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A family of ratio estimators for population mean in extreme ranked set sampling using two auxiliary variables

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Abstract

In this paper we have adopted the Khoshnevisan *et al.* (2007) family of estimators to extreme ranked set sampling (*ERSS*) using information on single and two auxiliary variables. Expressions for mean square error (*MSE*) of proposed estimators are derived to first order of approximation. Monte Carlo simulations and real data sets have been used to illustrate the method. The results indicate that the estimators under *ERSS* are more efficient as compared to estimators based on simple random sampling (*SRS*), when the underlying populations are symmetric.

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Keywords: Ratio estimator, ranked set sampling, extreme ranked set sampling.

1. Introduction

Ranked set sampling (*RSS*) was introduced by McIntyre (1952) and suggested using *RSS* as a costly efficient alternative as compared to *SRS*. Takahasi and Wakimoto (1968) developed the mathematical theory and proved that the sample mean of a ranked set sample is an unbiased estimator of the population mean and possesses smaller variance than the sample mean of a simple random sample with the same sample size. Samawi and Muttlak (1996) suggested the use of *RSS* to estimate the population ratio and showed that it gives more efficient estimates as compared to *SRS*. Samawi *et al.* (1996) introduced *ERSS* to estimate the population mean and showed that the sample mean under *ERSS*

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is an unbiased and is more efficient than the sample mean based on *SRS*. Samawi (2002) introduced the ratio estimation in estimating the population ratio using *ERSS* and showed that the ratio estimator under *ERSS* is an approximately unbiased estimator of the population ratio. Also in the case of symmetric populations ratio estimators under *ERSS* are more efficient than ratio estimators under *SRS*. Samawi and Saeid (2004) investigated the use of the separate and the combined ratio estimators in *ERSS*. Samawi *et al.* (2004) studied the use of regression estimator in *ERSS* and showed that for symmetric distributions, the regression estimator under *ERSS* is more efficient as compared to *SRS* and *RSS*.

In this paper, *SRS* and *ERSS* methods are used for estimating the population mean of the study variable Y by using information on the auxiliary variables X and Z.

The organization of this paper is as follows. Section 2 includes sampling methods like *SRS* and *ERSS*. In Section 3, main notations and results are given. Sections 4 and 5 comprise of a family of ratio estimators using single and two auxiliary variables. Section 6 describes of simulation and empirical studies and Section 7 finally provides the conclusion.

2. Sampling methods

2.1. Simple random sampling

In *SRS*, m units out of N units of a population are drawn in such a way that every possible combination of items that could make up a given sample size has an equal chance of being selected. In usual practice, a simple random sample is drawn unit by unit.

2.2. Ranked set sampling

RSS procedure involves selection of m sets, each of m units from the population. It is assumed that units within each set can be ranked visually at no cost or at little cost. From the first set of m units, the lowest ranked unit is selected; the remaining units of the sample are discarded. From the second set of m units, the second lowest ranked unit is selected and the remaining units are discarded. The procedure is continued until from the mth set, the mth ranked unit is selected. This completes one cycle of a ranked set sample of size m. The whole process can be repeated r times to get a ranked set sample of size $n = mr$.

2.3. Extreme ranked set sampling

Samawi *et al.* (1996) introduced a new variety of ranked set sampling, named as *ERSS* to estimate the population mean and have shown that *ERSS* gives more efficient estimates as compared to *SRS*.

In *ERSS*, m independent samples, each of m units are drawn from infinite population to estimate the unknown parameter. Here we assume that lowest and largest units of these samples can be detected visually with no cost or with little cost as explained by Samawi (2002). From the first set of m units, the lowest ranked unit is measured, similarly from the second set of m units, the largest ranked unit is measured. Again in the third set of m units the lowest ranked unit is measured and so on. The procedure continues until from $(m-1)$ units, $(m-1)$ units are measured. From the last mth sample, the selection of the unit depends whether m is even or not. It can be measured in two ways:

(i) If m is even then the largest ranked unit is to be selected; we denote such a sample with notation $ERSS_a$.
(ii) If m is odd then for the measurement of the mth unit, we take the average of the lowest and largest units of the mth sample; such a sample will be donated by $ERSS_b$ or we take the median of the mth sample; such a sample is denoted by $ERSS_c$.

The choice of a sample $ERSS_b$ will be more difficult as compared to the choices of $ERSS_a$ and $ERSS_c$ (see Samawi *et al.* 1996). The above procedure can be repeated r times to select an *ERSS* of size mr units.

3. Notations under *SRS* and *ERSS*

Let $(X_1,\ Y_1),(X_2,\ Y_2),\ldots,(X_m,\ Y_m)$ be a random sample from a bivariate normal distribution with probability density function $f(X,Y)$, having parameters $\mu_X,\ \mu_Y,\ \sigma_X,\sigma_Y$ and ρ. We assume that the ranking is performed on the auxiliary variable X for estimating the population mean (μ_Y). Let $(X_{11},\ Y_{11}),(X_{12},\ Y_{12}),\ldots,(X_{1m},\ Y_{1m})$, $(X_{21},\ Y_{21})$, $(X_{22},\ Y_{22}),\ldots,(X_{2m},\ Y_{2m})$,..., $(X_{m1},\ Y_{m1}),(X_{m2},\ Y_{m2}),\ldots,(X_{mm},\ Y_{mm})$ be m independent bivariate random vectors each of size m, $\left(X_{i(1)},\ Y_{i[1]}\right),\left(X_{i(2)},\ Y_{i[2]}\right),\ldots,\left(X_{i(m)},\ Y_{i[m]}\right)$ be the *RSS* for $i=1,2,\ldots m$. In *ERSS*, if m is even then $\left(X_{1(1)j},\ Y_{1[1]j}\right),\left(X_{2(m)j},\ Y_{2[m]j}\right)$, $\ldots,\left(X_{m-1(1)j},\ Y_{m-1[1]j}\right),\left(X_{m(m)j},\ Y_{m[m]j}\right)$, denoted by $ERSS_a$, and if m is odd then $\left(X_{1(1)j},\ Y_{1[1]j}\right),\left(X_{2(m)j},\ Y_{2[m]j}\right),\ldots,\left(X_{m-1(m)j},\ Y_{m-1[m]j}\right),\left(X_{m\left(\frac{m+1}{2}\right)j},\ Y_{m\left[\frac{m+1}{2}\right]j}\right)$, denoted by $ERSS_c$, for the jth cycle, where j=1,2,..., r.

