_+(S) < \Delta \leq \beta$,

$$\begin{aligned}\eta \geq & 2(\Delta-(1-4\beta))\left(1-\frac{13-47\beta}{15(D_+(S)+2\beta)}\right)\\ & -\left(\frac{64\beta+4}{35}-\frac{2(13-47\beta)}{15(D_+(S)+2\beta)}\right)D_+(S)\\ \geq & 2(\Delta-(1-4\beta))\left(1-\frac{13-47\beta}{45\beta}\right)-\left(\frac{64\beta+4}{35}-\frac{2(13-47\beta)}{45\beta}\right)\Delta=0,\end{aligned}$$

which is a contradiction. Thus, (iii) holds.

Therefore, we are left to prove (i), i.e. that S contains exactly one heavy edge and two heavy 3-cliques. We have already shown that S contains exactly two heavy 3-cliques. Since a 3-clique containing a heavy edge is heavy by Lemma 4.1.1 (iii), there is at most one heavy edge in S or else S has more than two heavy 3-cliques. Thus, it is enough to show that S contains exactly one heavy edge. For the remainder of the proof, assume that S does not contain any heavy edge and $D(S) < \Delta$. Let T_3 and T_4 be the heavy 3-cliques in S with $D_+(T_3) \leq D_+(T_4) \leq D_+(S) \leq \beta$. Here, we introduce a lower bound on $\eta(S)$ which differs from (6.16). Observe that (6.15) also gives an upper bound on $D_+(S)$. Then, from the definition of $\eta(S)$, we have

$$\begin{aligned}\eta(S) \geq & \widetilde{D}(S)-\frac{32\beta+2}{35}\left(\widetilde{D}(S)+\sum_{T\in\mathcal{K}_3(S)}D_+(T)\right)+\frac{13-47\beta}{15}\sum_{T\in\mathcal{K}_3(S)}\frac{D_+(T)}{D_+(T)+2\beta}\\ = & \frac{33-32\beta}{35}\widetilde{D}(S)-\sum_{T\in\mathcal{K}_3(S)}\left(\frac{32\beta+2}{35}-\frac{13-47\beta}{15(D_+(T)+2\beta)}\right)D_+(T)\\ \geq & \frac{33-32\beta}{35}\widetilde{D}(S)-2D_+(T_4)\left(\frac{32\beta+2}{35}-\frac{13-47\beta}{45\beta}\right)\\ = & \frac{33-32\beta}{35}\widetilde{D}(S)-\frac{2(288\beta^2+347\beta-91)}{315\beta}D_+(T_4). \qquad (6.17)\end{aligned}$$

Next, we claim that $\widetilde{D}(S) \geq D_+(T_4)$. Let $e_i = T_i \cap T_4$ be an edge of T_4 for $i =$

$1, 2, 3$. By Corollary 4.1.3 taking $S = T_4$ and $t = 2$, we obtain

$$\begin{aligned} D(e_1) + D(e_2) + D(e_3) \geq & 2 - 3\beta + D(T_4) \\ D(e_1) + D(e_2) \geq & 2 - 4\beta + D_+(T_4) \end{aligned}$$

as $D(e_3) \leq 3\beta$ and $D_-(T_4) = 2\beta$. By Lemma 4.1.1 (ii), we get

$$D(T_1) + D(T_2) \geq D(e_1) + D(e_2) - 2\beta \geq 2(1 - 3\beta) + D_+(T_4).$$

Hence, $\widetilde{D}(S) \geq D_+(T_4)$ by (6.14). Therefore, by (6.17)

$$\begin{aligned} \eta(S) \geq & \frac{(33 - 32\beta)D_+(T_4)}{35} - \frac{2(288\beta^2 + 347\beta - 91)}{315\beta} D_+(T_4) \\ = & \frac{91 - 242\beta - 288\beta^2}{105\beta} D_+(T_4) > 0, \end{aligned}$$

which is a contradiction This completes the proof of the claim. □

Thus, by Claim 6.4.1 (iv), $\beta > 0.241$ for the remainder of the proof, else $\gamma < 0$ and so all 4-cliques are not bad. If a 4-clique S contains at least three heavy 3-cliques, $\widetilde{\eta}(S) \geq 0$. In the following claim, we bound $\widetilde{\eta}(S)$ away from zero provided $D(S)$ is not too large.

Claim 6.4.2. *Suppose S is a 4-clique with at least three heavy 3-cliques. If S contains exactly three heavy 3-cliques, then*

$$\widetilde{\eta}(S) \geq 4\gamma\Delta/(\Delta + \beta) + 3\gamma(2\beta - 11(5\beta - 1)/16)/\beta.$$

If all 3-cliques in S are heavy and $D_+(S) \leq 2\beta - 11(5\beta - 1)/16$, then

$$\widetilde{\eta}(S) \geq 4\gamma\Delta/(\Delta + \beta) + 6\gamma(2\beta - 11(5\beta - 1)/16)/\beta.$$

Proof. Suppose S contains exactly three heavy 3-cliques. Recall that $\eta(S) = \widetilde{\eta}(S)D(S)$. Since not all 3-cliques in S are heavy, $D_+(S) \leq \beta$ and $D(S) \leq 2\beta$ by

Lemma 4.1.1 (*iii*). By (6.16), we obtain

$$\begin{aligned}
\eta(S) \geq & (5\beta - 1)\left(1 - \frac{13 - 47\beta}{15(D_+(S) + 2\beta)}\right) - \left(\frac{64\beta + 4}{35} - \frac{2(13 - 47\beta)}{15(D_+(S) + 2\beta)}\right) D_+(S) \\
\geq & (5\beta - 1)\left(1 - \frac{13 - 47\beta}{45\beta}\right) - \left(\frac{64\beta + 4}{35} - \frac{2(13 - 47\beta)}{45\beta}\right)\beta \\
\geq & \left(\frac{4\gamma\Delta}{\Delta + \beta} + \frac{3\gamma(2\beta - 11(5\beta - 1)/16)}{\beta}\right) 2\beta \\
\geq & \left(\frac{4\gamma\Delta}{\Delta + \beta} + \frac{3\gamma(2\beta - 11(5\beta - 1)/16)}{\beta}\right) D(S).
\end{aligned}$$

Thus, the first assertion of the claim is true.

Now, suppose all 4-cliques in S are heavy and $D_+(S) \leq 2\beta - 11(5\beta - 1)/16$. Note that $\widetilde{D}(S) = 2(5\beta - 1)$ by (6.14). Thus, (6.16) becomes

$$\begin{aligned}
\eta(S) \geq & 2(5\beta - 1)\left(1 - \frac{13 - 47\beta}{15(D_+(S) + 2\beta)}\right) - \left(\frac{64\beta + 4}{35} - \frac{2(13 - 47\beta)}{15(D_+(S) + 2\beta)}\right) D_+(S) \\
\geq & 2(5\beta - 1)\left(1 - \frac{13 - 47\beta}{15(4\beta - 11(5\beta - 1)/16)}\right) \\
& - \left(\frac{64\beta + 4}{35} - \frac{2(13 - 47\beta)}{15(4\beta - 11(5\beta - 1)/16)}\right)(2\beta - 11(5\beta - 1)/16) \\
\geq & \left(\frac{4\gamma\Delta}{\Delta + \beta} + \frac{3\gamma(2\beta - 11(5\beta - 1)/16)}{\beta}\right)(3\beta - 11(5\beta - 1)/16) \\
\geq & \left(\frac{4\gamma\Delta}{\Delta + \beta} + \frac{3\gamma(2\beta - 11(5\beta - 1)/16)}{\beta}\right) D(S)
\end{aligned}$$

as required. □

The structure of a bad 5-clique U is identified in the following claim. Recall that a 5-clique U is bad if $\zeta(U) = \sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S) < 0$. It turns out that a bad 5-clique U has similar structure to a bad 4-clique S, namely both have exactly one heavy edge e, and every heavy subclique contains e. We have shown in Claim 6.4.1 that $D_+(S) < \Delta$ and the right hand side tends to zero as β tends to 1/4. However, the upper bound on $D_+(U)$ that we obtain does not tend to zero as β tends to 1/4.

Claim 6.4.3. *Suppose U is a bad 5-clique. Then the following hold,* *(i)* *U contains exactly one heavy edge e and all heavy cliques in U contain e,*

(ii) $0 < D(U) < \beta - 11(5\beta - 1)/16$,

(iii) $D(S) < \beta - 11(5\beta - 1)/16$ *for* $S \in \mathcal{K}_4(U)$ *not heavy,*

(iv) $\zeta(U) > -3\gamma D(U)/(D(U) + \beta)$,

(v) $\widetilde{\zeta}(U) > -3\gamma/\beta$.

Proof. Let $u_1, \ldots, u_5$ be the vertices of U. Write S_i, η_i and $\widetilde{\eta}_i$ to be $U - u_i$, $\eta(S_i)$ and $\widetilde{\eta}(S_i)$ respectively for $1 \leq i \leq 5$. Similarly, we denote by $T_{i,j}$, the 3-clique $U - u_i - u_j$ for $1 \leq i < j \leq 5$. Since $\zeta(U) < 0$, U contains a bad 4-clique. Without loss of generality, S_1 is a bad 4-clique in U. Thus, $D_+(S_1) \leq \Delta$ by Claim 6.4.1 *(ii)*. Also, U contains at least one heavy edge, as S_1 contains one by Claim 6.4.1 *(i)*. Moreover, $\sum_{S \in \mathcal{K}_4^{bad}(U)} \widetilde{\eta}(S) > -b\gamma\Delta/(\Delta + \beta)$ by Claim 6.4.1 *(v)*.

For $2 \leq i \leq 5$, we have

$$D(S_i) \leq D(T_{1,i}) = D_+(T_{1,i}) + 2\beta \leq D_+(S_1) + 2\beta < \Delta + 2\beta \tag{6.18}$$

by Lemma 4.1.1 and Claim 6.4.1 *(ii)*. Note that $D(S_i) < \Delta + 2\beta < 3\beta - 11(5\beta - 1)/16$. If there exists a S_i containing at least three heavy 3-cliques, then $\widetilde{\eta}(S) \geq 4\gamma\Delta/(\Delta+\beta)$ by Claim 6.4.2, so $\zeta(U) > 0$ as $b \leq 4$. This is a contradiction as U is bad. Thus, all S_i contain at most two heavy 3-cliques. Hence, $D_+(S_i) \leq \beta$ for $1 \leq i \leq 5$, by Lemma 4.1.1 *(iii)*. Recall that a 3-clique containing a heavy edge is itself heavy by Lemma 4.1.1 *(iii)*. Therefore, U contains exactly one heavy edge, say u_4u_5, otherwise there exists a 4-clique containing at least three heavy 3-cliques namely the 4-clique contains two heavy edges. Furthermore, we may assume that $S_1, \ldots, S_b$ are the bad 4-cliques in U with $b \leq 3$.

Note that $D(U) = D_+(U) \geq D_+(S_i)$ by Lemma 4.1.1 *(iii)*. By Claim 6.4.1 *(iv)*,

$$\begin{aligned}\zeta(U) \geq \sum_{1 \leq i \leq b} \widetilde{\eta}_i > -\gamma \sum_{1 \leq i \leq b} D_+(S_i)/(D_+(S_i) + \beta) \\ \geq - b\gamma D_+(U)/(D_+(U) + \beta) \geq -3\gamma D(U)/(D(U) + \beta).\end{aligned}$$

Thus, *(iv)* of the claim is true. Also, *(v)* is an easy consequence of *(iv)*.

Next, we are going to show that (iii) implies (i) and (ii). Since $D(U) \le D(S_i)$, so (iii) easily implies (ii). If (iii) is true, then neither $T_{4,5}$, S_4 nor S_5 is not heavy. Recall that a clique containing a heavy subclique is itself heavy and each S_i contains at most two heavy 3-cliques. Therefore, for $1 \le i \le 3$, $T_{i,4}$ and $T_{i,5}$ are not heavy. Hence, (i) is true. Therefore, it is enough to prove (iii).

Suppose $D_-(T_{4,5}) > D_-(S_4) + D_-(S_5) + 11(5\beta - 1)/16$. Recall from the definition of D_- that $D_-(T_{4,5}) \le 2\beta$ and $D_-(S_4), D_-(S_5) \le \beta$. Thus, at least one of $D_-(S_4)$ and $D_-(S_5)$ is strictly less than 2β, i.e. one of S_4 and S_5 is not heavy. This implies that $T_{4,5}$ is not heavy by Lemma 4.1.1 (iii). If S_4 is not heavy, then by Lemma 4.1.1 (i) we have

$$D(S_4) + D_-(S_5) + 11(5\beta - 1)/16 < D_-(T_{4,5}) = D(T_{4,5}) \le D(S_4) + \beta, \quad (6.19)$$

so $D_-(S_5) \le \beta - 11(5\beta - 1)/16$. By a similar argument, $D_-(S_4) \le \beta - 11(5\beta - 1)/16$. This implies (iii).

