# PUBLIC EVALUATION of LARGE PROJECTS: variational inequalities, bilevel programming and complementarity. A Survey

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ABSTRACT

Large projects evaluation rises well known difficulties because -by definition- they modify the current price system; their public evaluation presents additional difficulties because they modify too existing shadow prices without the project.

This paper analyzes –first- the basic methodologies applied until late 80s., based on the integration of projects in optimization models or, alternatively, based on iterative procedures with information exchange between two organizational levels. New methodologies applied afterwards are based on variational inequalities, bilevel programming and linear or nonlinear complementarity. Their foundations and different applications related with project evaluation are explored.

As a matter of fact, these new tools are closely related among them and can treat more complex cases involving –for example- the reaction of agents to policies or the existence of multiple agents in an environment characterized by common functions representing demands or constraints on polluting emissions.

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## 1.-INTRODUCTION

This Survey refers to new methodologies adapted to problems generated by public evaluation of large projects. Its immediate motivation is related with the exploration of a set of topics related with climate change economics, published in a collective book in 2009[1]. One consequence of this exploration has been this new analysis of a previous line of research already focused on large project evaluation[2].

Until the 80s, the most widespread methods tended to be conceived as planning procedures -centralized or not- characterized by information exchange between two levels of decision; this characterization is especially true in the cases of Dantzig-Wolfe and Benders algorithms.

New methodologies included are variational inequalities, bilevel programming and linear or nonlinear complementarity; relevant applications are also analyzed. Special attention is dedicated to the difficulties created by discrete variables or -in other words- the problems posed by non-convexities.

These new tools can treat more complex cases than the traditional ones such is the case of the reaction of agents of belonging to a second level to the policies set by a first hierarchical level or the existence of multiple centers of decision in an environment of common constrains that may represent functions of joint demand or constraints on common pollution emissions. Finally, in an Appendix, the special topic of project programs and budged constraints is analyzed.

## 2.- MARGINAL PROJECTS EVALUATION

### 2.1.-Introduction to marginal projects evaluation

Let us suppose it is necessary to evaluate a new steel plant in a medium-size economy. The point of departure is a given set of prices, quantities, revenues, etc. in the economy. The eventual adoption of any of the investment alternatives available to the steel plant would change current set. Under these conditions ¿does it make sense to use the initial configuration to assess the alternatives ? If not ¿how to proceed?

In order to analyze this basic question we will assume is available a model of the economy in which we will proceed to integrate the model of the steel plant in order to

[1] VEGARA J.M. (1987) *Evaluación pública de grandes proyectos de inversión por integración en modelos macroeconómicos*, Instituto de Estudios Fiscales, Madrid

[2] Some partial results were presented to the *Technical Workshop on Cost-Benefir Analysis on climate change adaptation*, Spanish Climate Change Bureau-OECC, Universidad Autónoma de Madrid, September 2009, Madrid

evaluate its alternative designs. Specifically, let us consider the model of the economy is a multi-sectoral and multi-period optimization one; a model like one relative to Mexico included in GOREUX L.M., MANNE A . (1973) or the one relative to the Spanish economy included in SEBASTIAN C. (1976). See also WESTPHAL LL (1971), Ch.R. BLITZER PB CLARK, TAYLOR L. (1975), GOREUX L.M. (1977), JANSSEN J.M.L., PAU L.F. or STRASZAK A. (1979) [3].

The economy model –without the Project- is to find the $m_0$-vector $x_0$ of continuous variables such that:

$$\underset{x_0}{Max.}w = F_0(x_0) \tag{2.1}$$

$$A_{12}(x_0) \leq b_1 \tag{2.2}$$

$$x_0 \geq 0 \tag{2.3}$$

Let $A_{12}(.)$ be a $n_1$-vector of functions and $\hat{v}$ a $n_1$-vector be the vector of dual variables corresponding to constraints (2.2). Let us suppose the project impact on the economy is marginal and equal to a differential vector $db$ .; as it is well known, under this conditions and according to the economic interpretation of dual variables, the variation of the objective function is equal to $\hat{v}^T db$. Consequently, if a project is marginal, existing shadow prices can be used to evaluate project impact. Acceptation rule is $\hat{v}^T db > 0$.This is the conventional approach considering projects as "marginal perturbations". Large projects are not marginal by definitions. DRÈZE J. and STERN N. (1987)[4] have analyzed these perturbations in a framework including also "policies" defined as modifications of the parameters of the model (2.1.)-(2.3).

The project model

The need for public evaluation stems from the existence of project impacts on the global economy, mainly contributions to demand and resource consumption. Consequently, we must incorporate project impacts in the economy.

Let $x_1$ be a $m_1$-vector of variables specific to the project. They are continuous variables and subject to $n_2$ constraints, specific to the Project:

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[3] WESTPHAL L. (1971) *Planning Investments with Economies of Scale*, North Holland Pu., Amsterdam; BLITZER Ch.R., CLARK P.B., TAYLOR L. (1975) *Economy-wide Models and Development Planning*, Oxford University Press, New York; GOREUX L.M. (1977) *Interdependence in Planning. Multilevel Programming Studies of the Ivory Coast*, The Johns Hopkins University Press, Baltimore, London; JANSSEN J.M.L., PAU L.F., STRASZAK A.(1979) *Models and Decision Making in National Economies*, North-Holland, Amsterdam. As is well known, the World Bank developed many initiatives in this field.

[4] DRÈZE J., STERN, N. (1987) The Theory of Cost-Benefit Analysis, in AUERBACH.A.J., FELDSTEIN M. (1987), Volume II.

$$A_{21}(x_1) \leq b_2 \tag{2.4}$$

$$x_1 \geq 0 \tag{2.5}$$

<u>Centralized project evaluation</u>

First we will proceed to evaluate the project using a centralized approach. Let us consider the model called "principal" with the project incorporated:

$$\underset{x_o}{Max}.w = F_0(x_0) \tag{2.6}$$

$$A_{11}(x_1) + A_{12}(x_0) \leq b_1 \tag{2.7}$$

$$A_{21}(x_1) \qquad\qquad \leq b_2 \tag{2.8}$$

$$x_1, x_0 \geq 0 \tag{2.9}$$

In the objective function the only relevant variables are the central ones. In constraints (2.7), the vector of functions $A_{11}(x_1)$ expresses project impact on the global economy and constraints (2.8) are specific to the project. The above model, therefore, is adapted for a central project evaluation.

2.2.- Decentralized Projects evaluation

From model (2.6)-(2.9) we can construct a new model relative to the Project:

$$Max\ \mathrm{w}_1 = -\hat{v}^T A_{11}(x_1) \tag{2.10}$$

$$A_{21}(x_1) \leq b_2 \tag{2.11}$$

$$x_1 \geq 0 \tag{2.12}$$

Where vector $\hat{v}$ is the $n_1$-vector corresponding to the dual variables of central constraints (2.7) in the "principal model" (2.6)-(2.9). This is the problem corresponding to the Project Evaluation Center-PEC. We will call it the "reduced problem".

It can be proofed that -under some specific conditions- the optimum of "reduced" (2.10)-(2.12) belongs to the solution of the "principal" (2.6) - (2.9); that is to say, if project managers know a) its specific constraints; b) projects impacts on the economy and c) the optimal vector of optimal dual variables of global constraints (2.7) in the "principal" then the optimal solution of (2.10)-(2.12) coincides with the solution of

(2.6) - (2.9). Therefore, optimal shadow prices in the “principal” can be used to evaluate the project in the “reduced” (2.10)-(2.12).

However, as can be noticed, in the general case there is circularity in the procedure because shadow prices computation is simultaneous with the determination of optimal solution in the program (2.6)-(2.9). In the general case -with non marginal projects- there is no possibility for decentralized project evaluation without interaction with the model of the economy: decentralized evaluation is only possible when project impacts are marginal, that is, when $A_{11}(x_1)$ is a differential vector $db$ .

Certainly, this result is not surprising since otherwise it would be possible to determine the economic impact resulting from a non marginal project without knowing its optimal design.

Definition and conditions of separability[5]

A "reduced" problem (2.10)-(2.12) is called "separable" from a “principal” one (2.6)-(2.9) if every optimal solution of the “reduced” problem is generated by an optimal solution of the principal. Therefore -in the example considered- when any optimal solutions of the model used to evaluate the project in a decentralized way is generated by an optimal solution of the model used for a centralized evaluation. In this context therefore, project evaluation is a separability problem.

Separability of the reduced problem from the principal one is basically related to the uniqueness of the optimal solution of the reduced problem. A sufficient condition for a given optimal solution of the reduced problem to be unique is that its objective function be strictly convex. This is not case of linear optimization programs; for this reason there in no separability in this case [6].

Objective functions and constraints

Let us consider a graphical illustration of separability. The initial problem is to find the solution of:

$$\underset{x_1,x_2,x_3}{Max.w} == F_0(x_1, x_2) \tag{2.13}$$

$$a_1(x_1, x_2) \leq b_1 \tag{2.14}$$

[5] BESSIÈRE Fr. and .SAUTER E. proofed that if the various functions verify certain conditions -specially differentiability and convexity- the existence of a unique solution in the reduced problem is a sufficient condition of separability. BESSIERE F., SAUTER,E. (1968) Optimization and subopti-mization: the method of extended models in the non-linear case, *Management Science*, September 1968

[6] See BESSIERE F., SAUTER,E. (1968). Another common case of non-separability happens when there is degeneracy, as it is very common in transportation and networks problems. These issues have been analyzed in the context of optimal control theory in continuous or in discrete time; see ALBOUY M. (1972) *La régulation économique dans l´entreprise*, Vols.I, II, Dunod, Paris

$$a_2(x_1, x_2) \leq b_2 \quad (2.15)$$

$$a_3(x_1, x_2) \leq b_3 \quad (2.16)$$

$$x_1, x_2 \geq 0 \quad (2.17)$$

Let $(\hat{x}_1, \hat{x}_2)$ be the optimum. The graphic representation of this problem can be seen in Fig. 2.1.

