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ON RINGS WHOSE MODULES HAVE NONZERO HOMOMORPHISMS TO NONZERO SUBMODULES

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Abstract: We carry out a study of rings R for which $\mathrm{Hom}_R(M,N) \neq 0$ for all nonzero $N \leq M_R$. Such rings are called retractable. For a retractable ring, Artinian condition and having Krull dimension are equivalent. Furthermore, a right Artinian ring in which prime ideals commute is precisely a right Noetherian retractable ring. Retractable rings are characterized in several ways. They form a class of rings that properly lies between the class of pseudo-Frobenius rings, and the class of max divisible rings for which the converse of Schur's lemma holds. For several types of rings, including commutative rings, retractability is equivalent to semi-Artinian condition. We show that a Köthe ring R is an Artinian principal ideal ring if and only if it is a certain retractable ring, and determine when R is retractable.

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1. Introduction

Throughout this paper rings will have unit elements and modules will be right unitary. Following [**12**], an R-module M is called *retractable* if $\mathrm{Hom}_R(M,N) \neq 0$ for all nonzero submodules N of M. Semisimple modules and fully idempotent modules [**21**] are clearly retractable, and more generally self-projective modules with zero radical and essentially compressible modules are known to enjoy this property; see [**5**, 3.4] and [**19**, Theorem 3.1]. Retractable modules have appeared in different situations. For example, in the study of nonsingular modules satisfying one of the properties: CS, continuous, quasi continuous or having a Baer endomorphism ring [**16**, Theorem 22]. They have also been applied in the study of prime M-ideals that correspond to the isomorphism classes of indecomposable M-injective modules in $\sigma[M]$ [**3**, Theorems 2.10 and 6.7] and in the characterization of endomorphism rings of quasi-injective envelopes of polyform modules [**5**, 5.19]; see also, [**9**, Theorem 2.6], and [**24**, Section 2]. In [**21**], it is shown that the commutative rings over which every module is fully idempotent are exactly the semisimple rings.

Rings with all finitely generated modules retractable are characterized in [**8**], and finitely generated retractable modules over right FBN rings are characterized in [**18**] where the term slightly compressible is used for retractable.

In the present work, we shall consider *retractable rings* which are rings with all nonzero module retractable. Recall from [**7**], R is a right CPF ring if for all proper ideals I of R, any faithful R/I-module is a generator in Mod-R/I. Artinian principal ideal rings are CPF [**23**, 56.9(c)]. In Proposition 2.4 we show that the class of retractable rings properly lies between the class of right CPF rings and the class of divisible right max rings which are "CS" in the sense of Hirano and Park [**11**]. These are rings for which the converse of Schur's Lemma holds; see also [**10**]. Some equivalent conditions for a ring to be retractable are given in Theorem 2.2, where it is shown that retractable rings are precisely rings over which all torsion theories are hereditary. Over a retractable ring, a module is Artinian if and only if it is Noetherian and its second singular submodule is Artinian (Proposition 2.10). Retractable rings with Krull dimension and reduced retractable rings are characterized in Theorems 3.6 and 3.2. More generally, retractable rings R such that $R/\mathrm{J}(R)$ is reduced are shown to be left semi-Artinian, and they are precisely semi-Artinian if in addition $\mathrm{J}(R) \subseteq \mathrm{Cent}(R)$ (Theorem 3.4 and Corollary 3.5). A result of Köthe states that over an Artinian principal ideal ring R every right (left) R-module is a direct sum of cyclic right (left) R-modules (i.e. R is a *Köthe* ring) [**13**]. We investigate the converse of the Köthe theorem and as an application of our results, we show that a Köthe ring R is an Artinian principal ideal ring if and only if it is a retractable ring such that for any ring decomposition $\mathrm{Mat}_n(S) \times T \simeq R$ with local S, the ring S is Köthe (Theorem 3.10). The retractability of Köthe rings are then determined. Any unexplained terminology, and all the basic results on rings and modules that are used in the sequel can be found in [**2**], [**5**] and [**14**].

2. Retractability of modules

In this section we investigate the class of retractable rings in Theorem 2.2 and Propositions 2.4, 2.6 and study modules over retractable rings. A class $\mathcal{C}$ of R-modules is called *retractable* if X_R is retractable for all $X \in \mathcal{C}$. An R-module M is called *essentially retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for all essential submodules N of M; see [**22**] for more information about essentially retractable modules. For an R-module M_R, the injective hull of M is denoted by $\mathrm{E}(M_R)$ or simply $\mathrm{E}(M)$.

