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The new class of Kummer beta generalized distributions

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Abstract

Ng and Kotz (1995) introduced a distribution that provides greater flexibility to extremes. We define and study a new class of distributions called the Kummer beta generalized family to extend the normal, Weibull, gamma and Gumbel distributions, among several other well-known distributions. Some special models are discussed. The ordinary moments of any distribution in the new family can be expressed as linear functions of probability weighted moments of the baseline distribution. We examine the asymptotic distributions of the extreme values. We derive the density function of the order statistics, mean absolute deviations and entropies. We use maximum likelihood estimation to fit the distributions in the new class and illustrate its potentiality with an application to a real data set.

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1. Introduction

Beta distributions are very versatile and can be used to analyze different types of data sets. Many of the finite range distributions encountered in practice can be easily

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transformed into the standard beta distribution. In econometrics, quite often the data are analyzed by using finite-range distributions. Generalized beta distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. Eugene *et al.* (2002) proposed a general class of distributions based on the logit of a beta random variable by employing two parameters whose role is to introduce skewness and to vary tail weights.

Following Eugene *et al.* (2002), who defined the beta normal (BN) distribution, Nadarajah and Kotz (2004) introduced the beta Gumbel distribution (BGu), provided expressions for the moments, examined the asymptotic distribution of the extreme order statistics and performed maximum likelihood estimation (MLE). Nadarajah and Gupta (2004) defined the beta Fréchet (BF) distribution and derived analytical shapes of the probability density and hazard rate functions. Nadarajah and Kotz (2005) proposed the beta exponential (BE) distribution, derived the moment generating function (mgf), the first four moments, and the asymptotic distribution of the extreme order statistics and discussed MLE. Most recently, Pescim *et al.* (2010) and Paranaíba *et al.* (2011) have studied important mathematical properties of the beta generalized half-normal (BGHN) and beta Burr XII (BBXII) distributions. However, those distributions do not offer flexibility to the extremes (right and left) of the probability density functions (pdfs). Therefore, they are not suitable for analyzing data sets with high degrees of asymmetry.

Ng and Kotz (1995) proposed the Kummer beta distribution on the unit interval $(0,1)$ with cumulative distribution function (cdf) and pdf given by

$$F(x) = K \int_0^x t^{a-1}\,(1-t)^{b-1}\,\exp(-ct)dt,$$

and

$$f(x) = K x^{a-1}\,(1-x)^{b-1}\,\exp(-cx), \quad 0 < x < 1,$$

respectively, where $a > 0$, $b > 0$ and $-\infty < c < \infty$. Here,

$$K^{-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_1F_1(a;a+b;-c) \tag{1}$$

and

$${}_1F_1(a;a+b;-c) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}\,(1-t)^{b-1}\exp(-ct)dt = \sum_{k=0}^{\infty} \frac{(a)_k(-c)^k}{(a+b)_k\,k!}$$

is the confluent hypergeometric function (Abramowitz and Stegun, 1968), $\Gamma(\cdot)$ is the gamma function and $(d)_k = d(d+1)\ldots(d+k-1)$ denotes the ascending factorial. Independently, Gordy (1998) has also defined the Kummer beta distribution in relation

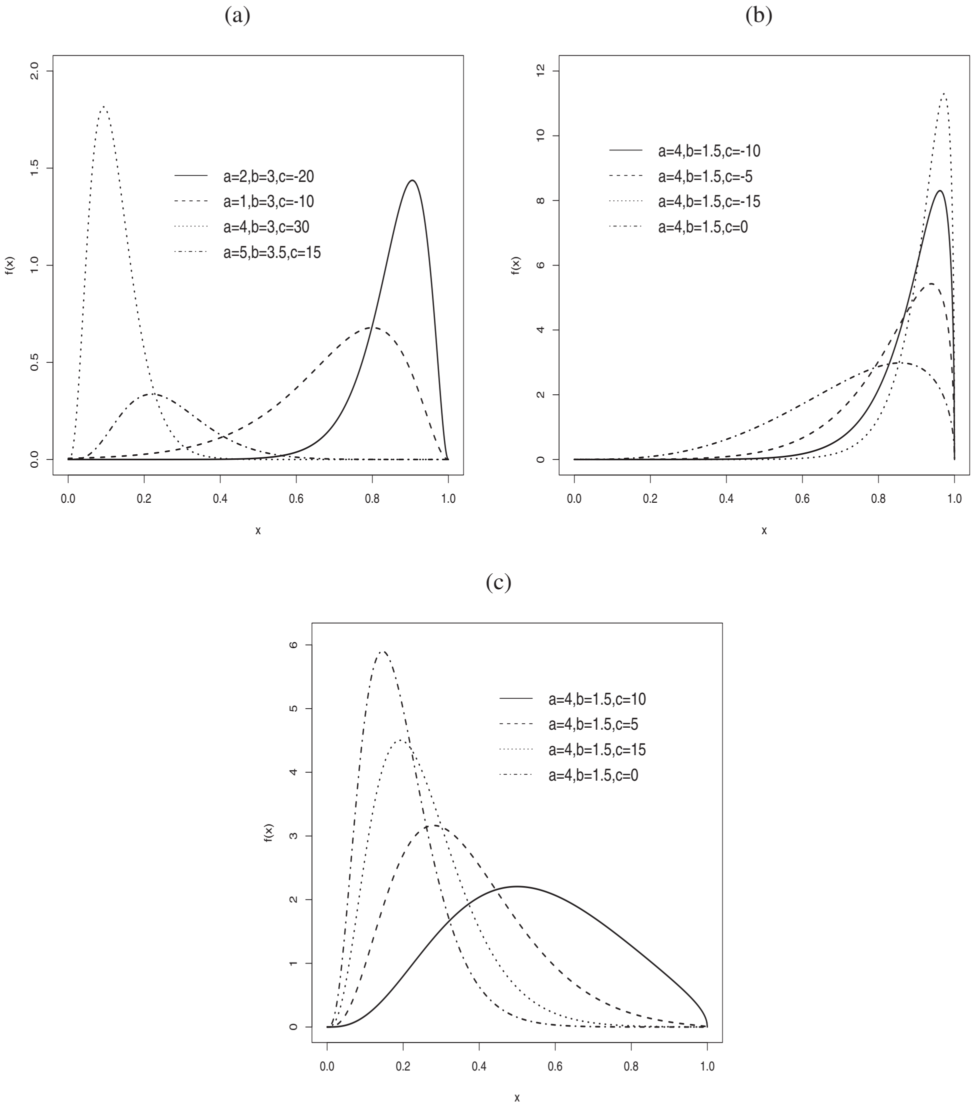

Figure 1: *Plots of the Kummer beta pdf for some parameter values.*

to the problem of common value auction. This distribution is an extension of the beta distribution. It yields bimodal distributions on finite range for $a < 1$ (and certain values of the parameter c). Plots of the Kummer beta pdf are displayed in Figure 1 for selected parameter values.

Consider starting from a parent continuous cdf $G(x)$. A natural way of generating families of distributions from a simple parent distribution with pdf $g(x) = dG(x)/dx$ is to apply the quantile function to a family of distributions on the interval $(0,1)$. We now use the same methodology of Eugene *et al.* (2002) and Cordeiro and de Castro (2011) to construct a new class of Kummer beta generalized (KBG) distributions. From an arbitrary parent cdf $G(x)$, the KBG family of distributions is defined by

$$F(x) = K \int_0^{G(x)} t^{a-1}\,(1-t)^{b-1}\exp(-c\,t)dt, \tag{2}$$

where $a > 0$ and $b > 0$ are shape parameters introducing skewness, and thereby promoting weight variation of the tails. The parameter $-\infty < c < \infty$ "squeezes" the pdf to the left or to the right.

The pdf corresponding to (2) can be expressed as

$$f(x) = K\,g(x)\,G(x)^{a-1}\,\{1-G(x)\}^{b-1}\exp\{-c\;G(x)\}\,, \tag{3}$$

where K is defined in (1).

The KBG family of distributions defined by (3) is an alternative family of models to the class of distributions proposed by Alexander *et al.* (2012). The shape parameter $c > 0$ in Alexander *et al.* (2012) together with $a > 0$ and $b > 0$ promotes the weight variation of the tails and adds flexibility. On the other hand, the parameter $-\infty < c < \infty$ of the proposed family offers flexibility to the extremes (left and/or right) of the pdfs. Therefore, the new family of distributions is suitable for analyzing data sets with high degrees of asymmetry.