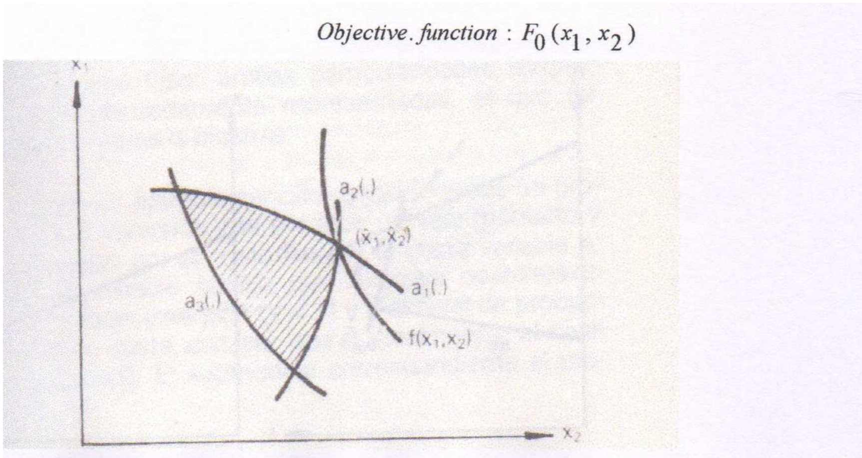


Figure 2.1.

¿Is it possible to get the same optimal solution by solving the problem without constraint (2.15)? According to separability theory the reduced problem will be:

$$\underset{x_1, x_2}{Max.}z = F_0(x_1, x_2) - \hat{v}_1\, a_1(x_1, x_2) \quad (2.18)$$

$$a_2(x_1, x_2) \leq b_2 \quad (2.19)$$

$$a_3(x_1, x_2) \leq b_3 \quad (2.20)$$

$$x_1, x_2 \geq 0 \quad (2.21)$$

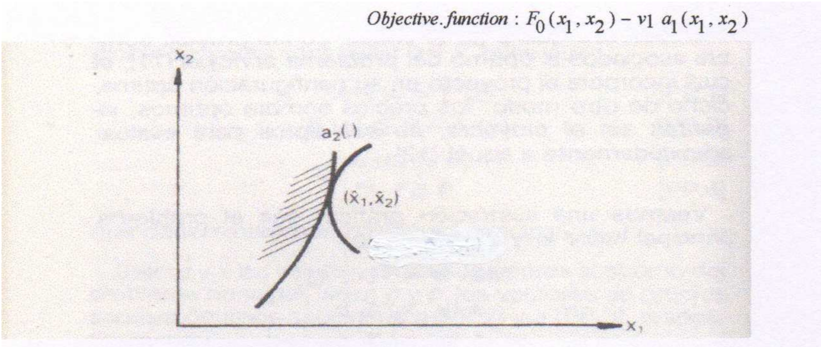


Figure 2.2

Numerical value of $\hat{v}_1$ is equal to the optimal value of the dual variable associated to the suppressed constraint. It is intuitive that in order to get the same optimal solution me need to modify the objective function.

Let us consider an economic interpretation of this example. Let us suppose the principal (2.13) - (2.17) models a firm maximizing its benefits and that constraint (2.14) express the limitation of carbon dioxide emissions imposed by the regulator. The result shows that it is possible to obtain the same optimal plan by imposing a tax equal to $\hat{v}_1$. The example illustrates the equivalence between a constraint and a suitable modification of the objective function. Obviously, the formal equivalence does not mean equivalence of the institutional conditions necessary for proper operation in both cases.

Next, we will analyze existing methodological alternatives in order to deal the problems raised by non-marginality.

## 3.-LARGE PROJECTS EVALUATION

In order to solve the problems generated by large project evaluation there are two basic methodologies. There are two different approaches:

a) evaluation by integration

b) iterative procedures

### 3.1.-Evaluation by integration

There are different applications of evaluation by integration. See WESTPHAL L.E.(1971)[7] and GOREUX, L.M.(1977)[8] .VEGARA J.M. (1987)[9] contains an

[7] WESTPHAL L. (1971) *Planning Investments with Economies of Scale*, Nort Holland Pu., Amsterdam

application to the Spanish economy. See also GOREUX L.M., MANNE A. (1977) or ECKAUS R.S., ROSENSTEIN-RODAN P.N.(1973) [10]. Section [4] in this Survey contains additional applications.

The approach by integration has the advantage that makes possible to use optimization models including binary variables so that it is feasible to deal problems associated with fixed costs, economies of scale and not-convexities in general, and also those related to the existence of alternatives among other highly relevant features.

The main limitation of this approach is that it assumes a single decision center for the global economy and, consequently, information centralization

3.2.-Iterative procedures

This approach is a direct application of different existing iterative algorithms applied to the solution of large mathematical programming problems. Basic algorithms are Dantzig-Wolfe and Benders.

Let as consider the linear version of the global economy with the project, (2.6)-(2.9):

$$\underset{x_0, x_1}{Max.} w = c_0^T x_0 \tag{3.1}$$

$$A_{11}x_1 + A_{12}x_0 = b_1 \tag{3.2}$$

$$A_{21}x_1 = b_2 \tag{3.3}$$

$$x_1, x_0 \geq 0 \tag{3.4}$$

The model includes only one Project but can be generalized to include several projects if using the initial form of Dantzig-Wolfe model [11].

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[8] Chapter 15 in MANNE A. (1977) *Interdependence in Planning*, The Johns Hopkins University Press, Baltimore and London

[9] VEGARA J.M.(1987) Evaluación pública de grandes proyectos de inversión por integración en modelos macroeconómicos, Instituto de Estudios Fiscales, Madrid. This model was used to evaluate a new integral steel plant in Spain by using the multisectoral and multiperiod model of the spanish economy contained in SEBASTIÁN C. (1976) *El crecimiento económico español 1974-1984: proyecciones mediante un modelo multisectorial de optimización*, Fundación del INI, Programa de Investigaciones Económicas, The integrated model MACROSID included binary variables in the steel plant submodel.

[10] GOREUX L.M., MANNE A. (1973) *Multilevel Planning: Case Studies in Mexico*, North Holland Pu.Co Amsterdam, London; ECKAUS, ROSENSTEIN-RODAN P.N.(eds.) *Analysis of development problems*, North Holland Pu, New York

[11] DANTZIG G.B, WOLFE D. (1961) The decomposition algorithm for linear programs, *Econometrica*, oct.1961, 29, 4. For additional contributions see WHINSTON A. (1964) Pricing Guides in Decentralized Organizations, in *New Perspectives in Organizational Research*, edited by COOPER W.W. at al., John Wiley & Sons, New York, where they relaxed linearity constraints. BAUMOL W.J., FABIAN T. (1964)

Let us consider only the constraints specific to the model:

$$A_{21}x_1 = b_2 \tag{3.5}$$

$$x_1 \geq 0 \tag{3.6}$$

The points belonging to (3.5)-(3.6) can be computed as convex linear combinations of the extreme points $x_{1k}$ :

$$x_1 = \sum_{j=1}^{k_1} \lambda_{1j} x_{1j} \tag{3.7}$$

$$\sum_{j=1}^{k_1} \lambda_{1j} = 1 \tag{3.8}$$

$$\lambda_j \geq 0 \,, \quad \forall j \tag{3.9}$$

By substituting vector $x_1$ in (3.1)-(3.4) we get a problem called the “extreme problem” in $x_0$ and $\lambda_{1j}$. The essential feature of Dantzig-Wolfe algorithm is that the extreme problem can be solved by using simplex method of linear programming without the initial totality of extreme points.

Let us consider the next problem (3.10)-(3.12) where ${}_k u$ corresponds to the $k$ iteration of the procedure:

$$\underset{x_1}{Max.} w^k = -{}_k u^T A_{11} x_1 \tag{3.10}$$

$$A_{21}x_1 = b_2 \tag{3.11}$$

$$x_1 \geq 0 \tag{3.12}$$

Any solution of this problem is an extreme point $x_{1j}$ that can be formulated as a convex linear combination of the relevant extreme points.

The algorithm proceeds as follows:

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Decomposition, Pricing for Decentralization and External Economies, *Management Science*, Vol.XI was focused on the issue of internalizing externalities and RUEFFLI T.W. (1971) A Generalized Goal Decomposition Model, *Management Science*, Vol.17, No.8 developed a three-level organization model

1.-CP computes a provisional price vector associated to constraints (3.2) and communicates vector $-_k u^T A_{11}$ to the CPE. This vector is a provisional evaluation of the unit net impact of the project.

2.- CEP solves (3.10) - (3.12) i.e. maximizes the net value of project contribution valued using the provisional prices and considering only their own constraints. CEP communicates to CP the new provisional plan, $x_{1\,j}$ and the associated value $w^k$ of the objective function to be used in the test of optimality.

3.-CP applies the test of optimality and if it is verified proceeds to compute the optimal solution as a linear convex combination of previous solutions.

In this case, therefore, there is no decision decentralization. Dantzig-Wolfe algorithm uses continuous variables and therefore cannot be used to deal with non convexities.

b.-<u>Benders algorithm</u> [12]

Let us consider the problem:

$$\underset{x_0,y}{Max}\ z = c_0^T x_0 \tag{3.8}$$

$$A_{11}\,y + A_{10}\,x_0 \le b_1 \tag{3.9}$$

$$x_0 \ge 0 \tag{3.10}$$

$$y \in Y \tag{3.11}$$

Variables $y$ are specific to the Project and can be binary: they must belong to a given set $Y$. Constraints (3.9) express Project impact on the economy. The objective function depends exclusively on central variables $x_0$. Let $\hat{z}$ be the optimal value of the objective function.

