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# On ratio and product methods with certain known population parameters of auxiliary variable in sample surveys

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**Abstract**

This paper proposes two ratio and product-type estimators using transformation based on known minimum and maximum values of auxiliary variable. The biases and mean squared errors of the suggested estimators are obtained under large sample approximation. Conditions are obtained under which the suggested estimators are superior to the conventional unbiased estimator, usual ratio and product estimators of population mean. The superiority of the proposed estimators are also established through some natural population data sets.

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*Keywords:* Study variate, auxiliary variate bias, mean squared error, simple random sampling without replacement

## 1. Introduction

The use of supplementary information on an auxiliary variable for estimating the finite population mean of the variable under study has played an eminent role in sampling theory and practices. Out of many ratio, product and regression methods of estimation are good illustrations in this context. When the correlation between the study variable $y$ and the auxiliary variable $x$ is positive (high), the ratio method of estimation is employed. On the other hand if this correlation is negative (high), the product method of estimation investigated by Robson (1957) and Murthy (1964), is quite effective.

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It is a well-established fact that the ratio estimator is most effective when the relation between $y$ and $x$ is straight line through the origin and the variance of $y$ about this line is proportional to $x$, for instance, see Cochran (1963). In many practical situations, the regression line does not pass through the origin. Also due to stronger intuitive appeal survey statisticians are more inclined towards the use of ratio and product estimators. Keeping these facts in mind several authors including Srivastava (1967, 1983), Reddy (1973,74), Walsh (1970), Gupta (1978), Vos (1980), Naik and Gupta (1991), Mohanty and Sahoo (1995), Sahai and Sahai (1985), Upadhyaya and Singh (1999), Srivenkataramana (1980), Bandyopadhyaya (1980), Mohanty and Das (1971), Srivenkataramana (1978), Sisodia and Dwivedi (1981) and Singh (2003) have suggested various modifications in ratio and product estimators.

Suppose we have population of $N$ identifiable units on which the two variates $y$ and $x$ are defined. For estimating the population mean $\overline{Y} = \sum_{i=1}^{N} y_i \Big/ N$ of the study variate $y$, a simple random sample of size $n$ is drawn without replacement. It is assumed that the population mean $\overline{X} = \sum_{i=1}^{N} x_i \Big/ N$ of the auxiliary variate $x$ is known. Then the classical ratio and product estimators of population mean $\overline{Y}$ are respectively defined by

$$\bar{y}_R = \bar{y}\left(\overline{X}/\bar{x}\right) \tag{1.1}$$

and

$$\bar{y}_p = \bar{y}\left(\bar{x}/\overline{X}\right) \tag{1.2}$$

where $\bar{y} = \sum_{i=1}^{n} y_i/n$ and $\bar{x} = \sum_{i=1}^{n} x_i/n$ are the sample means of variates $y$ and $x$ respectively.

Let $x_m$ and $x_M$ be the minimum and maximum values of a known positive variate $x$ respectively. Using these values (i.e. $x_m$ and $x_M$), Mohanty and Sahoo (1995) suggested to transform auxiliary variable $x$ to new variables $z$ and $u$ such that

$$z_i = \frac{x_i + x_m}{x_M + x_m} \tag{1.3}$$

and

$$u_i = \frac{x_i + x_M}{x_M + x_m}, \quad i = 1, 2, \ldots, N. \tag{1.4}$$

Using these transformed variables $z$ and $u$, Mohanty and Sahoo (1995) proposed the following ratio estimators for population mean $\overline{Y}$ as

$$t_{1R} = \bar{y}\left(\overline{Z}/\bar{z}\right) \tag{1.5}$$

and

$$t_{2R} = \bar{y}\,(\overline{U}/\bar{u}), \tag{1.6}$$

where

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i + x_m}{x_M + x_m}\right) = \left(\frac{\bar{x} + x_m}{x_M + x_m}\right) \quad \text{and} \quad \bar{u} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i + x_M}{x_M + x_m}\right) = \left(\frac{\bar{x} + x_M}{x_M + x_m}\right)$$

are sample means of $z$ and $u$ respectively, and

$$\overline{Z} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i + x_m}{x_M + x_m}\right) = \left(\frac{\overline{X} + x_m}{x_M + x_m}\right) \quad \text{and} \quad \overline{U} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i + x_M}{x_M + x_m}\right) = \left(\frac{\overline{X} + x_M}{x_M + x_m}\right)$$

are the population means of $z$ and $u$ respectively.

When the correlation between $y$ and $x$ is negative, the product estimator based on transformed variables $z$ and $u$ are defined by

$$t_{1p} = \bar{y}\,(\bar{z}/\overline{Z}) \tag{1.7}$$

and

$$t_{2p} = \bar{y}\,(\bar{u}/\overline{U}) \tag{1.8}$$

It is well known under simple random sampling without replacement (SRSWOR) that the mean squared error (or variance) of $\bar{y}$ is

$$\text{MSE}(\bar{y}) = Var(\bar{y}) = \theta\, S_y^2 = \theta\, \overline{Y}^2 C_y^2 \tag{1.9}$$

where $\theta = (N-n)/(nN)$ ,$C_y = \dfrac{S_y}{\overline{Y}}$: the coefficient of variation of the study variate $y$.

To the first degree of approximation, the biases and mean squared errors (MSEs) of the ratio-type estimators $\bar{y}_R$, $t_{1R}$, and $t_{2R}$, and product-type estimators $\bar{y}_p$, $t_{1p}$ and $t_{2p}$ are respectively given by

$$B(\bar{y}_R) = \theta\, \overline{Y}\, C_x^2 (1-K) \tag{1.10}$$

$$B(t_{1R}) = \theta\, \overline{Y}\,(C_x^2/C_1)\,\{(1/C_1) - K\} \tag{1.11}$$

$$B(t_{2R}) = \theta\, \overline{Y}\,(C_x^2/C_2)\,\{(1/C_2) - K\} \tag{1.12}$$

$$B(\bar{y}_p) = \theta\, \overline{Y}\, C_x^2 K \tag{1.13}$$

$$B(t_{1p}) = \theta\,\overline{Y}\,(C_x^2/C_1)K \tag{1.14}$$

$$B(t_{2p}) = \theta\overline{Y}\;(C_x^2/C_2)\,K \tag{1.15}$$

$$\text{MSE}(\bar{y}_R) = \theta\,\overline{Y}^2[C_y^2 + C_x^2(1-2K)] \tag{1.16}$$

$$\text{MSE}(t_{1R}) = \theta\,\overline{Y}^2[C_y^2 + (C_x^2/C_1)\;\{(1/C_1) - 2K\}] \tag{1.17}$$

$$\text{MSE}(t_{2R}) = \theta\,\overline{Y}^2[C_y^2 + (C_x^2/C_2)\;\{(1/C_2) - 2K\}] \tag{1.18}$$

$$\text{MSE}(\bar{y}_p) = \theta\,\overline{Y}^2[C_y^2 + C_x^2(1+2K)] \tag{1.19}$$

$$\text{MSE}(t_{1p}) = \theta\,\overline{Y}^2[C_y^2 + (C_x^2/C_1)\,\{(1/C_1) + 2K\}] \tag{1.20}$$

$$\text{MSE}(t_{2p}) = \theta\,\overline{Y}^2[C_y^2 + (C_x^2/C_2)\,\{(1/C_2) + 2K\}] \tag{1.21}$$

where $K = \rho\, C_y/C_x$, $\rho = S_{yx}/(S_x S_y)$ is the correlation coefficient between $y$ and $x$,