For each continuous G distribution (here and henceforth "G" denotes the baseline distribution), we associate the KBG-G distribution with three extra parameters a, b and c defined by the pdf (3). Setting $u = t/G(x)$ in equation (2), we obtain

$$\begin{aligned} F(x) &= K\,G(x)^a \int_0^1 u^{a-1}\,[1-G(x)\,u]^{b-1}\exp[-c\,G(x)\,u]\;du \\ &= \frac{K}{a}\,G(x)^a\,\Phi_1\,(a;1-b;a+1;-c\,G(x);G(x))\,, \end{aligned}$$

where Φ_1 is the confluent hypergeometric function of two variables defined by (Erdélyi *et al.*, 1953)

$$\Phi_1(a;b;c;x;y) = \sum_{j,m=0}^{\infty} \frac{(a)_{j+m}\,(b)_j}{(c)_{j+m}}\,x^j\,y^m$$

for $|x| < 1$ and $|y| < 1$.

Special generalized distributions can be generated as follow. The KBG-normal (KBGN) distribution is obtained by taking $G(x)$ in equation (3) to be the normal cdf. Analogously, the KBG-Weibull (KBGW), KBG-gamma (KBGGa) and KBG-Gumbel (KBGGu) distributions are obtained by taking $G(x)$ to be the cdf of the Weibull, gamma and Gumbel distributions, respectively. Hence, each new KBG-G distribution can be obtained from a specified G distribution. The Kummer beta distribution is clearly a basic example of the KBG distribution when G is the uniform distribution on $[0,1]$. The G distribution corresponds to $a = b = 1$ and $c = 0$. For $c = 0$, the KBG-G distribution reduces to the beta-G distribution proposed by Eugene *et al.* (2002). Further, for $b = 1$ and $c = 0$, the KBG-G distribution becomes the exponentiated-G distribution. One major benefit of the KBG family of distributions is its ability to fit skewed data that cannot be properly fitted by existing distributions.

We study some mathematical properties of the KBG family of distributions because it extends several widely-known distributions in the literature. The article is outlined as follows. Section 2 provides some special cases. In Section 3, we derive general expansions for the new pdf in terms of the baseline pdf $g(x)$ multiplied by a power series in $G(x)$. We can easily apply these expansions to several KBG distributions. In Section 4, we derive two simple expansions for moments of the KBG-G distribution as linear functions of probability weighted moments (PWMs) of the G distribution. The mean absolute deviations and Rényi entropy are determined in Sections 5 and 6, respectively. In Section 7, we provide some expansions for the pdf of the order statistics. Extreme values are obtained in Section 8. Some inferential tools are discussed in Section 9. In Section 10, we analyze a real data set using a special KBG distribution. Section 11 ends with some concluding remarks.

2. Special KBG generalized distributions

The KBG pdf (3) allows for greater flexibility of its tails and promotes variation of the tail weights to the extremes of the distribution. It can be widely applied in many areas of engineering and biological sciences. The pdf (3) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions. We now define some of the many distributions which arise as special sub-models within the KBG class of distributions.

2.1. KBG-normal

The KBGN pdf is obtained from (3) by taking $G(\cdot)$ and $g(\cdot)$ to be the cdf and pdf of the normal distribution, $\mathrm{N}(\mu, \sigma^2)$, so that

$$f(x) = \frac{K}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left\{\Phi\left(\frac{x-\mu}{\sigma}\right)\right\}^{a-1}\left\{1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right\}^{b-1}\exp\left\{-c\,\Phi\left(\frac{x-\mu}{\sigma}\right)\right\},$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, a and b are positive shape parameters, $c \in \mathbb{R}$, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. A random variable with the above pdf is denoted by $\mathrm{X} \sim \mathrm{KBGN}(a, b, c, \mu, \sigma^2)$. For $\mu = 0$ and $\sigma = 1$, we have the standard KBGN distribution. Further, the KBGN distribution with $a = 2$, $b = 1$ and $c = 0$ is the skew normal distribution with shape parameter equal to one (Azzalini, 1985).

2.2. KBG-Weibull

The cdf of the Weibull distribution with parameters $\beta > 0$ and $\alpha > 0$ is $G(x) = 1 - \exp\{-(\beta x)^\alpha\}$ for $x > 0$. Correspondingly, the KBG-Weibull (KGBW) pdf is

$$f(x) = K\,\alpha\,\beta^\alpha\,x^{\alpha-1}\,[1-\exp\{-(\beta x)^\alpha\}]^{a-1}\exp\{-c\,[1-\exp\{-(\beta x)^\alpha\}]-b(\beta x)^\alpha\},$$

where $x, a, b, \beta > 0$ and $c \in \mathbb{R}$. Let $\mathrm{KBGW}(a,b,c,\alpha,\beta)$ denote a random variable with this pdf. For $\alpha = 1$, we obtain the KBG-exponential (KBGE) distribution. $\mathrm{KBGW}(1,b,0,1,\beta)$ is an exponential random variable with parameter $\beta^* = b\,\beta$.

2.3. KBG-gamma

Let Y be a gamma random variable with cdf $G(y) = \Gamma_{\beta y}(\alpha)/\Gamma(\alpha)$ for y, α, $\beta > 0$, where $\Gamma(\cdot)$ is the gamma function and $\Gamma_z(\alpha) = \int_0^z t^{\alpha-1}\mathrm{e}^{-t}\mathrm{d}t$ is the incomplete gamma function. The pdf of a random variable X, say $\mathrm{X} \sim \mathrm{KBGGa}(a,b,c,\beta,\alpha)$, having the KBGGa distribution can be expressed as

$$f(x) = \frac{K\,\beta^\alpha\,x^{\alpha-1}\exp(-\beta x)}{\Gamma(\alpha)^{a+b-1}}\exp\left\{-c\,\frac{\Gamma_{\beta x}(\alpha)}{\Gamma(\alpha)}\right\}\Gamma_{\beta x}(\alpha)^{a-1}\left\{\Gamma(\alpha)-\Gamma_{\beta x}(\alpha)\right\}^{b-1}.$$

For $\alpha = 1$ and $c = 0$, we obtain the KBGE distribution. $\mathrm{KBGGa}(1,b,0,\beta,1)$ is an exponential random variable with parameter $\beta^* = b\,\beta$.

2.4. KBG-Gumbel

The pdf and cdf of the Gumbel distribution with location parameter $\mu > 0$ and scale parameter $\sigma > 0$ are given by

$$g(x) = \sigma^{-1}\exp\left\{\frac{x-\mu}{\sigma}-\exp\left(\frac{x-\mu}{\sigma}\right)\right\},\, x > 0,$$

and

$$G(x) = 1-\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\},$$

respectively. The mean and variance are equal to $\mu - \gamma\sigma$ and $\pi^2\sigma^2/6$, respectively, where $\gamma \approx 0.57722$ is the Euler's constant. By inserting these equations in (3), we obtain a KBGGu random variable, say $\mathrm{KBGGu}(a,b,c,\mu,\sigma)$.

Figure 2 displays some possible shapes of the four KBG pdfs. These plots show the great flexibility achieved with the new distributions.