Given $y = \overline{y}$ (3.12) the dual of (3.8)-(3.11) [13] is to find vector $u$ such that:

$$min.w = u^T(b_1 - A_{11}\overline{y}) \quad (3.13$$

$$u^T A_{11} \ge c_0^T \tag{3.14}$$

[12] BENDERS J.F.(1962) Partitioning procedures for solving mixed-variables programming models, <u>*Numerische Mathematic*</u>, Vo,4, 352-252

[13] The singularity of Benders algorithm is generated by this partial dualization of the global problem. This is particularly relevant in the presence of discrete variables.

$$u \geq 0 \tag{3.15}$$

The convex polyhedron of feasible solutions associated with the dual of (3.12)-(3.13) does not depend on $\bar{y}$; the polyhedron has $K$ vertex. The optimum will correspond to one vertex of the polyhedron, supposed unique. Therefore, our problem is equivalent to find vector ${}_k u$ such that:

$$\min_{\forall k \in K} .w = {}_k u^T (b_1 - A_{11} \bar{y})$$

In the optimum of primal and dual $\hat{w} = \hat{z}$ so that:

$$\underset{x \in X}{Max} .c_o^T x_o = \min_{\forall k \in K} .w = {}_k u^T (b_1 - A_{11} \bar{y})$$

where:

$$X = \left[ x_0 \,/\, Ax \leq (b - A_{11}\, \bar{y}), x_0 \geq 0 \right]$$

Outside the optimum the relation is: $c_0{}^T x_0 \geq u^T (b_1 - A_{11} \bar{y})$

Therefore, original problem (3.8)-(3.11) is equivalent to find $x_0$ and $y$ such that:

$$\underset{y}{Max} \left[ \max_{xo} c_0^T (b_1 - A_{11} y \right] \tag{3.16}$$

$$A_{11}\, y + A_{10}\, x_0 \leq b_1 \tag{3.17}$$

$$x_0 \geq 0 \tag{3.18}$$

$$y \in Y \tag{3.19}$$

Given the equality in the optimum between the values of primal and dual objective functions so that to solve:

$$\underset{y \in Y}{Max} \left[ \min_{\forall k \in K} {}_k u^k (b_1 - A_{11}\, \bar{y}) \right]$$

$$y \in Y$$

is equivalent to solve:

$$Max.z \tag{3.20}$$

$$z \leq \min_{\forall k \in K} {}_k u^T (b_1 - A_{11} \bar{y}) \tag{3.21}$$

$$y \in Y \tag{3.22}$$

Therefore, if the totality of extreme point were known, problem (a)-(b) would be equivalent to solve:

$$\min_{\forall k \in K} {}_k u^T (b_1 - A_{11}\bar{y}) \tag{3.23}$$

$$y \in Y \tag{3.24}$$

<u>The algorithm</u>

Initially all the vertices of (3.13) are un known: they have to be generate iteratively. The algorithm basically consists of the following steps:

1-CP determines an extreme point ${}_1u$ of:

$$\hat{u}^T A_{10} \geq c_0^T \tag{3.25}$$

$$u \geq 0 \tag{3.26}$$

This vector plays the role of a provisional price and it is communicate to the CEP, simultaneously with the value of "resources" ${}_1ub$ used by the CEP.

2 - The CEP solves the problem:

$$\underset{y}{Max}.\alpha \tag{3.27}$$

$$\alpha \leq {}_1u^T (b_1 - Ay) \tag{3.28}$$

$$y \in Y \tag{3.29}$$

Vector $\bar{y}$ is a “provisional project specification” which is communicated to CP.

3- CP solves its own problem taking “provisional project specification” as given:

$$\max_{xo} z = c_0^T x_0 \tag{3.30}$$

$$Ax_o = b_1 - A_{11}\bar{y} \tag{3.31}$$

$$x_0 \geq 0 \tag{3.32}$$

CP applies the optimality test. If the optimum has not yet been reached he must compute another provisional price vector $_k u$ using (3.13) - (3.14), generating another constraint (3.28) in Step 2 [14].

In Step 2, the CEP solves its own problem with one additional constraint (3.28) generating upper bounds for the objective function value. This constraints include not only prices but also quantities. The set of hyperplanes defined by constraints (3.28) generates "dome" constraining values of $z$ to be maximized in order to induce the CEP to compute the optimum.

A special interest of Benders algorithm derives its capacity to solve mathematical programming problems including discrete variables such as those associated with the presence non-convexities, fixed costs or the presence of alternative decisions, aspects that cannot be formulated in models with continuous variables [15]. This is possible because CP communicates constraints including prices and quantities. As in Dantzig-Wolfe- information is decentralized but the final decision is not [16].

Independently of their economic interpretation [17], the algorithms of Dantzig-Wolfe and Benders are used to solve optimization problems large because their approach involves dividing global problem into subproblems.

<u>Algorithmic-heuristic procedures</u>

A complementary approach is so-called algorithmic-heuristic procedures, developed initially by KORNAY J. (1969)[18] and based on alternating algorithmic and heuristic steps based on the knowledge and experience of planners. The basic idea is to multiply information transmitted in each step in order to accelerate convergence.

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[14] It is feasible to generate several vectors in each step in order to accelerate convergence by reducing the number of information exchanges between CP and CEP.

[15] See VIETORISZ T. (1963) Industrial Development Planning Models with Economies of Scale, *<u>Papers of the Regional Science Association</u>*, 12, 157-92 and also Decentralization and Project Evaluation under Economics of Scale and Indivisibilities, *<u>Industrialization and Productivity</u>*, New York, United Nations, Bulletin 12 1968, 25-58

[16] It is possible to integrate Dantzig-Wolfe and Benders to operate with more complex structures. See VEGARA JM, SEBASTIAN C. (1975).Project evaluation in a two level framework, *<u>Econometric Society World Congress</u>*, Toronto, september 1975

[17] See Chapters 3 and 7 de LASDON L.S. (1970) *<u>Optimization Theory for Large Systems</u>*, The MacMillan Co., New York US

[18] KORNAI J.(1969) Man-machine Planning, *<u>Economics of Planning</u>*, vol.9, 9, January. See also KORNAI J.(2006) *<u>By Force of Thought</u>*, The MIT Press, Cambridge USA

Model MACROSID –mentioned in Note 9- integrated in a multisectoral and multiperiod model of the spanish economy was solved by Jaume BARCELO y Antonio DE LECEA using this approach.

4.- MODELS WITH ENDOGENOUS PRICES

In a seminal paper SAMUELSON P. A. (1952)[19] introduced the idea that a particular optimization problem can generate the conditions corresponding to an equilibrium in a market. Specifically, maximizing the total surplus of producers and consumers in a partial equilibrium model is obtained as a condition the equality between price and marginal cost characteristic of competitive markets. The possibility to endogenize market prices makes possible to simulate market equilibria using mathematical optimization models with appropriate objective functions [20].

If demand is linear, the simplest formulation of total surplus is quadratic. Moreover, the dual variable of the constraint that expresses the relationship between supply and demand is equal to market price. The problem formulation can be extended without difficulty to the case of several products or several periods.Consider the inverse function of demand for the good $j$ :

$$p_j = (a_j - d_j q_j) \qquad (4.1)$$

Lel us $C(q_j)$ be the production cost of $q_j$. The algebraic expression of total surplus is::

$$S = 1/2(a_j - p_j)q_j + p_j q_j - C(q_j) \qquad (4.2)$$

or, taking into account the inverse demand of (4.1):

$$S = a_j q_j - 1/2 d_j q_j^2 - C(q_j) \qquad (4.2)$$

The simplest complete model is, therefore:

$$Max.S = a_j q_j - 1/2 d_j q_j^2 - C(q_j) \qquad (4.3)$$

subject to constraints expressing:

a) supply must be greater or equal to demand

b) used resources must not be greater than availabilities

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[19] SAMUELSON P.A. (1952) Spatial Price Equilibrium and Linear Programming, American Economic Review, Vol.42, pp.283-303

[20] See TAKAYAMA T., JUDGE,G.G. (1971) Spatial and Temporal Price and Allocation Models, North Holland, Amsterdam

c) variables must be nonnegative.

Taking into consideration K-K-T conditions we get:

$$p_j = C^{'}(q_j) \quad (4.5)$$

and dual variables associated with supply/demand constraints are equal to market prices.

DULOY J.H., NORTON R.D (1973)[21] applied this approach to mexican agriculture. This methodology can be used to analyse different forms of regulation, as can be seen in GREENBERG H.J., MURPHY F.H. (1985)[22] .

It is also interesting the model of the energy sector MARKAL: a linear programming model including different demand functions and existing technological alternatives; the model computes the emissions of greenhouse gases. The objective function to maximize is consumers and producers surplus; the model can be applied to a country level or to the world scale, differentiating eighteen regions. The general structure of the model can be seen in FISHBONE J.G., ABILOCH H.(1981) [23]. Greenhouse gas emissions can be constrained so that it is possible to analyze the impacts of this internalization policy on different demands and on production technologies including carbon sequestration.

There are MARKAL versions integrating international transactions including those related with emissions permits. See RAFAJ P., KYPREOS S., BARRETO L. (2005)[24].

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Next we will explore new tools and new possibilities made possible by:

-variational inequalities

-bilevel programming

-complementarity [25]

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[21] DULOY J.H., NORTON R.D (1973) CHAC, A programming model of mexican agriculture, pp.291-337 in GOREUX L.M., MANNE A.S (1973) *Multi-level planning: case studies in Mexico*, North Holland Pu. Co. Amsterdam

[22] GREENBERG H.J., MURPHY F.H.(1985) Computing Market Equilibria with Price Regulations Using Mathematical Programming, *Operations Research*, Vol.33, No.5. HAZELL P.B.R., NORTON R.D. (1986) [22] Survey on quadratic programming and applications in agriculture models is very interesting and useful.