$S_x^2 = \sum_{i=1}^{N} (x_i - \overline{X})^2/(N-1)$, $S_y^2 = \sum_{i=1}^{N} (y_i - \overline{Y})^2/(N-1)$, $S_{xy} = \sum_{i=1}^{N} (x_i - \overline{X})(y_i - \overline{Y})/(N-1)$,

$C_1 = \left(1 + \frac{x_m}{\overline{X}}\right), C_2 = \left(1 + \frac{x_M}{\overline{X}}\right)$ and $C_x = \frac{S_x}{\overline{X}}$: the coefficient of variation of the auxiliary variate $x$.

It is to be noted that the transformations (1.3) and (1.4) depend on both maximum $(x_M)$ and minimum $(x_m)$ values but the estimators $t_{1R}(t_{1P})$ and $t_{2R}(t_{2P})$ generated through these transformations depend only on maximum value $(x_M)$ and minimum value $(x_m)$ respectively. For instance,

$$\begin{aligned} t_{1R} &= \bar{y}\frac{\overline{Z}}{\bar{z}} \\ &= \bar{y}\frac{(\overline{X} + x_m)/(x_M + x_m)}{(\bar{x} + x_m)/(x_M + x_m)} \\ &= \bar{y}\frac{(\overline{X} + x_m)}{(\bar{x} + x_m)} \end{aligned} \tag{1.22}$$

In similar fashion it can be shown that the estimators $t_{1P}$ and $(t_{2R,} t_{2P,})$ depend only on $x_m$ and $x_M$ respectively.

Expressions (1.22) – (1.25) motivated authors to investigate some transformations which make use of both maximum value $(x_M)$ and minimum value $(x_m)$ and hence using such transformations the constructed estimators should also depend on $x_M$ and $x_m$. Some ratio- and product-type estimators of population mean $\overline{Y}$ have been suggested and their properties are studied. Numerical illustrations are given in support of the present study.

## 2. The suggested transformations and estimators

Let $x_m$ and $x_M$ be the minimum and maximum values of a known positive variate $x$ respectively. Using $x_m$ and $x_M$, it is suggested to transform the auxiliary variable $x$ to new variables '$a$' and '$b$' such that

$$a_i = x_M x_i + x_m^2 \tag{2.1}$$

and

$$b_i = (x_M - x_m)x_i + x_m^2 \qquad i = 1, 2, \ldots, N. \tag{2.2}$$

Using the transformed variates at (2.1) and (2.2) we define the following ratio-type estimators for population mean $\overline{Y}$ as

$$d_{1R} = \bar{y}\left(\frac{\overline{A}}{\bar{a}}\right) \tag{2.3}$$

$$d_{2R} = \bar{y}\left(\frac{\overline{B}}{\bar{b}}\right) \tag{2.4}$$

and the product-type estimators for $\overline{Y}$ as

$$d_{1p} = \bar{y}\left(\frac{\bar{a}}{\overline{A}}\right) \tag{2.5}$$

and

$$d_{2p} = \bar{y}\left(\frac{\bar{b}}{\overline{B}}\right) \tag{2.6}$$

where

$$\bar{a} = \sum_{i=1}^{n} a_i/n = x_M\bar{x} + x_m^2 \quad \text{and} \quad \bar{b} = \sum_{i=1}^{n} b_i/n = (x_M - x_m)\bar{x} + x_m^2$$

are the sample means of '$a$' and '$b$' respectively and

$$\overline{A}=\sum_{i=1}^{N}a_i/N=x_M\overline{X}+x_m^2 \quad \text{and} \quad \overline{B}=\sum_{i=1}^{N}b_i/N=(x_M-x_m)\overline{X}+x_m^2$$

are the population means of '$a$' and '$b$' respectively.

### 2.1. Biases and variances of ratio-type estimators $d_{1R}$ and $d_{2R}$

To obtain the biases and variances of $d_{1R}$ and $d_{2R}$, we write

$$\bar{y}=\overline{Y}(1+e_0)$$

$$\bar{x}=\overline{X}(1+e_1)$$

such that

$$E(e_0)=E(e_1)=0$$

and

$$\left.\begin{array}{l} E(e_0^2)=\theta C_y^2 \\ E(e_1^2)=\theta C_x^2 \\ E(e_0e_1)=\theta KC_x^2 \end{array}\right\} \tag{2.7}$$

Expressing $d_{1R}$and $d_{2R}$ in terms of $e$'s we have

$$\begin{aligned} d_{1R} &= \overline{Y}(1+e_0)\frac{\overline{A}}{\left\{x_M\overline{X}(1+e_1)+x_m^2\right\}} \\ &= \overline{Y}(1+e_0)\frac{\overline{A}}{\left\{x_M\overline{X}+x_m^2+x_M\overline{X}\,e_1\right\}} \\ &= \overline{Y}(1+e_0)\frac{\overline{A}}{\left\{\overline{A}+x_M\overline{X}\,e_1\right\}} \\ &= \overline{Y}(1+e_0)\left(1+\lambda_{(1)}e_1\right)^{-1} \end{aligned} \tag{2.8}$$

$$\begin{aligned} d_{2R} &= \overline{Y}(1+e_0)\frac{\overline{B}}{\left\{(x_M-x_m)\overline{X}(1+e_1)+x_m^2\right\}} \\ &= \overline{Y}(1+e_0)\frac{\overline{B}}{\left\{(x_M-x_m)\overline{X}+x_m^2+(x_M-x_m)\overline{X}\,e_1\right\}} \end{aligned}$$

$$= \overline{Y}(1+e_0)\frac{\overline{B}}{\{\overline{B}+(x_M-x_m)\overline{X}\,e_1\}}$$

$$= \overline{Y}(1+e_0)\left(1+\lambda_{(2)}\,e_1\right)^{-1} \tag{2.9}$$

where

$$\lambda_{(1)} = \frac{x_M\overline{X}}{x_M\overline{X}+x_m^2} = \frac{x_M\overline{X}}{\overline{A}} = \frac{(C_2-1)}{(C_2-1)+(C_1-1)^2} \tag{2.10}$$

and

$$\lambda_{(2)} = \frac{(x_M-x_m)\overline{X}}{(x_M-x_m)\overline{X}+x_m^2} = \frac{(x_M-x_m)\overline{X}}{\overline{B}} = \frac{(C_2-C_1)}{(C_2-C_1)+(C_1-1)^2} \tag{2.11}$$