Lemma 2.1. *The following statements are equivalent for a nonzero R-module M.*

(i) *M_R is essentially retractable.*
(ii) *There exists a nonzero $f \in \mathrm{Hom}_R(M, \mathrm{E}(M))$ such that $f(M)$ is an essentially retractable R-module.*

Proof: (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. Let K be an essential submodule in M_R. Then K is essential in $\mathrm{E}(M)$, and hence $K \cap f(M)$ is essential in $f(M)$. Thus there exists a nonzero homomorphism from $f(M)$ to $K \cap f(M)$. It follows that $\mathrm{Hom}_R(M, K) \neq 0$. □

Theorem 2.2. *For a ring R, the following statements are equivalent.*

(i) *R is a retractable ring.*
(ii) *Every nonzero R-module is essentially retractable.*
(iii) *Every essential extension of a cyclic R-module is essentially retractable.*
(iv) $\mathrm{Hom}_R(M, X) = 0 \Leftrightarrow \mathrm{Hom}_R(M, \mathrm{E}(X)) = 0$ *for all R-modules M and X.*
(v) *All torsion theories on R are hereditary.*

Proof: (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (ii). Note that if $0 \neq m \in M_R$, then mR essentially embeds in a suitable factor of M_R [**14**, Proposition 6.18].

(ii) $\Rightarrow$ (iv). If $\mathrm{Hom}_R(M, \mathrm{E}(X))$ is nonzero, then similar to the proof of Lemma 2.1, we have $\mathrm{Hom}_R(M, X) \neq 0$. The converse is clear.

(iv) $\Rightarrow$ (v). Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory on R, $N \leq M_R \in \mathcal{T}$ and $X \in \mathcal{F}$. Then $\mathrm{Hom}_R(M, X) = 0$, hence $\mathrm{Hom}_R(M, \mathrm{E}(X)) = 0$ by (iv). It follows that $\mathrm{Hom}_R(N, X) = 0$, proving that $N_R \in \mathcal{T}$.

(v) $\Rightarrow$ (i). Let $N \leq M_R$ and $(\mathcal{T}, \mathcal{F})$ be a torsion theory generated by M_R. By (v) $N_R \in \mathcal{T}$ and hence $\mathrm{Hom}_R(M, N) \neq 0$. □

In the following we collect more properties of modules over retractable rings. A module M_R is called *divisible* if $Mc = M$ for any right regular element $c \in R$ (i.e., $\text{r-ann}_R(c) = 0$). The ring R is called *right divisible* if the module R_R is divisible. It is well known that injective modules are divisible. If M is an R-module such that $(M/N)_R$ and N_R are divisible for some $N \leq M_R$, then it is easily seen that M_R is divisible.

Proposition 2.3. *Let R be a retractable ring and let M and N be nonzero R-modules.*

(i) *If $MI^n = 0$ for some ideal I of R and $n \geq 1$, then $Mc = M$ for every $c \in R$ which is right regular modulo I.*
(ii) $\mathrm{J}(M) \neq M$.
(iii) *If $M/\mathrm{J}(M)$ is a semisimple R-module and $\mathrm{Hom}_R(M, N) \neq 0$, then* $\mathrm{Hom}_R(N/\mathrm{J}(N), M/\mathrm{J}(M)) \neq 0$.
(iv) *The module M_R is nonsingular if and only if every nonzero submodule of M_R contains a nonzero injective projective submodule if and only if every nonzero submodule of M_R contains a nonzero projective submodule.*
(v) *If M_R is nonsingular, then* $\mathrm{J}(M) = 0$.

Proof: (i) We first show that any nonzero R-module is divisible. Let U_R be nonzero. Then $E = \mathrm{E}(U)$ is a retractable R-module by our assumption. So there is a proper submodule K of E such that U_R contains a submodule isomorphic to E/K. It follows that U_R contains a nonzero divisible submodule. Consequently, if $N = \sum\{K \leq M_R \mid K \text{ is divisible}\}$ for a given nonzero R-module M_R, then N is a nonzero divisible submodule of M_R. If M/N is nonzero, then it contains a nonzero divisible submodule A/N. It is easy to verify that A_R is also divisible and so it lies in N, a contradiction. Thus $M = N$ and M_R is divisible, as desired. Now let M_R be nonzero and $MI^n = 0$ for some ideal I of R and $n \geq 1$. Since R/I is a retractable ring, by the first part, MI^i/MI^{i+1} are divisible R/I-modules for $i = 0, 1, \ldots, n$ with $I^0 = R$. It follows that $Mc = M$ for every $c \in R$ which is right regular modulo I.

(ii) By [**23**, 14.9], M_R has a factor L such that $\mathrm{Soc}(L) \neq 0$. Thus by the retractable condition on L, we can deduce that M has a maximal submodule, proving that $\mathrm{J}(M) \neq M$.