Considering ranking on the auxiliary variable X, we use the following notations and results.

Let $E(X_i)=\mu_X,\ E(Y_i)=\mu_Y,\ Var(X_i)=\sigma_X^2,\ Var(Y_i)=\sigma_Y^2,\ E\left(X_{i(m)}\right)=\mu_{X(m)}$, $E\left(Y_{i[m]}\right)=\mu_{Y[m]}$, $E\left(X_{i(1)}\right)=\mu_{X(1)}$, $E\left(Y_{i[1]}\right)=\mu_{Y[1]}$, $Var\left(X_{i(1)}\right)=\sigma^2_{X(1)}$, $Var\left(Y_{i[1]}\right)=\sigma^2_{Y[1]}$,

$$Var\left(X_{i(m)}\right) = \sigma^2_{X(m)}, \qquad Var\left(Y_{i[m]}\right) = \sigma^2_{Y[m]},$$

$$E\left(X_{i\left(\frac{m+1}{2}\right)}\right) = \mu_{X\left(\frac{m+1}{2}\right)}, \qquad E\left(Y_{i\left[\frac{m+1}{2}\right]}\right) = \mu_{Y\left[\frac{m+1}{2}\right]},$$

$$Var\left(X_{i\left(\frac{m+1}{2}\right)}\right) = \sigma^2_{X\left(\frac{m+1}{2}\right)}, \quad Var\left(Y_{i\left[\frac{m+1}{2}\right]}\right) = \sigma^2_{Y\left[\frac{m+1}{2}\right]}$$

and

$$Cov\left(X_{i(h)}, Y_{i[k]}\right) = \sigma_{X(h)Y[k]}.$$

In *SRS* the sample means of variables X and Y are

$$\bar{X} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} X_{ij}$$

and

$$\bar{Y} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{ij}$$

In $ERSS_a$, the sample means of X and Y are

$$\bar{X}_{(a)} = \frac{1}{2}\left(\bar{X}_{(1)} + \bar{X}_{(m)}\right),$$

where

$$\bar{X}_{(1)} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} X_{2i-1(1)j}, \qquad \bar{X}_{(m)} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} X_{2i(m)j}$$

and

$$\bar{Y}_{[a]} = \frac{1}{2}\left(\bar{Y}_{[1]} + \bar{Y}_{[m]}\right),$$

where

$$\bar{Y}_{[1]} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} Y_{2i-1[1]j}, \qquad \bar{Y}_{[m]} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} Y_{2i[m]j}.$$

In $ERSS_c$, we define

$$\bar{X}_{(c)} = \frac{\sum_{j=1}^{r}\left(X_{1(1)j} + X_{2(m)j} + \cdots + X_{m-1(m)j} + X_{m\left(\frac{m+1}{2}\right)j}\right)}{mr} = \frac{\left(\frac{m-1}{2}\right)\left(\bar{X}'_{(1)} + \bar{X}'_{(m)}\right) + \bar{X}'_{\left(\frac{m+1}{2}\right)}}{m},$$

where

$$\bar{X}'_{(1)} = \frac{2}{r(m-1)} \sum_{j=1}^{r} \sum_{i=1}^{(m-1)/2} X_{2i-1(1)j}, \quad \bar{X}'_{(m)} = \frac{2}{r(m-1)} \sum_{j=1}^{r} \sum_{i=1}^{(m-1)/2} X_{2i(m)j},$$

$$\bar{X}'_{\left(\frac{m+1}{2}\right)} = \frac{1}{r} \sum_{j=1}^{r} X_{m\left(\frac{m+1}{2}\right)j}.$$

Also for Y, we have

$$\bar{Y}_{[c]} = \frac{\sum_{j=1}^{r} \left(Y_{1[1]j} + Y_{2[m]j} + \cdots + Y_{m-1[m]j} + Y_{m\left[\frac{m+1}{2}\right]j}\right)}{mr} = \frac{\left(\frac{m-1}{2}\right)\left(\bar{Y}'_{[1]} + \bar{Y}'_{[m]}\right) + \bar{Y}'_{\left[\frac{m+1}{2}\right]}}{m},$$

where

$$\bar{Y}'_{[1]} = \frac{2}{r(m-1)} \sum_{j=1}^{r} \sum_{i=1}^{(m-1)/2} Y_{2i-1[1]j}, \quad \bar{Y}'_{[m]} = \frac{2}{r(m-1)} \sum_{j=1}^{r} \sum_{i=1}^{(m-1)/2} Y_{2i[m]j},$$

$$\bar{Y}'_{\left[\frac{m+1}{2}\right]} = \frac{1}{r} \sum_{j=1}^{r} Y_{m\left[\frac{m+1}{2}\right]j}.$$

Similarly, in case of the two auxiliary variables X and Z, when ranking is done on Z, we use the following notations.