[23] FISHBONE J.G., ABILOCH H.(1981) MARKAL, a linear-programming model for energy systems analysis: Technical description of the BNL version, *International Journal of Energy Research*, Vol.5, Issue 4, pp.353-375

[24] RAFAJ P., KYPREOS S., BARRETO L. (2005) Flexible carbon mitigation policies: analysis with a global multi-regional MARKAL model. In HAURIE A., VIGUIER L.eds. (2005) *The Coupling of Climate and Economic Dynamics*, , Springer Dordrecht, Berlin

[25] This Survey does not include models like MERGE, used to evaluate greenhouse emission mitigation policies and including as objective function an aggregate welfare function and an aggregate production

There are strong relationships among these different tools and methodologies as can be seen in the Introduction of Volume II of FACCHINEI F. PANG J-S (2003) [26]. VI problems are the less structured formulation: they represent the most general formulation.

5.-VARIATIONAL INEQUALITIES .

5.1.-Introduction to VI

A finite variational inequalities problem $VI(F, K)$ is to find vectors $x^* \in K \subset \Re^n$ such that:

$$F(x^*)^T (x - x^*) \geq 0, \qquad \forall x \in K \tag{5.1}$$

where $F(x)$ is a continuous function $F : K \rightarrow \Re^n$ and $K$ is nonempty, closed convex set.

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function. See MANNE A., MENDELSOHN R.,RICHELS R. (1995) MERGE. A model for evaluating regional and global effects of GHG reduction policies, *Energy Policy*, Vol.23, No.1.

Neither includes POLES model family: a world scale simulation energy model, regionally desagregated and using recursive simulation methods in a parcial equilibrium framework. POLES models are focused on existing interactions among energy sectors and climate change; see HIDALGO I.(2005) Introducción a los modelos de sistemas energéticos, económicos y medio-ambientales: descripción y aplicaciones del modelos POLES, Revista de economía mundial, 2005, 33-75

As it is well known, the specificity of Integral Assessment Models-IAM consists in their capacity to analyze, first, existing interactions between economic activity –specially from energy sectors- and climate change and, second, the implications of policies useful to mitigate carbon dioxide emissions and other greenhouse emissions. IAM are optimization models such us RICE or DICE, or simulation models designed to deal with these complex interactions or to explore the consequences of parameter uncertainty. This is tge case model PAGE2002. See NORDHAUS W.D, YANG Z. (1996) A Regional Dynamic General-Equilibrium Model of Alternative Climate-Change Strategies, *The American Economic Review*, Vol.86, No.4, pp.741-765; NORDHAUS W.D. Rolling the DICE: (193) An optimal transition path for controlling greenhouse gases, Resources and Energy Economics,, 15, 27-50.NORDHAUS W.D., BOYER J.(2000) *Warming the World: Economic Models of Global Warming*, Internet edition; the book is published by MIT Press. HOPE C. (2006) The Marginal Impact f CO2 from PAGE2002: An Integrated Assessment Model Incorporating the IPCC´s Five Reasons for Concern, *The Integrated Asessment Journal*, Vol.6, Iss.1, .As a matter of fact, Integrated Assessment Models refer to policies.

[26] See FACCHINEI F., PANG J-S. (2003) Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer-Verlag, New York.

NAGURNEY´s geometric interpretation of inequality (5.1.1) states that a point $x$, belonging to a set $K$ is a solution of $VI(F,K)$ if and only if $F(x)$ forms a non-obtuse angle with vectors $(y-x)$ for any $y$ belonging to $K$[27].

* * *

A simple VI exemple is the relationship between price and excess supply in a simple partial equilibrium model:

$$E(p^*)(p^*-p)\geq 0 \qquad \forall p \ni [p_{\min}, p_{\max}]$$

where $E(p^*)$ is the excess demand function.

<u>Equation systems and VI</u>

Given the equations system $F(x)=0$ it is possible to formulate the following Proposition: let $K=\Re^n$ and let function $F:\Re^n \mapsto \Re^n$ be a given function. Vector $x^*\in\Re^n$ is a solution of $VI(F,\Re^n)$ if and only if $F(x^*)=0$.

Proof: if $F(x^*)=0$ then (5.1.1) is verified as an equality. Inversely, if $x^*$ satisfy (5.1.1) taking $x=x^*-F(x^*)$ this implies:

$$F(x^*)^T(-F(x^*)^T\geq 0 \text{ or } -\|F(x^*)\|^2\geq 0 \qquad (1.2)$$

As a result $F(x^*)=0$ and $x^*$ solves the equations systems.

<u>Relationship between VI and optimization problems</u>

Optimization problems with constraints can be formulated as VI problems. Let us consider next problem:

$$\min f(x) \qquad (5.2)$$

---

[27] Using a different notation, the problem can be formulated as find vectors $x^*\in K\subset\Re^n$ such that:

$$\langle F(x^*)^T,(x-x^*)\rangle\geq 0 \qquad \forall \in K$$

$$\forall x \in K \qquad (5.3)$$

It can be proofed that if $f$ is continuous and differentiable and $K$ is a closed, convex set then $x^*$ solves next VI problem:

$$grad.f(x)^T(x - x^*) \geq 0 \qquad \forall x \in K \qquad (5.4)$$

It can also be proofed that if $f(x)$ is a convex function and $x^*$ is a solution of $VI(\nabla f, K)$ then $x^*$ is a solution of the optimization problem (5.2)-(5.3).

Variational inequalities problems are also strongly connected with equilibrium problems. We will see it immediately[28].

Relations between VI and Mathematical Problems with Equilibrium Constraints-MPEC

As already emphasized there are strong relationships between VI and the other methodologies. Let us consider too the case of Mathematical Problems with Equilibrium Constraints-MPEC[29]. MPEC problems are optimization problems with two sets of variables $x \in \Re^n$ and $y \in \Re^m$ in which some or all constraints can be defined as variational inequalities. Vector $y$ is called the primary vector and vector $x$ is the parameter vector.

Let us take next two functions $f : \Re^{n+m} \rightarrow \Re$ and $F : \Re^{n+m} \rightarrow \Re^m$. Set $Z \subset R^{n+m}$ is non-empty and closed. For every $x \in \Re^n$, $C : \Re^n \rightarrow \Re^m$ is a set-valued map such that $C(x)$ is a closed convex set in $\Re^m$.

A MPEC takes the form:

$$\min_{x,y} f(x,y) \qquad (5.5)$$

$$(x,y) \in Z \qquad (5.6)$$

$$y \in S(x) \qquad (5.7)$$

---

[28] See KONNOV I.(2007) Equilibrium Models and Variational Inequalities, Elsevier Science, Amsterdam, The Netherlands. Existing relations between VI and Linear and Nonlinear Complementarirty will be analyzed in Section [7.1]

[29] LUO Z-Q, PANG J-S, RALPH D.(1996) Mathematical Programs with Equilibrium Constraints, Cambridge University Press, New York

where for each $x \in X$, $S(x)$ is the solution set of a VI problem defined by the pair $\langle F(x,y), C(x) \rangle$.

"Equilibrium constraints" makes reference to set $y \in S(x)$ and refers to the fact that we are interested in the cases in which these relations express equilibrium conditions modelled as variational inequalities.

5.2.-VI applications

The first VI problem was formulated by SIGNORINI in 1959. Afterwards, HARTMAN G.J.,. STAMPACCHIA G.(1966) introduced VI in mechanics. Years after, SMITH M.J.(1979) applied VI to network traffic problems, a field in which S.DAFERMOS and A.NAGURNEY [30] have been very active.

FACCHINEI F., PANG J-S (2003) identify as "source problems" those related with economic issues:

-Nash or Nash-Cournot equilibria

-oligopolistic market models of electric sector, specially when producers do not control transmission sector and sell energy to an independent operator

-general equilibrium models specially walrasian equilibrium

See also Chapter 10 of KONNOV I. (2007) [31] for additional applications. Sections [8.1] and [8.2] of this Survey include two applications of VI related with pollution emission permits markets, the first one, and with the energy system in USA, the second.

## 6.-BILEVEL PROGRAMMING

### 6.1.-Introduction to bilevel programming [32]

A linear bilevel problem is a hierarchical optimization problem. The first level or the leader´s problem is to find vector $y$ such that:

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[30] A. (1999) *Network Economics: A Variational Inequality Approach*, Kluwer Academic Pu. See also NAGURNEY A. Equilibrium modeling, analysis and computation: the contributions of Stella Dafermos (1991), *Operations Research*, 39, 9-12.

[31] KONNOV I.(2007) *Equilibrium Models and Variational Inequalities*, Elsevier Science, Amsterdam, The Netherlands

[32] In the 70s. CANDLER W., NORTON R. (1977) Multi-Level Programming and Development Policy, *Working Paper* No.258, World Bank, Washington DC (may 1977) published one of the first applications of VI in economics. See also, CANDLER W. TOWNSELY R. (1982) A Linear Two-Level Programming Problem, *Computer and Operations Research*, Vol.9, No.1, pp.59-76

$$\min_{y} .F(x,y) = c_1^T x + d_2^T y \qquad (6.1.1)$$

$$A_1 x + B_1 y \leq b_1 \qquad (6.1.2)$$

The second level or the follower problem is –given $y$ - find vector $x$ such that:

$$\min_{x} . f(x,y) = c_2^T x + d_2^T y \qquad (6.1.3)$$

$$A_2 x + B_2 y \leq b_2 \qquad (6.1.4)$$

Given the linear structure, once the leader selects $x$, $c_2^T x$ is a constant that does not play any role.