(iii) Let $0 \neq f \in \mathrm{Hom}_R(M, N)$. Then f induces

$$\bar{f} \colon M/\mathrm{J}(M) \to f(M)/f(\mathrm{J}(M)).$$

Thus $f(\mathrm{J}(M)) \subseteq \mathrm{J}(f(M)) \neq f(M)$ by (ii). Hence, by hypothesis, there exists a simple submodule S of $M/\mathrm{J}(M)$ such that S embeds in $N/f(\mathrm{J}(M))$. Now by retractable condition on $N/f(\mathrm{J}(M))$, $S \simeq N/K$ for some maximal submodule K of N. It follows that

$$\mathrm{Hom}_R(N/\mathrm{J}(N), M/\mathrm{J}(M)) \neq 0.$$

(iv) First note that if M_R is nonsingular, then by hypothesis there is a nonzero map $f \colon \mathrm{E}(M) \to M$. Thus $\mathrm{Ker}\, f$ is an essentially closed, and hence a direct summand of the injective R-module $\mathrm{E}(M)$. It follows that $\mathrm{Im}\, f$ is a nonzero injective submodule of M_R. Therefore, we can deduce that every nonzero nonsingular R-module contains a nonzero injective R-module. On the other hand, if m is any nonzero element of a nonsingular R-module M then $\text{r-ann}_R(m)$ is not an essential right ideal of R, and so there exists a right ideal A in R such that $mA \simeq A$. Consequently, if M_R is nonsingular, then every nonzero submodule of M_R contains a nonzero injective projective submodule. The proof is now completed by the fact that nonzero projective modules are not singular. To see this let $P_R \neq 0$, $P \oplus K = F$ and F_R be free with basis $\{e_i \mid i \in I\}$. If $P \simeq F/K$ is singular then for every $i \in I$, there exists an essential right ideal A_i of R such that $e_iA_i \subseteq K$. Now $N := \oplus_{i\in I} e_iA_i$ is an essential submodule of F_R with $N \cap P = 0$, contradiction. Therefore, P_R is not singular.

(v) Let M_R be nonsingular. If $0 \neq m \in \mathrm{J}(M)$, then by (iv) mR contains a nonzero direct summand of M_R, but mR is a small submodule of M_R, a contradiction. Thus $\mathrm{J}(M) = 0$. □

The following result together with Examples 3.9 show that the class of retractable rings properly lies between two classes of known rings. We first recall the necessary definitions. Following [**7**], a ring R is called *right CPF* if for all proper ideals I of R, any faithful R/I-module is a generator in Mod-R/I. Artinian principal ideal rings are known to be CPF [**23**, 56.9(c)]. Also in [**6**], the ring R is called *right HP* (Hirano-Park) if for every non-zero R-module M, the converse of Schur's Lemma holds (i.e., if $\mathrm{End}_R(M)$ is a division ring, then M_R is a simple module). More recent works on HP rings are cited in the references. Rings over which any non-zero module has a maximal submodule are called *right max* rings; see [**20**] for an excellent reference on the subject.

Proposition 2.4. (i) *Right CPF rings are retractable.*

(ii) *Any retractable rings is a right divisible, right max and HP ring.*

Proof: Part (i) follows from the definitions. For part (ii), note that R is a right max ring by Proposition 2.3(ii). Now if $\mathrm{End}_R(M)$ is a division ring and $0 \neq N \leq M_R$, then the existence of a nonzero map $M_R \to N_R$ implies that $N = M$. Thus M_R is simple and R is an HP ring. Applying Proposition 2.3(i) for $M = R$ and $I = 0$, we have that R_R is divisible. □

Lemma 2.5. *Being (essentially) retractable is a Morita invariant property.*

Proof: Just note that in the definition of the (essentially) retractable modules, only categorical terms are used; see [**2**, Proposition 21.6]. □

Proposition 2.6. *The class of retractable rings is closed under homomorphic image, Morita equivalence and finite product.*

Proof: Let $\mathcal{C}$ be the class of all retractable rings. Clearly, $\mathcal{C}$ is closed under homomorphic image and Morita equivalence by Lemma 2.5. Now suppose R_1 and R_2 are retractable rings and set $T = R_1 \oplus R_2$. If M is a T-module then $M = Me_1 \oplus Me_2$ where e_1 and e_2 are central orthogonal idempotents in T such that $e_1R_2 = e_2R_1 = 0$ and $e_1 + e_2 = 1_T$. Clearly Me_i is naturally an R_i-module for $i = 1, 2$. Now let $0 \neq m \in M$. We have $m = m1_T = me_1 + me_2$. Hence there is $i \in \{1, 2\}$ such that $me_i \neq 0$. So by our assumption, there exists a nonzero R_i-homomorphism $f_i \colon Me_i \to mR_ie_i$. Since $e_1R_2 = e_2R_1 = 0$, Me_i is a T-submodule of M and f_i is a T-module homomorphism. Now $f_i\pi_i$ is a nonzero T-module homomorphism from M to mT where $\pi_i \colon M \to Me_i$ is the natural projection. Hence M_T is retractable, and T is a retractable ring. □