We now assume that $\left|\lambda_{(1)}\,e_1\right| < 1$ and $\left|\lambda_{(2)}\,e_2\right| < 1$ so that we may expand $(1+\lambda_{(1)}e_1)^{-1}$ and $(1+\lambda_{(2)}e_1)^{-1}$ as a series in power of $\lambda_{(1)}e_1$ and $\lambda_{(2)}e_1$. Expanding right hand sides of (2.8) and (2.9), multiplying out and retaining terms of $e$'s to the second degree, we obtain

$$t_{1R} \cong \overline{Y}\left(1+e_0-\lambda_{(1)}e_1-\lambda_{(1)}e_1e_0+\lambda_{(1)}^2e_1^2\right)$$

or

$$(t_{1R}-\overline{Y}) = \overline{Y}\left(e_0-\lambda_{(1)}e_1-\lambda_{(1)}e_1e_0+\lambda_{(1)}^2e_1^2\right) \tag{2.12}$$

and

$$t_{2R} \cong \overline{Y}\left(1+e_0-\lambda_{(2)}e_1-\lambda_{(2)}e_1e_0+\lambda_{(2)}^2e_1^2\right)$$

or

$$(t_{2R}-\overline{Y}) = \overline{Y}\left(e_0-\lambda_{(2)}e_1-\lambda_{(2)}e_1e_0+\lambda_{(2)}^2e_1^2\right) \tag{2.13}$$

Taking expectations of both sides of (2.12) and (2.13) and using the results in (2.7) we get the biases of $d_{1R}$ and $d_{2R}$to the first degree of approximation respectively as

$$B(d_{1R}) = \theta\,\overline{Y}C_x^2\lambda_{(1)}(\lambda_{(1)}-K) \tag{2.14}$$

and

$$B(d_{2R}) = \theta\,\overline{Y}C_x^2\lambda_{(2)}(\lambda_{(2)}-K) \tag{2.15}$$

It follows from (2.14) and (2.15) that the biases $B(d_{1R})$ and $B(d_{2R})$ are negligible, if the sample size n is large enough.

Squaring both sides of (2.12) and (2.13) and retaining terms of $e$'s to the second degree we have

$$(d_{1R}-\overline{Y})^2=\overline{Y}^2\left(e_0^2+\lambda_{(1)}^2e_1^2-2\lambda_{(1)}e_0e_1\right) \tag{2.16}$$

and

$$(d_{2R}-\overline{Y})^2=\overline{Y}^2\left(e_0^2+\lambda_{(2)}^2e_1^2-2\lambda_{(2)}e_0e_1\right) \tag{2.17}$$

Taking expectation of both sides of (2.16) and (2.17) and using the results in (2.7), we get the MSEs of $d_{1R}$ and $d_{2R}$ to the first degree of approximation respectively as

$$\text{MSE}(d_{1R})=\theta\overline{Y}^2\left[C_y^2+\lambda_{(1)}C_x^2(\lambda_{(1)}-2K)\right] \tag{2.18}$$

and

$$\text{MSE}=(d_{2R})=\theta\overline{Y}^2\left[C_y^2+\lambda_{(2)}C_x^2(\lambda_{(2)}-2K)\right] \tag{2.19}$$

### 2.2. Biases and variances of product-type estimators

To obtain the biases and MSEs of $d_{1P}$ and $d_{2P}$, we express $d_{1P}$ and $d_{2P}$ in terms of $e$'s as

$$\begin{aligned} d_{1P} &= \overline{Y}(1+e_0)\frac{\left\{x_M\overline{X}(1+e_1)+x_m^2\right\}}{(x_M\overline{X}+x_m^2)} \\ &= \overline{Y}(1+e_0)\left\{1+\frac{x_M\overline{X}\,e_1}{(x_M\overline{X}+x_m^2)}\right\} \\ &= \overline{Y}(1+e_0)(1+\lambda_{(1)}e_1) \\ &= \overline{Y}(1+e_0+\lambda_{(1)}e_1+\lambda_{(1)}e_0e_1) \end{aligned}$$

or

$$(d_{1P}-\overline{Y})=\overline{Y}(e_0+\lambda_{(1)}\,e_1+\lambda_{(1)}e_0e_1) \tag{2.20}$$

$$\begin{aligned} d_{2P} &= \overline{Y}(1+e_0)\frac{\left\{(x_M-x_m)\overline{X}(1+e_1)+x_m^2\right\}}{\left\{(x_M-x_m)\overline{X}+x_m^2\right\}} \\ &= \overline{Y}(1+e_0)\left\{1+\frac{(x_M-x_m)\overline{X}\,e_1}{\left\{(x_M-x_m)\overline{X}+x_m^2\right\}}\right\} \\ &= \overline{Y}(1+e_0)(1+\lambda_{(2)}e_1) \\ &= \overline{Y}(1+e_0+\lambda_{(2)}e_1+\lambda_{(2)}e_0e_1) \end{aligned}$$

or

$$(d_{2P} - \overline{Y}) = \overline{Y}(e_0 + \lambda_{(2)}e_1 + \lambda_{(2)}e_0e_1), \tag{2.21}$$

where $\lambda_{(1)}$ and $\lambda_{(2)}$ are respectively given by (2.10) and (2.11).

Taking expectation of both sides of (2.19) and (2.20) and using the results in (2.7), we get the exact biases of $d_{1P}$ and $d_{2P}$ as

$$B(d_{1P}) = \theta\,\overline{Y}\,\lambda_{(1)}\,KC_x^2 \tag{2.22}$$

and

$$B(d_{2P}) = \theta\,\overline{Y}\,\lambda_{(2)}\,KC_x^2 \tag{2.23}$$

Squaring both sides of (2.20) and (2.21) and retaining terms of $e$'s to the second degree, and then taking expections, we get the MSEs of $d_{1P}$ and $d_{2P}$ respectively as

$$\text{MSE}(d_{1P}) = \theta\overline{Y}^2\left[C_y^2 + \lambda_{(1)}C_x^2(\lambda_{(1)} + 2K)\right] \tag{2.24}$$

and

$$\text{MSE}(d_{2P}) = \theta\overline{Y}^2\left[C_y^2 + \lambda_{(2)}C_x^2(\lambda_{(2)} + 2K)\right] \tag{2.25}$$

## 3. Comparison of biases

The absolute relative bias (ARB) of an estimator $t$ of the population mean $\overline{Y}$ is defined by

$$\text{ARB}(t) = \left|\frac{B(t)}{\overline{Y}}\right| \tag{3.1}$$

where $B(t)$ stands for bias of the estimator $t$.