Model (6.1.1)-(6.1.4) has some similarities with Dantzig-Wolfe model (3.1)-(3.4) but there is a crucial difference: in bilevel programming the second level has it own objective function.

* * *

Let us now consider where $X \subset \Re^{n_1}, Y \subset \Re^{n_2}$, $F : X \times Y \to \Re$, $G : X \times Y \to \Re^{m_1}$, $f : Y \to \Re$, $g : X \times Y \to \Re$. The second level problem or the follower´s problem is to find vector $x$ -given $y$ - such that [33]:

$$\min_{x} f[x,y] \qquad (6.1.5)$$

$$g(x,y) \leq 0 \qquad (6.1.6)$$

Vector $y$ defines the “environment” of the second level problem and is fixed by the first level. Let $\phi(y)$ be the solution set of (6.1.5)-(6.1.6) problem.

First level problem is to find vector $y$ solving:

[33] DEMPE S (2002) *Foundations of Bilevel Programming*, Kluwer Academic Publishers, Dordrecht. See also. DEMPÉ S.(2003) An Annotated Bibliography on Bilevel Programming and Mathematical Programs with Equilibrium Constraints, *Optimization*, 52, 33-359. Sea also COLSON B., MARCOTTE P., SAVARD G.(2007) An overview of bilevel optimization, *Annals of Operations research*, Vol.153, No.1, September, and also VICENTE L.N., CALAMAI P.H. (1994) Bilevel and Multilevel Programming: A Bibliographical Review, *Journal of Global Optimization*, 5, 291-306; COLSON B.,MARCOTTE P., SVARD g. (2005) Bilevel Programming: A Survey, *4OR*, 3, 87-107

$$\min_{y} F[x(y), y] \tag{6.1.7}$$

$$G[x(y), y] \leq 0 \tag{6.1.8}$$

$$x(y) \in \phi(y) \tag{6.1.9}$$

Figure 6.1 reflects this particular structure. Given this particular structure these problems are also called “mathematical programs with optimization problems in the constraints”.

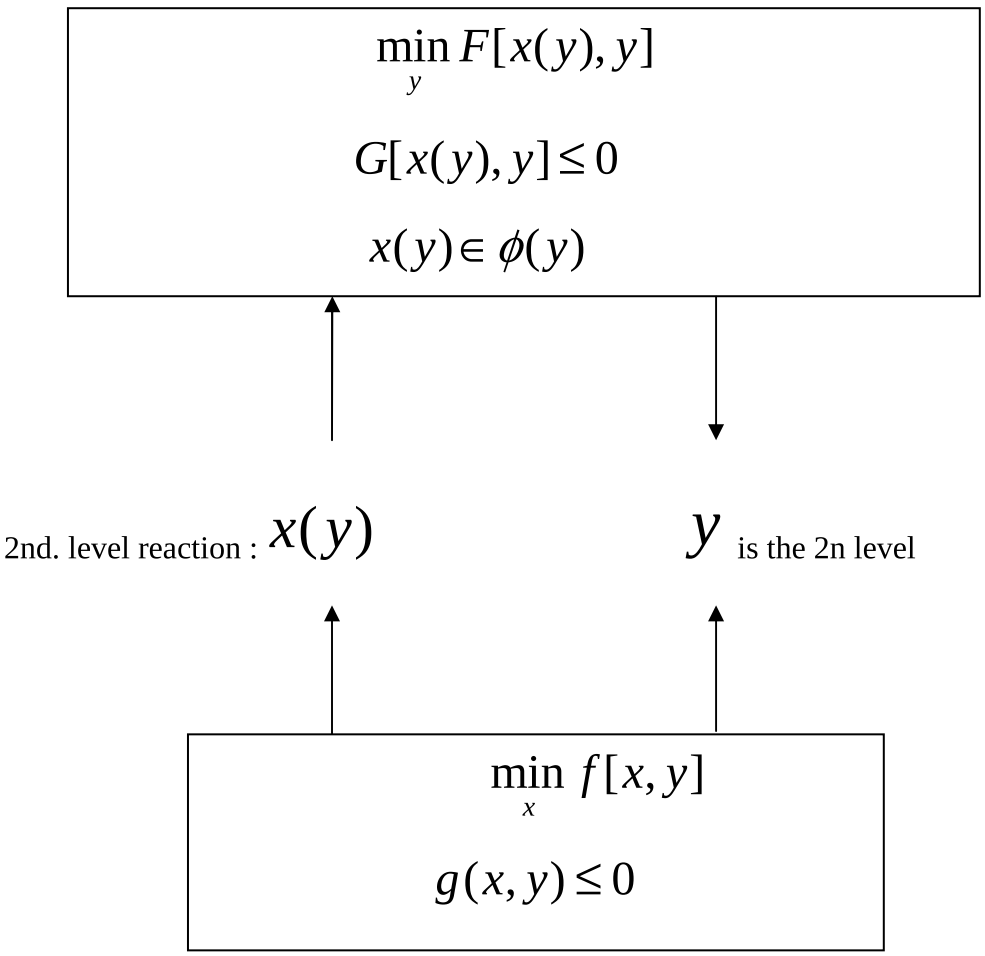


Figure 6.1

Let vector $u$ be dual variables corresponding to second level constraints (6.1.6). If regularity conditions are verified, K-K-T conditions corresponding to second level are necessary conditions for the optimum. Therefore, global model can we written in the form:

$$\min_{x,y,u} F[x(y), y] \quad (6.1.10)$$

$$G[x, y] \leq 0 \quad (6.1.11)$$

$$\nabla_x L(x, y, u) = 0 \quad (6.1.12)$$

$$u^T g(x, y) = 0 \quad (6.1.13)$$

$$u \geq 0 \quad (6.1.14)$$

If second level problem (6.1.5)-(6.1.6) is convex and has a unique solution problems bilevel problem (6.1.5)-(6.1.9) and (6.10)-(6.14) are equivalent.

Relation of BP with policies

BRINER A., AVRIEL M.(1999) [34] have emphasized, "policy analysis" consists on two interrelated problems, a) the choice of optimal policy from the point of view of the objectives, and b) the prediction of systems reaction. Bilevel models integrate both aspects.

In his 2009 "Presidential Address" to the *European Economic Association*, in Barcelona, Nicholas STERN emphasized public policy analysis has failed "to make non-marginal change central to analysis". The distinction between projects and policies can be seen in DRÈZE J., STERN, N. (1987) The Theory of Cost-Benefit Analysis [35],

Optimistic and pessimistic solutions

It is not obvious the second level problem posses a unique solution. If this is the case and -in order to simplify- we consider first level constraints does not depend on the decision of the follower, then there are two ways to face the above problem:

[34] BREINER A., AVRIEL M. (1999) Two-Stage Approach for Quantitative Policy Analysis Using Bilevel Programming, *Journal of Optimization Theory and Applications*, Vol..100, No.1, pp 15-27

[35] The distinction between projects and policies can be seen in DRÈZE J., STERN, N. (1987) The Theory of Cost-Benefit Analysis, el Vol.II de AUERBACH. A.J., FELDSTEIN M. (1987) *Handbook of Public Economics*, Vols.I and II North Holland Pu., New York, Vol.II de AUERBACH. A.J., FELDSTEIN M. (1987) *Handbook of Public Economics*, Vols. I and II North Holland Pu., New York

a) optimistic or weak version
b) the pessimistic version or strong

Optimistic version takes the form:

$$\varphi_o(y) = \min_x [F(x(y), y) : x(y) \in \psi(y)] \qquad (6.1.16)$$

where $\psi(y)$ is the solution set mapping of the leader´s problem. The leader will take this option if he anticipates the follower will support by taking among the set $x(y) \in \psi(y)$, the most convenient decisions for him, the leader. In other words, if he anticipates the follower –among their equivalent decisions- will take the one maximizing the leader´s´objective function.

Pessimistic solution is relevant when -on the contrary- cooperation with the leader is not allowed for institutional reasons or because he has risk aversion and, therefore, he wants to limit potential damages generated by follower´s decision.

Pessimistic solution takes de form:

$$\min_y [\varphi_p(y) : G(y) \leq 0] \qquad (6.17)$$

where:

$$\varphi_p(y) = \max_x [F(x(y), y) : x(y) \in \psi(y)] \qquad (6.14)$$

Generally $\varphi_p(y)$ and $\varphi_o(y)$ are discontinuous, non differentiable and non concave so that optimization is not easy.

The general model may include binary variables. See BARD J. (1998) Chapter .6 for the algorithmic difficulties rised by this aspect [36]. In this case, again, decentralization is not possible using only prices.

6.2.-Bilevel programming applications

Principal applications of BLP in economics are:

-Stackelberg games

-Cournot-Nash games

-principal-agent problems

[36] BARD J.F. (1998) *Practical Bilevel Optimization*, Kluwer Academic Pu. Dordrecht

-environmental economics.

See DEMPE S.(2002)[37]. Sections [8.3] and [8.4] in this Survey include two applications of BLP specially relevant from the point of view of our analysis and related with waste generation and taxes the fitst one, and with price support policy for biofuels the second.

## 7.-LINEAR AND NONLINEAR COMPLEMENTARITY.

### 7.1.-INTRODUCTION [38]

Let $q$ be a $n$-vector and $M$ a $nxn$ matrix. A linear complementarity problem is [39] to find vectors $z \in \Re^n$ and $w \in \Re^n$ such that :

$$w,\ z \geq 0 \qquad (7.1.1)$$

$$q + Mz = w \qquad (7.1.2)$$

$$z^T w = 0 \qquad (7.1.3)$$

LCP can be solved by using the simplex algorithm or some variant thereof. This approach makes possible to solve numerically problems with special structures that are of interest from various points of view.