In the following we investigate the retractability of the class $\sigma[M_R]$ when M_R is a locally Noetherian module. Recall from [**23**, 15] that $\sigma[M_R]$ is a full subcategory of the category Mod-R whose objects are submodules of modules which are generated by M_R. Also a module M_R is said to be *polyform* if $\mathrm{Hom}_R(M/N, \hat{M}) = 0$ for any $N \leq_e M_R$. Here $\hat{M}$ is the M-injective envelope of M_R in $\sigma[M_R]$. Alternatively, M_R is polyform if and only if $\mathrm{End}_R(\hat{M})$ is a regular ring [**5**, 4.9]. The class of polyform modules properly contains both the class of nonsingular and the class of semisimple modules. It is known that any submodule and any quasi-injective hull of a polyform module is again polyform.

Proposition 2.7. *Suppose that M_R is polyform such that nonzero direct summands of $\hat{M}$ are retractable R-modules. If M_R is locally Noetherian or it has acc (dcc) on direct summands, then M_R is semisimple.*

Proof: The first we show that every indecomposable submodule of M_R is a simple M-injective R-module. Let U be an indecomposable submodule of M_R and $0 \neq K \leq U$. Then $\hat{U}$, the M-injective hull of U, is a direct summand of $\hat{M}$, and so by our assumption $\hat{U}$ is retractable. Now similar to the proof of Proposition 2.3(iv), U contains a nonzero M-injective submodule of K. Therefore, $K = U$ by the indecomposable condition on U, as desired. Now if M_R has acc (dcc) on direct summands, then we are done by [**2**, Proposition 10.14]. Let M_R be a local noetherian.

By the first part and [**5**, Corollary 5.2(2)], we can deduce $\mathrm{Soc}(M)$ is an essential submodule of M. On the other hand, $\mathrm{Soc}(M)$ is an M-injective submodule of M by [**5**, 2.5(c)]. It follows that $\mathrm{Soc}(M) = M$. □

Corollary 2.8. *Over a right Noetherian ring R, a nonzero module M_R is semisimple if and only if it is polyform and the class $\sigma[M_R]$ is retractable.*

Proof: By Proposition 2.7. □

By Proposition 2.4 and the next lemma, we observe that if R is retractable, then every Artinian module is Noetherian. The converse will be investigated in Proposition 2.10.

Lemma 2.9. *Let M be a nonzero R-module and R is a right max ring. If every factor module of M has finite uniform dimension, then M_R is Noetherian.*

Proof: Just note that if $N \leq M_R$ is not finitely generated then by [**5**, 5.11], there exists a finitely generated submodule $K \leq N$ such that N/K has no maximal submodule, a contradiction. □

Proposition 2.10. *Over a retractable ring, a module is Artinian if and only if it is Noetherian and its second singular submodule is Artinian.*

Proof: The necessity follows from Proposition 2.4(ii) and Lemma 2.9. Let M_R be noetherian and $\mathrm{Z}_2(M)$ be Artinian. Note that $L := M/\mathrm{Z}_2(M)$ is nonsingular and hence a polyform module. Now apply Proposition 2.7 for the module L to deduce that L_R is a semisimple noetherian module. Since now L and $\mathrm{Z}_2(M)$ are Artinian, M_R is Artinian. □

3. Characterization of some classes of rings

In this section, we give new characterizations for semisimple Artinian rings and certain semi-Artinian rings in terms of retractable rings. Two important subclasses of the class of Artinian rings are the class of Artinian principal ideal rings and the class of rings over which every right (and left) module is a direct sum of cyclic modules (Köthe rings). Let $\mathcal{K}$ (resp. $\mathcal{AP}$) be the classes of Köthe (resp. Artinian principal ideal) rings. In [**13**], it is proved that $\mathcal{AP} \subseteq \mathcal{K}$ and it is asked *what the Köthe rings are*; see also [**17**, Appendix B, Problem 2.48]. Recently, in [**4**, Theorem 3.1], it is proved that normal Köthe rings are Artinian principal ideal rings. A restatement of our Corollary 3.8, gives $\mathcal{AP} \subseteq \mathcal{R}$ where $\mathcal{R}$ is the class of retractable rings. Hence, if a Köthe ring is an Artinian principal ideal ring then it must be a retractable ring. We first characterize when a (semi-)Artinian ring is retractable and then determine when Köthe rings are Artinian principal ideal rings.