The comparison of absolute relative biases of ratio-type and product-type estimators have been made and the conditions are displayed in Tables 3.1 and 3.2 respectively.

***Table 3.1:*** *Comparison of absolute relative biases of ratio-type estimators.*

| Estimator | Absolute Relative Bias of | |
|---|---|---|
| | $d_{1R}$ is less than | $d_{2R}$ is than |
| $\bar{y}_R$ | if<br>either $K > \left(1+\lambda_{(1)}\right)$<br>or $K < \frac{(1+\lambda_{(1)}^2)}{(1+\lambda_{(1)})}$ | if<br>either $K > \left(1+\lambda_{(2)}\right)$<br>or $K < \frac{(1+\lambda_{(2)}^2)}{(1+\lambda_{(2)})}$ |
| $t_{1R}$ | if<br>$\frac{(1+\lambda_{(1)}^2 C_1^2)}{C_1(1+\lambda_{(1)}C_1)} < K < \frac{(1+\lambda_{(1)}C_1)}{C_1}$ | if<br>either $\frac{(1+\lambda_{(2)}^2 C_1^2)}{C_1(1+\lambda_{(2)}C_1)} < K < \frac{(1+\lambda_{(2)}C_1)}{C_1}$,<br>$C_1 < \frac{1}{2}(1+C_2)$<br>or $K < \frac{(1+\lambda_{(2)}^2 C_1^2)}{C_1(1+\lambda_{(2)}^2 C_1)}$, $C_1 < \frac{1}{2}(1+C_2)$<br>or $K > \frac{(1+\lambda_{(2)} C_1)}{C_1}$, $C_1 > \frac{1}{2}(1+C_2)$ |
| $t_{2R}$ | if<br>$\frac{(1+\lambda_{(1)}^2 C_2^2)}{C_2(1+\lambda_{(1)}C_2)} < K < \frac{(1+\lambda_{(1)}C_1)}{C_2}$ | if<br>either $\frac{(1+\lambda_{(2)}^2 C_2^2)}{C_2(1+C_2\lambda_{(2)})} < K < \frac{(1+\lambda_{(2)}C_2)}{C_2}$,<br>$\lambda_{(2)}C_2 > 1$<br>or $K < \frac{(1+\lambda_{(2)}^2 C_2^2)}{C_2(1+\lambda_{(2)} C_2)}$, $\lambda_{(2)}C_2 > 1$<br>or $K > \frac{(1+\lambda_{(2)}C_2)}{C_2}$, $\lambda_{(2)}C_2 < 1$ |
| $d_{2R}$ | if<br>$\frac{(\lambda_{(1)}^2+\lambda_{(2)}^2)}{(\lambda_{(1)}+\lambda_{(2)})} < K < (\lambda_{(1)}+\lambda_{(2)})$ | — |

It can be easily proved that $d_{1P}$ has smaller absolute relative bias (ARB) than the conventional product estimator $\bar{y}_p$ but larger than that of Mohanty and Sahoo's (1995) estimators $t_{1p}$ and $t_{2p}$. Table 3.2 clearly indicates that the proposed estimator $d_{2P}$ has smaller absolute relative bias than the conventional product estimator $\bar{y}_P$ as the condition $\lambda_{(2)} < 1$ always holds.

***Table 3.2:*** *Comparison of absolute relative biases of product-type estimators.*

| Estimator | Absolute Relative Bias of $d_{2P}$ is less than |
|---|---|
| $\bar{y}_P$ | if $\lambda_{(2)} < 1$ |
| $t_{1P}$ | if $\lambda_{(2)} < \frac{1}{C_1}$, $C_1 > \frac{(1+C_2)}{2}$ |
| $t_{2P}$ | if $\left\|C_1^2 + C_1\,(C_2-3) - C_2\,(C_2-1)+1\right\| > 0$ |
| $d_{1P}$ | if $\lambda_{(2)} < \lambda_{(1)}$ |

## 4. Efficiency comparison

The efficiency comparisons of ratio-type ($d_{1R}$ and $d_{2R}$) and product-type ($d_{1P}$ and $d_{2P}$) estimators have been made with $\bar{y}$, $\bar{y}_R$, $t_{1R}$ and $t_{2R}$; and shown in Tables 4.1 and 4.2 respectively.

***Table 4.1:*** *Comparison of mean squared errors of ratio-type estimators.*

| Estimator | Mean squared error of | |
|---|---|---|
| | $d_{1R}$ | $d_{2R}$ |
| $\bar{y}$ | if $K > \frac{\lambda_{(1)}}{2}$ | if $K > \frac{\lambda_{(2)}}{2}$ |
| $\bar{y}_R$ | if $K < \frac{\left(1+\lambda_{(1)}\right)}{2}$ | if $K < \frac{(1+\lambda_{(2)})}{2}$ |
| $t_{1R}$ | if $K > \frac{\left(1+\lambda_{(1)}C_1\right)}{2C_1}$ | if<br>either $K < \frac{(1+C_1\,\lambda_{(2)})}{2C_1}$, $\lambda_{(2)} < \frac{1}{C_1}$<br>or $K > \frac{(1+C_1\,\lambda_{(2)})}{2C_1}$, $\lambda_{(2)} > \frac{1}{C_1}$ |
| $t_{2R}$ | if $K > \frac{\left(1+\lambda_{(2)}C_2\right)}{2C_2}$ | if<br>either $K < \frac{(1+\lambda_{(2)}C_2)}{2C_2}$, $\lambda_{(2)} < \frac{1}{C_2}$<br>or $K > \frac{(1+\lambda_{(2)}C_2)}{2C_2}$, $\lambda_{(2)} > \frac{1}{C_2}$ |