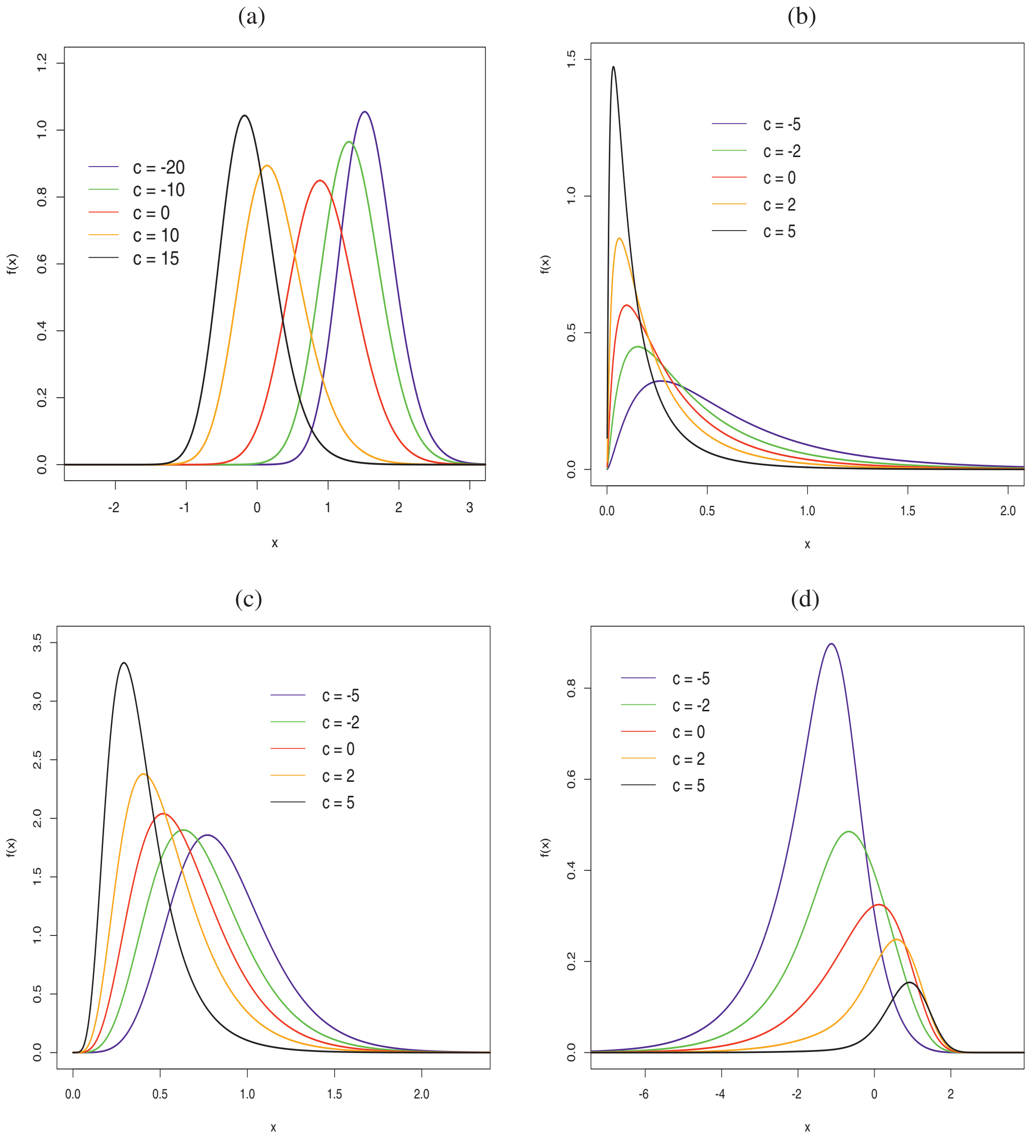

Figure 2: *(a) KBGN(8,2,c,0,1), (b) KBGW(5,3,c,0.5,4), (c) KBGGa(3,1.5,c,4,2) and (d) KBGGu(0.8,1,c,0,1) pdfs (the red lines represent the beta-G pdfs).*

3. Expansions for pdf and cdf

The cdf $F(x)$ and pdf $f(x) = dF(x)/dx$ of the KBG-G distribution are usually straightforward to compute given $G(x)$ and $g(x) = dG(x)/dx$. However, we provide expansions for these functions as infinite (or finite) weighted sums of cdf's and pdf's of exponentiated-G distributions. In the next sections, based on these expansions, we obtain some structural properties of the KBG-G distribution, including explicit expressions for moments, mean absolute deviations, pdf of order statistics and moments of order statistics.

Using the exponential expansion in (2), we write

$$F(x) = \sum_{i=0}^{\infty} w_i H_{a+i,b}(x), \tag{4}$$

where $w_i = [K B(a+i,b)(-c)^i]/i!$ and

$$H_{a,b}(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1} (1-t)^{b-1} dt$$

denotes the beta-G cdf with positive shape parameters a and b (Eugene *et al.*, 2002). Equation (4) reveals that the KBG-G cdf is a linear combination of beta-G cdf's. This result is important. It can be used to derive properties of any KBG-G distribution from those of beta-G distributions.

For $b > 0$ real non-integer, we have the power series representation

$$\{1 - G(x)\}^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} G(x)^j, \tag{5}$$

where the binomial coefficient is defined for any real. Expanding $\exp\{-c\,G(x)\}$ in power series and using (5) in equation (2), the KBG-G cdf can be expressed as

$$F(x) = \sum_{i,j=0}^{\infty} w_{i,j}\, G(x)^{a+i+j}, \tag{6}$$

where

$$w_{i,j} = \frac{K\,(-1)^{i+j}\, c^i}{i!\,(a+i+j)} \binom{b-1}{j}.$$

If b is an integer, the index i in the previous sum stops at $b-1$. If a is an integer, equation (6) reveals that the KBG-G pdf can be expressed as the baseline pdf multiplied by an infinite power series of its cdf.

If a is a real non-integer, we can expand $G(x)^{a+i+j}$ as follows

$$G(x)^{a+i+j} = \sum_{k=0}^{\infty} (-1)^k \binom{a+i+j}{k} [1 - G(x)]^k.$$

Then,

$$G(x)^{a+i+j} = \sum_{k=0}^{\infty}\sum_{r=0}^{k}(-1)^{k+r}\binom{a+i+j}{k}\binom{k}{r}G(x)^{r}.$$

Further, equation (2) can be rewritten as

$$F(x) = \sum_{i,j,k=0}^{\infty}\sum_{r=0}^{k} t_{i,j,k,r}\, G(x)^{r}, \tag{7}$$

where

$$t_{i,j,k,r} = t_{i,j,k,r}(a,b,c) = (-1)^{k+r}\binom{a+i+j}{k}\binom{k}{r}w_{i,j}$$

and $w_{i,j}$ is defined in (6). Replacing $\sum_{k=0}^{\infty}\sum_{r=0}^{k}$ by $\sum_{r=0}^{\infty}\sum_{k=r}^{\infty}$ in equation (7), we obtain

$$F(x) = \sum_{r=0}^{\infty} b_r\, G(x)^{r}, \tag{8}$$

where the coefficient $b_r = \sum_{i,j=0}^{\infty}\sum_{k=r}^{\infty} t_{i,j,k,r}$ denotes a sum of constants.

Expansion (8), which holds for any real non-integer a, expresses the KBG-G cdf as an infinite weighted power series of G. If b is an integer, the index i in (7) stops at $b-1$.

We also note that the cdf of the KBG family can be expressed in terms of exponentiated-G cdfs. We have

$$F(x) = \sum_{r=0}^{\infty} b_r\, V_r(x), \tag{9}$$

where $V_r = G(x)^r$ is an exponentiated-G cdf (Exp-G cdf for short) with power parameter r.

The corresponding expansions for the KBG pdf are obtained by simple differentiation of (6) for $a > 0$ integer

$$f(x) = g(x)\sum_{i,j=0}^{\infty} w_{i,j}^{*}\, G(x)^{a+i+j-1}, \tag{10}$$

where $w_{i,j}^{*} = (a+i+j)\, w_{i,j}$. Analogously, from equations (8) and (9), for $a > 0$ real non-integer, we obtain

$$f(x) = g(x)\sum_{r=0}^{\infty} b_r^{*}\, G(x)^{r}, \tag{11}$$

and

$$f(x)=\sum_{r=0}^{\infty} c_r\, v_{r+1}(x), \tag{12}$$

where $b_r^*=(r+1)\,b_{r+1}$ and $c_r=b_{r+1}$ for $r=0,1\ldots$, and $v_{r+1}=(r+1)\,g(x)\,G(x)^r$ denotes the Exp-G pdf with parameter $r+1$. Equation (12) reveals that the KBG-G pdf is a linear combination of Exp-G pdfs. This result is important to derive properties of the KBG-G distribution from those of the Exp-G distribution.

Mathematical properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar *et al.* (1995) for exponentiated Weibull, Gupta *et al.* (1998) for exponentiated Pareto, Gupta and Kundu (2001) for exponentiated exponential and Nadarajah and Gupta (2007) for exponentiated gamma.

Equations (10)-(12) are the main results of this section. They play an important role in this paper.