$$E\left(Y_{i[m]}\right) = \mu_{Y[m]}, \quad E\left(X_{i[m]}\right) = \mu_{X[m]}, \quad E\left(Z_{i(m)}\right) = \mu_{Z(m)},$$

$$E\left(Y_{i[1]}\right) = \mu_{Y[1]}, \quad E\left(X_{i[1]}\right) = \mu_{X[1]}, \quad E\left(Z_{i(1)}\right) = \mu_{Z(1)},$$

$$Var\left(Y_{i[1]}\right) = \sigma^2_{Y[1]}, \quad Var\left(X_{i[1]}\right) = \sigma^2_{X[1]}, \quad Var\left(Z_{i(1)}\right) = \sigma^2_{Z(1)},$$

$$Var\left(Y_{i[m]}\right) = \sigma^2_{Y[m]}, \quad Var\left(X_{i[m]}\right) = \sigma^2_{X[m]}, \quad Var\left(Z_{i(m)}\right) = \sigma^2_{Z(m)},$$

$$E\left(Y_{i\left[\frac{m+1}{2}\right]}\right) = \mu_{Y\left[\frac{m+1}{2}\right]}, \quad E\left(X_{i\left[\frac{m+1}{2}\right]}\right) = \mu_{X\left[\frac{m+1}{2}\right]}, \quad E\left(Z_{i\left(\frac{m+1}{2}\right)}\right) = \mu_{Z\left(\frac{m+1}{2}\right)},$$

$$Var\left(Y_{i\left[\frac{m+1}{2}\right]}\right) = \sigma^2_{Y\left[\frac{m+1}{2}\right]}, \quad Var\left(X_{i\left[\frac{m+1}{2}\right]}\right) = \sigma^2_{X\left[\frac{m+1}{2}\right]}, \quad Var\left(Z_{i\left(\frac{m+1}{2}\right)}\right) = \sigma^2_{Z\left(\frac{m+1}{2}\right)},$$

$Cov\left(X_{i[h]}, Y_{i[k]}\right) = \sigma_{X[h]Y[k]}$, $Cov\left(X_{i[h]}, Z_{i(k)}\right) = \sigma_{X[h]Z(k)}$ and $Cov\left(Y_{i[h]}, Z_{i(k)}\right) = \sigma_{Y[h]Z(k)}$.

In *SRS* the sample means of variables X, Y and Z are

$$\bar{X} = \frac{1}{mr} \sum_{j=1}^{r} \sum_{i=1}^{m} X_{ij}, \qquad \bar{Y} = \frac{1}{mr} \sum_{j=1}^{r} \sum_{i=1}^{m} Y_{ij} \qquad \text{and} \qquad \bar{Z} = \frac{1}{mr} \sum_{j=1}^{r} \sum_{i=1}^{m} Z_{ij}.$$

In $ERSS_a$, the sample means of X, Y and Z are

$$\bar{X}_{[a]} = \frac{1}{2}\left(\bar{X}_{[1]} + \bar{X}_{[m]}\right),$$

where

$$\bar{X}_{[1]} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} X_{2i-1[1]j}, \qquad \bar{X}_{[m]} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} X_{2i[m]j}, \qquad \bar{Y}_{[a]} = \frac{1}{2}\left(\bar{Y}_{[1]} + \bar{Y}_{[m]}\right),$$

where

$$\bar{Y}_{[1]} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} Y_{2i-1[1]j}, \qquad \bar{Y}_{[m]} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} Y_{2i[m]j} \qquad \text{and} \qquad \bar{Z}_{(a)} = \frac{1}{2}\left(\bar{Z}_{(1)} + \bar{Z}_{(m)}\right),$$

where

$$\bar{Z}_{(1)} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} Z_{2i-1(1)j}, \qquad \bar{Z}_{(m)} = \frac{2}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m/2} Z_{2i(m)j}.$$

In $ERSS_c$, the sample means for X, Y and Z are

$$\bar{X}_{[c]} = \frac{\left(\frac{m-1}{2}\right)\left(\bar{X}'_{[1]} + \bar{X}'_{[m]}\right) + \bar{X}'_{\left[\frac{m+1}{2}\right]}}{m},$$

where

$$\bar{X}'_{[1]} = \frac{2}{r(m-1)}\sum_{j=1}^{r}\sum_{i=1}^{(m-1)/2} X_{2i-1[1]j}, \qquad \bar{X}'_{[m]} = \frac{2}{r(m-1)}\sum_{j=1}^{r}\sum_{i=1}^{(m-1)/2} X_{2i[m]j},$$

$$\bar{X}'_{\left[\frac{m+1}{2}\right]} = \frac{1}{r}\sum_{j=1}^{r} X_{m\left[\frac{m+1}{2}\right]j}, \qquad \bar{Y}_{[c]} = \frac{\left(\frac{m-1}{2}\right)\left(\bar{Y}'_{[1]} + \bar{Y}'_{[m]}\right) + \bar{Y}'_{\left[\frac{m+1}{2}\right]}}{m},$$

where

$$\bar{Y}'_{[1]} = \frac{2}{r(m-1)}\sum_{j=1}^{r}\sum_{i=1}^{(m-1)/2} Y_{2i-1[1]j}, \qquad \bar{Y}'_{[m]} = \frac{2}{r(m-1)}\sum_{j=1}^{r}\sum_{i=1}^{(m-1)/2} Y_{2i[m]j},$$

$$\bar{Y}'_{\left[\frac{m+1}{2}\right]} = \frac{1}{r}\sum_{j=1}^{r} Y_{m\left[\frac{m+1}{2}\right]j} \qquad \text{and} \qquad \bar{Z}_{(c)} = \frac{\left(\frac{m-1}{2}\right)\left(\bar{Z}'_{(1)} + \bar{Z}'_{(m)}\right) + \bar{Z}'_{\left(\frac{m+1}{2}\right)}}{m},$$

where

$$\bar{Z}'_{(1)} = \frac{2}{r(m-1)} \sum_{j=1}^{r} \sum_{i=1}^{(m-1)/2} Z_{2i-1(1)j}, \qquad \bar{Z}'_{(m)} = \frac{2}{r(m-1)} \sum_{j=1}^{r} \sum_{i=1}^{(m-1)/2} Z_{2i(m)j},$$

$$\bar{Z}'_{\left(\frac{m+1}{2}\right)} = \frac{1}{r} \sum_{j=1}^{r} Z_{m\left(\frac{m+1}{2}\right)j}.$$