***Table 4.2:*** *Comparison of mean squared errors of product-type estimators.*

| Estimator | Mean squared error of | |
|---|---|---|
| | $d_{1P}$ is less than | $d_{2P}$ is less than |
| $\bar{y}$ | if $K < -\frac{\lambda_{(1)}}{2}$ | if $K < -\frac{\lambda_{(2)}}{2}$ |
| $\bar{y}_P$ | if $K > -\frac{(1+\lambda_{(1)})}{2}$ | if $K > -\frac{(1+\lambda_{(2)})}{2}$ |
| $t_{1P}$ | if $K < -\frac{(1+\lambda_{(1)}C_1)}{2C_1}$ | if<br>either $K < -\frac{1}{2}\frac{(1+\lambda_{(2)}C_1)}{C_1}, \quad \lambda_{(2)} > \frac{1}{C_1}$<br>or $K > -\frac{1}{2}\frac{(1+\lambda_{(2)}C_1)}{C_1}, \quad \lambda_{(2)} < \frac{1}{C_1}$ |
| $t_{2P}$ | if $K < -\frac{(1+\lambda_{(1)}C_2)}{2C_2}$ | if<br>either $K < -\frac{1}{2}\frac{(1+\lambda_{(2)}C_2)}{C_2}, \quad \lambda_{(2)} > \frac{1}{C_2}$<br>or $K > -\frac{1}{2}\frac{(1+\lambda_{(2)}C_2)}{C_2}, \quad \lambda_{(2)} < \frac{1}{C_2}$ |

Table 4.1 exhibits that the ratio type estimator $d_{1R}$ is better than $\bar{y}$, $\bar{y}_R$, $t_{1R}$ and $t_{2R}$ if

$$\frac{\left(1+\lambda_{(1)}C_1\right)}{2C_1} < K < \frac{(1+\lambda_{(1)})}{2} \tag{4.1}$$

We also note that the estimator $d_{1R}$ is more efficient than $d_{2R}$ if

$$K > \frac{\left(\lambda_{(1)}+\lambda_{(2)}\right)}{2} \tag{4.2}$$

It is observed from Table 4.1 that the product-type estimator $d_{1P}$ is more efficient than $\bar{y}$, $\bar{y}_P$, $t_{1P}$ and $t_{2P}$ if

$$-\frac{(1+\lambda_{(1)})}{2} < K < -\frac{(1+\lambda_{(1)}C_1)}{2C_1} \tag{4.3}$$

Further it can be proved that the product-type estimator $d_{1P}$ is better than the product-type estimator $d_{2P}$ if

$$K < -\frac{\left(\lambda_{(1)}+\lambda_{(2)}\right)}{2} \tag{4.4}$$

## 5. Unbiased versions of the suggested estimators

In this section we will obtain the unbiased versions of the suggested estimators in Section 2, using two well known procedures: (i) Interpenetrating subsamples design and (ii) Jack-knife technique.

### *5.1. Interpenetrating sub-sample design*

Let the sample in the form of n independent interpenetrating subsamples be drawn. Let $y_i$ and $x_i$ be unbiased estimates of the population totals $Y(=N\overline{Y})$ and $X(=N\overline{X})$ respectively based on the $i^{\text{th}}$ independent interpenetrating subsample, $i=1,2,\ldots,n$. We now consider following ratio and product-type estimators of the population mean $\overline{Y}$:

$$d_1 = \bar{y}\left(\overline{A}/\bar{a}\right) \tag{5.1}$$

$$d_{1n} = \left(\overline{A}/n\right)\sum_{i=1}^{n}\left(y_i/a_i\right) \tag{5.2}$$

$$d_2 = \bar{y}\left(\overline{B}/\bar{b}\right) \tag{5.3}$$

$$d_{2n} = \left(\overline{B}/n\right)\sum_{i=1}^{n}\left(y_i/b_i\right) \tag{5.4}$$

$$d_3 = \bar{y}\left(\bar{a}/\overline{A}\right) \tag{5.5}$$

$$d_{3n} = \sum_{i=1}^{n} y_i\, a_i \Big/ \left(n\overline{A}\right) \tag{5.6}$$

$$d_4 = \bar{y}\left(\bar{b}/\overline{B}\right) \tag{5.7}$$

and

$$d_{4n} = \sum_{i=1}^{n} y_i\, b_i \Big/ \left(n\overline{B}\right) \tag{5.8}$$

where $\bar{a}$, $\bar{b}$, $\overline{A}$, $\overline{B}$, $a_i$ and $b_i$ are same as defined in Section 2.

It is easy to verify that

$$B(d_{1n}) = nB(d_1) \tag{5.9}$$

$$B(d_{2n}) = nB(d_2) \tag{5.10}$$

$$B(d_{3n}) = nB(d_3) \tag{5.11}$$

and

$$B(d_{4n}) = nB(d_4) \tag{5.12}$$

Thus we get the following ratio and product-type unbiased estimators of $\overline{Y}$ as

$$d_{1u} = \frac{(nd_1 - d_{1n})}{(n-1)} \tag{5.13}$$

$$d_{2u} = \frac{(nd_2 - d_{2n})}{(n-1)} \tag{5.14}$$

$$d_{3u} = \frac{(nd_3 - d_{3n})}{(n-1)} \tag{5.15}$$

$$d_{4u} = \frac{(nd_4 - d_{4n})}{(n-1)} \tag{5.16}$$

The properties of these unbiased estimators ($d_{ju}$, $j = 1$ to 4) can be studied on the lines of Murthy and Nanjamma (1959).

**Remark 5.1.** In the case of simple random sampling without replacement (SRSWOR), let $y_i$ and $x_i$ denote respectively the $y$ and $x$ values of the sample of unit, $i = 1, 2, \ldots, n$. We have

$$\begin{aligned} d_1 &= \bar{y}\left(\overline{A}/\bar{a}\right) \\ d_{1n} &= \left(\overline{A}/n\right) \sum_{i=1}^{n} \left(y_i/a_i\right) \\ d_2 &= \bar{y}\left(\overline{B}/\bar{b}\right) \end{aligned}$$

$$d_{2n} = \left(\overline{B}/n\right) \sum_{i=1}^{n} \left(y_i/b_i\right)$$

$$d_3 = \bar{y} \left(\bar{a}/\overline{A}\right)$$

$$d_{3n} = \sum_{i=1}^{n} y_i\, a_i / \left(n\overline{A}\right)$$

$$d_4 = \bar{y} \left(\bar{b}/\overline{B}\right)$$

and

$$d_{4n} = \sum_{i=1}^{n} y_i\, b_i / \left(n\overline{B}\right)$$

It can be shown under SRSWOR scheme that the following ratio-type estimators are unbiased for population mean $\overline{Y}$ as

$$d_{1u}^{*} = \frac{n\,(N-1)}{N\,(n-1)} \bar{y} \left(\frac{\overline{A}}{\bar{a}}\right) - \frac{(N-n)}{N\,(n-1)} \frac{\overline{A}}{n} \sum_{i=1}^{n} \left(y_i/a_i\right) \tag{5.17}$$

$$d_{2u}^{*} = \frac{n\,(N-1)}{N\,(n-1)} \bar{y} \left(\frac{\overline{B}}{\bar{b}}\right) - \frac{(N-n)}{N\,(n-1)} \frac{\overline{B}}{n} \sum_{i=1}^{n} \left(y_i/b_i\right) \tag{5.18}$$

$$d_{3u}^{*} = \frac{n\,(N-1)}{N\,(n-1)} \bar{y} \left(\frac{\bar{a}}{\overline{A}}\right) - \frac{(N-n)}{N\,(n-1)} \frac{1}{n\overline{A}} \sum_{i=1}^{n} y_i a_i \tag{5.19}$$

$$d_{4u}^{*} = \frac{n\,(N-1)}{N\,(n-1)} \bar{y} \left(\frac{\bar{b}}{\overline{B}}\right) - \frac{(N-n)}{N\,(n-1)} \frac{1}{n\overline{B}} \sum_{i=1}^{n} y_i b_i \tag{5.20}$$