4. Moments and generating function

4.1. Moments

The sth moment of the KBG-G distribution can be expressed as an infinite weighted sum of PWMs of order (s,q) of the parent G distribution from equation (10) for a integer and from (11) for a real non-integer. We assume that Y and X follow the baseline G and KBG-G distributions, respectively. The sth moment of X can be expressed in terms of the (s,q)th PWMs of Y, say $\tau_{s,q}=\mathrm{E}[Y^s\,G(Y)^q]$ (for $q=0,1,\ldots$), as defined by Greenwood *et al.* (1979). The moments $\tau(s,q)$ can be derived for most parent distributions.

For an integer a, we have

$$\mu_s'=\mathrm{E}(X^s)=\sum_{i,j=0}^{\infty} w_{i,j}^*\,\tau_{s,a+i+j-1}.$$

For a real non-integer a, we can write from (11)

$$\mu_s'=\sum_{r=0}^{\infty} b_r^*\,\tau_{s,r}.$$

So, we can calculate the moments of any KBG-G distribution as infinite weighted sums of PWMs of the G distribution.

Alternatively, we can express μ_s' from (11) in terms of the baseline quantile function $Q(u)=G^{-1}(u)$. We have

$$\mu_s' = \sum_{r=0}^{\infty} b_r^* \int x^s \, g(x) \, G(x)^r \, dx.$$

Setting $u = G(x)$ in the last equation, we obtain

$$\mu_s' = \sum_{r=0}^{\infty} b_r^* \int_0^1 u^r \, Q(u)^s \, dt.$$

Now, we express moments of KBG distributions from equation (12) in terms of moments of Exp-G distributions. Let Y_{r+1} have the Exp-G pdf $v_{r+1} = (r+1)\, g(x)\, G(x)^r$ with power parameter $(r+1)$. As a first example, consider G the Weibull distribution with scale parameter $\lambda > 0$ and shape parameter $\gamma > 0$. If Y_{r+1} has the exponentiated Weibull distribution, its moments are

$$\mathrm{E}(Y^s) = \frac{(r+2)}{\lambda^s} \Gamma\left(\frac{s}{\gamma} + 1\right) \sum_{i=0}^{\infty} \frac{(-r)_i}{i!\,(i+1)^{(s+\gamma)/\gamma}},$$

where $(a)_i = a(a+1)\ldots(a+i-1)$ denotes the ascending factorial. From this expectation and equation (12), the sth moment of the KBG-Weibull distribution is

$$\mu_s' = \lambda^{-s} \Gamma\left(\frac{s}{\gamma} + 1\right) \sum_{r,i=0}^{\infty} \frac{(r+2)\, c_r\, (-r)_i}{i!\,(i+1)^{(s+\gamma)/\gamma}}.$$

For a second example, take the Gumbel distribution with cdf $G(x) = 1 - \exp\{-\exp(-\frac{x-\mu}{\sigma})\}$. The moments of Y_{r+1} having the exponentiated Gumbel distribution with parameter $(r+1)$ can be obtained from Nadarajah and Kotz (2006) as

$$\mathrm{E}(Y_{r+1}^s) = (r+1) \sum_{i=0}^{s} \binom{s}{i} \mu^{s-i} (-\sigma)^i \left(\frac{\partial}{\partial p}\right)^i \left[(r+1)^{-p}\, \Gamma(p)\right] \Big|_{p=1}.$$

From the last equation and (12), the sth moment of the KBG-Gumbel (KBGGu) distribution becomes

$$\mu_s' = \sum_{r=0}^{\infty} c_r\, (r+1) \sum_{i=0}^{s} \binom{s}{i} \mu^{s-i} (-\sigma)^i \left(\frac{\partial}{\partial p}\right)^i \left[(r+1)^{-p}\, \Gamma(p)\right] \Big|_{p=1}.$$

Finally, as a third example, consider the standard logistic cdf $G(x) = [1+\exp(-x)]^{-1}$. We can easily obtain the sth moment of the KBG-logistic (KBGL) distribution as

$$\mu_s' = \sum_{r=0}^{\infty} c_r \left(\frac{\partial}{\partial t}\right)^s B\,(t+(r+1), 1-t) \Big|_{t=0}.$$

4.2. Generating function

Let $X \sim$ KBG-$G(a,b,c)$. We provide four representations for the mgf $M(t) = \mathrm{E}[\exp(tX)]$ of X. Clearly, the first one is

$$M(t) = \sum_{s=0}^{\infty} \frac{\mu_s'}{s!} t^s,$$

where $\mu_s' = \mathrm{E}(X^s)$. The second one comes from

$$\begin{aligned} M(t) &= K\,\mathrm{E}\left[\exp\left[tX - c\,G(X)\right] G^{a-1}(X)\,\{1 - G(X)\}^{b-1}\right] \\ &= K \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \mathrm{E}\left[\frac{\exp(tX - Uc)}{U^{-(a+j-1)}}\right], \end{aligned}$$

where U is a uniform random variable on the unit interval. Note that X and U are not independent.

A third representation for $M(t)$ is obtained from (12)

$$M(t) = \sum_{i=0}^{\infty} c_i\, M_{i+1}(t),$$

where $M_{i+1}(t)$ is the mgf of $Y_{i+1} \sim$ Exp-$G(i+1)$. Hence, for any KBG-G distribution, $M(t)$ can be immediately determined from the mgf of the G distribution.

A fourth representation for $M(t)$ can be derived from (11) as

$$M(t) = \sum_{i=0}^{\infty} b_i^*\, \rho(t,i), \tag{13}$$

where $\rho(t,r) = \int_{-\infty}^{\infty} \exp(tx)\, g(x)\, G(x)^r dx$ can be expressed in terms of the baseline quantile function $Q(u)$ as

$$\rho(t,a) = \int_0^1 u^a \exp\left[t\, Q(u)\right] du. \tag{14}$$

We can obtain the mgf of several KBG distributions from equations(13) and (14). For example, the mgfs of the KBG-exponential (KBGE) (with parameter λ), KBGL and KBG-Pareto (KBGPa) (with parameter $\nu > 0$) are easily calculated as

$$M(t) = \sum_{i=0}^{\infty} b_i^*\, B\left(i+1, 1-\lambda t^{-1}\right), \quad M(t) = \sum_{i=0}^{\infty} b_i^*\, B\left(i+1, 1-t\right),$$

and

$$M(t) = \exp(-t) \sum_{i,p=0}^{\infty} \frac{b_i^* t^p}{p!} B\left(i+1, 1-p\nu^{-1}\right),$$

respectively.

Clearly, four representations for the characteristic function (chf) $\phi(t) = E[\exp(\mathrm{i}tX)]$ of the KBG-G distribution are immediately obtained from the above representations for the mgf by $\phi(t) = M(\mathrm{i}t)$, where $\mathrm{i} = \sqrt{-1}$.

5. Mean absolute deviations

Let $X \sim$ KBG-$G(a,b,c)$. The mean absolute deviations about the mean ($\delta_1(X)$) and about the median ($\delta_2(X)$) can be expressed as

$$\begin{aligned}
\delta_1(X) &= \mathrm{E}\left(\left|X-\mu_1'\right|\right) = 2\mu_1' F\left(\mu_1'\right) - 2T\left(\mu_1'\right), \\
\delta_2(X) &= \mathrm{E}\left(|X-M|\right) = \mu_1' - 2T(M),
\end{aligned} \tag{15}$$

respectively, where $\mu_1' = \mathrm{E}(X)$, $F(\mu_1')$ comes from (2), $M = \mathrm{Median}(X)$ denotes the median determined from the nonlinear equation $F(M) = 1/2$, and $T(z) = \int_{-\infty}^{z} x f(x) dx$. Setting $u = G(x)$ in (11) yields

$$T(z) = \sum_{r=0}^{\infty} b_r^* T_r(z), \tag{16}$$

where the integral $T_r(z)$ can be expressed in terms of $Q(u) = G^{-1}(u)$ by

$$T_r(z) = \int_0^{G(z)} u^r Q(u)\, du. \tag{17}$$

The mean absolute deviations of any KBG distribution can be computed from equations (15)-(17). For example, the mean absolute deviations of the KBGE (with parameter λ), KBGL and KBGPa (with parameter $\nu > 0$) are immediately calculated using

$$T_r(z) = \lambda^{-1}\Gamma(r+2) \sum_{j=0}^{\infty} \frac{(-1)^j \left[1-\exp(-j\lambda z)\right]}{\Gamma(r+2-j)(j+1)!},$$

$$T_r(z) = \frac{1}{\Gamma(z)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(r+j+1) \left[1-\exp(-jz)\right]}{(j+1)!}$$

and

$$T_r(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^j \binom{r+1}{j} \binom{j}{k}}{(1-kv)} z^{1-kv},$$

respectively.