Recall from [**14**, 11.9] a ring R is said to be *right Goldie* if R has ascending chain condition on right annihilators and the uniform dimension of R_R is finite. Left Goldie rings are defined similarly. Semiprime right Goldie rings are known to be right nonsingular. In [**11**, Proposition 11], it is shown that right nonsingular HP rings with finite uniform dimension are semisimple Artinian. Hence, by Proposition 2.4, retractable semiprime right Goldie rings are precisely semisimple Artinian rings. In the following, we obtain a similar result for retractable semiprime left (right) Goldie rings.

Proposition 3.1. (i) *Retractable domains are precisely division rings.*

(ii) *The ring R is a semiprime left (right) Goldie retractable ring if and only if R is a semisimple Artinian ring if and only if R is a right nonsingular retractable ring with acc (dcc) on direct summand right ideals.*

Proof: (i) This follows from Proposition 2.3(i).

(ii) Suppose that R is semiprime left Goldie and let I be an essential left ideal of R. Then I contains a regular element x and so $Rx = R$, by Proposition 2.3(i). It follows that ${}_RR$ has no proper essential left ideals, proving that R is a semisimple Artinian ring. The second equivalence is obtained by Proposition 2.7. □

A ring R is said to be *reduced* if R has no nonzero nilpotent elements. A reduced ring which is a regular ring is called *strongly regular*; see [**23**, 3.11] for more information. A ring R is said to be *right (left) semi-Artinian* if every nonzero right (left) R-module has a nonzero socle, and R is called *semi-Artinian* if it is right and left semi-Artinian. In [**8**, Theorem 2.7], it is shown that for a commutative ring, the semi-Artinian condition implies the retractable condition. The converse follows by [**15**, Theorem 8]. We will give a generalization of this result in Theorem 3.4.

Theorem 3.2. *A ring is reduced and retractable if and only if it is a (right) semi-Artinian strongly regular ring.*

Proof: For the sufficiency, note that since R is strongly regular, R is reduced, right ideals in R are two sided and cyclic R-modules are flat. Hence, all simple R-modules are injective by [**14**, Corollary 3.6A]. It follows that the semi-Artinian ring R is retractable. Conversely, let R be a reduced retractable ring, $0 \neq a \in R$, $M = aR$, and $I = \text{r-ann}_R(a)$.

Since R is a reduced ring, I is an ideal of R and so we have $I \cap aR = 0$. It follows that a is right regular modulo I. Hence $Ma = M$ by Proposition 2.3, proving that R is a regular ring. Now R is strongly regular by the reduced condition on R.

To show that R is semi-Artinian, we will show that every cyclic R-module contains an injective R-module [**5**, 15.11]. Suppose now $B \leq R_R$. Since R is a strongly regular ring, B is an ideal of R and the ring R/B is (right) nonsingular. By hypothesis, R/B is also a retractable ring and so it contains an injective R/B-module by Proposition 2.3(iv). On the other hand, R/B is a flat left R-module and so by [**14**, Corollary 3.6(A)], every injective right R/B-module is injective as a right R-module, as desired. □

Following [**2**, p. 314], a non-empty subset Y of R is called *left T-nilpotent* provided for each sequence $y_1, y_2, y_3, \ldots$ of elements of Y there exists a positive integer n such that $y_1 y_2 \ldots y_n = 0$.

Proposition 3.3. *Let R be a ring with $\mathrm{J}(R) \subseteq \mathrm{Cent}(R)$. Then R is a retractable ring if and only if $R/\mathrm{J}(R)$ is a retractable ring and $\mathrm{J}(R)$ is a T-nilpotent ideal.*

Proof: ($\Rightarrow$) This follows from Proposition 2.4 and [**2**, Remark 28.5].

($\Leftarrow$) Let M_R be a nonzero R-module and $J = \mathrm{J}(R)$. By Theorem 2.2, we shall show that M_R is essentially retractable. If $MJ = 0$ then M is an R/J-module and we are done. If $MJ \neq 0$, then there exists $r \in J$ such that $MrJ = 0$ but $Mr \neq 0$. Since $J \subseteq \mathrm{Cent}(R)$, Mr is an R/J-module and so it is essentially retractable as an R/J-module as well as R-module. Now multiplication by r defines a nonzero homomorphism f in $\mathrm{End}_R(M)$ such that $f(M) = Mr$. Thus M_R is essentially retractable by Lemma 2.1. □

A ring R is called *normal* if all idempotent elements in R are central.