To the first degree of approximation, it can be shown that

$$Var\,(d_{1u}^{*}) = Var\,(d_{1R}) \tag{5.21}$$

$$Var\,(d_{2u}^{*}) = Var\,(d_{2R}) \tag{5.22}$$

$$Var\,(d_{3u}^{*}) = Var\,(d_{1p}) \tag{5.23}$$

and

$$Var\left(d^*_{4u}\right) = Var\left(d_{2p}\right). \tag{5.24}$$

Thus the unbiased estimators $d^*_{1u}$, $d^*_{2u}$, $d^*_{3u}$ and $d^*_{4u}$ are to be preferred over biased estimators $d_{1R}$, $d_{2R}$, $d_{1p}$ and $d_{2p}$ respectively.

## 5.2. Jack-knife technique

We may take $n = 2m$ and split the sample at random into two subsamples of $m$ units each. Let $\bar{y}_i, \bar{x}_i$ $(i = 1,2)$ be unbiased estimators of population mean $\bar{Y}$ and $\bar{X}$ respectively based on the subsamples and $\bar{y}, \bar{x}$ the means based on the entire sample. Thus $\left(\bar{a}_i, \bar{b}_i; i = 1,2\right)$ are unbiased estimators based on the sub-samples and $\left(\bar{a}, \bar{b}\right)$ the means based on the entire sample i.e.,

$$\begin{aligned}
\bar{a}_i &= \left(x_M \bar{x}_i + x_m^2\right), \\
\bar{b}_i &= \left\{(x_M - x_m)\, \bar{x}_i + x_m^2\right\}, \\
\bar{a} &= \left(\bar{x} x_M + x_m^2\right),
\end{aligned}$$

and

$$\bar{b} = \left\{(x_M - x_m)\, \bar{x} + x_m^2\right\},$$

Thus motivated by Quenoulle (1956) we define the following ratio and product-type unbiased estimators of population mean $\bar{Y}$ as

$$d^{(u)}_{1J} = \frac{(2N-n)}{N} d_1 - \frac{(N-n)}{2N} \left\{d^{(1)}_1 + d^{(2)}_1\right\} \tag{5.25}$$

$$d^{(u)}_{2J} = \frac{(2N-n)}{N} d_2 - \frac{(N-n)}{2N} \left\{d^{(1)}_2 + d^{(2)}_2\right\} \tag{5.26}$$

$$d^{(u)}_{3J} = \frac{(2N-n)}{N} d_3 - \frac{(N-n)}{2N} \left\{d^{(1)}_3 + d^{(2)}_3\right\} \tag{5.27}$$

and

$$d^{(u)}_{4J} = \frac{(2N-n)}{N} d_4 - \frac{(N-n)}{2N} \left\{d^{(1)}_4 + d^{(2)}_4\right\} \tag{5.28}$$

where $d_1$, $d_2$, $d_3$ and $d_4$ are same as defined in Section 5, and

$$d_1^{(i)} = \bar{y}_i\left(\bar{A}/\bar{a}_i\right), \quad d_2^{(i)} = \bar{y}_i\left(\bar{B}/\bar{b}_i\right), \quad d_3^{(i)} = \bar{y}_i\left(\bar{a}_i/\bar{A}\right)$$

and

$$d_4^{(i)} = \bar{y}_i\left(\bar{b}_i/\bar{B}\right), \ (i = 1,2).$$

Following the procedure outlined in Sukhatme and Sukhatme [1970, pp. 161-165], it can be shown to the first degree of approximation that the variance expressions of $d_{lJ}^{(u)}$, $(l = 1,2,3,4)$ and variance expressions of $d_{1R}$, $d_{2R}$, $d_{1p}$ and $d_{2p}$ respectively are same.

Thus we advocate that one can prefer the unbiased estimators $d_{lJ}^{(u)}$, $(l = 1,2,3,4)$ as compared to biased estimators $d_{1R}$, $d_{2R}$, $d_{1p}$ and $d_{2p}$.

## 6. Empirical study

### *6.1. When the variates y and x are positively correlated*

To see the performances of the suggested estimators $d_{1R}$ and $d_{2R}$ over $\bar{y}$, $\bar{y}_R$, $t_{1R}$ and $t_{2R}$, we have considered eight natural population data sets. Descriptions of the populations are given below:

***Table 6.1:*** *Description of populations.*

| Pop. No. | Source | $N$ | $n$ | $Y$ | $X$ | $\rho$ | $C_x$ | $C_y$ | $C_1$ | $C_2$ | $K$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Sahoo and Swain (1987) | 4 | 2 | Unit: (0.2,0.6, 0.9,0.8) | Unit: (0.1,0.2, 0.3,0.4) | 0.87 | 0.51 | 0.49 | 1.4 | 2.6 | 0.84 |
| 2 | Murthy (1967), p. 422 (13-44) | 12 | 4 | Number of cattle (Survey) | Number of cattle (Census) | 0.98 | 1.05 | 0.99 | 1.23 | 4.49 | 0.92 |
| 3 | Murthy (1967), p. 398 (1-12) | 12 | 4 | Number of Absentees | Number of Workers | 0.80 | 0.52 | 0.63 | 1.35 | 2.52 | 0.96 |
| 4 | Panse and Sukhatme (1967), p. 118 (1-25) | 25 | 10 | Parental plot mean (mm) | Parental plant value (mm) | 0.53 | 0.07 | 0.03 | 1.83 | 2.15 | 0.62 |
| 5 | Panse and Sukhatme (1967), p. 118 (1-20) | 20 | 8 | Parental plot mean (mm) | Parental plant value (mm) | 0.56 | 0.07 | 0.04 | 1.83 | 2.15 | 0.29 |
| 6 | Panse and Sukhatme (1967), p. 118 (1-10) | 10 | 4 | Progeny mean (mm) | Parental plant value (mm) | 0.44 | 0.07 | 0.05 | 1.92 | 2.13 | 0.31 |
| 7 | Singh and Chaudhary p. 176 (1-10) | 10 | 4 | No. of Cows in milk (Survey) | No. of Cows in milk (Census) | 0.97 | 0.63 | 0.58 | 1.26 | 2.81 | 0.89 |
| 8 | Singh and Chaudhary p. 306 | 10 | 4 | No. of inhabitants ('000) in 1980-81 | No. of inhabitants ('000) in 1981-82 | 0.88 | 0.64 | 0.60 | 1.53 | 3.64 | 0.82 |
| 9 | Samford (1962), p. 61 (1-9) | 9 | 3 | Acreage under oats in 1957 | Acreage of crops and gross in 1947 | 0.07 | 0.10 | 0.29 | 1.86 | 2.12 | 0.19 |