An alternative representation for $T(z)$ can be derived from (12) as

$$T(z) = \int_{-\infty}^{z} x f(x) dx = \sum_{r=0}^{\infty} c_r J_{r+1}(z), \tag{18}$$

where

$$J_{r+1}(z) = \int_{-\infty}^{z} x\, v_{r+1}(x) dx. \tag{19}$$

Equation (19) is the basic quantity to compute mean absolute deviations of Exp-G distributions. Hence, the KBG mean absolute deviations depend only on the quantity $J_{r+1}(z)$. So, alternative representations for $\delta_1(X)$ and $\delta_2(X)$ are

$$\delta_1(X) = 2\mu_1' F(\mu_1') - 2 \sum_{r=0}^{\infty} c_r J_{r+1}(\mu_1') \quad \text{and} \quad \delta_2(X) = \mu_1' - 2 \sum_{r=0}^{\infty} c_r J_{r+1}(M).$$

A simple application is provided for the KBGW distribution. The exponentiated Weibull pdf with parameter $r+1$ is given by

$$v_{r+1}(x) = (r+1)\, d\, \beta^d x^{d-1} \exp\left\{-(\beta x)^d\right\} \left[1 - \exp\left\{-(\beta x)^d\right\}\right]^r$$

for $x > 0$. Then,

$$\begin{aligned} J_{r+1}(z) &= (r+1)\, d\, \beta^d \int_0^z x^d \exp\left\{-(\beta x)^d\right\} \left[1 - \exp\left\{-(\beta x)^d\right\}\right]^r dx \\ &= r\, d\, \beta^d \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \int_0^z x^d \exp\left[-(k+1)(\beta x)^d\right] dx. \end{aligned}$$

We calculate the last integral using the incomplete gamma function $\gamma(\alpha, x) = \int_0^x w^{\alpha-1} \mathrm{e}^{-\mathrm{w}} \mathrm{dw}$ for $\alpha > 0$. Then,

$$J_{r+1}(z) = (r+1)\, \beta^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k \binom{r}{k}}{(k+1)^{1+d^{-1}}} \gamma\left(1 + d^{-1}, (k+1)(\beta z)^d\right).$$

Equations (16) and (18) are the main results of this section. These equations can be applied to Bonferroni and Lorenz curves defined for a given probability p by

$$B(p) = \frac{T(q)}{p\mu_1'} \quad \text{and} \quad L(p) = \frac{T(q)}{\mu_1'},$$

where $\mu_1' = \mathrm{E}(X)$ and $q = F^{-1}(p)$.

6. Entropies

An entropy is a measure of variation or uncertainty of a random variable X. The most popular measures of entropy are the Shannon entropy (Shannon, 1951) and the Rényi entropy.

6.1. Shannon entropy

The Shannon entropy (Shannon, 1951) is defined by $\mathrm{E}\{-\log[f(X)]\}$. Let X has the pdf (3). We can write

$$\begin{aligned}
\mathrm{E}\{-\log[f(X)]\} &= -\log(K) - \mathrm{E}\{\log[g(X)]\} + (1-a)\mathrm{E}\{\log[G(X)]\} \\
&\quad +(1-b)\mathrm{E}\{\log[1-G(X)]\} + c\,\mathrm{E}[G(X)] \\
&= -\log K - \mathrm{E}\{\log[g(X)]\} + (a-1)\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{E}\left\{[1-G(X)]^k\right\} \\
&\quad +(b-1)\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{E}\left[G^k(X)\right] + c\,\mathrm{E}[G(X)] \\
&= -\log(K) - \mathrm{E}\{\log[g(X)]\} + (a-1)\sum_{k=1}^{\infty}\frac{K(a,b+k,c)}{kK(a,b,c)} \\
&\quad +(b-1)\sum_{k=1}^{\infty}\frac{K(a+k,b,c)}{kK(a,b,c)} + \frac{cK(a+1,b,c)}{K(a,b,c)}, \qquad (20)
\end{aligned}$$

where $K = K(a,b,c)$ is given by (1). The only unevaluated term in (20) is $\mathrm{E}\{\log[g(X)]\}$.

6.2. Rényi entropy

The Rényi entropy is given by

$$\mathfrak{I}_R(\xi) = \frac{1}{1-\xi}\log\left[\int_{-\infty}^{\infty} f^{\xi}(x)dx\right], \ \xi > 0 \text{ and } \xi \neq 1.$$

The integral can be expressed as

$$\int_{-\infty}^{\infty} f^{\xi}(x)dx = K^{\xi}\int_{-\infty}^{\infty} g^{\xi}(x)\, G^{\xi(a-1)}(x)\; [1-G(x)]^{\xi(b-1)}\, \exp[-\xi\, c\, G(x)]\, dx.$$

Expanding the exponential and the binomial terms and changing variables, we obtain

$$\int_{-\infty}^{\infty} f^{\xi}(x)dx = K^{\xi}\sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(c\,\xi)^{i}}{i!}\binom{\xi(b-1)}{j} I_{i,j}(\xi), \tag{21}$$

where $I_{i,j}(\xi)$ denotes the integral

$$I_{i,j}(\xi) = \int_{0}^{1} g^{\xi-1}\,(Q(u))\;\, u^{i+j+\xi(a-1)}\, du,$$

to be calculated for each KBG-model. For the KBGE (with parameter λ), KBGL and KBGPa (with parameter ν), we obtain

$$I_{i,j}(\xi) = \lambda^{\xi-1}B\,(i+j+\xi(a-1)+1,\xi)\,,\;\; I_{i,j}(\xi) = B\,(i+j+\xi\, a\,,\,\xi)\,,$$

and

$$I_{i,j}(\xi) = \nu^{\xi-1}\,B\left(i+j+\xi(a-1)+1,\nu^{-1}(\xi-1)+\xi\right),$$

respectively. Equation (21) is the main result of this section.

7. Order statistics

Order statistics have been used in a wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials.

Suppose $X_1,\ldots,X_n$ is a random sample from a continuous distribution and let $X_{1:n} < \cdots < X_{i:n}$ denote the corresponding order statistics. There has been a large amount of work relating to moments of order statistics $X_{i:n}$. See Arnold *et al.* (1992), David and Nagaraja (2003) and Ahsanullah and Nevzorov (2005) for excellent accounts. It is well-known that

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)}F(x)^{i-1}\,\{1-F(x)\}^{n-i},$$

where $B(\cdot,\cdot)$ denotes the beta function. Using the binomial expansion in the last equation, we have

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \tag{22}$$

We now provide an expression for the pdf of KBG order statistics as a function of the baseline pdf multiplied by infinite weighted sums of powers of $G(x)$. Based on this result, we express the ordinary moments of the order statistics of any KBG-G distribution as infinite weighted sums of the PWMs of the G distribution.

Replacing (8) in equation (22), we have

$$F(x)^{i+j-1} = \left(\sum_{r=0}^{\infty} b_r u^r\right)^{i+j-1}, \tag{23}$$

where $u = G(x)$ is the baseline cdf.