Finally, suppose $D_-(T_{4,5}) \le D_-(S_4) + D_-(S_5) + 11(5\beta - 1)/16$. Applying Corollary 5.1.2 to S_4 and S_5 taking $t = 3$, and adding the two inequalities together, we obtain

$$\begin{aligned}\sum_{1 \le i \le 3} (D_-(T_{i,4}) + D_-(T_{i,5})) &\ge 4(1 - 2\beta) - 2(D_-(T_{4,5}) - D_-(S_4) - D_-(S_5)) \\ &\ge 4(1 - 2\beta) - 11(5\beta - 1)/8 \\ &= 6(1 - 3\beta) + 5(5\beta - 1)/8 \qquad (6.20)\end{aligned}$$

Next, we bound each summand in the left hand side of (6.20) separately. Recall that each S_i contains at most two heavy 3-cliques. Since S_i contains a heavy edge for $1 \le i \le 3$, it has exactly two heavy 3-cliques. Thus, $T_{i,4}$ and $T_{i,5}$ are both non-heavy for $1 \le i \le 3$. Also, $D_+(S_i) \le \beta$ by Lemma 4.1.1 (iii). For $1 \le i \le b$, we have

$$D(T_{i,4}) + D(T_{i,5}) < 2(\Delta + \beta) = 2(1 - 3\beta) + (\Delta - (1 - 4\beta))$$

by Claim 6.4.1 (iii). For $b < i \le 3$, $\widetilde{D}(S_i) = D(T_{i,4}) + D(T_{i,5}) - 2(1 - 3\beta)$

by (5.12). By (6.16), we have

$$\begin{aligned}\eta_i \geq& \left(1-\frac{13-47\beta}{15(D_+(S_i)+2\beta)}\right)\widetilde{D}(S_i)-\left(\frac{64\beta+4}{35}-\frac{2(13-47\beta)}{15(D_+(S_i)+2\beta)}\right)D_+(S_i)\\ \geq& \left(1-\frac{13-47\beta}{45\beta}\right)\widetilde{D}(S_i)-\left(\frac{64\beta+4}{35}-\frac{2(13-47\beta)}{45\beta}\right)\beta\\ =&\frac{92\beta-13}{45\beta}(D(T_{i,4})+D(T_{i,5})-2(1-3\beta))-\gamma\beta.\end{aligned}$$

Hence,

$$D(T_{i,4})+D(T_{i,5})\leq 2(1-3\beta)+\frac{45\beta}{92\beta-13}(\eta_i+\gamma\beta) \tag{6.21}$$

for $b<i\leq 3$. Therefore, (6.20) becomes

$$5(5\beta-1)/8\leq 2b(\Delta-(1-4\beta))+\frac{45\beta}{92\beta-13}\sum_{b<i\leq 3}\eta_i+(3-b)\frac{45\gamma\beta^2}{92\beta-13}. \tag{6.22}$$

If $b=3$, the above inequality becomes

$$5(5\beta-1)/8 \leq 6(\Delta-(1-4\beta))<5(5\beta-1)/8$$

which is a contradiction. Thus, $b\leq 2$. Recall that $D(S_i)\leq\beta+D_+(S_i)\leq 2\beta$ for $1\leq i\leq 3$ and $\sum_{1\leq i\leq b}\widetilde{\eta}_i>-b\gamma\Delta/(\Delta+\beta)$. Since U is bad, $\sum_{b<i\leq 3}\widetilde{\eta}(S_i)<b\gamma\Delta/(\Delta+\beta)$. Thus, $\sum_{b<i\leq 3}\eta_i\leq 2\beta b\gamma\Delta/(\Delta+\beta)$, so (6.20) becomes

$$5(5\beta-1)/8 < 2b(\Delta-(1-4\beta))+\frac{45\beta}{92\beta-13}\frac{2\beta b\gamma\Delta}{\Delta+\beta}+(3-b)\frac{45\gamma\beta^2}{92\beta-13}.$$

It can be checked that for $b=1,2$, the right hand side is strictly less than $5(5\beta-1)/8$, which is a contradiction. This completes the proof of the claim. □

Now suppose U is a 5-clique with exactly one heavy edge. Moreover, $D(S)+D(S')\geq 2\beta-11(5\beta-1)/8$, where S and S' are the two 4-cliques that do not contain the heavy edge. This means that U is not bad, as it fails condition (iii) of Claim 6.4.3. Thus, $\widetilde{\zeta}(U)\geq 0$. The next claim shows that $\widetilde{\zeta}(U)$ is strictly greater than zero by a positive amount provided $D(U)$ is not too large. The proof is very similar to the proof of Claim 6.4.3.

Claim 6.4.4. *Let U be a 5-clique. Suppose U has exactly one heavy edge and $D(U) \leq 2\beta - 11(5\beta - 1)/16$. Futhermore, suppose that $D(S) + D(S') \geq 2\beta - 11(5\beta - 1)/8$, where S and S' are the two 4-cliques that does not contain the heavy edge. Then, $\widetilde{\zeta}(U) \geq 3\gamma/\beta$.*

Proof. Write $c = 11(5\beta - 1)/16$. Let $u_1, \ldots, u_5$ be the vertex of U and u_4u_5 be the heavy edge. Let $S_i = U - u_i$ for $1 \leq i \leq 5$ and $T_{i,j} = U - u_i - u_j$ for $1 \leq i < j \leq 5$. Thus, $D(S_4) + D(S_5) \geq 2\beta - 2c$. Also, notice that $D(S_i) \leq D(U) \leq 2\beta - c$ for $1 \leq i \leq 5$. If there exists a 4-clique S_i in U containing at least three heavy 3-cliques, then

$$\widetilde{\eta}(S_i) \geq \frac{4\gamma\Delta}{\Delta + \beta} + \frac{3\gamma(2\beta - c)}{\beta} \geq \frac{4\gamma\Delta}{\Delta + \beta} + \frac{3\gamma}{\beta}D(U)$$

by Claim 6.4.2. Thus, $\zeta(U) \geq 3\gamma D(U)/\beta$, because there are at most four bad 4-cliques S' in U and $\widetilde{\eta}(S') > -\gamma\Delta/(\Delta + \beta)$ by Claim 6.4.1. Hence, $\widetilde{\zeta}(U) \geq 3\gamma/\beta$.

Suppose S_i contains at most two heavy 3-cliques for $1 \leq i \leq 5$. Recall that $D(S) \leq D(U) \leq 2\beta - c$, so $D_+(S_i) \leq \beta - c$ for $1 \leq i \leq 5$. If $D_-(T_{4,5}) > D_-(S_4) + D_-(S_5) + c$, then $T_{4,5}$ is not heavy and one of S_4 and S_5 is also not heavy. By (6.19), we have $D(S_i) < \beta - c$ for $i = 4, 5$, which contradicts the hypothesis of the claim.

Finally, suppose $D_-(T_{4,5}) \leq D_-(S_4) + D_-(S_5) + c$. Let b be the number of bad 4-cliques in U. Without loss of generality, we may assume that $S_1, \ldots, S_b$ are the bad 4-cliques in U if $b > 0$. By the same argument in the proof of Claim 6.4.3, we obtain (6.22). Note that $D_+(S_i) \leq \beta - c$ for $1 \leq i \leq 3$. For $b < i \leq 3$, the equivalent version of (6.21) is

$$D(T_{i,4}) + D(T_{i,5}) \leq 2(1 - 3\beta) + \frac{45\beta}{92\beta - 13}\left(\eta(S_i) + \gamma(\beta - c)\right).$$

Thus, we have the following stronger version of (6.22):

$$5(5\beta-1)/8 \leq 2b(\Delta-(1-4\beta))+\frac{45\beta}{92\beta-13}\sum_{b<i\leq 3}\eta(S_i)+(3-b)\frac{45\gamma\beta(\beta-c)}{92\beta-13}. \tag{6.23}$$

Again, the case $b=3$ implies a contradiction, so $b \leq 2$. Since $D(U) \leq 2\beta - c$, it is enough to show that $\zeta(U)=\sum_{1\leq i\leq 5}\widetilde{\eta}(S_i) \geq 3\gamma(2\beta-c)/\beta$. Observe that this would follow from the inequality

$$\sum_{b<i\leq 3}\eta(S_i) \geq (2\beta-c)\left(\frac{b\gamma\Delta}{\Delta+\beta}+\frac{3\gamma}{\beta}(2\beta-c)\right).$$

Suppose this inequality fails, so (6.22) becomes

$$\begin{aligned}5(5\beta-1)/8 <& 2b(\Delta-(1-4\beta))+\frac{45\beta}{92\beta-13}(2\beta-c)\left(\frac{b\gamma\Delta}{\Delta+\beta}+\frac{3\gamma}{\beta}(2\beta-c)\right)\\&+(3-b)\frac{45\gamma\beta(\beta-c)}{92\beta-13}.\end{aligned}$$

It can be checked that for $0 \leq b \leq 2$, the right hand side is strictly less than $5(5\beta-1)/8$, which is a contradiction. The proof of the claim is complete. □

Note that a bad 5-clique U is contained in a 6-clique as $D(U) > 0$. The next claim shows the relationship between the number of heavy edges and bad 5-cliques in a 6-clique. In fact, the statement is identical to Claim 6.3.2. Hence, the proof is omitted.

Claim 6.4.5. *Suppose W is a 6-clique containing $h \geq 2$ heavy edges and b bad 5-cliques. Then $b \leq 2h/(h-1) = 2+2/(h-1)$. Moreover, if there exist two heavy edges sharing a common vertex, then $b \leq 3$.* □

Now we show that for every 6-clique W, $\sum_{U\in\mathcal{K}_5(W)}\widetilde{\zeta}(U) \geq 0$.

Claim 6.4.6. *Suppose W is a 6-clique. Then $\sum_{U\in\mathcal{K}_5(W)}\widetilde{\zeta}(U) \geq 0$.*

Proof. We may assume that W contains at least one bad 5-clique or there is nothing to prove. Since a bad 5-clique contains a heavy edge by Claim 6.4.3 (i),

W also contains at least one heavy edge. We claim that $D(U) \leq 2\beta - 11(5\beta-1)/16$ for all 5-cliques in W. If U is bad, then it is true by Claim 6.4.3 (ii). If U is not bad,

$$D(U) \leq D(U \cap U^b) = \beta + D_+(U \cap U^b) \leq D_+(U^b) < 2\beta - 11(5\beta-1)/16 \quad (6.24)$$

where U^b is a bad 5-clique in W. We separate cases by the number of heavy edges in W.

First, suppose W contains two heavy edges e and e'. If e and e' share a common vertex, there are at most three bad 5-cliques in W by Claim 6.4.5. Thus, $\sum_{U \in \mathcal{K}_5^{bad}(W)} \widetilde{\zeta}(U) > -9\gamma/\beta$ by Claim 6.4.3 (v). Note that there are three 5-cliques U containing both e and e'. Thus, it is enough to show that $\widetilde{\zeta}(U) \geq 3\gamma/\beta$ for each such U. Within U, there exists a 4-clique S containing both e and e'. Hence, S contains at least three heavy 3-cliques. By Claim 6.4.2, $\widetilde{\eta}(S) \geq 4\gamma\Delta/(\Delta+\beta) + 3\gamma(2\beta - 11(5\beta-1)/16)$. There are at most four bad 4-cliques S' in U with $\widetilde{\eta}(S') \geq -\gamma\Delta/(\Delta+\beta)$ by Claim 6.4.1 (v). This implies $\zeta(U) \geq 3\gamma(2\beta - 11(5\beta-1)/16)/\beta$, i.e. $\widetilde{\zeta}(U) \geq 3\gamma/\beta$ as required.

If e and e' are vertex disjoint, there are at most four bad 5-cliques in W by Claim 6.4.5. Thus, $\sum_{U \in \mathcal{K}_5^{bad}(W)} \widetilde{\zeta}(U) > -12\gamma/\beta$. Note that there are two 5-cliques U containing both e and e'. Thus, it is enough to show that $\widetilde{\zeta}(U) \geq 6\gamma/\beta$ for such U. Within U, there exists a 4-clique S containing both e and e'. Moreover, all 3-cliques in S are heavy. Thus, $\widetilde{\eta}(S) \geq 4\gamma\Delta/(\Delta+\beta) + 6\gamma(2\beta - 11(5\beta-1)/16)$ by Claim 6.4.2. By a similar argument, $\widetilde{\zeta}(U) \geq 6\gamma/\beta$.

Suppose W contains exactly one heavy edge. Let $w_1, \dots, w_6$ be the vertices of W and w_5w_6 be the heavy edge. Write $U_i = W - w_i$ for $1 \leq i \leq 6$. Since a bad 5-clique contains a heavy edge by Claim 6.4.3 (i), if U_i is a bad 5-clique, then $i \leq 4$. Without loss of generality, $U_1, \dots, U_b$ are the bad 5-cliques in W. Similarly, denote by $S_{i,j}$ the 4-clique $W - w_i - w_j$ for $1 \leq i < j \leq 6$, so $\sum_{U \in \mathcal{K}_5(W)} \widetilde{\zeta}(U) \geq 0$.

Let g be the number of 5-cliques which satisfy the hypothesis of Claim 6.4.4. Clearly if U_i satisfies the hypothesis of Claim 6.4.4 then U_i is not bad, so $b < i \leq 4$. In addition, if $b \leq g$ then $\sum_{U \in \mathcal{K}_5(W)} \widetilde{\zeta}(U) > 0$ by Claim 6.4.3 (v) and Claim 6.4.4. As a consequence, $g \leq 1$, and if $g = 1$ then we may assume that U_4 is such a

5-clique. Therefore for $1 \leq i \leq 4-g$,

$$D(S_{i,5}) + D(S_{i,6}) \leq 2(\beta - 11(5\beta - 1)/16)$$

by Claim 6.4.3 (*iii*) and Claim 6.4.4.