4. Proposed estimators using the single auxiliary variable

4.1. A family of ratio estimators using $ERSS_a$

Following Khoshnevisan *et al.* (2007), we propose a family of ratio estimators in $ERSS_a$ using the single auxiliary variable, when ranking is performed on the auxiliary variable X and is given by

$$\hat{\bar{Y}}_{ERSS_a} = \bar{Y}_{[a]} \left[\frac{a\mu_X + b}{\alpha\left(a\,\bar{X}_{(a)} + b\right) + (1-\alpha)(a\mu_X + b)} \right]^g, \tag{1}$$

where α and g are suitable constants, also a and b are either real numbers or functions of known parameters for the auxiliary variable X, like coefficient of variation (C_X) or coefficient of kurtosis (β_{2X})or standard deviation (S_X) or coefficient of correlation (ρ_{YX}).

Using bivariate Taylor series expansion, we have

$$\left(\hat{\bar{Y}}_{ERSS_a} - \mu_Y\right) \cong \frac{1}{2}\left[\bar{Y}_{[1]} - E\left(\bar{Y}_{[1]}\right)\right] + \frac{1}{2}\left[\bar{Y}_{[m]} - E\left(\bar{Y}_{[m]}\right)\right] - \frac{\mu_Y(a\alpha g)\left[\bar{X}_{(1)} - E\left(\bar{X}_{(1)}\right)\right]}{2(a\mu_X + b)}$$
$$- \frac{\mu_Y(a\alpha g)\left[\bar{X}_{(m)} - E\left(\bar{X}_{(m)}\right)\right]}{2(a\mu_X + b)}. \tag{2}$$

Solving (2) and using assumption of symmetry of distribution, the approximate MSE of $\hat{\bar{Y}}_{ERSS_a}$ is given by

$$MSE\left(\hat{\bar{Y}}_{ERSS_a}\right) \cong \frac{1}{mr}\left(\sigma^2_{Y[1]} + w^2\sigma^2_{X(1)} - 2w\sigma_{X(1)Y[1]}\right), \tag{3}$$

where $w = \dfrac{\mu_Y(a\alpha g)}{(a\mu_X + b)}$.

Minimizing (3) with respect to w, we get the optimum value of w i.e.

$$w_{(opt)} = \frac{\sigma_{X(1)Y[1]}}{\sigma^2_{X(1)}}.$$

The minimum MSE of $\hat{\hat{Y}}_{ERSS_a}$ is given by

$$MSE_{\min}\left(\hat{\hat{Y}}_{ERSS_a}\right) \cong \frac{\sigma^2_{Y[1]}\left(1-\rho^2_{X(1)Y[1]}\right)}{mr}, \tag{4}$$

where $\rho^2_{X(1)Y[1]} = \dfrac{\sigma^2_{X(1)Y[1]}}{\sigma^2_{X(1)}\sigma^2_{Y[1]}}$.

Note that the minimum MSE in (4) is equal to the MSE of the traditional regression estimator based on single auxiliary variable under $ERSS_a$.

4.2. A Family of ratio estimators using $ERSS_c$

We propose the same family of ratio estimators in $ERSS_c$ as

$$\hat{\hat{Y}}_{ERSS_c} = \bar{Y}_{[c]}\left[\frac{a\mu_X+b}{\alpha\left(a\,\bar{X}_{(c)}+b\right)+(1-\alpha)\left(a\mu_X+b\right)}\right]^g, \tag{5}$$

where α, g, $a \neq 0$ and b are defined earlier.

Using bivariate Taylor series expansion, we have

$$\begin{aligned}\left(\hat{\hat{Y}}_{ERSS_c}-\mu_Y\right) \cong\ & \frac{(m-1)}{2m}\left[\bar{Y}'_{[1]}-E\left(\bar{Y}'_{[1]}\right)\right]+\frac{(m-1)}{2m}\left[\bar{Y}'_{[m]}-E\left(\bar{Y}'_{[m]}\right)\right] \\ & +\frac{\left[\bar{Y}'_{\left[\frac{m+1}{2}\right]}-E\left(\bar{Y}'_{\left[\frac{m+1}{2}\right]}\right)\right]}{m}-\frac{\mu_Y\,(a\alpha g)\,(m-1)\left[\bar{X}'_{(1)}-E\left(\bar{X}'_{(1)}\right)\right]}{2m\,(a\mu_X+b)} \\ & -\frac{\mu_Y\,(a\alpha g)\,(m-1)\left[\bar{X}'_{(m)}-E\left(\bar{X}'_{(m)}\right)\right]}{2m\,(a\mu_X+b)}-\frac{\mu_Y\,(a\alpha g)\left[\bar{X}'_{\left(\frac{m+1}{2}\right)}-E\left(\bar{X}'_{\left(\frac{m+1}{2}\right)}\right)\right]}{m\,(a\mu_X+b)}.\end{aligned} \tag{6}$$

Using the assumption of symmetry of distribution, the approximate MSE of $\hat{\hat{Y}}_{ERSS_c}$, is given by

$$MSE\left(\hat{\hat{Y}}_{ERSS_c}\right) \cong \frac{1}{mr}\left(\frac{(m-1)\,\sigma^2_{Y[1]}+\sigma^2_{Y\left[\frac{m+1}{2}\right]}}{m}+w^2\frac{(m-1)\,\sigma^2_{X(1)}+\sigma^2_{X\left(\frac{m+1}{2}\right)}}{m}\right.$$

$$\left.-2w\frac{(m-1)\,\sigma_{X(1)Y[1]}+\sigma_{X\left(\frac{m+1}{2}\right)Y\left[\frac{m+1}{2}\right]}}{m}\right), \tag{7}$$

where w is defined earlier.