We use the identity $(\sum_{k=0}^{\infty} a_k x^k)^n = \sum_{k=0}^{\infty} d_{k,n} x^k$ (see Gradshteyn and Ryzhik, 2000), where

$$d_{0,n} = a_0^n \qquad \text{and} \qquad d_{k,n} = (ka_0)^{-1} \sum_{m=1}^{k} [m(n+1)-k]\, a_m\, d_{k-m,n}$$

(for $k = 1,2,\ldots$) in equation (23) to obtain

$$F(x)^{i+j-1} = \sum_{r=0}^{\infty} d_{r,i+j-1}\, G(x)^r, \tag{24}$$

where

$$d_{0,i+j-1} = b_0^{i+k-1} \quad \text{and} \quad d_{r,i+j-1} = (kb_r)^{-1} \sum_{m=1}^{r} [(i+j)m-r]\, b_m\, d_{r-m,i+j-1}.$$

For real non-integer a, inserting (11) and (24) into equation (22) and changing indices, we rewrite $f_{i:n}(x)$ for the KBG distribution in the form

$$f_{i:n}(x) = \frac{g(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{u,v=0}^{\infty} b_u^* d_{u,i+j-1}\, G(x)^{u+v}. \tag{25}$$

For an integer a, we obtain from equations (10), (22) and (24)

$$f_{i:n}(x) = \frac{g(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{p,q,u=0}^{\infty} w_{p,q}^* d_{u,i+j-1}\, G(x)^{a+p+q+u-1}. \tag{26}$$

Equations (25) and (26) immediately yield the pdf of KBG order statistics as a function of the baseline pdf multiplied by infinite weighted sums of powers of $G(x)$. Hence, the moments of KBG-G order statistics can be expressed as infinite weighted sums of PWMs of the G distribution. Clearly, equation (26) can be expressed as linear combinations of Exp-G pdfs. So, the moments and the mgf of KBG order statistics follow immediately from linear combinations of those quantities for Exp-G distributions.

8. Extreme values

If $\overline{X} = (X_1 + \cdots + X_n)/n$ denotes the mean of a random sample from (3), then by the usual central limit theorem $\sqrt{n}(\overline{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$.

Firstly, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t\to\infty} \frac{1 - G(t + xh(t))}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t\to\infty} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t\to\infty} \frac{[1 + xh'(t)] f(t + xh(t))}{f(t)} \\
&= \lim_{t\to\infty} \frac{[1 + xh'(t)] g(t + xh(t))}{g(t)} \left[\frac{G(t + xh(t))}{G(t)}\right]^{a-1} \\
&\quad \times \left[\frac{1 - G(t + xh(t))}{1 - G(t)}\right]^{b-1} \exp\{cG(t) - cG(t + xh(t))\} \\
&= \exp(-bx)
\end{aligned}$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} \Pr\{a_n (M_n - b_n) \leq x\} = \exp\{-\exp(-bx)\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Secondly, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a $\beta > 0$ such that

$$\lim_{t\to\infty} \frac{1-G(tx)}{1-G(t)} = x^{\beta}$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\to\infty} \frac{xf(tx)}{f(t)} \\
&= \lim_{t\to\infty} \frac{xg(tx)}{g(t)} \left[\frac{G(tx)}{G(t)}\right]^{a-1} \left[\frac{1-G(tx)}{1-G(t)}\right]^{b-1} \exp\{cG(t) - cG(tx)\} \\
&= x^{b\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} \Pr\{a_n(M_n - b_n) \le x\} = \exp\left(-x^{b\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Thirdly, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a $\alpha > 0$ such that

$$\lim_{t\to-\infty} \frac{G(tx)}{G(t)} = x^{\alpha}$$

for every $x < 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t\to-\infty} \frac{F(tx)}{F(t)} &= \lim_{t\to-\infty} \frac{xf(tx)}{f(t)} \\
&= \lim_{t\to\infty} \frac{xg(tx)}{g(t)} \left[\frac{G(tx)}{G(t)}\right]^{a-1} \left[\frac{1-G(tx)}{1-G(t)}\right]^{b-1} \exp\{cG(t) - cG(tx)\} \\
&= x^{a\beta}.
\end{aligned}$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty} \Pr\{a_n(M_n - b_n) \le x\} = \exp\{-(-x)^{a\alpha}\}$$

for some suitable norming constants $a_n > 0$ and b_n.

The same argument applies to min domains of attraction. That is, F belongs to the same min domain of attraction as that of G.

9. Inference

Let γ be the p-dimensional parameter vector of the baseline distribution in equations (2) and (3). We consider independent random variables $X_1,\ldots,X_n$, each X_i following a KBG-G distribution with parameter vector $\theta=(a,b,c,\gamma)$. The log-likelihood function, $\ell=\ell(\theta)$, for the model parameters is

$$\begin{aligned}\ell(\theta) &= n\log(K)+\sum_{i=1}^{n}\log g(x_i;\gamma)-c\sum_{i=1}^{n}G(x_i;\gamma)\\ &\quad +(a-1)\sum_{i=1}^{n}\log\{G(x_i;\gamma)\}+(b-1)\sum_{i=1}^{n}\log\{1-G(x_i;\gamma)\}. \end{aligned}\tag{27}$$

The elements of score vector are given by

$$\frac{\partial\ell(\theta)}{\partial a}=\frac{n}{K}\frac{\partial K}{\partial a}+\sum_{i=1}^{n}\log\{G(x_i;\gamma)\},$$

$$\frac{\partial\ell(\theta)}{\partial b}=\frac{n}{K}\frac{\partial K}{\partial b}+\sum_{i=1}^{n}\log\{1-G(x_i;\gamma)\},$$

$$\frac{\partial\ell(\theta)}{\partial c}=\frac{n}{K}\frac{\partial K}{\partial c}-\sum_{i=1}^{n}G(x_i;\gamma),$$

and

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial\gamma_j} &= \sum_{i=1}^{n}\left[\frac{1}{g(x_i;\gamma)}\frac{\partial g(x_i;\gamma)}{\partial\gamma_j}-c\frac{\partial g(x_i;\gamma)}{\partial\gamma_j}+\frac{(a-1)}{G(x_i;\gamma)}\frac{\partial G(x_i;\gamma)}{\partial\gamma_j}\right.\\ &\qquad\left.+\frac{(b-1)}{1-G(x_i;\gamma)}\frac{\partial G(x_i;\gamma)}{\partial\gamma_j}\right]\end{aligned}$$

for $j=1,\ldots,p$, where

$$\frac{\partial K}{\partial a}=-\frac{\left\{[\psi(a)-\psi(a+b)]\,{}_1F_1(a,a+b,-c)+\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial a}\right\}}{B(a,b)\,[{}_1F_1(a,a+b,-c)]^2},$$

$$\frac{\partial K}{\partial b}=-\frac{\left\{[\psi(b)-\psi(a+b)]\,{}_1F_1(a,a+b,-c)+\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial b}\right\}}{B(a,b)\,[{}_1F_1(a,a+b,-c)]^2},$$

$$\frac{\partial K}{\partial c}=\frac{a\,{}_1F_1(a+1,a+b+1,-c)}{(a+b)B(a,b)\,{}_1F_1(a,a+b,-c)},$$

$$\frac{\partial\, {}_1F_1(a,a+b,-c)}{\partial a} = -\left[\psi(a)-\psi(a+b)\right] {}_1F_1(a,a+b,-c) - \sum_{k=0}^{\infty} \frac{(a)_k(-c)^k}{k!(a+b)_k} \left[\psi(a+b+k)-\psi(a+k)\right],$$

and

$$\frac{\partial\, {}_1F_1(a,a+b,-c)}{\partial b} = \psi(a+b)\, {}_1F_1(a,a+b,-c) + \sum_{k=0}^{\infty} \frac{(a)_k(-c)^k}{k!(a+b)_k} \psi(a+b+k).$$

These partial derivatives depend on the specified baseline distribution. Numerical maximization of the log-likelihood above was accomplished by using the RS method (Rigby and Stasinopoulos, 2005) available in the R contributed gamlss package (Stasinopoulos and Rigby, 2007; R Development Core Team, 2009).

For interval estimation of each parameter in $\boldsymbol{\theta} = (a,b,c,\boldsymbol{\tau}^T)^T$, and tests of hypotheses, we require the expected information matrix. Interval estimation for the model parameters can be based on standard likelihood theory. The elements of the information matrix for (27) are given in the Appendix. Under suitable regularity conditions, the asymptotic distribution of the MLE, $\widehat{\theta}$, is multivariate normal with mean vector θ and covariance matrix estimated by $\{-\partial^2\ell(\theta)/\partial\theta\partial\theta^T\}$ at $\theta = \widehat{\theta}$. The required second derivatives were computed numerically.