By applying Corollary 5.1.2 to both U_5 and U_6 taking $t = 4$, and adding the two inequalities together, we obtain

$$\begin{aligned}
2(2-5\beta) \leq & \sum_{1 \leq i \leq 4} (D_-(S_{i,5}) + D_-(S_{i,6})) + 2D_-(S_{5,6}) \\
2(2-6\beta) \leq & \sum_{1 \leq i \leq 4-g} (D(S_{i,5}) + D(S_{i,6})) + \sum_{4-g < i \leq 4} (D(S_{i,5}) + D(S_{i,6})) \\
\leq & 2(4-g)(\beta - 11(5\beta-1)/16) + 2g\beta \\
11(4-g)(5\beta-1)/16 \leq & 2(5\beta - 1).
\end{aligned}$$

Since $g \leq 1$, the above inequality leads to a contradiction. □

Recall that in proving Lemma 6.1.2, it is sufficient to show that $\sum_{S \in \mathcal{K}_4} \eta(S) \geq 0$. For $U \in \mathcal{K}_5$ with $D(U) = 0$, $\zeta(U) \geq 0$ by Claim 6.4.3. Therefore,

$$\begin{aligned}
n^2 \sum_{S \in \mathcal{K}_4} \eta(S) = & n \sum_{U \in \mathcal{K}_5} \sum_{S \in \mathcal{K}_4(U)} \widetilde{\eta}(S) = n \sum_{U \in \mathcal{K}_5} \zeta(U) \\
= & \sum_{W \in \mathcal{K}_6} \sum_{U \in \mathcal{K}_5(W)} \widetilde{\zeta}(U) + n \sum_{U \in \mathcal{K}_4 : D(U) = 0} \zeta(U) \geq \sum_{W \in \mathcal{K}_6^{bad}} \sum_{U \in \mathcal{K}_5(W)} \widetilde{\zeta}(U) > 0.
\end{aligned}$$

The inequality follows from Claim 6.4.6. The proof of Lemma 6.1.2 is complete. □

Chapter 7

$k_r(n,\delta)$ for $\delta > 4n/5$

In this chapter, we look at Conjecture 3.1.1 for $p \geq 5$. As we have mentioned after Corollary 5.2.2 in Section 5.2.1, it is more difficult to prove Conjecture 3.1.1 for $\beta = 1/p - \epsilon$ than for $\beta = 1/(p+1) + \epsilon$, with small $\epsilon > 0$ depending on p. This is because in order to prove the case $\beta = 1/p - \epsilon$, we require a generalisation of the proof of Theorem 5.2.3. Without such generalisation, we are going to prove the following theorem.

Theorem 7.a. *Let $0 < \beta < 1/5$ and $p = \lceil \beta^{-1} \rceil - 1$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then*

$$\frac{k_s(G)}{g_s(\beta)n^s} \geq \frac{k_t(G)}{g_t(\beta)n^t}$$

for $r(\beta) \leq t < s \leq p+1$. Moreover, the following three statements are equivalent:

(i) Equality holds for some $r(\beta) \leq t < s \leq p+1$.

(ii) Equality holds for all $r(\beta) \leq t < s \leq p+1$.

(iii) The pair (n,β) is feasible and G is a member of $\mathcal{G}(n,\beta)$.

The definition of $r(\beta)$ is given in Section 7.1. One important fact of $r(\beta)$ is that for each integer p there exists $\beta_p > 1/(p+1)$ such that $r(\beta_p) = 2$ for $1/(p+1) \leq \beta < \beta_p$. Hence, Theorem 1.2.4 (stated below) follows easily from Theorem 7.a as $k_2(n,(1-\beta)n) \geq (1-\beta)n^2/2$.

Theorem 1.2.4. *For positive integers p, there exists* $1/(p+1) < \beta_p \le 1/p$ *such that for all* $1/(p+1) \le \beta < \beta_p$ *and integers n,* $\delta = (1-\beta)n$ *and r,*

$$k_r(n,\delta) \ge g_r(\beta)n^r$$

holds. Moreover, for $3 \le r \le p+1$ *equality holds if and only if* (n,β) *is feasible, and the extremal graphs are members of* $\mathcal{G}(n,\beta)$.

In Section 7.2, we evaluate $k_{p+1}(n,(1-\beta)n)$, that is, the largest r such that $k_r(n,(1-\beta)n) > 0$.

Theorem 7.b. *Let* $0 < \beta < 1$ *and* $p = \lceil \beta^{-1} \rceil - 1$. *Suppose G is a graph of order n with minimum degree* $(1-\beta)n$. *Then, for any integer* $2 \le t \le p$,

$$\frac{k_{p+1}(G)}{g_{p+1}(\beta)n^{p+1}} \ge \frac{k_t(G)}{g_t(\beta)n^t}.$$

Moreover, for $t = 2$*, equality holds if and only if* (n,β) *is feasible and G is a member of* $\mathcal{G}(n,\beta)$.

The main idea is to bound $\sum_{T\in\mathcal{K}_t(S)} D_-(T)$ above and below for $S \in \mathcal{K}_s$, where s is no longer always equal to $t+1$ (as in Corollary 5.1.2).

7.1 $k_r(n,\delta)$ for $\delta > 4n/5$

In this section, our aim is to prove Theorem 7.a. The proof closely follows Section 5.2.1.

First, we are going to generalise Lemma 5.2.1. Before stating the lemma, we are going to define $A_t(\beta)$ and $B_t(\beta)$ for $\beta < 1/5$. They will be used as the coefficients for the terms involving D_+. We stress that neither $A_t(\beta)$ nor $B_t(\beta)$ defined below is optimal, but they are chosen for their nice expressions. Let $0 < \beta < 1/5$ and $p = \lceil \beta^{-1} \rceil - 1$ as before. For $2 \le t \le p$, define

$$\begin{aligned} A_t(\beta) =& (t-1)((p+1)\beta - 1)C_t(\beta), \text{ and} \\ B_t(\beta) =& ((p+1)\beta - 1)C_t(\beta), \end{aligned}$$

where $C_j(\beta)$ satisfies the recurrence

$$C_t(\beta) + 1 = (p - t + 1)\beta C_{t-1}(\beta), \tag{7.1}$$

with the initial condition $C_p(\beta) = 0$ for $0 < \beta < 1/5$. Explicitly,

$$C_{p-j}(\beta) = \frac{1}{j!\beta^j} \sum_{0 \le i < j} i!\beta^i \qquad \text{if } 0 \le j \le p-2.$$

It is easy to see that $C_t(\beta)$ is a non negative function for $0 < \beta < 1/5$, and so are $A_t(\beta)$ and $B_t(\beta)$. Note that

$$A_t(\beta) = (t-1)B_t(\beta). \tag{7.2}$$

Next, we define the integer $r(\beta)$ to be the smallest integer $r \ge 2$ such that for $t = r, \ldots, p$

$$A_t(\beta) < 1 \text{ and } B_t(\beta) < (p-t)\beta \tag{7.3}$$

hold. Since $C_p(\beta) = 0$, $r(\beta) \le p$. Let

$$\beta_p = \sup\{\beta_0 \le 1/p : r(\beta) = 2 \text{ for all } 1/(p+1) \le \beta < \beta_0\}.$$

Observe that $A_t(\beta)$, $B_t(\beta)$ and $C_t(\beta)$ are right continuous functions of β. Moreover, both $A_t(\beta)$ and $B_t(\beta)$ tend to zero as β tends $1/(p+1)$ from above, so $\beta_p > 1/(p+1)$.

Next, we state an analogue of Lemma 5.2.1 with the additional condition that $r(\beta) \le t \le p$. Its proof is very similar to the proof of Lemma 5.2.1. The condition $r(\beta) \le t \le p$ allows us to use (7.3).

Lemma 7.1.1. *Let $0 < \beta < 1/5$ and $p = \lceil \beta^{-1} \rceil - 1$. Let t be an integer with $r(\beta) \le t \le p$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$ and $S \in \mathcal{K}_{t+1}(G)$. Then*

$$\widetilde{D}(S) + B_t(\beta) \sum_{T \in \mathcal{K}_t(S)} \frac{D_+(T)}{D(T)} \ge A_{t+1}(\beta) D_+(S). \tag{7.4}$$

Moreover, if equality holds, then S is not heavy and $d(v) = (1-\beta)n$ for all $v \in S$.

Proof. First we show (7.4) holds. Fix β and write A_t, B_t and C_t for $A_t(\beta)$, $B_t(\beta)$ and $C_t(\beta)$ respectively. Corollary 5.1.2 states that $\widetilde{D}(S) \geq 0$, so we may assume that S is heavy or else 7.4 holds. Recall that Corollary 4.1.3 states

$$\widetilde{D}(S) + \sum_{T \in \mathcal{K}_t(S)} D_+(T) \geq (t-1)D_+(S). \tag{7.5}$$

If S does not contain a heavy t-clique, then (7.3) becomes

$$\widetilde{D}(S) \geq (t-1)D_+(S) \geq D_+(S) \geq A_{t+1}D_+(S),$$

where the last inequality follows from (7.3) if $t < p$ and $A_{p+1} = 0$. Therefore, we may assume that S contains at least one heavy t-clique. Let $T_0 \in \mathcal{K}_t(S)$ with $D(T_0) = \max\{D(T) : T \in \mathcal{K}_t(S)\}$. By substituting (7.5) into (7.4), it is sufficient to show that

$$f = \left(1 - \frac{B_t}{D(T_0)}\right)\widetilde{D}(S) - \left(A_t - \frac{(t-1)B_t}{D(T_0)}\right)D_+(S)$$

is non-negative. As S is heavy, we have explicitly that

$$\widetilde{D}(S) = \sum_{T \in \mathcal{K}_t(S)} D_-(T) - (2 - (t+1)\beta + (t-1)(p-t)\beta). \tag{7.6}$$

First suppose that $D_+(S) \leq 1-p\beta$. Since T_0 is assumed to be heavy, $D(T_0) > (p-t+1)\beta > B_t$ by Lemma 4.1.1 (*iv*) and (7.3). On the other hand, Lemma 4.1.1 (*ii*) gives $D(T_0) \leq D(S) + \beta = (p-t+1)\beta + D_+(S) \leq 1 - (t-1)\beta$. Hence,

$$\begin{aligned} D(T_0)A_t - (t-1)B_t \leq& (1-(t-1)\beta)A_t - (t-1)B_t \\ =& -(t-1)^2((p+1)\beta - 1)C_t < 0. \end{aligned}$$

Thus, $A_t - (t-1)B_t/D(T_0) < 0$. Therefore, $f > 0$ by considering the coefficients of $\widetilde{D}(S)$ and $D_+(S)$. Hence, we may assume $D_+(S) > 1 - p\beta$. We address different cases separately depending on the number of heavy t-cliques in S.

Suppose all t-cliques are heavy. Thus, $\widetilde{D}(S) = 2((p+1)\beta - 1)$ by (7.6), because

$D_-(T) = (p - t + 1)\beta$ for all $T \in \mathcal{K}_t(S)$ by Lemma 4.1.1 (*iv*). Furthermore, we may assume that $2((p + 1)\beta - 1) \leq (t - 1)D_+(S)$, otherwise (7.4) is true as $\widetilde{D}(S) = 2((p+1)\beta - 1) \geq (t-1)D_+(S) \geq D_+(S) \geq A_t D_+(S)$ by (7.3). Note that $D(T_0) \leq 1$ and $D_+(S) = D(S) - (p - t)\beta \leq 1 - (p - t)\beta$. Thus,

$$\begin{aligned} f =& 2((p+1)\beta - 1)\left(1 - \frac{B_t}{D(T_0)}\right) - \left(A_t - \frac{(t-1)B_t}{D(T_0)}\right) D_+(S) \\ \geq& 2((p+1)\beta - 1)(1 - B_t) - (A_t - (t-1)B_t)D_+(S) \\ \overset{\text{by (7.2)}}{\geq}& 2((p+1)\beta - 1)(1 - B_t) \overset{\text{by (7.3)}}{>} 2((p+1)\beta - 1)(1 - (p-t)\beta) > 0, \end{aligned}$$

which implies (7.4). Thus, we may assume that there is at least one t-clique T in S that is not heavy. We further claim that $D_+(S) \leq \beta$, so $D_+(T) \leq \beta$. Otherwise, Lemma 4.1.1 (*iii*) implies $D_+(T) > 0$ for all $T \in \mathcal{K}_t(S)$, which is a contradiction.

Suppose there are at most $t - 2$ heavy t-cliques in S, sayprecisely $t - 1 - i$ of them for $1 \leq i \leq t-2$. Note that $\widetilde{D}(S) \geq iD_+(S)$ by (7.5) and Lemma 4.1.1 (*iii*). Hence

$$\begin{aligned} f &\geq \left(1 - \frac{B_t}{D(T_0)}\right) iD_+(S) - \left(A_t - \frac{(t-1)B_t}{D(T_0)}\right) D_+(S) \\ &\geq \left(1 + \frac{(t-2)B_t}{D(T_0)} - A_t\right) D_+(S) \geq (1 - A_t)\, D_+(S) \overset{(7.3)}{>} 0, \end{aligned}$$

so (7.4) holds.