The CLP was in the beginning a way of unifying mathematical linear, nonlinear programming and bimatricial games.

Quadratic programming.

[37] DEMPE S. (20029kkpñ) *Foundations of Bilevel Programming*, Kluwer Academic Publishers, Dordrecht, Cap.12. DEMPE S. (2003) An Annotated Bibliography on Bilevel Programming and Mathematical Programs with Equilibrium Constraints, *Optimization*, 52, 33-359

[38] LCP were formulated during the 40s.; the field was not consolidated until the 60s. COTTLE R.W., PANG J-S., STONE R.E.(2009) *The Linear Complemetarity Problem*, SIAM, Philadelphia.

[39] COTTLE R.W., PANG J-S., STONE R.E.(2009) *The Linear Complemetarity Problem*, SIAM, Philadelphia.

Let us consider the problem, find vector $x \in \Re^n$ such that:

$$\min_x . z = c^T x + 1/2 x^T Q x \qquad (7.1.4)$$

$$Ax \geq b \qquad (7.1.5)$$

$$x \geq 0 \qquad (7.1.6)$$

where $Q \in \Re^{nxn}$ is a symmetric matrix, $c \in R^n, A \in \Re^{nxn}$ and $b \in \Re^m$. Let us $\hat{x}$ be a local optimum of problem (7.1.4)-(7.1.6); then there is a vector of dual variables, $u$, such that the pair $x, u$ satisfies K-K-T conditions. Problem (7.1.4)-(7.1.6) is therefore equivalent to solve LCP(q, M), (7.1.1)-(7.1.3) taking:

$$q = \begin{bmatrix} c \\ -b \end{bmatrix} \qquad M = \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$$

Obviously, if matrix Q is a zero matrix, the problem is linear.

<u>Non linear complementarity and VI problems</u>

Complementarity is a particular case of VI as can be seen considering the problem find vector $x^*$ such that:

$$x^* \geq 0 \qquad (7.1.7)$$

$$f(x^*) \geq 0 \qquad (7.1.8)$$

$$(x^*)^T f(x) = 0 \qquad (7.1.9)$$

## 7.3.-<u>COMPLEMENTARITY APPLICATIONS</u> [40].

There are numerous applications of complementarity in economics. See, in particular, FERRIS, M.C., PANG J.S. (1997) survey [41] and the Cap.10 of KONNOV; see also

[40] FERRIS M.C., PANG J.S.(1997) Engineering and economic applications of complementarity problems, <u>SIAM Rev</u>. Vol.39, no. 4

[41] FERRIS M.C., PANG J.S.(1997) Engineering and economic applications of complementarity problems, <u>*SIAM* Review</u>. Vol.39, no. 4. During the 70s there were relevant applications of VI to economics: TAKAYAMA T., HASHIMOTO,H. (1984) A comparative Study of Linear Complementarity

MURTY K.G., YO F-T (1997) MURTY[42] and references included in FACHINEI F., PANG J-S.(1997) [43].

Applications outlined above can be grouped around the following themes:
-traffic and congestion;
-network design;
-walrasian general equilibrium problems;
-invariant capital stock;
-models of non-cooperative games, especially the prisoner's dilemma and
- oligopolistic markets.

Sections [8.45] and [8.6] contain two relevant applications of related with the gaz sector in USA, the first, and with the energy sector in Spain, the second.

## 8.-APPLICATION; SELECTED EXEMPLES

### 8.1.-Polution and emissions permits markets.

From the perspective of this survey is of particular interest NEGURNEY A, DHANDA KK (2000) on the environment and emissions permit markets[44], an enlarged version of NAGURNEY N., DHANDA KK (1996).

The model includes multiproduct firms producing the same products in oligopolistic markets and generating various polluting emissions. Companies operating in the permits market have a global target defined by the governement [45]. The model takes the form of a VI system.

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Models and Linear Programming Models in Multiregional Investment Analysis, *World Bank*, Division Working Paper No. 1984-1 and HANSEN T., MANNE A.(1974) Equilibrium and Linear Complementarity. An Economy with Institutional Constraint on Prices, World Bank are two good exemples.

[42] Cap.10 KONNOV I.(2007) *Equilibrium Models and Variational Inequalities*, Elsevier Science, Amsterdam, The Netherlands.. MURTY K.G., YO F-T (1997) *Linear complementarity, linear and nonlinear complementarity*. Internet Edition

[43] See Section [1.4] in FACCHINEI F., PANG J-S. (2003) *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York

[44] NAGURNEY N.,DHANDA K.K.(1996) A variational inequality approach for marketable pollution permits, *Computational Economics*, Vol.4. No.4 (363-384); NAGOURNEY A., DHANDA K.K. (2000) Marketable pollution permits in oligopolistic markets with transaction costs, *Operations Research*, 48, 3, 424

[45] Vease MONTGOMERY W.D. (1972) Markets in licenses and efficient pollution control programs, *Journal of Economic Theory*, 5, 747-756; STAVINS R.N. (1995) Transaction costs and tradeable permits, *Journal of Environmental Economics*, 29, 133-148.

Companies receive an initial permits allocation and they can participate in permits transactions; he market is supposed to be perfectly competitive. This is a mechanism inducing firms to internalize the externalities generated by emissions but leaving each company the decision on how best to respond to the price of permits set at the market.

Oligopolistic firms maximize their profits taking into account production costs and abatement costs associated to emissions reduction and also to permit prices in the market. Equilibrium is a non-cooperative Nash-Cournot game [46].

K-K-T conditions associated to companies optimization problem, taken together with conditions expressing market equilibria in markets -including in emission permits markets- is a VI problem. Finally, the article includes numerical examples of the model, analyzes its qualitative properties of the same and also presents an adapted algorithm.

8.2.- Energy policy models for the United States.

After the oil embargo, the USA administration developed Project Independence Evaluation System-PIES in order to represent the energy systems of the country and to assess policies adapted to different scenarios [47]. The model included production, processing, conversion, distribution, transportation and consumption activities; its main inconvenient is its static nature and its very limited analysis of impacts on the environment.

Further development resulted in the National Energy Modeling System-NEMS for the period 1990-2020. See EIA (2009) [48]. NEMS solved principal PIES shortcomings. The model is articulated on various regionalized models. NEMS includes, specifically:

-demand module for residential, commercial, industrial and transportation demands considered in terms of nonlinear functions;

-supply module, expressing supply curves for different types of fuels: oil, gas, coal and renewable energies;
-conversion / transmission model associated with the power sector and refineries.

This module is formed by linear programs whose objective include market prices that are also included in demand and supply modules;

-finally, the model includes a macroeconomic module and an international one. The first one connects NEMS with the rest of the economy, generating key economic projections

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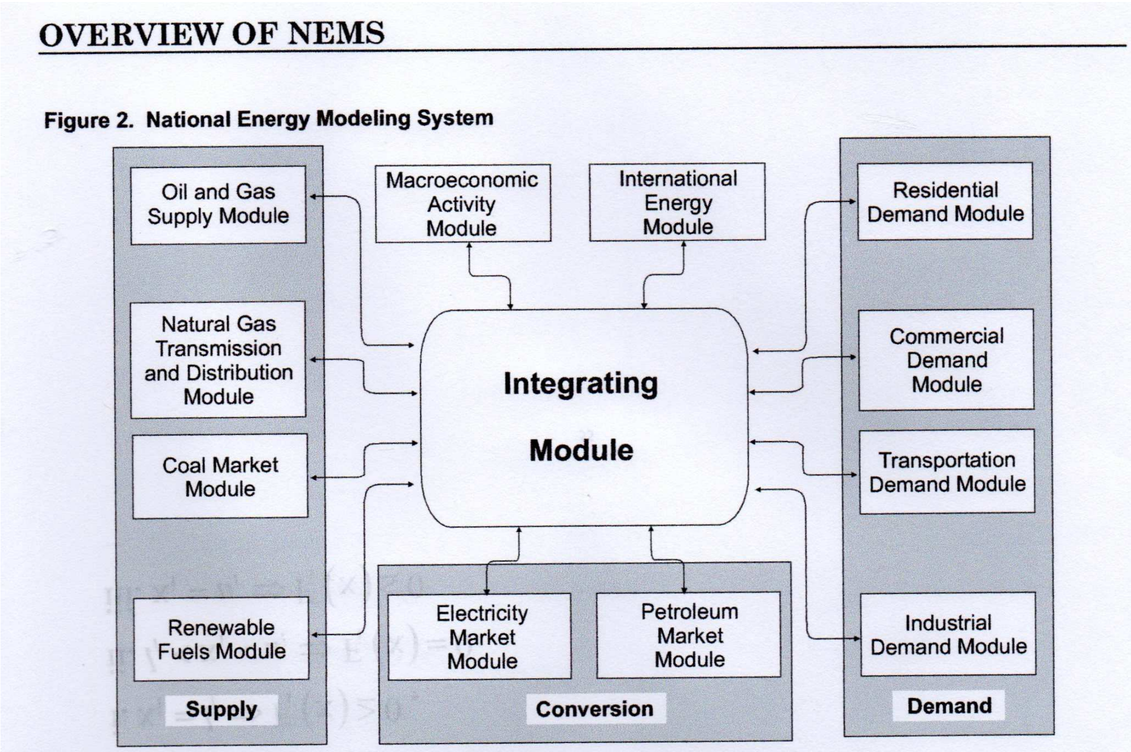
[46] TIROLE J.(1989) *The Theory of Industrial Organization*, The MIT Press, Cambridge USA

[47] See OGAN W. (1975) Energy Policy Models for Project Independence, *Computers & Operations Research*, Vol.2, pp.251-271 and AHN,B-H (1979) *Computation of Market Equilibria for Policy Analysis*, Garland Publishing, Inc. New York & London

[48] See www.eia.doe.gov, specially EIA (2009) The National Energy Modelling System, EIA, Washington. Recommendations for the design and development of NEMS contained in the report *The National Energy Modeling System*, National Academy Press, are very interesting.

that determine the supply and demand of energy derived from the various assumptions of potential growth of the economy. The international module provides supply curves or import prices of various fuels.