Consider two nested KBG-G distributions: a KBG-G_A distribution with parameters $\theta_1,\ldots,\theta_r$ and maximized log-likelihood $-2\,\ell(\widehat{\theta}_A)$; and, a KBG-$G_B$ distribution containing the same parameters $\theta_1,\ldots,\theta_r$ plus additional parameters $\theta_{r+1},\ldots,\theta_p$ and maximized log-likelihood $-2\,\ell(\widehat{\theta}_B)$, the models being identical otherwise. For testing the KBG-G_A distribution against the KBG-G_B distribution, the likelihood ratio statistic (LR) is equal to $w = -2\,\{\ell(\widehat{\theta}_A) - \ell(\widehat{\theta}_B)\}$. It has an asymptotic χ^2_{p-r} distribution.

We compare non-nested KBG-G distributions by using the Akaike information criterion given by $\mathrm{AIC} = -2\,\ell(\widehat{\theta}) + 2p^*$ and the Bayesian information criterion defined by $\mathrm{BIC} = -2\,\ell(\widehat{\theta}) + p^*\log(\theta)$, where p^* is the number of model parameters. The distribution with the smallest value for any of these criteria (among all distributions considered) is usually taken as the one that gives the best description of the data.

10. Application-Ball bearing fatigue data

In this section, we shall compare the fits of the KBGW, beta Weibul (BW), Birnbaum-Saunders (BS) and Weibull distributions to the data set studied by Lieblein and Zelen (1956). They described the data from fatigue endurance tests for deep-groove ball bearings. The main objective of the study was to estimate parameters in the equation relating bearing life to load. The data are a subset of $n = 23$ bearing failure times for units tested at one level of stress reported by Lawless (1982). Because of the lower

Table 1: *MLEs and information criteria for the ball bearing data.*

Model	d	β	a	b	c	AIC	BIC
KBGW	1.5040	0.0456	15.9411	0.1972	12.7943	223.9	229.6
BW	1.5254	0.0435	3.3335	0.2032	0	233.7	238.3
Weibull	2.1018	0.0122	1	1	0	231.3	233.6
	α	β	-	-	-		
BS	0.5391	62.9794	-	-	-	230.2	232.5

bound on cycles (or time) to fail at zero, the distributional shape is typical of reliability data.

Firstly, in order to estimate the model parameters, we consider the MLE method discussed in Section 9. We take initial estimates of d and β as those obtained by fitting the Weibull distribution. All computations were performed using the statistical software R. Table 1 lists the MLEs of the parameters and the values of the following statistics: AIC and BIC as discussed before. The results indicate that the KBGW model has the smallest values for these statistics among all fitted models. So, it could be chosen as the most suitable model.

A comparison of the proposed distribution with some of its sub-models using LR statistics is shown in Table 2. The p-values indicate that the proposed model yields the best fit to the data set. In order to assess if the model is appropriate, we plot in Figure 3 the histogram of the data and the fitted KBGW, BW, Weibull and BS pdfs. We conclude that the KBGW distribution is a suitable model for the data.

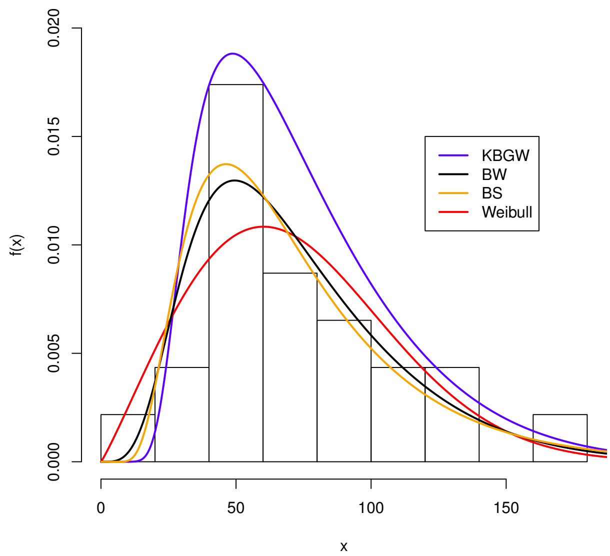

Figure 3: *Fitted KBGW, BW, Weibull and BS pdfs for the ball bearing data.*

Table 2: *LR statistics for the ball bearing data.*

Model	Hypotheses	Statistic w	p-value
KBGW vs BW	$H_0 : c = 0$ vs $H_1 : H_0$ is false	11.85	0.00057
KBGW vs Weibull	$H_0 : a = b = 1$ and $c = 0$ vs $H_1 : H_0$ is false	13.45	0.00375

Secondly, we apply formal goodness-of-fit tests in order to verify which distribution gives the best fit to the data. We consider the Cramér-Von Mises (W^*) and Anderson-Darling (A^*) statistics. In general, the smaller the values of the statistics, W^* and A^*, the better the fit to the data. Let $H(x;\boldsymbol{\theta})$ denote a cdf, where the form of H is known but $\boldsymbol{\theta}$ (a k-dimensional parameter vector, say) is unknown. To obtain the statistics, W^* and A^*, we proceed as follows: (i) compute $v_i = H(x_i;\widehat{\boldsymbol{\theta}})$, where the x_i's are in ascending order, $y_i = \Phi^{-1}(\cdot)$ is the standard normal quantile function and $u_i = \Phi\{(y_i - \overline{y})/s_y\}$, where $\overline{y} = n^{-1}\sum_{i=1}^n y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^n (y_i - \overline{y})^2$; (ii) calculate $W^2 = \sum_{i=1}^n \{u_i - (2i-1)/(2n)\}^2 + 1/(12n)$ and $A^2 = -n - n^{-1}\sum_{i=1}^n \{(2i-1)\log(u_i) + (2n+1-2i)\log(1-u_i)\}$ and (iii) modify W^2 into $W^* = W^2(1+0.5/n)$ and A^* into $A^* = A^2(1+0.75/n+2.25/n^2)$. For further details, the reader is referred to Chen and Balakrishnan (1995). The values of the statistics, W^* and A^*, for all fitted models are given in Table 3. Thus, according to these formal tests, the KBGW model fits the data better than other models. These results illustrate the flexibility of the KBGW distribution and the necessity for the additional shape parameters.

Table 3: *Goodness-of-fit tests for the ball bearing data.*

Model	Statistic	
	W^*	A^*
KBGW	0.00507	0.19916
BW	0.20587	0.57785
Weibull	0.13587	0.34791
BS	0.02298	0.34791

11. Conclusions

Following the idea of the class of beta generalized distributions and the distribution due to Ng and Kotz (1995), we define a new family of Kummer beta generalized (KBG) distributions to extend several widely known distributions such as the normal, Weibull, gamma and Gumbel distributions. For each continuous G distribution, we define the corresponding KBG-G distribution using simple formulae. Some mathematical properties of the KBG distributions are readily obtained from those of the parent distributions. The moments of any KBG-G distribution can be expressed explicitly in terms of infinite weighted sums of probability weighted moments (PWMs) of the G distribution.

The same happens for the moments of order statistics of the KBG distributions. We discuss maximum likelihood estimation and inference on the parameters. We consider likelihood ratio statistics and goodness-of-fit tests to compare the KBG-G model with its baseline model. An application to real data shows the feasibility of the proposed class of models. We hope this generalization may attract wider applications in statistics.

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Appendix: elements of the information matrix

The elements of this matrix for (27) can be worked out as:

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial a^2}\right) = -\frac{n}{K}\mathrm{E}\left[\frac{1}{K}\left(\frac{\partial K}{\partial a}\right) - \frac{\partial^2 K}{\partial a^2}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial b\partial c}\right) = -\frac{n}{K}\mathrm{E}\left[\frac{1}{K}\left(\frac{\partial K}{\partial b}\right)\left(\frac{\partial K}{\partial c}\right) - \frac{\partial^2 K}{\partial b\partial c}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial c^2}\right) = -\frac{n}{K}\mathrm{E}\left[\frac{1}{K}\left(\frac{\partial K}{\partial c}\right) - \frac{\partial^2 K}{\partial c^2}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial a\partial b}\right) = -\frac{n}{K}\mathrm{E}\left[\frac{1}{K}\left(\frac{\partial K}{\partial a}\right)\left(\frac{\partial K}{\partial b}\right) - \frac{\partial^2 K}{\partial a\partial b}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial a\partial c}\right) = -\frac{n}{K}\mathrm{E}\left[\frac{1}{K}\left(\frac{\partial K}{\partial a}\right)\left(\frac{\partial K}{\partial c}\right) - \frac{\partial^2 K}{\partial a\partial c}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial b^2}\right) = -\frac{n}{K}\mathrm{E}\left[\frac{1}{K}\left(\frac{\partial K}{\partial b}\right) - \frac{\partial^2 K}{\partial b^2}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial a\partial\gamma_j}\right) = -\sum_{i=1}^{n}\mathrm{E}\left[\frac{1}{G(x_i;\gamma)}\frac{\partial G(x_i;\gamma)}{\partial\gamma_j}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial b\partial\gamma_j}\right) = -\sum_{i=1}^{n}\mathrm{E}\left[\frac{1}{1-G(x_i;\gamma)}\frac{\partial G(x_i;\gamma)}{\partial\gamma_j}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial c\partial\gamma_j}\right)=\sum_{i=1}^{n}\mathrm{E}\left[\frac{\partial g\left(x_i;\gamma\right)}{\partial\gamma_j}\right],$$