Theorem 3.4. *If R is a retractable ring such that $R/\mathrm{J}(R)$ is reduced then R is a left semi-Artinian ring and $R/\mathrm{J}(R)$ is a right semi-Artinian normal ring. The converse holds if $\mathrm{J}(R) \subseteq \mathrm{Cent}(R)$.*

Proof: Let R be a retractable ring and $R/\mathrm{J}(R)$ be reduced. By Theorem 3.2, $R/\mathrm{J}(R)$ is a (right) semi-Artinian strongly regular ring. By Proposition 2.4, R is a right max ring and so $\mathrm{J}(R)$ is a right T-nilpotent. It follows that R is left semi-Artinian [**20**, Lemma 2.12]. The last statement is true because reduced rings are normal.

Suppose now that $J = \mathrm{J}(R) \subseteq \mathrm{Cent}(R)$, R is left semi-Artinian and $R/\mathrm{J}(R)$ is a right semi-Artinian normal ring. Because R is left semi-Artinian, J is a T-nilpotent ideal [**2**, Remark 28.5(2)]. Thus by Proposition 3.3, it is enough to show that R/J is a retractable ring. Since R/J is semi-Artinian normal ring with zero Jacobson radical, it is a regular ring by [**1**, Corollary 1.4]. Hence R/J is a strongly regular ring by our assumption. The proof is now completed by Theorem 3.2. □

Corollary 3.5. *Suppose that R is a ring Morita equivalent to a ring S such that $S/\mathrm{J}(S)$ is a reduced ring and $\mathrm{J}(S) \subseteq \mathrm{Cent}(S)$. Then R is a retractable ring if and only if R is a semi-Artinian ring.*

Proof: By Theorem 3.4, the ring S is retractable if and only if it is semi-Artinian. Hence, the proof is completed by Proposition 2.6 and the fact that being semi-Artinian is a Morita invariant property. □

Following [**5**, Section 6], the *Krull dimension* of a module M_R is denoted by $\text{K-dim}(M_R)$ and $\text{K-dim}(R_R)$ is called the *right Krull dimension* of R. Clearly, $\text{K-dim}(M_R) \leq 0$ if and only if M_R is Artinian. Noetherian modules have Krull dimension and modules with Krull dimension are known to have finite uniform dimension [**5**, 6.2].

Theorem 3.6. *The following statements are equivalent for a ring R.*

(i) *R is a retractable ring and every cyclic R-module has finite uniform dimension.*
(ii) *R is a right Noetherian retractable ring.*
(iii) *R is a retractable ring with right Krull dimension.*
(iv) *R is a finite product of matrix rings over right Artinian local rings.*
(v) *R is a right Artinian for which the product of any two prime ideal commutes.*
(vi) *R is a right Artinian retractable ring.*

Proof: (i) ⇒ (ii). This follows from Proposition 2.4 and Lemma 2.9.

(ii) ⇒ (iii). This is obtained by [**5**, 6.2].

(iii) ⇒ (iv). By a well known result $\text{K-dim}(R_R) = \mathrm{Sup}\{\text{right K-dim}(R/P) \mid P$ is a prime ideal of $R\}$. Also by [**5**, Theorem 6.6], R/P is a prime right Goldie ring and so it is Artinian by Proposition 3.1. Thus R is a right Artinian ring. By [**2**, Proposition 10.17], we may suppose that R is indecomposable as a ring. Thus R_R has no non-trivial fully invariant direct summand. On the other hand, by [**2**, Proposition 28.13], $R_R \simeq e_1R^{(A_1)} \oplus \cdots \oplus e_nR^{(A_n)}$ for some $n \geq 1$ where each e_iRe_i is a local ring. Now the indecomposable condition on R with an application of

Proposition 2.3(iii) for $M = e_iR$ and $N = e_jR$ ($i \neq j$) imply that $R_R \simeq e_iR^{(A_i)}$, hence $R \simeq \mathrm{M}_{A_i}(e_iRe_i)$ for some i. The proof is complete.

(iv) $\Rightarrow$ (v). This follows by the fact that in any right Artinian local ring the Jacobson radical is the unique prime ideal.

(v) $\Rightarrow$ (vi). Suppose that every two prime ideal in R commute together, M is a nonzero R-module and S is a simple submodule of M with $P_1 := \mathrm{ann}_R(S)$. We shall show that $\mathrm{Hom}_R(M, S)$ is nonzero. Let $J = \mathrm{J}(R)$, since R is right Artinian, $J^k = 0$ for some $k \geq 1$. We claim that $MP_1 \neq M$. If $MP_1 = M$, then R is not a local ring (otherwise, $P_1 = J$ is a nilpotent ideal and so $MP_1 \neq M$). Therefore, suppose that $P_1, \ldots, P_n$ ($n \geq 2$) are all distinct prime (maximal) ideals in R. Then we have $S(P_2 \ldots P_n)^k \subseteq M(P_2 \ldots P_n)^k = MP_1^k(P_2 \ldots P_n)^k \subseteq MJ^k = 0$. It follows that $P_i \subseteq P_1$ for some $i \geq 2$, a contradiction. Therefore $MP_1 \neq M$, as claimed. Now M/MP_1 is a nonzero module over the semisimple ring R/P_1 and so there exists a nonzero homomorphism from M/MP_1 to the (unique) simple R/P_1-module S. This shows that $\mathrm{Hom}_R(M, S)$ is nonzero.