Now suppose there are either $t - 1$ or t heavy t-cliques in S. We are going to show that in both cases $\widetilde{D}(S) \geq 2(D_+(S) - (1 - p\beta))$. First, assume that S has $t - 1$ heavy t-cliques. Let T_1 and T_2 be the two non-heavy t-cliques in S. By Lemma 4.1.1 (*iii*), $D_-(T_i) = D(T_i) \geq D(S) = D_+(S) + (p-t)\beta > 0$ for $i \in \{1, 2\}$. Thus, (7.6) becomes

$$\begin{aligned} \widetilde{D}(S) =& \sum D_-(T) - (2 - (t+1)\beta + (t-1)(p-t)\beta) \\ \geq& 2(D_+(S) - (1 - p\beta)). \end{aligned}$$

Secondly, assume that all but one of the t-cliques in S are heavy. Recall that since not all t-cliques in S are heavy, $D_+(S) \leq \beta$ by Lemma 4.1.1 (*iii*). Let T_1 be the non-heavy t-clique in S. By Lemma 4.1.1 (i) we have $D(T_1) \geq D(S) - (s-t)\beta = D_+(S) + (p-t)\beta > 1 - t\beta$. By (7.6),

$$\widetilde{D}(S) \geq (p+1)\beta - 1 + D_+(S) - (1-p\beta) \geq 2(D_+(S) - (1-p\beta)).$$

Recall that our aim is to show $f \geq 0$. By substituting $\widetilde{D}(S) \geq 2(D_+(S) - (1-p\beta))$ and multiplying both side by $D(T_0)$, it is enough to show that

$$\begin{aligned} D(T_0)f \geq & 2(D(T_0) - B_t)(D_+(S) - (1-p\beta)) - (D(T_0)A_t - (t-1)B_t)D_+(S) \\ = & 2(D(T_0) - B_t)(D_+(S) - (1-p\beta)) + (1 - D(T_0))D_+(S)A_t \qquad \text{by (7.2)} \\ = & 2((p-t+1)\beta + D_+(T_0) - B_t)(D_+(S) - (1-p\beta)) \\ & + (1 - (p-t+1)\beta - D_+(T_0))D_+(S)A_t \end{aligned}$$

is non-negative for $0 < D_+(T_0) \leq D_+(S)$ and $1 - p\beta \leq D_+(S) \leq \beta$. By considering the Hessian matrix, we deduce that all stationary points are saddle points. Thus, it enough to check the inequality on the boundary. In fact, it is sufficient to check at the extreme values of $D_+(T_0)$ and $D_+(S)$, because say if $D_+(T_0)$ is fixed, then we take the extremal values of $D_+(S)$, and if $D_+(T) = D_+(S)$, then the above inequality is a concave function. If $D_+(T_0) = 0$ and $D_+(S) = 1 - p\beta$, we have

$$D(T_0)f \geq (1 - (p-t+1)\beta)(1-p\beta)A_t > 0.$$

If $D_+(T_0) = 0$ and $D_+(S) = \beta$, we have

$$\begin{aligned} D(T_0)f \geq & 2((p-t+1)\beta - B_t)((p+1)\beta - 1) + (1 - (p-t+1)\beta)\beta A_t \\ \overset{\text{by (7.3)}}{>} & 2\beta((p+1)\beta - 1) + (1 - (p-t+1)\beta)\beta A_t > 0. \end{aligned}$$

If $D_+(T_0) = D_+(S) = 1 - p\beta$, $D(T_0)f \geq (t-1)\beta(1-p\beta)A_t > 0$. Finally, if $D_+(T_0) = D_+(S) = \beta$,

$$\begin{aligned} D(T_0)f &\geq 2((p-t+2)\beta - B_t)((p+1)\beta - 1) + (1-(p-t+2)\beta)\beta A_t \\ &\overset{\text{by (7.3)}}{>} 2((p+1)\beta - 1)\beta + (1-(p-t+2)\beta)\beta A_t > 0. \end{aligned}$$

It can be checked that if equality holds, then $D_+(S) = 0$, so S is not heavy. Thus, $D_+(T) = 0$ for $T \in \mathcal{K}_t(S)$ by Lemma 4.1.1. Furthermore, equality in (7.5) implies equality in Corollary 4.1.3 as $D_+(T) = 0 = D_+(S)$. Hence, $d(v) = (1-\beta)n$ for $v \in S$. This completes the proof of the lemma. □

Now, we are going to prove Theorem 7.a. The proof is the same as the the proof of Theorem 4 with an additional argument that deals with the D_+ terms.

Proof of Theorem 7.a. Fix β and we write A_t, B_t, C_t and g_t for $A_t(\beta)$, $B_t(\beta)$, $C_t(\beta)$ and $g_t(\beta)$ respectively. Actually, we are going to show that

$$\frac{k_s}{g_s n^s} \geq \frac{k_t}{g_t n^t} + \frac{1 - t\beta - B_t}{(1-t\beta)(p-t+1)\beta g_t n^t} \sum_{T \in \mathcal{K}_t} D_+(T), \tag{7.7}$$

for $r(\beta) \leq t < s \leq p+1$. Observe that $1 - t\beta - B_t > 1 - p\beta > 0$ by (7.3), so the above inequality implies inequality in the theorem. Moroever, it is sufficient to prove the case when $s = t+1$. We proceed by induction on t from above.

Lemma 7.1.1 (after rearrangement) gives a lower bound on $\widetilde{D}(S)$ for $S \in \mathcal{K}_{t+1}$. In addition, Lemma 5.1.3 gives a upper bound on $\sum_{S \in \mathcal{K}_{t+1}} \widetilde{D}(S)$. Thus, we have

$$\begin{aligned} &\big(t - 1 + (p-2t+2)(t+1)\beta\big)k_{t+1} + (t-1-A_t) \sum_{S \in \mathcal{K}_{t+1}} D_+(S) \geq \\ &(1-t\beta)(p-t+1)\beta n k_t + (t-1)(t+2)\frac{k_{t+2}}{n} + (1-t\beta-B_t)n \sum_{T \in \mathcal{K}_t} D_+(T) \end{aligned} \tag{7.8}$$

for $r(\beta) \leq t \leq p$. Suppose $t = p$. Recall that by definition

$$g_{p+1}(\beta) = (1-(p-1)\beta)(1-p\beta)\beta^{p-1}/2 \text{ and } g_{p+2}(\beta) = 0.$$

Thus, (4.7) with $t = p$ becomes

$$g_{p+1}(\beta) = \frac{(1 - p\beta)\beta}{p - 1 - (p+1)(p-2)\beta} g_p(\beta).$$

By (7.8) with $t = p$, we have

$$(p - 1 - (p-2)(p+1)\beta)k_{p+1} \geq (1 - p\beta)\beta n k_p + (1 - p\beta - B_p)n \sum_{S \in K_p} D_+(T)$$
$$\frac{k_{p+1}}{g_{p+1}(\beta)n^{p+1}} \geq \frac{k_p}{g_p(\beta)n^p} + \frac{1 - p\beta - B_p}{(1 - p\beta)\beta g_p(\beta)n^p} \sum_{T \in \mathcal{K}_p} D_+(T)$$

Hence, (7.7) is true for $t = p$. Suppose $2 \leq t < p - 1$, so by the induction hypothesis (7.8) becomes

$$\begin{aligned}
&(t - 1 + (t+1)(p - 2t + 2)\beta)k_{t+1} + (t - 1 - A_t) \sum_{S \in \mathcal{K}_{t+1}} D_+(S) \\
\geq &(1 - t\beta)(p + 1 - t)\beta n k_t + (t-1)(t+2)\frac{k_{t+2}}{n} + (1 - t\beta - B_t)n \sum_{T \in \mathcal{K}_t} D_+(T) \\
\geq &(1 - t\beta)(p + 1 - t)\beta n k_t + (1 - t\beta - B_t)n \sum_{T \in \mathcal{K}_t} D_+(T) \\
+&(t-1)(t+2)g_{t+2}(\beta)\left(\frac{k_{t+1}}{g_{t+1}(\beta)} + \frac{1 - (t+1)\beta - B_{t+1}}{(1 - (t+1)\beta)(p-t)\beta g_{t+1}(\beta)} \sum_{S \in \mathcal{K}_{t+1}} D_+(S)\right).
\end{aligned} \tag{7.9}$$

Notice that if we ignore terms with D_+, the inequality above is identical to (4.11) in the proof of Theorem 4. It turns out that if we rearrange (7.9) in the same way as in the proof of Theorem 4, then we would obtain (7.7) with an additional term $c \sum D_+(S)$, where c is a constant depending on p, t and β. Therefore, it is sufficient to show that the terms with $\sum D_+(S)$ in (7.9) can be removed, which is equivalent to showing

$$\frac{(t-1)(t+2)g_{t+2}(1 - (t+1)\beta - B_{t+1})}{(1 - (t+1)\beta)(p-t)\beta g_{t+1}} - (t - 1 - A_t). \tag{7.10}$$

is non-negative. By (4.6), $(t+2)g_{t+2}(\beta) \geq (1 - (t+1)\beta)g_{t+1}(\beta)$. Hence, (7.10) is at least

$$\begin{aligned}
&\frac{(t-1)(1-(t+1)\beta - B_{t+1})}{(p-t)\beta} - (t-1-A_t)\\
=&\frac{(t-1)(1-(t+1)\beta-((p+1)\beta-1)C_{t+1})}{(p-t)\beta} - (t-1)(1-((p+1)\beta-1)C_t)\\
=&\frac{(t-1)((p+1)\beta-1)}{(p-t)\beta}\left((p-t)\beta C_t - C_{t+1} - 1\right) \overset{\text{by (7.1)}}{=} 0.
\end{aligned}$$

Hence, we can remove the terms with $\sum D_+(S)$ and so (7.7) holds. The proof of inequality in the theorem is complete.

If equality holds in Theorem 7.a, then equality holds in (7.7). Moreover, $\sum D_+(T)$ must be zero as $(1-t\beta-B_t)>0$. Therefore, G is heavy-free, so we are done by Theorem 4. □

It is easy to see that Theorem 7.a implies Theorem 1.2.4 as $r(\beta)=2$ for $1/(p+1)\le\beta<\beta_p$.

Recall that β_p is not optimal. Ideally, we want to generalise Theorem 5.2.3 and show that

$$\frac{k_{p+1}}{g_{p+1}(\beta)n^{p+1}} \ge \frac{k_p}{g_p(\beta)n^p} + \frac{1-p\beta-\widetilde{B}_p(\beta)}{(1-p\beta)\beta g_p(\beta)n^p}\sum_{T\in\mathcal{K}_p} D_+(T) \tag{7.11}$$

for some $\widetilde{B}_p(\beta)\ge 0$. Then, we can redefine $C_t(\beta)$ for $2\le t\le p$ with the initial condition that $\widetilde{B}_p(\beta) = ((p+1)\beta-1)C_t(\beta)$. As a consequence, we would obtain a better β_p. However, as we saw in Chapter 6, it is unlikely that we would obtain $\beta_p = 1/p$ just by proving (7.11).

In fact, defining $A_t(\beta)$ and $B_t(\beta)$ for a general β is where the main difficulty lies in proving Conjecture 3.1.1. In fact, by analysing the proof of Theorem 7.a, one can obtain a set of inequalities that needs to be satisfied by $A_t(\beta)$ and $B_t(\beta)$ for $2\le t\le p$. Thus, one can determine the functions $A_t(\beta)$ and $B_t(\beta)$. However, in the proof of Theorem 5.2.3, there are many computations in the proof of Theorem 5.2.3, (even more in proof of Lemma 6.1.2 in Section 6.4). Determining that whether $A_t(\beta)$ and $B_t(\beta)$ satisfy all the required inequalities is difficult as both $A_t(\beta)$ and $B_t(\beta)$ are likely to involve binomial coefficients. Therefore, what we mean is that the main obstacle in proving Conjecture 3.1.1 is to define functions $A_t(\beta)$ and $B_t(\beta)$, which have nice expressions to work with.

7.2 Counting $(p+1)$-cliques

In this section, we are going to prove Theorem 7.b. This is equivalent to showing that Conjecture 3.1.1 holds when $r = p+1$, that is, $k_{p+1}(n,(1-\beta)n) \geq g_{p+1}(\beta)n^{p+1}$.

In all previous sections, our aim is to bound $\sum_{S\in\mathcal{K}_s}\sum_{T\in\mathcal{K}_t(S)} D_-(T)$ with $s = t+1$ above and below using Corollary 5.1.2 and Proposition 3.3.1 respectively. In this section, s is no longer always equal to $(t+1)$. A lower bound on $\sum_{S\in\mathcal{K}_s}\sum_{T\in\mathcal{K}_t(S)} D_-(T)$ can be obtained using Lemma 4.1.2 and mimicking the proof of Lemma 5.1.1. Note that

$$\sum_{S\in\mathcal{K}_s}\sum_{T\in\mathcal{K}_t(S)} D_-(T) = \sum_{T\in\mathcal{K}_t} f(T)D_-(T),$$

where $f(T)$ is the number of s-cliques containing a t-clique T. In order to bound $\sum_{S\in\mathcal{K}_s}\sum_{T\in\mathcal{K}_t(S)} D_-(T)$ above, an obvious approach would be to apply Proposition 3.3.1 to $\sum_{T\in\mathcal{K}_t} f(T)D_-(T)$. However, $f(T)$ is not a two-valued function for $G \in \mathcal{G}(n,\beta)$ and $s > t+1$ even if (n,β) is feasible. Thus, the upper bound obtained by Proposition 3.3.1 is unlikely be sharp.

The following observation allows us to overcome this problem. For $s = t+2$, observe that

$$\begin{aligned} 2\sum_{S\in\mathcal{K}_{t+2}}\sum_{T\in\mathcal{K}_t(S)} D_-(T) &= \sum_{S\in\mathcal{K}_{t+2}}\sum_{U\in\mathcal{K}_{t+1}(S)}\sum_{T\in\mathcal{K}_t(U)} D_-(T) \\ &= n\sum_{U\in\mathcal{K}_{t+1}} D(U) \sum_{T\in\mathcal{K}_t(U)} D_-(T). \end{aligned}$$

Note that $\sum D_-(T)$ is two-valued for $G \in \mathcal{G}(n,\beta)$ and (n,β) feasible. Explicitly

$$\sum_{T \in \mathcal{K}_t(U)} D_-(T)$$
$$= \begin{cases} 2(1-t\beta) & \text{if } |V(U) \cap V_0| = 0,1 \\ (1-t\beta)(t+2)(t+1) + ((p+1)\beta - 1)t(t-1) & \text{if } |V(U) \cap V_0| = 2. \end{cases}$$

Hence, this suggests that $\sum_{S \in \mathcal{K}_s} \sum_{S' \in \mathcal{K}_{s-1}(S')} \cdots \sum_{T' \in \mathcal{K}_{t+1}(U)} \sum_{T \in \mathcal{K}_t(T')} D_-(T)$ may be more appropriate than $\sum_{S \in \mathcal{K}_{t+2}} \sum_{T \in \mathcal{K}_t(S)} D_-(T)$.

For positive integers $t \le s$, define the function $\phi_t^s : \mathcal{K}_s \to \mathbb{R}$ to be

$$\phi_t^s(S) = \begin{cases} D_-(S) & \text{if } t = s, \text{ and} \\ \sum\{\phi_t^{s-1}(U) : U \in K_{s-1}(S)\} & \text{if } t < s \end{cases}$$

for $S \in \mathcal{K}_s$. Note that $\phi_t^s(S) = (s-t)! \sum_{T \in \mathcal{K}_t(S)} D_-(T)$, because each $T \in \mathcal{K}_t(S)$ misses $s-t$ vertices of S, so each T appears exactly $(s-t)!$ times in the summation. For $G \in \mathcal{G}(n,\beta)$ with (n,β) feasible,

$$\phi_t^s(S) = \begin{cases} (s-t)!(1-t\beta) & \text{if } |V(S) \cap V_0| = 0,1 \\ (1-t\beta)s!/t! + ((p+1)\beta - 1)(s-2)!/(t-2)! & \text{if } |V(S) \cap V_0| = 2 \end{cases}$$

for $S \in \mathcal{K}_s$. We define the function Φ_t^s to be analogous of D_-. Define $\Phi_t^s(S) = \min\{\phi_t^s(S), \varphi_t^s\}$ for $S \in \mathcal{K}_s$ and $2 \le t \le s \le p+1$, where

$$\varphi_t^s = (1-t\beta)s!/t! + ((p+1)\beta - 1)(s-2)!/(t-2)!.$$

In the next lemma, we investigate the lower bound on $\Phi_t^s(S)$ for $S \in \mathcal{K}_s$.