These modules are interconnected by the “interaction module” that plays a central role in the iterative numerical algorithm that solves supply and demand modules to achieve a balance between price and quantity among different production and demand sectors. See Figure 8.1. The algorithm used is the Gauss-Seidel method simulating a walrasian auctioneer in the role of determining the equilibrium prices.




Source: www.eia.doe.gov.

Figure 8.1

NEMS uses the fact that many constraints for the model takes the form of NCP / VI and use dual variables as prices in another module. The overall pattern can be solved by combining equations demand and K-K-T conditions of the conversion modules / transmission.

The model can be used to analyze -among others topics- issues such as sector reactions to policies to mitigate carbon dioxide emissions such as taxes or the establishment of emission permit markets or changes in the conditions of world oil or natural gas markets. Periodical reports are produced at the request of political institutions such as the White House or Congress.

GABRIEL,S.,KYDES A.,WHITMAN,P.(2001) contains the reformulation of NEMS in terms of NLC/VI [49]; their approach is better adapted for a simultaneous, non sequential approach.

8.3.-Waste generation and taxes

The application of bilevel programming contained in AMOUZEGAR MA, K. MOSHIRVAZIRI (1999) is particularly interesting from the standpoint of this Survey. The model also includes binary variables [50].

The problem is to decide the capacity and location of treatment plants for hazardous waste in California. The first approach is based on the conventional approach, based on an mixed variable, integrated model of a single-level, minimizing the total cost of the system. This approach does not take into the crucial consideration that companies have their own objective function.

The second model is based on a bilevel approach in which the Central Authority-CA may introduce taxes that encourage businesses to reduce waste generation. The objective function of the CA is to minimize the total cost and the second level prolem is a linear program with continuous and integer in which some of the objective function coefficients of the second level is determined by the first level, in particular using taxes.

It will be noted the aggregate of all enterprises constitutes the second level: this is one of the limits of this model from the point of view of considering it as a tool to explore the implications of the policy decided by the CA.

Another notable application is contained in DEMPE S., KALASHNINOV.V, RIOS-MERCADO R. (2005) [51] concerning the gas sector in USA.

8.4.- Price support policy for bifuels

[49] GABRIEL,S.,KYDES A.,WHITMAN,P.(2001) The National Energy Modelling System: A Large-Scale Energy-Economic Equilibrium Model, *Operations Research* Vol.4, No.1, january-february 2001,pp 14-25.

[50] AMOUZEGAR M.A., MOSHIRVAZIRI K. (1999) Determining optimal control policies: An application of bilevel programming, *European Journal of Operational Research*, 119, pp.100-120

[51] DEMPE S.,KALASHNIKOV V. RÍOS-MERCADO R. (2005) [51] Discrete Bilevel Programming: Application to a Natural Gas Cash-Out Problem, *European Journal of Operational Research*, 16, 2. DEMPE S. (2002) *Foundations of Bilevel Programming*, Kluwer Academic Publishers, Dordrecht, Cap.12,; DEMPE S. (2003) An Annotated Bibliography on Bilevel Programming and Mathematical Programs with Equilibrium Constraints, *Optimization*, 52, 33-359

[51] BARD J.F. (1998) *Practical Bilevel Optimization*, Kluwer Academic Pu. Dordrecht

The purpose of these application of bilevel programming presented in DEMPE (2002) [52] is to reduce pollution caused by conventional fuels through a policy of encouraging the production and use of biofuels. In order to get this objective government will reduce the price of non-food agricultural products used by the petrochemical industry producing biofuels.

The main instrument used by the government for this purpose are tax credits to reduce the price paid for the petrochemical industry for non-food agricultural products and, simultaneously, to devote a minimum area for such production. The government determines the new price minimizing total value of tax credits. Therefore, there is a conflict between the government and farmers while the industry is neutral.

In this context of industry neutrality, the price paid by industry to farmers for non-food agricultural products must not exceed the sum of tax credits received by the industry per unit of biofuel plus biofuel market price plus market price of byproducts minus the cost of converting one unit of non-food agriculture product into biofuels and the expected profit per unit corresponding to the biofuel industry.

Farmers seek to maximize their profits under the new conditions and their key decisions is related to their production of food and nonfood products to produce biofuels and the maintenance of land fallow, according to the policy of the European Union. Farmers are also subject to other constraints on land availability, generated by agronomic criteria or reflecting EU policies that lead to different subsidies.

This is a bilevel programming problem. The government is the first level and farmers are the second level. There is a common variable: the price paid by industry to farmers for non-food agriculture products used in the production of biofuels by the industry.

8.5.-A complementarity model of natural gaz markets

The model of the gas industry in the USA is an application of linear complementarity including only continuous variables; designed for a three years time horizon cannot deal for increases in capacity [53].

The model has a regional structure and includes a network of pipelines, defined by directed arcs, connecting regions. Demand differentiate three seasons. Operator groups considered are: a) pipelines operators; b) production operators managing exploration and gas production; c) marketers selling to residential, commercial, industrial and electrical sectors; d) storage operators, e) peak demand operators and finally f) consumers. All the agents operate in competitive environments except marketers.

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[52] Chapter 12 of DEMPE S.(2002) *Foundations of Bilevel Programming*, Kluwer Academic Publishers, Dordrecht.

[53] See GABRIEL S. , KIET S., ZHUANG J. (2005) A Mixed Complementarity-Based Equilibrium Model of Natural Gas Markets, *Operations Research*. Vol.53, September-october, pp.799-818. A Mixed LCP includes also free variables and equality constraints. See KONNOV I.V.. VOLOTSKAYA E.O. (2002) Mixed Variational Inequalities and Economic Equilibrium, *Journal of Applied Mathematics*, 2:6, 289-31

Pipelines belong to a regulated market with only one company. Producer and consumer regions are at the ends of arcs of the pipeline. Production operators are located at the network nodes and manage exploration and production in competitive markets. Conditions exist for clearing at each node.

K-K-T conditions and market clearing conditions define each operator. Global model is a mixed nonlinear complementarity problem. The authors analyze existence and uniqueness of solutions and explore some numerical results. On this basis the authors have developed the model applied [54].

8.6.-The European Emissions Trading Directive and the Spanish Electricity Sector

An application of complementarity approach to the Spanish electrical sector is contained in LINARES P. et al. (2006) [55]. The application models the sector taking into account that companies react to an aggregate demand curve and are also subject to a constraint expressing demand of emission permits as a function of permits price. Firms maximize their profits so that there is no single objective function but as many as companies in the sector[56].

Income of companies depends on the price of electricity and also on transactions in the emissions permits markets. The authors formalize the model taking into account the oligopolistic behavior of companies.

The global model is:

$$\underset{q_1 p^e p^f}{Max}\; w_1 = \pi_1(q_1, p^e, p^p) \ldots \; \underset{q_F p^e p^f}{Max}\; w_F = \pi_F(q_F, p^e, p^p) \quad (8.6.1)$$

$$h_1(q_1) \le 0 \qquad \ldots \qquad h_F(q_F) \le 0 \qquad (8.6.2)$$

$$q_1 \ge 0 \qquad \ldots \qquad q_F \ge 0 \qquad (8.6.3)$$

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[54] The model applied to the natural gas sector in USA is published in GABRIEL S., KIET S., ZHUANG J. (2005) A large-scale linear complementarity mode of the North American natural gas market, *Energy Economics*, 27, 639-665

[55] LINARES P., SANTOS F.J., VENTOSA M., LAPIEDRA L.(2006) Impacts of the European Emissions Trading Scheme Directive and Permit Assignment Methods on the Spanish Electricity Sector, *Energy Journal*, Vol.21, No.1

[56] The model takes into account that European emissions market has not specific emissions constraints for the electrical sector. The presentation in this Survey does not include this special feature.

Electricity demand function is:

$$p^e = \overline{p}^e - \alpha^e \sum_f^F q_f^e \qquad (8.6.4)$$

and permit demand function is:

$$p^p = \overline{p}^p - \alpha^p \sum_f^F q_f^p \qquad (8.6.5)$$

Constraints (8.4)-(8.5) plus K-K-T conditions corresponding to optimization problems (8.6.1)-(8.6.3) constitute a non linear complementarity problem.

As already mentioned, in this model companies have their own objective function and his behavior is subject to their global effects at the sector level. This capacity to incorporate constraints common to the various agents is very powerful. .

Dual variables in the model have the conventional mathematical and economic interpretation so that they can be used to evaluate marginal changes.

9.-CONCLUSIONS

Until the 80's, the methods discussed in this Survey tended to be considered as planning procedures, centralized or not; this was especially true in the case of Dantzig-Wolfe and Benders algorithms. This trend was most likely due to narrow proximity and strong interactions -existing during this initial period - between the communities of researchers belonging to the fields of mathematical programming and of economic theory.

As it is well known, during the initial period mathematical programming was strongly intertwined with economic theory, as has been pointed out by SCARF H. (1990) [57] In this sense, it is well known the relevant role played in the origins of mathematical programming not only DANTZIG but also other researchers such as ABRAMOVITZ, ARROW, CHENERY, HURWICZ, KOOPMANS, SAMUELSON and UZAWA among others [58]. This close connection no longer exists today: both fields seem to be developed by two different scientific communities.