Also (7) can be written as

$$MSE\left(\hat{\hat{Y}}_{ERSS_c}\right) \cong \frac{1}{mr}\left[\sigma^{2*}_{Y[1]}+w^2\sigma^{2*}_{X(1)}-2w\sigma^{*}_{X(1)Y[1]}\right], \tag{8}$$

where

$$\sigma^{2*}_{Y[1]}=\frac{(m-1)\,\sigma^2_{Y[1]}+\sigma^2_{Y\left[\frac{m+1}{2}\right]}}{m}, \qquad \sigma^{2*}_{X(1)}=\frac{(m-1)\,\sigma^2_{X(1)}+\sigma^2_{X\left(\frac{m+1}{2}\right)}}{m}$$

and

$$\sigma^{*}_{X(1)Y[1]}=\frac{(m-1)\,\sigma_{X(1)Y[1]}+\sigma_{X\left(\frac{m+1}{2}\right)Y\left[\frac{m+1}{2}\right]}}{m}.$$

The minimum MSE of $\hat{\hat{Y}}_{ERSS_c}$ at the optimum value of w given by $w_{(opt)}=\dfrac{\sigma^{*}_{X(1)Y[1]}}{\sigma^{2*}_{X(1)}}$ is

$$MSE_{\min}\left(\hat{\hat{Y}}_{ERSS_c}\right) \cong \frac{\sigma^{2*}_{Y[1]}\left(1-\rho^{2*}_{X(1)Y[1]}\right)}{mr}, \tag{9}$$

where $\rho^{2*}_{X(1)Y[1]}=\dfrac{\sigma^{2*}_{X(1)Y[1]}}{\sigma^{2*}_{X(1)}\sigma^{2*}_{Y[1]}}$.

Note that the minimum MSE in (9) is of similar form to the MSE of the regression estimator based on the single auxiliary variable under $ERSS_c$. Also from (1) and (5), several different forms of ratio and product estimators can be generalized by taking different values of α,g, a and b. It is to be noted that for g = +1 and $g=-1$, we can make the ratio and product family of estimators respectively under $ERSS_a$ and $ERSS_c$ using the single auxiliary variable.

5. Proposed estimators using the two auxiliary variables

5.1. A family of ratio estimators in $ERSS_a$

Following Khoshnevisan *et al.* (2007), we propose a family of ratio estimators in $ERSS_a$ using information on the two auxiliary variables, when ranking is performed on the auxiliary variable Z.

$$\hat{\hat{Y}}'_{ERSS_a} = \bar{Y}_{[a]}\left[\frac{a\mu_X+b}{\alpha_1\left(a\,\bar{X}_{[a]}+b\right)+(1-\alpha_1)(a\mu_X+b)}\right]^{g_1}\left[\frac{c\mu_Z+d}{\alpha_2\left(c\,\bar{Z}_{(a)}+d\right)+(1-\alpha_2)(c\mu_Z+d)}\right]^{g_2}, \quad (10)$$

where α_1, α_2, g_1 and g_2 are suitable constants, also a, b, c and d are either real numbers or functions of known parameters for the auxiliary variables X and Z respectively.

Using multivariate Taylor series expansion, we have

$$\left(\hat{\hat{Y}}'_{ERSS_a}-\mu_Y\right) \cong \frac{1}{2}\left[\bar{Y}_{[1]}-E\left(\bar{Y}_{[1]}\right)\right]+\frac{1}{2}\left[\bar{Y}_{[m]}-E\left(\bar{Y}_{[m]}\right)\right]-\frac{\mu_Y(a\alpha_1 g_1)\left[\bar{X}_{[1]}-E\left(\bar{X}_{[1]}\right)\right]}{2(a\mu_X+b)} - \frac{\mu_Y(a\alpha_1 g_1)\left[\bar{X}_{[m]}-E\left(\bar{X}_{[m]}\right)\right]}{2(a\mu_X+b)}-\frac{\mu_Y(c\alpha_2 g_2)\left[\bar{Z}_{(1)}-E\left(\bar{Z}_{(1)}\right)\right]}{2(c\mu_Z+d)} - \frac{\mu_Y(c\alpha_2 g_2)\left[\bar{Z}_{(m)}-E\left(\bar{Z}_{(m)}\right)\right]}{2(c\mu_Z+d)}. \quad (11)$$

Squaring both sides, taking expectation of (11) and using assumption of symmetry of distribution, the MSE of $\hat{\hat{Y}}'_{ERSS_a}$ is given by

$$MSE\left(\hat{\hat{Y}}'_{ERSS_a}\right) \cong \frac{1}{mr}\left(\sigma^2_{Y[1]}+w_1^2\sigma^2_{X[1]}+w_2^2\sigma^2_{Z(1)}-2w_1\sigma_{X[1]Y[1]}-2w_2\sigma_{Y[1]Z(1)}+2w_1w_2\sigma_{X[1]Z(1)}\right). \quad (12)$$