To assess the biasedness of the ratio-type estimators $\bar{y}_R$, $t_{1R}$, $t_{2R}$, $d_{1R}$ and $d_{2R}$, we have computed the following quantities for the population given in Table 6.1 using the formulae:

$$B_1 = \left| \frac{B(\bar{y}_R)}{\theta \bar{Y} C_x^2} \right| = |(1-K)| \tag{6.1}$$

$$B_2 = \left|\frac{B(t_{1R})}{\theta \overline{Y} C_x^2}\right| = \frac{1}{C_1}\left|\left(\frac{1}{C_1} - K\right)\right| \tag{6.2}$$

$$B_3 = \left|\frac{B(t_{2R})}{\theta \overline{Y} C_x^2}\right| = \frac{1}{C_2}\left|\left(\frac{1}{C_2} - K\right)\right| \tag{6.3}$$

$$B_4 = \left|\frac{B(d_{1R})}{\theta \overline{Y} C_x^2}\right| = \lambda_{(1)}\left|\left(\lambda_{(1)} - K\right)\right| \tag{6.4}$$

$$B_5 = \left|\frac{B(d_{2R})}{\theta \overline{Y} C_x^2}\right| = \lambda_{(2)}\left|\left(\lambda_{(2)} - K\right)\right| \tag{6.5}$$

The findings are listed in Table 6.2.

***Table 6.2:*** *Values of $B_1$, $B_2$, $B_3$, $B_4$ and $B_5$.*

| Values of $B_i$'s $i = 1$ to 5 | Population | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $B_1$ | 0.1600 | 0.0826 | 0.0433 | 0.7399 | 0.7087 | 0.6951 | 0.1109 | 0.1767 | 0.8079 |
| $B_2$ | 0.0898 | 0.0847 | 0.1602 | 0.1554 | 0.1397 | 0.1128 | 0.0781 | 0.1125 | 0.1852 |
| $B_3$ | 0.1752 | 0.1547 | 0.2125 | 0.0946 | 0.0812 | 0.0772 | 0.1897 | 0.1507 | 0.1318 |
| $B_4$ | 0.0628 | 0.0668 | 0.0178 | 0.2227 | 0.2091 | 0.1534 | 0.0708 | 0.0702 | 0.2460 |
| $B_5$ | 0.0374 | 0.0657 | 0.0299 | 0.0175 | 0.0081 | 0.0209 | 0.0644 | 0.0489 | 0.0171 |

Table 6.2 exhibits that the proposed estimator $d_{2R}$ has least bias for all data sets except in population III considered here. In population III, the proposed estimator $d_{1R}$ has least bias. Using the following formulae:

$$PRE\,(\bar{y}_R, \bar{y}) = \frac{MSE\,(\bar{y})}{MSE\,(\bar{y}_R)} \times 100 = \left[1 + \left(\frac{C_x}{C_y}\right)^2 (1 - 2K)\right]^{-1} \times 100 \tag{6.6}$$

$$PRE\,(t_{1R}, \bar{y}) = \frac{MSE\,(\bar{y})}{MSE\,(t_{1R})} \times 100 = \left[1 + \frac{1}{C_1}\left(\frac{C_x}{C_y}\right)^2 \left(\frac{1}{C_1} - 2K\right)\right]^{-1} \times 100 \tag{6.7}$$

$$PRE\,(t_{2R}, \bar{y}) = \frac{MSE\,(\bar{y})}{MSE\,(t_{2R})} \times 100 = \left[1 + \frac{1}{C_2}\left(\frac{C_x}{C_y}\right)^2 \left(\frac{1}{C_2} - 2K\right)\right]^{-1} \times 100 \tag{6.8}$$

$$PRE\,(d_{1R},\bar{y}) = \frac{MSE(\bar{y})}{MSE(d_{1R})} \times 100 = \left[1 + \left(\frac{C_x}{C_y}\right)^2 \lambda_{(1)}(\lambda_{(1)} - 2K)\right]^{-1} \times 100 \quad (6.9)$$

and

$$PRE\,(d_{2R},\bar{y}) = \frac{MSE(\bar{y})}{MSE(d_{2R})} \times 100 = \left[1 + \left(\frac{C_x}{C_y}\right)^2 \lambda_{21)}(\lambda_{(2)} - 2K)\right]^{-1} \times 100 \quad (6.10)$$

We have computed the percent relative efficiencies (PREs) of $\bar{y}_R$, $t_{1R}$, $t_{2R}$, $d_{1R}$ and $d_{2R}$ with respect to usual unbiased estimator $\bar{y}$ and compiled in Table 6.3.

***Table 6.3:*** *Percent relative efficiencies of $\bar{y}_R$, $t_{1R}$, $t_{2R}$, $d_{1R}$ and $d_{2R}$ with respect to $\bar{y}$.*

| Estimator | $PRE(.,\bar{y})$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Population | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $\bar{y}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\bar{y}_R$ | 383.33 | 2279.92 | 273.92 | 33.62 | 39.24 | 55.15 | 1263.21 | 380.08 | 92.90 |
| $t_{1R}$ | 399.65 | 2063.93 | 252.32 | 94.69 | 107.82 | 110.63 | 1313.15 | 382.20 | 98.99 |
| $t_{2R}$ | 218.13 | 169.80 | 161.47 | 112.07 | 125.30 | 115.91 | 249.18 | 175.31 | 99.49 |
| $d_{1R}$ | 419.76 | 2421.29 | 274.71 | 78.95 | 90.93 | 104.63 | 1408.39 | 426.60 | 98.41 |
| $d_{2R}$ | 425.54 | 2430.62 | 274.35 | 136.19 | 145.40 | 120.65 | 1428.98 | 432.85 | 100.41 |

Table 6.3 shows that the proposed estimator $d_{2R}$ has largest gain in efficiency for all population data sets except in population III, where the proposed estimator $d_{1R}$ has maximum gain in efficiency. We also note that the proposed estimator $d_{1R}$ dominates over the estimators ($\bar{y}$, $\bar{y}_R$, $t_{1R}$ and $t_{2R}$) in population I, II, III, IV, VII and VIII. Thus the proposed estimators $d_{1R}$ and $d_{2R}$ are to be preferred over other estimators.