(vi) $\Rightarrow$ (i). This is well known. □

An R-module M is called *finitely annihilated* provided there exist a positive integer n and elements $m_i \in M$ ($1 \leq i \leq n$) such that $A := \mathrm{ann}_R(M) = \cap_i \mathrm{ann}_R(m_i)$, equivalently there exists an embedding $\theta\colon R/A \to M^{(n)}$. It is well known that a ring R is right Artinian if and only if every right R-module is finitely annihilated. Hence, if the ring R/A is right Artinian, then M_R is finitely annihilated.

Corollary 3.7. *Let R be a retractable ring. Then a nonzero R-module M is finitely annihilated with finite uniform dimensional factors if and only if M_R is finitely generated and the ring $R/\mathrm{ann}_R(M)$ is a finite product of matrix rings over right Artinian local rings.*

Proof: The sufficiency is clear. Conversely, by Proposition 2.4 and Lemma 2.9, M_R is Noetherian. Hence $R/\mathrm{ann}_R(M)$ is a right Noetherian ring by our assumption. The proof is now completed by Theorem 3.6. □

Corollary 3.8. *Right Artinian principal right ideal rings are retractable.*

Proof: By [**23**, 56.3] and Theorem 3.6. □

Examples 3.9. (i) If R is any ring and e is an idempotent element of R such that $eR \cap \text{l-ann}_R(e)$ contains a nonzero right ideal I then eR is not retractable as an R-module and consequently R is not a retractable ring. To see this let $0 \neq f \in \mathrm{Hom}_R(eR, I)$, then $f(eR) = f(e)eR = 0$, a contradiction.

(ii) Suppose that A and B are rings and ${}_AM_B$ is a nonzero bimodule. Let $R = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$, and $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then $\left[\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right]$ lies in $eR \cap \text{l-ann}_R(e)$. So by (i), R is not a retractable ring. Thus one may easily produce semi-Artinian rings which are not retractable.

(iii) For any ring R, the ring $R[x]$ is never retractable (Proposition 2.4(ii)).

(iv) There exists a retractable ring which is not CPF. Suppose that $S = \mathbb{Q}[x_i \mid i \in \mathbb{N}]$ and I is the ideal of S generated by the subset $\{x_i x_j, x_k^{k+1} \mid i \neq j,\, k \in \mathbb{N}\}$ and $R = S/I$. Then it is easy to verify that R is a local ring with $\mathrm{J}(R) = J = \langle \bar{x}_i \mid i \in \mathbb{N} \rangle$. In view of Proposition 3.3, to show that R is retractable, we shall show that J is T-nilpotent. Let $f_i \in J$ and $f_1 \in A := \langle \bar{x}_1, \ldots, \bar{x}_n \rangle$. Thus $f_1 f_2 \ldots f_{n+1} \in AJ^n = 0$. To proof that R is not a CPF ring, consider the faithful R-module $M = \bigoplus_i R/(\oplus_{j \neq i} \bar{x}_j R)$. If M_R is generator, then R must be embedded in $M_R^{(k)}$ for some $k \geq 1$, but every element in $M^{(k)}$ has nonzero annihilator, a contradiction. Hence M_R is not generator, and so R is not a CPF ring.

(v) There exists a divisible, HP, max ring which is not retractable. Let $R = \mathbb{Q}^{\mathbb{N}}$ be the countable product of $\mathbb{Q}$ and $I = \mathbb{Q}^{(\mathbb{N})}$. Then it is well known that R is a self-injective regular ring such that $\mathrm{Soc}(R/I)_R = 0$. Thus R is a divisible, max ring and it is an HP ring by [**11**, Corollary 15], but R is not a retractable ring by Corollary 3.5.

A characterization of Artinian principal ideal rings in [**23**, 56.9], shows that a ring R is Artinian principal ideal ring if and only if such is $\mathrm{Mat}_n(R)$. If S is a ring, we say that S is a *matrix ring direct summand (matrix rds) of* R whenever $\mathrm{Mat}_n(S) \times T \simeq R$ for some ring T and $n \geq 1$.