Lemma 7.2.1. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let s and t be integers with $2 \le t \le s \le p+1$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then,*

$$\Phi_t^s(S) \ge (1-t\beta)s!/t! + \big(D_-(S) - (1-s\beta)\big)(s-2)!/(t-2)!$$

for $S \in \mathcal{K}_s$.

Proof. We fix β and t and proceed by induction on s. This inequality holds in the trivial cases $s = t$ and $s = t+1$ by Corollary 5.1.2. Suppose that $s \geq t+2$ and that the lemma is true for $t, \ldots, s-1$. Hence

$$\begin{aligned}
\phi_t^s(S) &= \sum_{T \in \mathcal{K}_{s-1}(S)} \phi_t^{s-1}(T) \geq \sum_{T \in \mathcal{K}_{s-1}(S)} \Phi_t^{s-1}(T) \\
&\geq \sum_{T \in \mathcal{K}_{s-1}(S)} \big[(1-t\beta)(s-1)!/t! + \big(D_-(T) - (1-(s-1)\beta)\big)(s-3)!/(t-2)!\big] \\
&=(1-t\beta)s!/t! + \left(\sum_{T \in \mathcal{K}_{s-1}(S)} D_-(T) - s(1-(s-1)\beta)\right)(s-3)!/(t-2)! \\
&\geq(1-t\beta)s!/t! + \big(2 - s\beta + (s-2)D_-(S) - s(1-(s-1)\beta)\big)(s-3)!/(t-2)! \\
&=(1-t\beta)s!/t! + \big(D_-(S) - (1-s\beta)\big)(s-2)!/(t-2)!,
\end{aligned}$$

where the last inequality comes from Corollary 5.1.2 with $t = s-1$. The right hand side is increasing in $D_-(S)$. In addition, the right hand side equals to φ_t^s only if $D_-(S) = (p-s+1)\beta$. Thus, the proof of the lemma is complete. □

Now, we bound $\sum_{S \in \mathcal{K}_s} \Phi_t^s(S)$ from above using Proposition 3.3.1 to obtain the next lemma. The proof is essentially a straightforward application of Proposition 3.3.1 with an algebraic check.

Lemma 7.2.2. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let s and t be integers with $2 \leq t \leq s \leq p+1$. Suppose G is a graph of order n with minimum degree $(1-\beta)n$. Then,*

$$\begin{aligned}
\sum_{S \in \mathcal{K}_s} \Phi_t^s(S) \leq& \varphi_t^{s-1} s k_s + 2((p+1)\beta - 1) \sum_{i=t+1}^{s-1} \left(\frac{(i-3)!}{(t-2)!} k_i n^{s-i} \prod_{j=i}^{s-1} (1-j\beta) \right) \\
&+ \left((t+1)k_{t+1} - (p-t+1)\beta k_t n\right) n^{s-t-1} \prod_{j=t}^{s-1} (1-j\beta).
\end{aligned}$$

Proof. Fix β and t. We proceed by induction on s. Suppose $s = t+1$. Note that $\Phi_t^{t+1}(S) \leq \sum_{T \in \mathcal{K}_t(S)} D_-(T)$. By Proposition 3.3.1, taking $\mathcal{A} = \mathcal{K}_t$, $f = D_-$,

$g = D$, $m = 1 - t\beta$ and $M = (p-t+1)\beta$,

$$\begin{aligned}
&\sum_{S\in\mathcal{K}_{t+1}} \Phi_t^{t+1}(S) \leq \sum_{S\in\mathcal{K}_{t+1}} \sum_{T\in\mathcal{K}_t(S)} D_-(T) = n\sum_{T\in\mathcal{K}_t} D(T)D_-(T)\\
\leq&(p-t+1)\beta n \sum_{T\in\mathcal{K}_t} D(T) + (1-t\beta)n \sum_{T\in\mathcal{K}_t} D_-(T) - (1-t\beta)(p-t+1)\beta n k_t\\
\leq&(t+1)(1-(p-2t+1)\beta)k_{t+1} - (1-t\beta)(p-t+1)\beta n k_t.
\end{aligned}$$

Hence the lemma is true for $s = t+1$. Now assume that $s \geq t+2$ and the lemma is true up to $s-1$. By Proposition 3.3.1 taking $\mathcal{A} = \mathcal{K}_t$, $f = \Phi_t^{s-1}$, $g = D$, $M = \varphi_t^{s-1}$ and $m = 1-(s-1)\beta$, we have

$$\begin{aligned}
&\sum_{S\in\mathcal{K}_s} \Phi_t^s(S) = n \sum_{T\in\mathcal{K}_{s-1}} D(T)\Phi_t^{s-1}(T)\\
\leq&\varphi_t^{s-1} \sum_{T\in\mathcal{K}_{s-1}} nD(T) + (1-(s-1)\beta)n \sum_{T\in\mathcal{K}_{s-1}} \Phi_t^{s-1}(T) - \varphi_t^{s-1}(1-(s-1)\beta)nk_{s-1}\\
=&\varphi_t^{s-1} s k_s + (1-(s-1)\beta)n \sum_{T\in\mathcal{K}_{s-1}} \Phi_t^{s-1}(T) - \varphi_t^{s-1}(1-(s-1)\beta)nk_{s-1}\\
\leq&\varphi_t^{s-1} s k_s - \left(\varphi_t^{s-1} - (s-1)\varphi_t^{s-2}\right)(1-(s-1)\beta)nk_{s-1}\\
&+ 2((p+1)\beta - 1) \sum_{i=t+1}^{s-2} \left(\frac{(i-3)!}{(t-2)!} k_i n^{s-i} \prod_{j=i}^{s-1} (1-j\beta) \right)\\
&+ \left((t+1)k_{t+1} - (p-t+1)\beta k_t n\right) n^{s-t-1} \prod_{j=t}^{s-1}(1-j\beta).
\end{aligned}$$

The last inequality is due to the induction hypothesis on $\sum \Phi_t^{s-1}(T)$. In addition, it is easy to show that

$$(s-1)\varphi_t^{s-2} - \varphi_t^{s-1} = 2((p+1)\beta - 1)(s-4)!/(t-2)!.$$

Thus, the proof of the lemma is complete. □

Now we are ready to prove Theorem 7.*b*. The proof is very similar to the proof of Theorem 4 with lots of algebraic checks.

Proof of Theorem 7.b. We fix β and write g_t to be $g_t(\beta)$. We proceed by induction on t from above. The theorem is true for $t = p$ by Theorem 7.a as $r(\beta) \leq p$. Hence, we may assume $t < p$. Recall that D_- is a zero function on $\mathcal{K}_{p+1}$. Summing Lemma 7.2.1 taking $s = p + 1$ over $S \in \mathcal{K}_{p+1}$ yields

$$\sum_{S \in K_{p+1}} \Phi_t^{p+1}(S) \geq ((1 - t\beta)(p+1)!/t! - (1 - (p+1)\beta)(p-1)!/(t-2)!)\, k_{p+1}. \tag{7.12}$$

By Lemma 7.2.2 and the induction hypothesis,

$$\begin{aligned}\sum \Phi_t^{p+1}(S) \leq & \varphi_t^p(p+1)k_{p+1} + 2((p+1)\beta - 1) \sum_{i=t+1}^{p} \left(\frac{(i-3)!}{(t-2)!} k_i n^{p+1-i} \prod_{j=i}^{p} (1 - j\beta) \right) \\ & + ((t+1)k_{t+1} - (p-t+1)\beta n k_t)\, n^{p-t} \prod_{j=t}^{p} (1 - j\beta) \\ \leq & \varphi_t^p(p+1)k_{p+1} + 2((p+1)\beta - 1) \frac{k_{p+1}}{g_{p+1}} \sum_{i=t+1}^{p} \left(g_i \frac{(i-3)!}{(t-2)!} \prod_{j=i}^{p} (1 - j\beta) \right) \\ & + \left((t+1) \frac{k_{p+1}}{g_{p+1}} g_{t+1} - (p-t+1)\beta n k_t \right) n^{p-t} \prod_{j=t}^{p} (1 - j\beta)\end{aligned}$$

Thus, after substituting the inequality into (7.12) and rearranging, we have $\lambda_t k_{p+1} \geq (p-t+1)\beta k_t n^{p-t+1} \prod_{j=t}^{p}(1 - j\beta)$, where

$$\begin{aligned}\lambda_t = & (t+1)g_{t+1} \prod_{j=t}^{p} (1 - j\beta) \\ & + \frac{2((p+1)\beta - 1)}{(t-2)!} \left((p-2)! g_{p+1} + \sum_{i=t+1}^{p} (i-3)! g_i \prod_{j=i}^{p} (1 - j\beta) \right).\end{aligned}$$

Hence, it is enough to show that $\lambda_t = (p-t+1)\beta g_t \prod_{j=t}^{p}(1 - j\beta)$; that is,

$$\begin{aligned}& ((p-t+1)\beta g_t - (t+1)g_{t+1}) \prod_{j=t}^{p} (1 - j\beta) \\ = & \frac{2((p+1)\beta - 1)}{(t-2)!} \left((p-2)! g_{p+1} + \sum_{i=t+1}^{p} (i-3)! g_i \prod_{j=i}^{p} (1 - j\beta) \right).\end{aligned} \tag{7.13}$$

We prove (7.13) by induction on t from above. By (4.7) with $t = p$, the left hand side of (7.13) becomes

$$\begin{aligned}(\beta g_p - (p+1)g_{p+1})(1-p\beta) =& \left(\frac{p-1-(p+1)(p-2)\beta}{1-p\beta} - (p+1)\right)(1-p\beta)g_{p+1} \\ =& 2((p+1)\beta - 1)g_{p+1}.\end{aligned}$$

Thus, (7.13) is true for $t = p$. Suppose $t < p$. Notice that the right hand side of (7.13) is equal to

$$\begin{aligned}
&\frac{2((p+1)\beta-1)}{(t-2)!}\left((p-2)!g_{p+1} + \sum_{i=t+1}^{p}(i-3)!g_i\prod_{j=i}^{p}(1-j\beta)\right) \\
=&(t-1)\frac{2((p+1)\beta-1)}{(t-1)!}\left((p-2)g_{p+1} + \sum_{i=t+2}^{p}(i-3)g_i\prod_{j=i}^{p}(1-j\beta)\right) \\
&+ 2((p+1)\beta-1)g_{t+1}\prod_{j=t+1}^{p}(1-j\beta) \\
\overset{\text{by (7.13)}}{=}&(t-1)\left((p-t)\beta g_{t+1} - (t+2)g_{t+2}\right)\prod_{j=t+1}^{p}(1-j\beta) \\
&+ 2((p+1)\beta-1)g_{t+1}\prod_{j=t+1}^{p}(1-j\beta) \\
=&\Big(((p-t+2)(t+1)\beta-2)g_{t+1} - (t-1)(t+2)g_{t+2}\Big)\prod_{j=t+1}^{p}(1-j\beta) \\
=&\Big((t-1+(p-2t+2)(t+1)\beta)g_{t+1} - (t-1)(t+2)g_{t+2}\Big)\prod_{j=t+1}^{p}(1-j\beta) \\
&-(t+1)g_{t+1}\prod_{j=t}^{p}(1-j\beta) \\
\overset{\text{by (4.7)}}{=}&\left((p-t+1)\beta g_t - (t+1)g_{t+1}\right)\prod_{j=t}^{p}(1-j\beta).
\end{aligned}$$

Hence, (7.13) is true for $2 \le t \le p$. Therefore, the equality of the theorem is true.

Now suppose that equality holds, so equality holds in (7.12). Therefore, $D(S) = D_-(S) = 0$ for all $S \in \mathcal{K}_{p+1}$. Thus, G is K_{p+2}-free. By Theorem 4

(n, β) is feasible and $G \in \mathcal{G}(n, \beta)$. This completes the proof of the theorem. $\square$

Chapter 8

Further directions

In this chapter, we discuss variants of $k_r(n,\delta)$. In the next section, we discuss $k_r^{reg}(n,\delta)$, that is, an analogue of $k_r(n,\delta)$ for regular graphs. In Section 8.2, we show that if (n,β) is not feasible, then $k_r(n,(1-\beta)n) \leq g_r(\beta)n^r + o(n^r)$. Thus, the lower bound on $k_r(n,(1-\beta)n)$ given in Conjecture 3.1.1 is asymptotically sharp.

In Section 8.3, we take a brief look at the natural opposite problem, that is, determining the maximum number of r-cliques in graphs of order n with maximum degree Δ.

Finally, in Section 8.4, we replace the condition on the minimum degree by the minimum $\sum_{v\in V(e)} d(v)$ over all edges e.

8.1 Cliques in regular graphs, $k_r^{reg}(n,\delta)$

Recall that $k_r^{reg}(n,\delta)$ is the minimum number of r-cliques in δ-regular graphs of order n. In Chapter 2, we evaluated $k_3(n,\delta)$ for $2n/5 < \delta \leq n/2$. We would like to extend the result for $\delta > n/2$. Since G is δ-regular, n or δ is even.