Minimizing $MSE\left(\hat{\hat{Y}}'_{ERSS_a}\right)$ with respect to w_1 and w_2, the optimum values of w_1 and w_2, are given by

$$w_{1(opt)} = \frac{\sigma^2_{Z(1)}\sigma_{X[1]Y[1]}-\sigma_{X[1]Z(1)}\sigma_{Y[1]Z(1)}}{\sigma^2_{X[1]}\sigma^2_{Z(1)}-\sigma^2_{X[1]Z(1)}}$$

and

$$w_{2(opt)} = \frac{\sigma^2_{X[1]}\sigma_{Y[1]Z(1)} - \sigma_{X[1]Z(1)}\sigma_{X[1]Y[1]}}{\sigma^2_{X[1]}\sigma^2_{Z(1)} - \sigma^2_{X[1]Z(1)}}.$$

Substituting the optimum values of w_1 and w_2 in (12), we get

$$MSE_{\min}\left(\hat{\bar{Y}}'_{ERSS_a}\right) \cong \frac{\sigma^2_{Y[1]}\left(1 - R^2_{Y[1].X[1]Z(1)}\right)}{mr}, \tag{13}$$

where $R^2_{Y[1].X[1]Z(1)} = \frac{\rho^2_{X[1]Y[1]} + \rho^2_{Y[1]Z(1)} - 2\rho_{X[1]Y[1]}\rho_{Y[1]Z(1)}\rho_{X[1]Z(1)}}{1 - \rho^2_{X[1]Z(1)}}$ is the multiple correlation coefficient of $Y[1]$ on $X[1]$ and $Z(1)$ in $ERSS_a$. The minimum MSE of $\hat{\bar{Y}}'_{ERSS_a}$ is equal to the MSE of the regression estimator when using the two auxiliary variables.

5.2. A family of ratio estimators in $ERSS_c$

We propose a following family of estimators in $ERSS_c$ using the two auxiliary variables X and Z as

$$\hat{\bar{Y}}'_{ERSS_c} =$$

$$\bar{Y}_{[c]}\left[\frac{a\mu_X + b}{\alpha_1\left(a\bar{X}_{[c]} + b\right) + (1-\alpha_1)(a\mu_X + b)}\right]^{g_1}\left[\frac{c\mu_Z + d}{\alpha_2\left(c\bar{Z}_{(c)} + d\right) + (1-\alpha_2)(c\mu_Z + d)}\right]^{g_2}, \tag{14}$$

where α_1,α_2,g_1, g_2, a, b, c and d are suitable constants as described earlier.

Using multivariate Taylor series expansion, we have

$$\begin{aligned}\left(\hat{\bar{Y}}'_{ERSS_c} - \mu_Y\right) \cong{}& \frac{(m-1)}{2m}\left[\bar{Y}'_{[1]} - E\left(\bar{Y}'_{[1]}\right)\right] + \frac{(m-1)}{2m}\left[\bar{Y}'_{[m]} - E\left(\bar{Y}'_{[m]}\right)\right] \\ & - \frac{1}{m}\left[\bar{Y}'_{\left[\frac{m+1}{2}\right]} - E\left(\bar{Y}'_{\left[\frac{m+1}{2}\right]}\right)\right] - \frac{\mu_Y(a\alpha_1 g_1)(m-1)\left[\bar{X}'_{[1]} - E\left(\bar{X}'_{[1]}\right)\right]}{2m(a\mu_X + b)} \\ & - \frac{\mu_Y(a\alpha_1 g_1)(m-1)\left[\bar{X}'_{[m]} - E\left(\bar{X}'_{[m]}\right)\right]}{2m(a\mu_X + b)} \\ & - \frac{\mu_Y(c\alpha_2 g_2)(m-1)\left[\bar{Z}'_{(1)} - E\left(\bar{Z}'_{(1)}\right)\right]}{2m(c\mu_Z + d)}\end{aligned}$$

$$-\frac{\mu_Y\left(c\alpha_2 g_2\right)(m-1)\left[\bar{Z}'_{(m)}-E\left(\bar{Z}'_{(m)}\right)\right]}{2m\left(c\mu_Z+d\right)}$$

$$-\frac{\mu_Y\left(a\alpha_1 g_1\right)\left[\bar{X}'_{\left[\frac{m+1}{2}\right]}-E\left(\bar{X}'_{\left[\frac{m+1}{2}\right]}\right)\right]}{m\left(a\mu_X+b\right)}-\frac{\mu_Y\left(c\alpha_2 g_2\right)\left[\bar{Z}'_{\left(\frac{m+1}{2}\right)}-E\left(\bar{Z}'_{\left(\frac{m+1}{2}\right)}\right)\right]}{m\left(c\mu_Z+d\right)}. \quad (15)$$

Squaring, taking expectation and using assumption of symmetry of distribution, we have

$$MSE\left(\hat{\bar{Y}}'_{ERSS_c}\right)\cong\frac{1}{mr}\left(\frac{(m-1)\,\sigma^2_{Y[1]}+\sigma^2_{Y\left[\frac{m+1}{2}\right]}}{m}+w_1^2\frac{(m-1)\,\sigma^2_{X[1]}+\sigma^2_{X\left[\frac{m+1}{2}\right]}}{m}\right.$$

$$+w_2^2\frac{(m-1)\,\sigma^2_{Z(1)}+\sigma^2_{Z\left(\frac{m+1}{2}\right)}}{m}-2w_1\frac{(m-1)\,\sigma_{X[1]Y[1]}+\sigma_{X\left[\frac{m+1}{2}\right]Y\left[\frac{m+1}{2}\right]}}{m}$$

$$\left.-2w_2\frac{(m-1)\,\sigma_{Y[1]Z(1)}+\sigma_{Y\left[\frac{m+1}{2}\right]Z\left(\frac{m+1}{2}\right)}}{m}+2w_1w_2\frac{(m-1)\,\sigma_{X[1]Z(1)}+\sigma_{X\left[\frac{m+1}{2}\right]Z\left(\frac{m+1}{2}\right)}}{m}\right). \quad (16)$$