Finally, from Tables 6.2 and 6.3 we recommend the use of the proposed estimator $d_{2R}$ in practice as it has largest gain in efficiency and also fewer bias in all population data sets except in population III, where the proposed estimator $d_{1R}$ has largest gain in efficiency as well as less bias and hence $d_{1R}$ is to be recommended for this population data set.

### 6.2. When the variates y and x are negatively correlated

To assess the biasdeness and efficiency of the product-type estimators $\bar{y}_p$, $t_{1p}$, $t_{2p}$, $d_{1p}$ and $d_{2p}$ we have considered natural population data sets.

***Table 6.4:*** *Description of the populations.*

| Pop. No. | Source | $N$ | $n$ | $Y$ | $X$ | $\rho$ | $\mathbf{C_x}$ | $\mathbf{C_y}$ | $C_1$ | $C_2$ | $K$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Maddla, G.S. (1977), p. 96 | 16 | 4 | Capita Consumption | Deflated price | −0.97 | 0.24 | 0.17 | 1.68 | 2.39 | −0.68 |
| 2 | Gupta, S.P. and Gupta, A. (1999) p. 65 | 5 | 2 | Artificial Population | | −0.96 | 0.52 | 0.51 | 1.43 | 2.74 | −0.93 |

To observe the biasedness of the estimators $\bar{y}_p$, $t_{1p}$, $t_{2p}$, $d_{1p}$ and $d_{2p}$, we use the following formulae:

$$B_1^* = \left| \frac{B(\bar{y}_p)}{\theta \bar{Y} C_x^2} \right| = |K| \tag{6.11}$$

$$B_2^* = \left| \frac{B(\bar{y}_{1p})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{K}{C_1} \right| \tag{6.12}$$

$$B_3^* = \left| \frac{B(t_{2p})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{K}{C_2} \right| \tag{6.13}$$

$$B_4^* = \left| \frac{B(d_{1p})}{\theta \bar{Y} C_x^2} \right| = \lambda_{(1)} |K| \tag{6.14}$$

$$B_5^* = \left| \frac{B(d_{2p})}{\theta \bar{Y} C_x^2} \right| = \lambda_{(2)} |K| \tag{6.15}$$

The quantities $B*'s\,(i = 1 \text{ to } 5)$ have been computed and findings are given in Table 6.5.

***Table 6.5:*** *Values of $B_1^*$, $B_2^*$, $B_3^*$, $B_4^*$ and $B_5^*$.*

| Population | Values of $B_i^*$'s, $i$ = 1 to 5 | | | | |
|---|---|---|---|---|---|
| | $B_1^*$ | $B_2^*$ | $B_3^*$ | $B_4^*$ | $B_5^*$ |
| **1** | 0.6814 | 0.4043 | 0.2843 | 0.5099 | 0.4104 |
| **2** | 0.9338 | 0.6508 | 0.3409 | 0.8422 | 0.8156 |

Using the following formulae:

$$PRE\left(\bar{y}_p,\bar{y}\right)=\frac{MSE\left(\bar{y}\right)}{MSE\left(\bar{y}_P\right)}\times 100=\left[1+\left(\frac{C_x}{C_y}\right)^2\left(1+2K\right)\right]^{-1}\times 100 \tag{6.16}$$

$$PRE\left(t_{1p},\bar{y}\right)=\frac{MSE\left(\bar{y}\right)}{MSE\left(t_{1P}\right)}\times 100=\left[1+\left(\frac{C_x}{C_y}\right)^2\frac{1}{C_1}\left(\frac{1}{C_1}+2K\right)\right]^{-1}\times 100 \tag{6.17}$$

$$PRE\left(t_{2p},\bar{y}\right)=\frac{MSE\left(\bar{y}\right)}{MSE\left(t_{2P}\right)}\times 100=\left[1+\left(\frac{C_x}{C_y}\right)^2\frac{1}{C_2}\left(\frac{1}{C_2}+2K\right)\right]^{-1}\times 100 \tag{6.18}$$

$$PRE\left(d_{1p},\bar{y}\right)=\frac{MSE\left(\bar{y}\right)}{MSE\left(\bar{y}_P\right)}\times 100=\left[1+\left(\frac{C_x}{C_y}\right)^2\lambda_{(1)}\left(\lambda_{(1)}+2K\right)\right]^{-1}\times 100 \tag{6.19}$$

and

$$PRE\left(d_{2p},\bar{y}\right)=\frac{MSE\left(\bar{y}\right)}{MSE\left(\bar{y}_P\right)}\times 100=\left[1+\left(\frac{C_x}{C_y}\right)^2\lambda_{(2)}\left(\lambda_{(2)}+2K\right)\right]^{-1}\times 100 \tag{6.20}$$

We have computed the percent relative efficiencies (PREs) of $\bar{y}_p$, $t_{1p}$, $t_{2p}$, $d_{1p}$ and $d_{2p}$ with respect to usual unbiased estimator $\bar{y}$ and the results are shown in Table 6.6.

***Table 6.6:*** *Percent relative efficiencies of* $\bar{y}_p$, $t_{1p}$, $t_{2p}$, $d_{1p}$ *and* $d_{2p}$ *with respect to* $\bar{y}$.

| Estimators | | $\bar{y}$ | $\bar{y}_p$ | $t_{1p}$ | $t_{2p}$ | $d_{1p}$ | $d_{2p}$ |
|---|---|---|---|---|---|---|---|
| $PRE(.,\bar{y})$ | Population 1 | 100.00 | 390.97 | 1578.36 | 524.73 | 1764.62 | 1658.49 |
| | Population 2 | 100.00 | 1133.69 | 701.62 | 236.13 | 1181.21 | 1143.86 |

Tables 6.5 and 6.6 show that the proposed estimators $d_{1p}$ and $d_{2p}$ are more efficient (with substantial gain) than usual unbiased estimator $\bar{y}$, product estimator $\bar{y}_p$ and the estimators $t_{1p}$ and $t_{2p}$ reported by Sahoo and Mohanty (1995), but these two estimators ($d_{1p}$ and $d_{2p}$) are more biased than $t_{1p}$ and $t_{2p}$. Thus if the variance / MSE's criterion of judging the performance of the estimators are adopted and also the biasedness of the estimators are not of primary concern then the proposed estimators $d_{1p}$ and $d_{2p}$ are recommended for their use in practice.

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