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[57] SCARF H.E.(1990) Mathematical Programming and Economic Theory, *Operations Research*, vol 38, No.3, may-june

[58] See ABRAMOVITZ M. et al. (1959) The allocation ef economic resoouces, Stanford University Press, Stanford USA. ARROW,K. "Optimization, Decentralization and Internal Pricing in Business Firms, in *Contributions to Scientific Research in Management*, UCLA, Western Data Processing Center, 1959. ARROW K. (1987) Oral History I: an Interview, in FEIWEL G.R.(1987) *Arrow and the Ascent of Modern Economic Theory*. Macmillan, London. ARROW K., HURWICZ L. (1960) *Decentralization and Computation in Resource Allocation, in Essays in Economic and Econometrics*, edited by PFOUTS R.W., The University of North Carolina Press, Chapel Hill. CHENERY H.B. (1959) The Interdependence of Investment Decisions. ABRAMOVIYZ M. et al. (1959). KOOPMANS T.C. (1951).*Activity analysis of production and allocation*, Lohn Wiley & Sons, New York USA. SAMUELSON P.A. (1949) Market

It should be noted that a relevant limitation of methodologies of the first period is that the corresponding optimization techniques can treat, basically, with problems characterized by a single objective function or using it in order to generate partial equilibrium prices. New methodologies open different possibilities related to: a) market equilibrium, b) the existence of several agents operating at the same level or in a hierarchical framework, c) the incorporation of the reactions of an agent in the second level with its own objective function, and d) the inclusion of constraints common to several agents.

The conventional approach to constrained optimization continues to be relevant to many fields of application [59]. Special algorithms of the first stage retain their relevance but basically play a role as computation procedures efficient for large problems, without focusing on the economic interpretation of the process: Dantzig-Wolfe Benders algorithms are still relevant from this perspective [60].

Annex 1.- AN NUMERICAL EXEMPLE OF SEPARABILITY

Let us consider the problem:

$$Min. z = x_1^2 + x_2^2 - 6x_1 - 3x_2$$

$$x_1^2 + x_2^2 \leq 4$$
$$x_1 - x_2 \leq 0$$

The optimum is $(\sqrt{2}, \sqrt{2})$ and $\hat{u}_1 = 3/2$.

According to Chapter [2.] if we want to suppress the second constraint the new modified objective function will be:

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Mechanism and Maximization, Rand Corporation, P-69. UZAWA H. (1960) Market Mechanisms and Mathematical Programming, *Econometrica*, Vol.28, 4 october, in ABRAMOVIYZ M. et al..(1959).

[59] See the applications included in KALSER H.K., MESSER K.D. (2012) *Mathematical Programming for Agricultural, Environmental and Resource Economics*, J.Wiley or related with environment in GREENBERG H.J.(1995) Mathematical Programming Models for Environmental Quality Control, *Operations Research*, Vol.43, No.4, july-august. See also the applications to energy, telecommunications, transportation, water reservoirs, air pollution and agriculture in PARDALOS P.M, RESENDE , G.C. ed.(2002) *Handbook of Applied optimization*, Oxford University Press, Oxford UK

[60] DAVID FULLER J., CHUNG W. (2008) Benders decomposition for a class of a variational inequalities, *European Journal of Operational Research*, 185 (2008). O bien FULLER J.D., CHUNG W.(2005) Dantzig-Wolfe Decomposition of Variational Inequalities, *Computational Economics*, Volume 25, Number 4

$$Min.z = x_1^2 + x_2^2 - 6x_1 - 3x_2 - \hat{u}_1(x_1 - x_2)$$

$$x_1^2 + x_2^2 \leq 4$$

Or in numerical terms:

$$Min.z = x_1^2 + x_2^2 - (15/2)x_1 - (3/2)x_2$$

$$x_1^2 + x_2^2 \leq 4$$

The optimum is again $(\sqrt{2}, \sqrt{2})$.

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Annex 2.- PROGRAMS OF PROJECTS AND BUDGET CONSTRAINTS

The analysis of budget constraints requires binary variables in order to specify existing alternatives and their impact on budgetary constraints. PEARCE D., ATKINSON G., MOURATO S. (2006)[61] contains an heuristic discussion of this topic; however, they don't emphasize the need to apply a formal combinatorial approach. Let us consider two examples consider, the first one with three non-exclusive projects with a budget constraint equal to 100. Choice criteria is Net Present Value [62].

Table 1

| Project | Investment | NPV |
|---|---|---|
| 1 | 100 | 100 |
| 2 | 50 | 60 |
| 3 | 50 | 70 |

If we classify projects according to selection criteria, the decreasing order of projects will be 1-3-2 and if the choice of project 1 is made, total budget is used and NPV will

[61] PEARCE D., ATKINSON G., MOURATO S.(2006) *Cost-Benefit Analysis and the Environment*, OECD, Paris

[62] PEARCE D., ATKINSON G., MOURATO S. (2006) Table 4.1, p.69

be 100. On the contrary, by selecting projects 2 and 3, budget will we exhausted but now NPV would be 130 with a total NPV equal to 130. The selection process can not be sequential

Selection process cannot be sequential. Next, let us consider another example:

| Project | Cost-C | Gross profits P | NPV | ratio P/C | ratio NPV/C |
|---|---|---|---|---|---|
| 1 | 100 | 200 | 100 | 2.0 | 1.0 |
| 2 | 50 | 110 | 60 | 2.2 | 1.2 |
| 3 | 60 | 120 | 70 | 2.0 | 1.17 |

Let us suppose budget constraint is equal to 115. Choosing first project 1 there is no room for any other project so that total Net Value is equal to 100. Conversely, if chosen projects are 2 and 3, Net Value is equal to 130.

This kind of problems can be formulated in terms of binary programs. Let $I_i$ be the investment of project $i$, $R_i$ its profitability, and $P$ the total budget. Problem formulation is:

$$Max.w = \sum_{i=1}^{3} R_i X_i$$

$$\sum_{i=1}^{3} I_i X_i \leq P$$

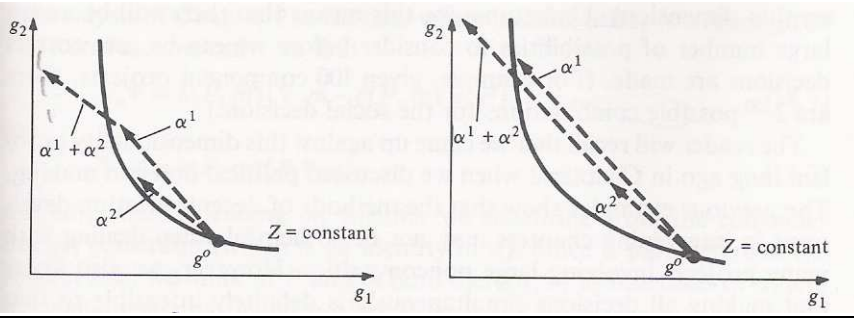
$$X_i binarias$$

This is one example of. so called "knapsack problem" well known in the field of Operations Research. [63].The already indicated possibility to incorporate binary variables in bielevel programs[64] is an open way to deal with budget constraints.

[63] PLANE D.R., McMILLAN jr C. (1971) *Discrete Optimization*, Prentice-Hall, Englewood Cliffs. See also WEINGARTNER H.M. (1966) Capital Budgeting of Interrelated Projects: Survey and Synthesis, *Management Science*, No.7, pp.486-516 and by the same author (1967) *Mathematical programming and the analysis of capital budgeting problems* , Markham Pu.Co

## Relevance of evaluation order

Evaluation results are not independent of the order in which projects are considered. STARRET D.A. (1988) [65] formulated a simple graphic example with two major projects in which the decision depends on the order in which projects.




Case 1 Case 2

Figure A.2.1

Figure A.2.1 shows a case where two non marginal projects, $a_1$ and $a_2$, are evaluated. When they are evaluated independently –Case 1- both are acceptable but became unacceptable when they are taken together. Acceptance may depend on the order in which they are evaluated, as can be seen in Case 2: if $a_1$ is first evaluated, then $a_2$ should be rejected and vice versa.

## Annex 3. LINEAR COMPLEMENTARITY: A NUMERICAL EXEMPLE

Let us consider next problem:

$$\min w = -10x_1 + 40x_2 + 20x_3$$

[64] Chapter 6, BARD J.F. (1998) *Practical Bilevel Optimization*, Kluwer Academic Pu. Dordrecht

[65] STARRET D.A. (1988) *Foundations of public economics*, Cambridge University Press. 234-236.

$$18x_1 - x_2 + 3x_3 - y_1 = 20$$

$$-3x_1 + 2x_2 - 13x_3 - y_2 = 30$$

$$x_1, x_2, x_3 y_1, y_2 \geq 0$$

K-K-T conditions are:

$$18u_1 - 3u_2 + v_1 = -10$$

$$-u_1 + 2u_2 + v_2 = 40$$

$$3u_1 - 13u_2 + v_3 = 20$$

$$x_j, u_j, y_i, v_i \geq 0 \quad \forall i, \forall j$$

$$u_i y_i = x_j v_j = 0 \quad \forall i, \forall j$$

Therefore, the problem is equivalent to find $w$ and $z$ such that:

$$w,\ z \geq 0$$

$$q + Mz = w$$

$$z^T w = 0$$

with:

$$q = \begin{bmatrix} c \\ -b \end{bmatrix}$$

$$M = \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$$

being:

$$M = \begin{bmatrix} 0 & 0 & 0 & -18 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 & 13 \\ 18 & -1 & 3 & 0 & 0 \\ -3 & 2 & -13 & 0 & 0 \end{bmatrix} \qquad q = \begin{bmatrix} -10 \\ 40 \\ 20 \\ -16 \\ 12 \end{bmatrix} \qquad z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ u_1 \\ u_2 \end{bmatrix}$$

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