Recall that $\mathcal{G}(n,\beta)$ is conjectured to be extremal for $k_r(n,(1-\beta)n)$ if (n,β) is feasible. Moreover, all graphs in $\mathcal{G}(n,\beta)$ are in fact regular graphs. Thus, if (n,β) is feasible, then Conjecture 3.1.1 would imply that $k_r^{reg}(n,(1-\beta)n) = g_r(\beta)n^r$ and $\mathcal{G}(n,\beta)$ is the extremal family. Therefore, it is natural to conjecture that $\mathcal{G}(n,\beta)$ is also the extremal family for $k_r^{reg}(n,(1-\beta)n)$ when (n,β) is not feasible with n or δ even. In summary, we conjecture the following for regular graphs.

Conjecture 8.1.1. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let n and βn be positive integers not both odd. Then, for any positive integer r*

$$k_r^{reg}(n, (1-\beta)n) = g_r(\beta)n^r + \binom{p-1}{r-3}\beta^{r-3}k_3^{reg}(n', \delta')n^{r-3},$$

holds, where $n' = (1-(p-1)\beta)n$ and $\delta' = (1-p\beta)n$. Moreover, when the equality holds and $2 \leq r \leq p+1$, the extremal graphs are members of $\mathcal{G}(n, \beta)$.

Notice that $\delta' \leq n'/2$. Thus, $k_3^{reg}(n', \delta')$ was investigated in Chapter 2. Recall that $k_3^{reg}(n', \delta') > 0$ only if n' is odd. By the construction from Chapter 2, $k_r^{reg}(n', \delta') \leq \delta'(3\delta' - n' - 1)/4$, which is of order n^2. Therefore, the difference between $k_r^{reg}(n, \delta)$ and $k_r(n, \delta)$ is of order at most n^{r-1}. Hence, $k_r^{reg}(n, \delta)$ and $k_r(n, \delta)$ are asymptotically the same.

8.2 (n, β) not feasible

Equality holds in Conjecture 3.1.1 (that is $k_r(n, (1-\beta) = g_r(\beta)n^r,)$ only if (n, β) is feasible. In this section, we discuss the situation when (n, β) is not feasible. Our aim is to show that $k_r(n, (1-\beta)n) \leq g_r(\beta)n^r + o(n^r)$ for (n, β) not feasible. From the definition of feasibility, we can deduce that if (n, β) is not feasible, then

(a) both n and $(1-\beta)n$ are odd, or

(b) $(1-(p-1)\beta)n$ is odd..

Condition (a) means that $\mathcal{G}(n, \beta)$ is not well defined, because all graphs in $\mathcal{G}(n, \beta)$ are $(1-\beta)n$-regular graphs of order n. Condition (b) says that even if $\mathcal{G}(n, \beta)$ is well defined, then $G[V_0]$ is not necessarily triangle-free by a theorem of Andrásfai, Erdős and Sós [3], (Theorem 2.1.1), because $G[V_0]$ is regular with $|V_0| = (1-(p-1)\beta)n$ odd. In fact, (a) is a subcase of (b), as βn is even. Suppose that n and $(1-\beta)n$ are not both odd. By the construction of $\mathcal{G}(n, \beta)$ and the discussion in the previous section, we have $k_r(n, (1-\beta)n) \leq k_r^{reg}(n, (1-\beta)n) \leq g_r(\beta)n^r + o(n^r)$. However, this does not deal with the case when both n and $(1-\beta)n$ are odd.

Let n and βn be positive integers. Recall that $p = \lceil \beta^{-1} \rceil - 1$, so $1/(p+1) \leq \beta < 1/p$. We now define a family $\mathcal{G}'(n, \beta)$ of graphs such that each member G' can be obtained by the construction below. We define the graph $G' = (V, E)$ with the following properties. There is a partition of V into $V_0, V_1, \ldots, V_{p-1}$ with $|V_0| = (1 - (p-1)\beta)n$ and $|V_i| = \beta n$ for $i = 1, \ldots, p-1$. For $0 \leq i < j \leq p-1$, $G'[V_i, V_j]$ is a complete bipartite graph. For $1 \leq j \leq p-1$, $G'[V_j]$ is empty. For $i = 0$, $G'[V_0]$ is an edge minimal triangle-free graph of order $(1 - (p-1)\beta)n$ with minimum degree $(1 - p\beta)n$. Then, $\mathcal{G}'(n, \beta)$ is the set of the graphs which can be obtained by the above construction.

Observe that $\mathcal{G}'(n, \beta)$ is defined as long as both n and $(1-\beta)n$ are positive integers, whereas $\mathcal{G}(n, \beta)$ requires n and $(1-\beta)n$ to not both be odd. The structures of the members of $\mathcal{G}(n, \beta)$ and $\mathcal{G}'(n, \beta)$ are similar. Note that if (n, β) is feasible, then $\mathcal{G}(n, \beta) = \mathcal{G}'(n, \beta)$. However, for (n, β) not feasible with n and $(1-\beta)n$ not both odd, members of $\mathcal{G}'(n, \beta)$ are not necessarily regular, but members of $\mathcal{G}'(n, \beta)$ are.

Let $G' \in \mathcal{G}'(n, \beta)$. Suppose (n, β) is not feasible, so $|V_0|$ is odd. Recall that $G'[V_0]$ is triangle-free with minimum degree $(1 - p\beta)n$. If $G'[V_0]$ is bipartite, then $e(G'[V_0]) = (1 - p\beta)n(|V_0| + 1)/2$ by considering the larger partition class. Hence, $e(G'[V_0]) \leq ((1 - (p-1)\beta)n + 1)(1 - p\beta)n/2$. Therefore, there are

$$\begin{aligned}&\binom{p-1}{r}\beta^r n^r + \binom{p-1}{r-1}(1-(p-1)\beta)\beta^{r-1}n^r + \binom{p-1}{r-2}\beta^{r-2}n^{r-2}e(G'[V_0])\\ \leq & g_r(\beta)n^r + \binom{p-1}{r-2}\beta^{r-2}(1-p\beta)n^{r-1}/2\end{aligned}$$

r-cliques in $G' \in \mathcal{G}'(n, \beta)$. This means that if (n, β) is not feasible, then $k_r(n, (1-\beta)n)$ is also at most $g_r(\beta)n^r + \theta(n^{r-1})$.

However, we are unable to determine $k_r(n, (1-\beta)n)$ if (n, β) not feasible. For example, pick n and β such that the integers k and l, where $4k + 2l + 1 = |V_0| = (1 - (p-1)\beta)n$ and $2k = \delta(G[V_0]) = (1 - p\beta)n$, satisfy the hypothesis of Theorem 2.1.2. Since $\delta(G[V_0]) > 2|V_0|/5$, $G'[V_0]$ is bipartite by Theorem 2.1.1 for $G' \in \mathcal{G}'(n, \beta)$. Thus, $e(G'[V_0]) = (n' + 1)(1 - \beta')n'/2$. Depending on r, $\mathcal{G}(n, \beta)$ would be preferred over $\mathcal{G}'(n, \beta)$. Thus, it is possible that there exists a class of intermediate extremal families of graphs depending on r, β and n, which achieve

$k_r(n, \beta)$.

8.3 Maximum number of cliques

Recall that $k_r(n, \delta)$ is the minimum number of r-cliques in graphs of order n with minimum degree δ. Now, we ask the natural opposite question. Let $h_r(n, \Delta)$ be the maximum number of r-cliques in graphs for a given n and maximum degree Δ. The graph complement of $G \in \mathcal{G}(n, \beta)$ is a natural candidate for $h_r(n, \beta n - 1)$. It turns out that this is true for $r = 3$, because the following theorem of Goodman[23] states that the sum of triangles in a graph and its complement is completely determined by its degree sequence.

Theorem 8.3.1 (Goodman [23])**.** *Suppose G is a graph of order n. Then*

$$k_3(G) + k_3(\overline{G}) = \frac{1}{2} \sum_{v \in V(G)} \left(d(v) - \frac{n-1}{2} \right)^2 + \frac{n(n-1)(n-5)}{24}.$$

Therefore, evaluating $h_3(n, \beta n - 1)$ is equivalent to evaluating $k_3(n, (1-\beta)n)$. In particular, for $1/(p+1) \leq \beta \leq 1/p$,

$$h_3(n, \beta n - 1) \leq ((1-2\beta)n + 1)^2 n/8 + n(n-1)(n-5)/24 - g_3(\beta)n^3.$$

Next, we look at $h_r(n, \Delta)$ for $r \geq 4$. Thomason [41] showed that $k_r(G) + k_r(\overline{G})$ for $r \geq 4$ is not uniquely determined by it degree sequences, even if $k_t(G)$ and $k_t(\overline{G})$ are known for all $t < r$. Nevertheless, it is natural to assume that the extremal graph for $h_r(n, \beta n - 1)$ is $\{\overline{G} : G \in \mathcal{G}(n, \beta)\}$.

Conjecture 8.3.2. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let n and βn be positive integers. Let $G \in \mathcal{G}(n, \beta)$ Then, $h_r(n, \beta n - 1) \leq k_r(\overline{G})$ for positive integers r. Moreover, for $3 \leq r \leq p+1$ equality holds if and only if (n, β) is feasible and the extremal graphs are the complements of the members of $\mathcal{G}(n, \beta)$.*

As mentioned in the introduction (Chapter 1), determining the upper bound on $k_r(G)$ for given e is a special case of the Kruskal-Katona Theorem. Hence, $h_r(n, \Delta)$ can also be viewed as a variant of the Kruskal-Katona Theorem.

Notice that in all the evaluations of $k_3(n,\delta)$, we need to know that values of $k_r(n,\delta)$ for $3 < r \leq p+1$ (e.g. Theorem 4). Thus, $k_3(n,\delta)$ is considered to be the most difficult case. Hence, if we can solve $h_3(n,\beta n-1)$ for all $0 \leq \beta \leq 1$, then we have $k_3(n,(1-\beta)n)$. It would lead to a new approach to prove Conjecture 3.1.1.

8.4 A degree sum condition

Let T be a t-clique. Denote by $\sigma(T)$ the sum of the degrees of the vertices in T. Define $\sigma_t(G)$ to be the minimal $\sigma(T)$ for all $T \in \mathcal{K}_t(G)$. Clearly, $\delta(G) \geq (1-\beta)n$ implies $\sigma_t(G) \geq t(1-\beta)n$ for all t.

We would like to replace the condition on the minimum degree with $\sigma_t(G) \geq t(1-\beta)n$. We further restrict to the case when $r > t$ and conjecture that the same result holds.

Conjecture 8.4.1. *Let $0 < \beta < 1$ and $p = \lceil \beta^{-1} \rceil - 1$. Let r and t be integers with $1 \leq t < r$. Let n and βn be positive integers. Suppose G is a graph of order n with $\sigma_t(G) = t(1-\beta)n$. Then,*

$$k_r(G) \geq g_s(\beta)n^r$$

holds. Moreover, equality holds if and only if (n,β) is feasible and the extremal graphs are members of $\mathcal{G}(n,\beta)$.

We could try to tackle this conjecture by the same method as before, replacing the assumption of $\delta(G) = (1-\beta)n$ by $\sigma_t(G) \geq t(1-\beta)n$. We hope that the corresponding result would also be true. However, Lemma 4.1.1 (i) holds, but not (ii). Lemma 4.1.1 (ii) states that $D(S) \geq D(T) - (s-t)\beta$ for $S \in \mathcal{K}_s(G)$, $T \in \mathcal{K}_t(S)$ and $s \leq t$. Suppose $s = t+1$ and all but one vertices of S join to every other vertex and the remaining vertices have degree $(1-t\beta)n + t$. Then, clearly $D(S) = 1 - t\beta < D(T) - \beta$, where T is the set of vertices with degree $n-1$. The correct analogue of Lemma 4.1.1 (ii) would be $D(S) \geq D(T) - \max\{r, s-t\}\beta$. Nevertheless, one can check that all the remaining results in both Section 3.3, Section 4.1 and Section 4.2 hold by examining the proof. In summary, we can prove the conjecture for $p = 2$ (i.e. $\sigma_2(G) \leq 4n/3$) and for K_{p+2}-free graphs.

Theorem 8.4.2 (A degrees sum condition of Theorem 3)**.** *Let* $1/3 < \beta < 1/2$*. Suppose* G *is a graph of order* n *with* $\sigma_2(G) = 2(1-\beta)n$*. Then*

$$k_3(G) \geq (1-2\beta)\beta n k_2(G).$$

Furthermore, equality holds if and only if (n, β) *is feasible and* G *is a member of* $\mathcal{G}(n, \beta)$*.*

Theorem 8.4.3 (A degrees sum condition of Theorem 4)**.** *Let* $0 < \beta < 1$ *and* $p = \lceil \beta^{-1} \rceil - 1$*. Let* r *and* t *be integers with* $r \leq t$*. Suppose* G *is a* K_{p+2}*-free graph of order* n *with* $\sigma_r(G) = r(1-\beta)n$*. Then,*

$$k_r(G) \geq g_r(\beta) n^r \tag{8.1}$$

holds. Moreover, equality holds if and only if (n, β) *is feasible and the extremal graphs are members of* $\mathcal{G}(n, \beta)$*.*

Once again, the difficulties lie in handling the heavy cliques. Since we have a weaker version of Lemma 4.1.1 (ii), our argument might not hold even for the case $p = 3$. Nevertheless, by modifying the arguments in Section 7.1, one would hope that there exists a constant $\beta_{p,t} > 1/(p+1)$ depending only on p and t, such that Conjecture 8.4.1 holds for $1/(p+1) < \beta < \beta_{p,t}$. The proof of Conjecture 3.1.1 could be modified to prove Conjecture 8.4.1.

Chapter 9

Constrained Ramsey numbers for rainbow matching

9.1 Introduction

A *k-edge colouring* of a graph G is an edge colouring of G with exactly k colours. A graph G is *monochromatic* if all its edges have the same colour. For integers s and t, the *Ramsey number* $R(s,t)$ is the minimum number N such that for every 2-edge colouring of K_N with two colours, say with colours red and blue, there is a red monochromatic K_s or a blue monochromatic K_t. The existence of $R(s,t)$ was first proved by Ramsey [38] and rediscovered by Erdős and Szekeres [16]. Similarly, $R(s_1,\dots,s_k)$ is the minimum number N such that every k-edge colouring of K_N with colours $c_1,\dots,c_k$ contains a monochromatic copy of K_{s_i} of colour c_i for some i. If $s_i = s$ for $i = 1,\dots,k$, we simply write $R_k(s)$.