$$\begin{aligned}\mathrm{E}\left(-\frac{\partial^2\ell(\theta)}{\partial\gamma_k\partial\gamma_j}\right) &= \sum_{i=1}^{n}\mathrm{E}\left[\frac{1}{g^2\left(x_i;\gamma\right)}\frac{\partial^2 g\left(x_i;\gamma\right)}{\partial\gamma_j\partial\gamma_k}\right]+c\sum_{i=1}^{n}\mathrm{E}\left[\frac{\partial^2 g\left(x_i;\gamma\right)}{\partial\gamma_j\partial\gamma_k}\right]+\\ &\sum_{i=1}^{n}\mathrm{E}\left[\frac{(a-1)}{G^2\left(x_i;\gamma\right)}\frac{\partial^2 G\left(x_i;\gamma\right)}{\partial\gamma_j\partial\gamma_k}\right]+\sum_{i=1}^{n}\mathrm{E}\left[\frac{(1-b)}{\left\{1-G\left(x_i;\gamma\right)\right\}^2}\frac{\partial^2 G\left(x_i;\gamma\right)}{\partial\gamma_j\partial\gamma_k}\right]\end{aligned}$$

for $j=1,\ldots,p$, where

$$\begin{aligned}\frac{\partial^2 K}{\partial a^2} &= -\left\{\frac{[\psi'(a)-\psi'(a+b)]}{{}_1F_1(a,a+b,-c)}+\frac{[\psi(a)-\psi(a+b)]^2}{[{}_1F_1(a,a+b,-c)]^2}\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial a}\right.\\ &+\frac{1}{[{}_1F_1(a,a+b,-c)]^2}\frac{\partial^2\,{}_1F_1(a,a+b,-c)}{\partial a^2}+\frac{[\psi(a)-\psi(a+b)]^2}{{}_1F_1(a,a+b,-c)}\\ &+\frac{2}{[{}_1F_1(a,a+b,-c)]^2}\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial a}\\ &\left.+\frac{1}{[{}_1F_1(a,a+b,-c)]^3}\left(\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial a}\right)^2\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 K}{\partial b^2} &= -\left\{\frac{[\psi'(b)-\psi'(a+b)]}{{}_1F_1(a,a+b,-c)}+\frac{[\psi(b)-\psi(a+b)]^2}{[{}_1F_1(a,a+b,-c)]^2}\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial b}\right.\\ &+\frac{1}{[{}_1F_1(a,a+b,-c)]^2}\frac{\partial^2\,{}_1F_1(a,a+b,-c)}{\partial b^2}+\frac{[\psi(b)-\psi(a+b)]^2}{{}_1F_1(a,a+b,-c)}\\ &+\frac{2}{[{}_1F_1(a,a+b,-c)]^2}\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial b}\\ &\left.+\frac{1}{[{}_1F_1(a,a+b,-c)]^3}\left(\frac{\partial\,{}_1F_1(a,a+b,-c)}{\partial b}\right)^2\right\},\end{aligned}$$

$$\frac{\partial^2 K}{\partial c^2}=-\left\{\frac{a(a+1)_1F_1(a+2,a+b+2,-c)}{(a+b)B(a,b)_1F_1(a,a+b,-c)}+\frac{a^2\left[{}_1F_1(a+1,a+b+1,-c)\right]^2}{(a+b)^2B(a,b)\left[{}_1F_1(a,a+b,-c)\right]^2}\right\},$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial a \partial b} = & -\Bigg\{ \frac{[\psi'(a+b)]}{{}_1F_1(a,a+b,-c)} + \frac{[\psi(b)-\psi(a+b)]}{[{}_1F_1(a,a+b,-c)]^2} \frac{\partial\, {}_1F_1(a,a+b,-c)}{\partial a} \\
& + \frac{1}{[{}_1F_1(a,a+b,-c)]^2} \frac{\partial^2\, {}_1F_1(a,a+b,-c)}{\partial a \partial b} + \left[\frac{\psi(b)-\psi(a+b)}{{}_1F_1(a,a+b,-c)} \right]^2 \\
& + \frac{2\,[\psi(a)-\psi(a+b)]}{[{}_1F_1(a,a+b,-c)]^2} \frac{\partial\, {}_1F_1(a,a+b,-c)}{\partial a} \\
& + \frac{2}{[{}_1F_1(a,a+b,-c)]^3} \frac{\partial\, {}_1F_1(a,a+b,-c)}{\partial b} \frac{\partial\, {}_1F_1(a,a+b,-c)}{\partial a} \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial a \partial c} = & \; {}_1F_1(a+1,a+b+1,-c) + \frac{a}{(a+b)} + a\,[\psi'(a)-\psi'(a+b)] \\
& + \frac{a}{{}_1F_1(a,a+b,-c)} \frac{\partial\, {}_1F_1(a+1,a+b+1,-c)}{\partial a} \\
& + \frac{a}{{}_1F_1(a+1,a+b+1,-c)} \frac{\partial {}_1F_1(a+1,a+b+1,-c)}{\partial a},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial b \partial c} = & \; {}_1F_1(a+1,a+b+1,-c) + \frac{a}{(a+b)} + a\,[\psi'(b)-\psi'(a+b)] \\
& + \frac{a}{{}_1F_1(a,a+b,-c)} \frac{\partial\, {}_1F_1(a+1,a+b+1,-c)}{\partial b} \\
& + \frac{a}{{}_1F_1(a+1,a+b+1,-c)} \frac{\partial {}_1F_1(a+1,a+b+1,-c)}{\partial b},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\, {}_1F_1(a,a+b,-c)}{\partial a^2} = & -\left[\psi'(a+b) - \psi'(a) + \{\psi(a)-\psi(a+b)\}^2 \right] {}_1F_1(a,a+b,-c) \\
& - \sum_{k=0}^{\infty} \frac{(a)_k(-c)^k}{k!(a+b)_k} \, [-2\psi(a)\psi(a+k) + 2\psi(a+b)\psi(a+k) \\
& + 2\,\psi(a)\psi(a+b+k) - 2\psi(a+b)\psi(a+b+k) + \psi^2(a+k) \\
& - 2\,\psi(a+k)\psi(a+b+k) + \psi^2(a+b+k) \\
& + \psi'(a+k) - \psi'(a+b+k)]\,,
\end{aligned}
$$

$$\begin{aligned}\frac{\partial^2 {}_1F_1(a,a+b,-c)}{\partial b^2} &= -\left[\psi'(a+b)-\psi^2(a+b)\right] {}_1F_1(a,a+b,-c)\\ &\quad -\sum_{k=0}^{\infty}\frac{(a)_k(-c)^k}{k!(a+b)_k}\left[-2\psi(a+b)\psi(a+b+k)\right.\\ &\quad \left.-\psi'(a+b+k)+\psi^2(a+b+k)\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 {}_1F_1(a,a+b,-c)}{\partial a\partial b} &= \left[\psi'(a+b)-\psi^2(a+b)-\psi(a)\psi(a+b)\right] {}_1F_1(a,a+b,-c)\\ &\quad -\sum_{k=0}^{\infty}\frac{(a)_k(-c)^k}{k!(a+b)_k}\left[2\psi(a+b)\psi(a+b+k)-\psi^2(a+b+k)\right.\\ &\quad -\psi(a+k)\psi(a+b)+\psi(a+k)\psi(a+b+k)\\ &\quad \left.-\psi(a)\psi(a+b+k)+\psi(a+b+k)\right].\end{aligned}$$

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