The above expression can be written as

$$MSE\left(\hat{\bar{Y}}'_{ERSS_c}\right)\cong$$

$$\frac{1}{mr}\left[\sigma^{2*}_{Y[1]}+w_1^2\sigma^{2*}_{X[1]}+w_2^2\sigma^{2*}_{Z(1)}-2w_1\sigma^{*}_{X[1]Y[1]}-2w_2\sigma^{*}_{Y[1]Z(1)}+2w_1w_2\sigma^{*}_{X[1]Z(1)}\right], \quad (17)$$

where

$$w_1=\frac{\mu_Y\left(a\alpha_1 g_1\right)}{\left(a\mu_X+b\right)},\qquad w_2=\frac{\mu_Y\left(c\alpha_2 g_2\right)}{\left(c\mu_Z+d\right)},\qquad \sigma^{2*}_{Y[1]}=\frac{(m-1)\,\sigma^2_{Y[1]}+\sigma^2_{Y\left[\frac{m+1}{2}\right]}}{m},$$

$$\sigma^{2*}_{X[1]}=\frac{(m-1)\,\sigma^2_{X[1]}+\sigma^2_{X\left[\frac{m+1}{2}\right]}}{m},\qquad \sigma^{2*}_{Z(1)}=\frac{(m-1)\,\sigma^2_{Z(1)}+\sigma^2_{Z\left(\frac{m+1}{2}\right)}}{m},$$

$$\sigma^{*}_{X[1]Y[1]}=\frac{(m-1)\,\sigma_{X[1]Y[1]}+\sigma_{X\left[\frac{m+1}{2}\right]Y\left[\frac{m+1}{2}\right]}}{m},$$

$$\sigma^*_{Y[1]Z(1)} = \frac{(m-1)\,\sigma_{Y[1]Z(1)} + \sigma_{Y\left[\frac{m+1}{2}\right]Z\left(\frac{m+1}{2}\right)}}{m}$$

and

$$\sigma^*_{X[1]Z(1)} = \frac{(m-1)\,\sigma_{X[1]Z(1)} + \sigma_{X\left[\frac{m+1}{2}\right]Z\left(\frac{m+1}{2}\right)}}{m}.$$

Using (17), the optimum values of w_1 and w_2 are given by

$$w_{1(opt)} = \frac{\sigma^{2*}_{Z(1)}\sigma^*_{X[1]Y[1]} - \sigma^*_{X[1]Z(1)}\sigma^*_{Y[1]Z(1)}}{\sigma^{2*}_{X[1]}\sigma^{2*}_{Z(1)} - \sigma^{2*}_{X[1]Z(1)}}$$

and

$$w_{2(opt)} = \frac{\sigma^{2*}_{X[1]}\sigma^*_{Y[1]Z(1)} - \sigma^*_{X[1]Z(1)}\sigma^*_{X[1]Y[1]}}{\sigma^{2*}_{X[1]}\sigma^{2*}_{Z(1)} - \sigma^{2*}_{X[1]Z(1)}}.$$

Substituting the optimum values of w_1 and w_2 in (17), we get the minimum MSE of $\hat{\hat{Y}}'_{ERSS_c}$, which is given by

$$MSE_{\min}\left(\hat{\hat{Y}}'_{ERSS_c}\right) \cong \frac{\sigma^{2*}_{Y[1]}\left(1 - R^{2*}_{Y[1].X[1]Z(1)}\right)}{mr}, \tag{18}$$

where

$$R^{2*}_{Y[1].X[1]Z(1)} = \frac{\rho^{2*}_{X[1]Y[1]} + \rho^{2*}_{Y[1]Z(1)} - 2\rho^*_{X[1]Y[1]}\rho^*_{Y[1]Z(1)}\rho^*_{X[1]Z(1)}}{\left(1 - \rho^{2*}_{X[1]Z(1)}\right)}$$

is the multiple correlation coefficient of $Y[1]$ on $X[1]$ and $Z(1)$ in $ERSS_c$. The expression given in (18) is equal to the MSE of the regression estimator when using the two auxiliary variables under $ERSS_c$.

Note: For different choices of g_1 and g_2 in (10) and (14), we have

$g_1 = g_2 = 1$,	ratio estimator,
$g_1 = g_2 = -1$,	product estimator,
$g_1 = 1$ and $g_2 = -1$,	ratio-product estimator,
$g_1 = -1$ and $g_2 = 1$,	product-ratio estimator.

6. Simulation study

A simulation study has been made to examine the performance of the considered estimators in *SRS* and *ERSS* for estimating the population mean, when ranking is done on the auxiliary variables X and Z separately. Following Samawi (2002), bivariate random observations were generated from bivariate normal distribution having parameters $\mu_X = 6$, $\mu_Y = 3$, $\sigma_X = \sigma_Y = 1$ and $\rho_{XY} = \pm 0.99$, ± 0.95, ± 0.90, ± 0.70 and ± 0.50. Using 4000 simulations, estimates of *MSE*s for ratio estimators were computed as given in Tables 1-5 (see Appendix). We consider $m(r)$ as 4(2), 4(4), 5(2), 6(2) and 6(4) respectively to study the performances of the ratio estimators under *SRS*, $ERSS_a$ and $ERSS_c$.

Further simulation has also been done for the same family of ratio estimators using the two auxiliary variables. For this trivariate random observations were generated from trivariate normal distribution having parameters $\mu_X = 6$, $\mu_Y = 3$, $\mu_Z = 8$, $\sigma_X = \sigma_Y = \sigma_Z = 1$ and for different values of ρ_{XY}. The correlation coefficients between (Y, Z) and (X, Z) are assumed to be $\rho_{YZ} = 0.70$ and $\rho_{XZ} = 0.60$ respectively as shown in Tables 6-8, with different sample sizes m and different cycles r. Again 4000 simulations have been made to study the performances of a family of the ratio estimators using the two auxiliary variables.