If an edge colouring of K_N uses infinitely many colours, then it is possible to avoid monochromatic K_s for $s \geq 3$. Nevertheless, there exists a well-structured edge coloured complete subgraph in K_N. For example, there may exist a complete subgraph that is *rainbow* (i.e every edge has a distinct colour). If we let the vertices of G be $v_1,\dots,v_n$, then a *lexicographically coloured* (or *colexicographically coloured*) G is an edge colouring such that the edge v_iv_j has colour c_i for $i <$

j (or $i > j$ respectively) with c_i distinct. A lexicographically coloured finite graph becomes colexicographically coloured if the ordering on the vertex set is reversed, and vice versa. Erdős and Rado [18] proved that for every $n \geq 3$, there exists an integer $N(n)$ such that every edge colouring of K_N contains a complete subgraph K_n with one of the mentioned edge colourings. This result is also known as the canonical Ramsey theorem.

Theorem 9.1.1 (Canonical Ramsey Theorem [18]). *For every positive integer s, there exists an integer $N(s) > 0$ with the following property. For each integer $N \geq N(s)$ and every edge colouring of K_N, there exists a K_s in K_N such that it is either monochromatic, rainbow, lexicographically coloured or colexicographically coloured.*

The Ramsey number $R(G, H)$ for graphs G and H is the minimum number N such that every 2-edge colouring of K_N, with colours red and blue say, contains a red monochromatic G or a blue monochromatic H. Similarly, we define $R(G_1, \ldots, G_k)$ and $R_k(G)$. For a graph G and an edge colouring of K_N with N sufficiently large, by the canonical Ramsey theorem we can deduce that there exists a subgraph isomorphic to G with one of the four stated edge colourings. Recall that both lexicographically and colexicographically colourings depend on the ordering of the vertex set. Thus, they are not preserved under vertex relabelling, so we focus our attentions to monochromatic and rainbow subgraphs. The *constrained Ramsey number* $f(S, T)$ of graphs S and T is the minimum number N such that any edge colouring of K_N with any number of colours contains a monochromatic S or a rainbow T. It is also called the *rainbow Ramsey number* or the *monochromatic-rainbow Ramsey number* in literature. However, $f(S, T)$ does not exist for every choice of S and T. Using the canonical Ramsey theorem, it is easy to characterise the pairs (S, T) for which $f(S, T)$ exists. For the sake of completeness, we give the proof below.

Proposition 9.1.2. *For graphs S and T, $f(S, T)$ exists if and only if S is a star or T is acyclic.*

Proof. First we show that if S is a star or T is acyclic, then $f(S, T)$ exists. Consider an edge colouring of K_N with vertex set $\{v_1, \ldots, v_N\}$ with N sufficiently large. By the canonical Ramsey theorem, there exists a complete subgraph K_n that is well-coloured. If it is monochromatic or rainbow, we are done providing that $n \geq \max\{|S|, |T|\}$. Without loss of generality, we may assume that K_n that is lexicographically coloured. Otherwise, reverse the ordering of vertex set. If S is a star, then we are done by considering the centre of the star to be the smallest element in K_n. Hence, we may assume S is not a star and T is acyclic. We are going to show that there exists an ordering of the vertex set $V(T)$ such that if T is lexicographically coloured then T is rainbow. It is enough to show that such a vertex ordering exists for T connected, so T is a tree. First, we root T at an arbitrary vertex. Order the vertex set of T such that if v_i is at a level larger than v_j, then $i \leq j$. It is easy to verify that if T is lexicographic coloured, then T is rainbow. Thus, there exists a monochromatic S or a rainbow T in K_n, so $f(S,T)$ exists.

Finally, suppose neither S is a star nor T is acyclic. We are going to show that there is no monochromatic S nor rainbow T in a lexicographic coloured of K_N for all N. Since T is not acyclic, T contains a cycle. However no cycle in K_N is rainbow (nor monochromatic), so there does not exist a rainbow T in K_N. By similar argument, we may assume that S is also acyclic. Since S is not a star, there exist two vertex disjoint edges in S. Observe that any two vertex disjoint edges have different colours in K_N. Therefore, K_N does not contain a monochromatic S. This completes the proof. □

From now on, we assume that either S is a star or T is acyclic. Thus, $f(S,T)$ exists. Notice that $f(S,T)$ is at least $R_{t-1}(S)$, where $t = e(T)$. This is because by definition of $R_{t-1}(S)$, there exists a $(t-1)$-edge colouring C of $K_{R_{t-1}(S)-1}$ without a monochromatic S. Since C uses at most $t-1$ colours, no subgraphs isomorphic to T are rainbow.

Various people studied $f(S,T)$ for different cases of S and T. Alon, Jiang, Miller and Pritikin [2] studied the case when S is a star and T is a complete graph and evaluated that $f(K_{1,s}, K_t) = \Theta((s-1)t^3/\ln t)$. Notice that $f(K_{1,s}, T)$ is very closely related to an $(s-1)$-good colouring. An edge colouring is called *m-good*, if for every vertex v there are at most m different coloured edges incident to v. Thus, $f(K_{1,s}, T)$ is the minimum number N such that every $(s-1)$-good colouring of K_N

contains a rainbow T. On the other hand, $f(S, K_{1,t})$ is the local $(t-1)$-Ramsey number of a graph T that was first introduced by Gyárfás, Lehel, Schelp and Tuza [26]. The local $(t-1)$-Ramsey number of a graph T is the Ramsey number of T restricted to edge colourings for which to each vertex v there are at most $t-1$ edges with distinct colours incident to v. Jamison, Jiang, Ling [28] studied $f(S, T)$ when S and T are both trees. Further, they conjectured that if S and T are paths of lengths s and t respectively, then $f(P_{s+1}, P_{t+1}) = \Theta(st)$. Wagner [46] showed this is $f(P_{s+1}, P_{t+1}) \leq O(s^2 t)$, and later on, Loh and Sudakov [32] showed that $f(P_{s+1}, P_{t+1}) \leq O(st \log t)$. Gyárfás, Lehel and Schelp [25] and independently Thomason and Wagner [43] showed that $f(G, P_{t+1})$ is equal to $R_{t-1}(G)$ for $t \leq 4$.

9.2 Rainbow matching $f(S, tK_2)$

From now on, T is a t-matching tK_2, that is t vertex disjoint edges. As mentioned above $f(S, tK_2) \geq R_{t-1}(S)$. Let C be an edge colouring of a complete graph containing a rainbow $(t-1)K_2$ but not a rainbow tK_2. Clearly, after removing the vertices of the rainbow $(t-1)K_2$, the resulting graph has at most $t-1$ colours. Moreover, the colours in the resulting graph are precisely the colours in the rainbow $(t-1)K_2$. Hence, it is an easy exercise to show that $f(S, tK_2) \leq R_{t-1}(S) + 2t - 2$ for $t \geq 3$, which was first proved by Truszczyński [44]. Therefore, $f(S, tK_2)$ is asymptotically equal to $R_{t-1}(S)$ providing $R_{t-1}(S)$ is not linear in t.

Next suppose that S is also a matching but of size s. Cockayne and Lorimer [11] evaluated $R_{t-1}(sK_2)$. In fact, they evaluated $R(s_1K_2, \ldots, s_kK_2)$ for positive integers $s_1, \ldots, s_k$.

Theorem 9.2.1 (Cockayne and Lorimer [11]). *Let $s_1, \ldots, s_k$ be positive integers. Then,*

$$R(s_1K_2, \ldots, s_kK_2) = \max\{s_1, \ldots, s_k\} + 1 + \sum_{i=1}^{k}(s_i - 1).$$

In particular, $R_k(sK_2) = (s-1)(k+1) + 2$.

Hence, $f(sK_2, tK_2) \geq R_{t-1}(sK_2) = (s-1)t + 2$. Bialostocki and Voxman [4] showed that if $s = t \geq 3$, then equality holds. Later, Eroh [20] extended the result to $s > t \geq 2$ and conjectured that the equality holds for all $s \geq 3$ and $t \geq 2$. Eroh [21] also investigated the case when S is a star.

We are going to generalise their results and evaluate $f(S, tK_2)$ exactly for almost all graphs S and all integers $t \geq 2$. From now on, we say that two graphs G and H are disjoint to mean vertex disjoint. We state Theorem 1.3.2 below.

Theorem 1.3.2. *Suppose S is a graph of order at least 5 and $R_{k+1}(S) \geq R_k(S)+3$ for all positive integers k. Then, $f(S, tK_2) = R_{t-1}(S)$ for all integers $t \geq 2$.*

Here, we give an outline of the proof. Suppose the theorem is false for some $t > 2$. Let $t > 2$ be the minimal counterexample and $N = R_{t-1}(S)$. Hence, there exists an edge colouring C of K_N that contains neither a monochromatic S nor a rainbow tK_2. By the minimal counterexample, there exists a rainbow $(t-1)K_2$, say $W = \{e_i : 1 \leq i \leq t-1\}$ in G. For a subgraph $G \subset K_N$, the colour set $C(G)$ is defined to be the set of colours $C(e)$ for $e \in E(G)$. Without loss of generality, $C(W) = \{c_i : 1 \leq i \leq t-1\}$ with $C(e_i) = c_i$ for $1 \leq i \leq t-1$. Since $N = R_{t-1}(S)$ and C does not contain a monochromatic S, C uses at least t colours and so there exists an edge f_1 of colour not in $C(W)$. Also, $V(f_1) \cap V(W) \neq \emptyset$ or else $W + f_1$ forms a rainbow tK_2. Assume that $V(f_1) \cap V(e_1) \neq \emptyset$ and $V(f_1) \cap V(e_i) = \emptyset$ for $2 \leq i \leq t-1$. If there exists an edge f_2 of colour c_1 disjoint from both f_1 and e_1, then $V(f_2) \cap V(e_i) \neq \emptyset$ for some i, else $f_1, f_2, e_2, e_3, \ldots, e_{t-1}$ form a rainbow tK_2. Also, assume that $V(f_2) \cap V(e_i) = \emptyset$ for all i except $i = 2$. Repeat this argument and during each step we always assume that there exists an edge f_i with the following properties. The edge f_i is coloured c_{i-1} and it is disjoint from e_j unless $j = i$. Moreover, the set of f_i are disjoint. The edge f_t is disjoint from $W \cup \{f_i : 1 \leq i \leq t-1\}$. More importantly, $\{f_i : 1 \leq i \leq t\}$ is a rainbow tK_2, which is a contradiction. In our assumption, $C(f_i) = c_i$ for all i, but it is not always true. This problem can be overcome if we relax the condition to $C(f_i) = c_j$ for $j \leq i$. However, it is possible that $|V(f_i) \cap V(W)| = 2$, that is, there exists $j_1 \neq j_2$ such that $|V(e_{j_1}) \cap V(f_i)| = 1 = |V(e_{j_2}) \cap V(f_i)|$. Thus, our principle is to always pick f_i such that $|V(f_i) \cap V(W)|$ is minimal. We now give a formal argument of the proof.

Proof of Theorem 1.3.2. Let N be $R_{t-1}(S)$. From our previous observation, $f(S, tK_2) \geq N$. Hence, it is sufficient to show that $f(S, tK_2) \leq N$.

First we consider the case $t = 2$. Let C be an edge colouring of $K_{|S|}$ that does not contain a rainbow copy of $2K_2$. Suppose there exists a path P with three vertices whose edge colours are distinct. Since $|S| \geq 5$, there is an edge e disjoint from P. Then, there exists a rainbow $2K_2$ in $P \cup \{e\}$, contradicting the assumption on C. Hence, $K_{|S|}$ is monochromatic, so $f(S, 2K_2) = |S| = R_1(S)$ and the theorem is true for $t = 2$.

Suppose the theorem is false and let $t > 2$ be the minimal counterexample. Let C be an edge colouring of K_N that contains neither a monochromatic S nor a rainbow tK_2. First, we show that there always exists at least one candidate for f_i in the claim below. The claim below states that given a vertex set U not too large containing a rainbow matching pK_2, we can always find an edge e disjoint from U coloured by a member of $C(pK_2)$. The proof of the claim follows easily from the proof of Lemma 2.2 in [20].

Claim 9.2.2. *Let $U \subset V(K_N)$ satisfy $|U| \leq 3p$ for positive integer $p < t$. Suppose U contains a rainbow pK_2 with colours $c_1, \ldots, c_p$. Then, there exists an edge e in K_N with $V(e) \cap U = \emptyset$ and $C(e) \in \{c_1, \ldots, c_p\}$.*

Proof of Claim. Since

$$\begin{aligned} |U| \leq 3p &= R_{t-1}(S) - (R_{t-1}(S) - 3p) \leq R_{t-1}(S) - R_{t-p-1}(S) \\ &= R_{t-1}(S) - f(S, (t-p)K_2), \end{aligned}$$

we may assume there exists a rainbow copy of $(t-p)K_2$ in the graph induced by the vertex set $V(K_N) \backslash U$. If $C((t-p)K_2) \cap \{c_1, \ldots, c_p\} = \emptyset$, then $(t-p)K_2$ together with the rainbow pK_2 in U forms a rainbow tK_2. Thus, the claim is true. □

Next, we set up notation to keep track of the f_i. Let $\mathcal{A}$ be the subset of $\{0, 1, 2\}^t \backslash \{1, 2\}^t$ consisting of sequences $a = (a_1, \ldots, a_t)$, whose zeros all come at the end; that is, $a_i = 0$ implies $a_{i+1} = 0$ for $1 \leq i \leq t$. Let $z(a) = \min\{i : a_i = 0\}$. Thus, $a_i \neq 0$ for $i < z(a)$ and $a_i = 0$ for $i \geq z(a)$. We say that $a \in \mathcal{A}$ is a *matching sequence* if

(*a*) there exists a set $W = \{e_i : 1 \leq i \leq t-1\}$ of independent edges which is rainbow coloured, that is, W is a rainbow $(t-1)K_2$;

(b) there exists a sequence $f_1, \ldots f_z$, $z = z(a)$, of independent edges not in W with $|V(f_i) \cap W| = a_i$ for $1 \leq i \leq z$;

(c) for every edge $e_i \in W$, $1 \leq i \leq t-1$, there exists at most one f_j, $1 \leq j \leq z$, such that $V(e_i) \cap V(f_j) \neq \emptyset$;

(d) $C(f_1) \notin \{C(e_i) : e_i \in W\}$;

(e) for $2 \leq i \leq z$, $C(f_i) \in \{C(e_j) : e_j \in W_{i-1}\}$ where W_{i-1} is the set of edges $e_j \in W$ that share a vertex with one of $f_1, \ldots, f_{i-1}$.