Theorem 3.10. *Let R be a Köthe ring. Then R is an Artinian principal ideal ring if and only if R is a retractable ring and every local ring which is a matrix rds of R is a Köthe ring.*

Proof: (⇒) By Corollary 3.8 R is a retractable ring. Suppose that S is a matrix rds of R. By hypothesis and [**23**, 56.9], S is an Artinian principal ideal ring. Hence S is a Köthe ring by [**13**].

(⇐) Since R is a Köthe ring, it is Artinian. Hence, by Theorem 3.6, $R \simeq \bigoplus_i \mathrm{Mat}_{n_i}(R_i)$ such that each R_i is local and a matrix rds of R. Thus by hypothesis, each R_i is a local Köthe ring. The proof is completed by [**4**, Theorem 3.1]. □

Corollary 3.11. *The following statements are equivalent.*

(i) $\mathcal{K} = \mathcal{AP}$.
(ii) $\mathcal{K} \subseteq \mathcal{R}$ *and for any local ring* R, *if* $\mathrm{Mat}_n(R) \in \mathcal{K}$ *for some* $n \geq 1$, *then* $R \in \mathcal{K}$.

Proof: This follows from Theorem 3.10. □

Now we shall consider when a Köthe ring is a retractable ring. First we state a lemma; note that the equivalence (i) ⇔ (ii) of below was obtained in [**8**, Proposition 2.2].

Lemma 3.12. *Let* I *be a proper right ideal in a ring* R. *Then the following statements are equivalent.*

(i) *The cyclic* R*-module* R/I *is retractable.*
(ii) *For any right ideal* J, *either* $J \subseteq I$ *or there exists* $x \in J \setminus I$ *such that* $xI \subseteq I$.
(iii) *For each* $x \in R \setminus I$, *there exists* $r \in R$ *such that* $xr \notin I$ *and* $xrI \subseteq I$.

Proof: (i) ⇒ (ii). If $J \subseteq I$, then we are through. Hence, let $a \in J \setminus I$, then by (i), there exists a nonzero homomorphism $f\colon R/I \to (aR+I)/I$ with $f(1+I) = ar + I$ for some $r \in R$. We now have $xI \subseteq I$, where $x = ar \in J \setminus I$.

(ii) ⇒ (iii). Suppose that $x \in R \setminus I$. Since the right ideal $J = xR + I$ properly contains I, by (ii) there exists $y \in J \setminus I$ such that $yI \subseteq I$. Write $y = xr + i$ for some $r \in R$ and $i \in I$. Then $xr \notin I$ as $y \notin I$, and $xrI \subseteq I$ because $yI \subseteq I$.

(iii) ⇒ (i). Suppose $I < J \leq R_R$. Pick $j \in J \setminus I$, then by (iii) there exists $r \in R$ with $jr \notin I$ and $jrI \subseteq I$. Let $x = jr$ then the left multiplication by x yields a nonzero homomorphism from R/I to J/I. □

A module M_R is called *co-cyclic* if it has a simple essential submodule.

Theorem 3.13. *Let* R *be a right Köthe ring. The following statements are equivalent.*

(i) *The ring* R *is retractable.*
(ii) $M \neq MP$ *for any nonzero cyclic co-cyclic* R*-module* M *with* $P \in \mathrm{Ass}(M)$.
(iii) *For any right ideal* $I \leq R$ *such that* $(R/I)_R$ *is co-cyclic and every* $x \in R \setminus I$, *there exists* $r \in R$ *such that* $xr \notin I$ *and* $xrI \subseteq I$.

Proof: Note that if Y_R is retractable, then for any module X_R, the R-module $Y \oplus X$ is always essentially retractable by Lemma 2.1. Therefore, since R is Köthe, by Theorem 2.2, we see that R is retractable if and only if nonzero cyclic R-modules are retractable. Let M_R be a cyclic R-module. Since R is right Artinian, M is retractable if and only if $\mathrm{Hom}_R(M, S) \neq 0$ for any simple submodule $S \leq M_R$. If S is not an essential submodule of M, it can be essentially embedded in some factor of M. Thus we can conclude that every nonzero cyclic R-modules is retractable if and only if every nonzero cyclic co-cyclic R-modules is retractable. Hence, (i) $\Leftrightarrow$ (iii) by Lemma 3.12. For (i) $\Leftrightarrow$ (ii) note that if M is a nonzero cyclic co-cyclic R-module and $P \in \mathrm{Ass}(M)$. Then $P = \mathrm{ann}_R(S)$ where S is the unique simple submodule of M. Since now R/P is a simple Artinian ring, we have $\mathrm{Hom}_R(M, S) \neq 0$ if and only if $\mathrm{Hom}_R(M, R/P) \neq 0$ if and only if $M \neq MP$. The proof is complete. □

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