Notice that if the all-zero sequence is a matching sequence then we are done, because $W + f_1$ is a rainbow copy of tK_2. To show that matching sequences exist, let $\mathcal{B}$ be the set of initial sequences from the sequences in $\mathcal{A}$, that is, $\mathcal{B}$ is the subset of $\{\Lambda\} \cup \bigcup_{k=0}^{t} \{0,1,2\}^k \cup \mathcal{A}$ consisting of sequences $b = (b_1, \ldots, b_l)$ whose zeros all come at the end, if any, (where Λ is the empty sequence of length 0). For $b = (b_1, \ldots, b_l) \in \mathcal{B}$, define $z(b) = \min\{i : b_i = 0\}$, if this set is non-empty, and otherwise define $z(b) = l$. We say that $b = (b_1, \ldots, b_l) \in \mathcal{B}$ is a *partial matching sequence* if it also satisfies properties $(a) - (e)$.

We claim that every partial matching sequence can be extended to a matching sequence (and, in particular, matching sequences exist because $\Lambda \in \mathcal{B}$). Let b be a partial matching sequence of length l. If $l = t$, b is also a matching sequence. Thus, it is enough to show that b can be extended to a partial matching sequence of length $l+1$ for $1 \leq l < t$. First, suppose that b is the empty sequence. There exists a rainbow $(t-1)K_2$ in K_N as $N \geq f(S, (t-1)K_2)$; call it W. Since $N = f(S, tK_2) = R_{t-1}(S)$, there are at least t colours in K_N. Therefore, there exists an edge f_1 not in W with $C(f_1) \neq C(e_i)$ for $1 \leq i \leq t-1$. Hence, this defines a partial matching sequence of length 1. Now, suppose $b = (b_1, \ldots, b_l)$ with $1 \leq l < t$ and W and $f_1, \ldots, f_z$ and $z = z(b)$ are defined by properties $(a) - (e)$. If $b_l = 0$, then $(b_1, \ldots, b_l, 0, \ldots, 0)$ is a matching sequence with the same W and $f_1, \ldots, f_z$. If $b_l \neq 0$, $z = l$. Let $U = V(W_l) \cup \bigcup_{i=1}^{l} V(f_i)$. Observe that $|U| \leq 3|W_l|/2$, and W_l is rainbow coloured from the construction. There exists an edge f_{l+1} independent of W_l with $C(f_{l+1}) \in C(W_l)$ by Claim 9.2.2. Thus, there exists a partial matching sequence $(b_1, \ldots, b_l, b_{l+1})$ with $b_{l+1} = |V(f_{l+1}) \cap W|$. This proves the claim.

Let $\mathcal{A}_M$ be this set of matching sequences. Let $a = (a_1, \ldots, a_t)$ be the lexicographically minimal element of $\mathcal{A}_M$. Thus, for any $(a'_1, \ldots, a'_t) \in \mathcal{A}_M \backslash \{a\}$, $a_l < a'_l$ where $l = \min\{i : a_i \neq a'_i\}$. Define W, $f_1, \ldots, f_z$ with $z = z(a)$ satisfying $(a) - (e)$. Recall that we always choose f_i with $|V(f_i) \cap V(W)|$ minimal. As mentioned before, if a is the all-zero sequence, we are done. But if a is not the all zero sequence, properties (c) and (e) imply there exists integers $j < t+1$ and $l < z$ with $C(f_z) = C(e_j)$ and $|V(f_l) \cap V(e_j)| = a_l \neq 0$. Set $W' = W - e_j + f_z$. We define $b = (a_1, \ldots, a_{l-1}, a_l - 1)$. This is a partial matching sequence with W', $f_1, \ldots, f_l$. By the previous claim, b extends to a matching sequence a'. But a' is lexicographically less than a contradicting the choice of a. This completes the proof of the theorem. □

9.3 Consequences of Theorem 1.3.2

First, we are going to identify those graphs that fail to satisfy the hypothesis of Theorem 1.3.2.

Proposition 9.3.1. *Let S be a graph of order at least 5 with no isolated vertex. If $R_{k+1}(S) < R_k(S) + 3$ for some positive integers k, then S is bipartite and one of its vertex classes has size at most 3.*

Proof. Fix k and let $N = R_k(S) - 1$. Let C be a k-edge colouring of K_N that does not contain a monochromatic S. If S is not bipartite, then define an edge colouring of K_{2N} as follows: take two copies of K_N each with edge colouring C and join the vertices between the two copies with a new colour. Since S is bipartite, there is no monochromatic S in K_{2N}. Therefore, $R_{k+1}(S) \geq 2N + 1 = 2R_k(S) - 1 \geq R_k(S) + |S| - 1 > R_k(S) + 3$. Suppose that S is bipartite with each vertex class of size of at least 4. We add three new vertices x, y, z,to K_N and join x, y, z to all other vertices (including x, y, z) with a new colour. It is easy to see that the resulting graph does not contain a monochromatic S, so $R_{k+1}(S) \geq R_k(S) + 3$. □

From the proof of Theorem 1.3.2, we can deduce the following result.

Corollary 9.3.2. *Let S be a graph with no isolated vertex satisfying one of the following:*

(i) S is not bipartite and $|S| \geq 5$, or

(ii) S is bipartite with each vertex classes of size at least 4, or

(iii) S is bipartite and $e(S) > 12$.

Then, there exists an integer $t_0(S)$ such that $f(S, tK_2) = R_{t-1}(S)$ for all $t \geq t_0(S)$.

Proof. By Proposition 9.3.1 and Theorem 1.3.2, the corollary is true for (i) and (ii) with $t_0(S) = 1$. Suppose S satisfies (iii). We may also assume that one of its vertex classes has size at most 3, or else S satisfies (ii). Since $e(S) > 12$, $\Delta(S) \geq 5$ and so $K_{1,5} \subset S$. By a result of Burr and Roberts [8], $R_k(K_{1,5}) \geq 4k+1$ and so $R_k(S) \geq 4k + 1$. Suppose $f(S, 2K_2) = R_1(S) + c$ for some constant c. By mimicking the proof of Theorem 1.3.2, we can deduce that $R_{t-1}(S) \leq f(S, tK_2) \leq R_{t-1}(S) + \max\{0, c+2-t\}$ for $t \geq 2$. Thus, $f(S, tK_2) = R_{t-1}(S)$ for $t \geq c+2$. □

Now we apply Theorem 1.3.2 to graphs G with known $R_k(S)$. Unfortunately, the only graphs with known $R_k(S)$ are star and matching. For S a matching, $R_k(S)$ is already mentioned before(see Theorem 9.2.1). For S is star, Burr and Roberts [8] proved that

$$R_k(K_{1,s}) = \begin{cases} (s-1)k + 1 & \text{if } s \text{ is even and } k \text{ is odd,} \\ (s-1)k + 2 & \text{otherwise.} \end{cases}$$

Hence, together with Theorem 1.3.2, we have the following corollary.

Corollary 9.3.3. *For any integers $s \geq 4$ and $t \geq 2$,*

$$f(sK_2, tK_2) = (s-1)t + 2.$$

For any integers $s \geq 5$ and $t \geq 2$,

$$f(K_{1,s}, tK_2) = \begin{cases} (s-1)(t-1)+1 & \text{if } s \text{ is even and } t \text{ is odd,} \\ (s-1)(t-1)+2 & \text{otherwise.} \end{cases}$$

□

This corollary improves both results of Eroh [21, 20]. Unfortunately, we are unable to prove $f(3K_2, tK_2) = 2t + 2$ as conjectured by Eroh. Instead, we have managed to show a weaker result.

Proposition 9.3.4. *For any positive integer $t \geq 3$, $f(3K_2, tK_2) \leq 3t - 1$.*

Proof. We proceed by induction on t. Bialostocki and Voxman [4] proved the case for $t = 3$. Thus, we may assume $t > 3$. Let $N = 3t - 1$. Let C be an edge colouring of K_N with neither a monochromatic $3K_2$ nor a rainbow tK_2. Let $c_1, c_2, \ldots$ be the colours of C. For an integer i, G_i denotes the subgraph induced by the edges with colour c_i. Clearly, G_i does not contain a copy of $3K_2$. Modifying the proof of Claim 9.2.2, we have the following claim.

Claim 9.3.5. *For any edge e and vertex v, there exists an edge e' independent of e and v with $C(e) = C(e')$.*

Now we claim that G_i is either a subgraph of K_5 if it is connected or else it is isomorphic to two disjoint triangles. First suppose G_i is connected. We say a path in G_i is *maximal* if it cannot be extended to a longer path. Since G_i is $3K_2$-free, no path has length greater than 4. By Claim 9.3.5, any edge of e in G_i can be extended to a $2K_2$. Hence, all maximal paths have length exactly 4. Let $p_1p_1 \ldots p_5$ be a maximal path in G_i. In fact, p_1p_5 is also an edge in G_i by Claim 9.3.5 taking $e = p_2p_3$ and $v = p_4$. Hence, $p_1p_2 \ldots p_5$ forms a 5-cycle in G_i. Thus, G_i is a subgraph of K_5, for otherwise, there exists a $3K_2$ in G_i. Next, suppose G_i is not connected. It is easy to see that G_i has exactly two components with each component being either a star or a triangle. Suppose one of the components is a star. By taking the centre of the star as v and any edge in the other component as e, we obtain a contradiction to Claim 9.3.5. Hence, G_i is an union of two disjoint triangles. In summary, G_i is either a subgraph of K_5 or isomorphic to $2K_3$.

Since $N > f(3K_2, (t-1)K_2)$, there is a rainbow $(t-1)K_2$, $W := \{x_iy_i : C(x_iy_i) = c_i \text{ for } 1 \leq i \leq t-1\}$. Let $H = K_N \backslash V(W)$. Clearly, $|H| = t+1$ and $e(H) = t(t+1)/2$. Notice that H is coloured only with colours $c_1, \ldots, c_{t-1}$, or else there exists a rainbow tK_2. Also, H_i is a subgraph of a triangle, so $e(H_i) \leq 3$ for $1 \leq i \leq t-1$. The number of colours used in H is at least $e(H)/3 = t(t+1)/6 > t-1$ as $t > 3$. This is a contradiction and completes the proof of the proposition. □

So far, the rainbow subgraph is always a matching. We would like to know whether $f(S,T) = R_{t-1}(S)$ is still true for T with a similar structure to a matching. It turns out that this is true if $T = P_3 \cup (t-2)K_2$ or $T = P_4 \cup (t-3)K_2$.

Corollary 9.3.6. *For a graph S and an integer t,*

$$f(S, P_3 \cup (t-2)K_2) \leq \max\{f(S,tK_2), 2t+1\} \qquad \text{if } t \geq 2,$$
$$f(S, P_4 \cup (t-3)K_2) \leq f(S,tK_2) \qquad \text{if } t \geq 3.$$

Moreover, if S satisfies the hypothesis in Theorem 1.3.2, then equality holds.

Proof. First, we evaluate $f(S, P_3 \cup (t-2)K_2)$. It is easy to see that $f(S, P_3) = |S| \leq f(S, 2K_2)$, so it is true for $t = 2$. Thus, we may assume that $t \geq 3$. Let $N = \max\{f(S,tK_2), 2t+1\}$. Consider an edge colouring C of K_N without a monochromatic S. Since $N \geq f(S,tK_2)$, there exists a rainbow tK_2, say $\{x_iy_i : C(x_iy_i) = c_i \text{ for } 1 \leq i \leq t\}$. Let w be a vertex not in $\{x_i, y_i : 1 \leq i \leq t\}$. Note $C(x_iw) = c_i$ for $i = 1, \ldots t$, otherwise there exists a rainbow $P_3 \cup (t-2)K_2$. However, $x_1wx_2 \cup \{x_iy_i : 3 \leq i \leq t\}$ is a rainbow $P_3 \cup (t-2)K_2$. This is a contradiction and proves the first assertion of the corollary.

Now suppose that $N = f(S,tK_2)$ with $t \geq 3$. Let C be an edge colouring of K_N without a monochromatic S. We are going to show that there exists a rainbow $P_4 \cup (t-3)K_2$. There exists a rainbow tK_2, $\{x_iy_i : C(x_iy_i) = c_i \text{ for } 1 \leq i \leq t\}$. Clearly, if the graph induced by x_i, y_i, x_j, y_j contains an edge not coloured by either c_i nor c_j for some $1 \leq i < j \leq t$, then there exists a rainbow $P_4 \cup (t-3)K_2$. Actually, it is sufficient to show that there exists a rainbow P_4 with colours c_1, c_2, c_3 in $\{x_1, x_2, x_3, y_1, y_2, y_3\}$. Without loss of generality, $C(x_1x_2) = c_1$. Then

y_2y_3 must be coloured by c_2. If $C(x_1x_3) = c_1$, $x_1x_3y_3y_2$ is rainbow. If $C(x_1x_3) = c_3$, $x_3x_1x_2y_2$ is rainbow. Hence, the proof is complete. □

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