From Tables 1-5 (see Appendix), it is noted that all considered ratio estimators using the one auxiliary variable (X) perform better under *ERSS* as compared to *SRS* for different values of ρ_{XY}. In the case of using the two auxiliary variables X and Z (see Tables 6-8, Appendix), for $r = 1$ and $r = 2$, *ERSS* again gives more precise estimates as compared to *SRS*. Also as we increase r=1 to $r = 2$, the *MSE* values of each estimator decreases under both *SRS* and *ERSS* schemes.

6.1. Empirical study

In this section, we have illustrated the performance of various estimators of population mean under *SRS* and *ERSS* through natural data sets. *ERSS* performs better than *SRS* in case of symmetric populations. In order to generate the symmetric data from positively skewed data, we have taken the logarithm of the study variable (Y) and the auxiliary variables (X and Z).

Table 9 provides the estimated *MSE* values of all considered estimators using the single auxiliary variable (X) based on 4000 samples drawn with replacement. It is immediate to observe that the proposed estimators under *ERSS* perform better than the estimators based on *SRS*. Among all estimators, the estimator $\hat{\bar{Y}}_{1ERSS_a}$ is more efficient for all values of m.

Table 10 gives the estimated *MSE* values of all considered estimators using the two auxiliary variables (X and Z) based on 4000 samples drawn with replacement. The proposed estimators under *ERSS* also perform better than the estimators based on *SRS*. For this data set, the estimator $\hat{\bar{Y}}'_{1ERSS_a}$ has the smaller *MSE* values than other considered estimators $\hat{\bar{Y}}'_{iERSS_a}$ $(i = 2, 3, 4)$.

7. Conclusion

In the present paper, we have studied the problem of estimating the population mean using single and two auxiliary variables in *ERSS*, when we have known information about the population parameters. A given family of estimators includes several ratio type estimators, which have also been adopted by different authors in *SRS*. We examined the effect of transformations on the same family of estimators in *ERSS*. From Tables 1-5, the estimators $\hat{\bar{Y}}_{4ERSS_a}$ and $\hat{\bar{Y}}_{4ERSS_c}$, with $a = \alpha = g = 1$ and $b = S_X$, perform better than all other estimators when $\rho_{XY} < 0$. In Tables 1-5 for $\rho_{XY} > 0$, the estimator $\hat{\bar{Y}}_{3ERSS_a}$, with $a = \beta_{2X}$, $\alpha = g = 1$ and $b = C_X$, generally give more precise estimates as compared to other estimators. In case of two auxiliary variables (see Tables 7 and 8), the ratio estimators $\hat{\bar{Y}}'_{3ERSS_a}$ and $\hat{\bar{Y}}'_{3ERSS_c}$, with choices $\alpha_1 = \alpha_2 = g_1 = g_2 = 1$, $a = \beta_{2X}$, $b = C_X$, $c = \beta_{2Z}$ and $d = C_Z$, are efficient in all other estimators for all values of ρ_{XY} with different sample size m. Finally, it is recommended to use *ERSS* over *SRS* in symmetric populations, in order to get more precise estimates of population mean.

References

Khoshnevisan, M., Singh, R., Chauhan, P., Sawan, N. & Smarandache, F. (2007). A general family of estimators for estimating population mean using known value of some population parameter(s). *Far East Journal of Theoretical Statistics*, 22, 181-191.

McIntyre, G. A. (1952). A method of unbiased selective sampling using ranked sets. *Australian Journal Agriculture Research*, 3, 385-390.

Murthy, M. N. (1967). *Sampling Theory and Methods*. Statistical Publishing Society, Calcutta, India.

Samawi, H. M. (2002). On ratio estimation using extreme ranked set samples. *Abhath Al-Yarmouk, Basic Sciences and Engineering*, 11, 815-827.

Samawi, H. M., Ahmed, M. S. & Abu-Dayyeh, W. (1996). Estimating the population mean using extreme ranked set sample. *Biometrical Journal*, 38(5), 577-586.

Samawi, H. M., Al-Samarraie, A. Y. A. & Al-Saidy, O. M. (2004). On regression estimators using extreme ranked set samples. *Journal of Science of Technology Sultan, Qaboos Universtiy*, 9, 67-86.

Samawi, H. M. & Muttlak, H. A. (1996). Estimation of ratio using rank set sampling. *Biometrical Journal*, 38, 753-764.

Samawi, H. M. & Saeid, L. J. (2004). Stratified extreme ranked set sample with application to ratio estimators. *Journal of Modern Applied Statistical Methods*, 3(1) 117-133.

Takahasi, K. & Wakimoto, K. (1968). On unbiased estimates of the population mean based on the sample stratified by means of ordering. *Annals of Institute of Statistical Mathematics*, 20, 1-31.

Appendix

Table 1: *MSE values of different estimators using SRS and ERSS for* $m = 4, r = 2$.

Estimator	ρ_{XY}	0.99	0.9	0.8	0.7	0.5	−0.99	−0.9	−0.8	−0.7	−0.5
$\hat{\hat{Y}}_{1ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{(a)}+\rho_{XY}}\right]$	*SRS*	0.0421445	0.0459502	0.0536968	0.0768756	0.0966342	0.3431534	0.332503	0.3220496	0.2717683	0.2336323
	ERSS	0.0213953	0.0280724	0.0375345	0.0680077	0.0953278	0.1614094	0.1560954	0.1592598	0.1694472	0.1643586
$\hat{\hat{Y}}_{2ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+C_X}{\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.0342081	0.0394587	0.0459384	0.0698305	0.0957964	0.2810607	0.2736063	0.2645737	0.2302537	0.2125556
	ERSS	0.0186795	0.0259662	0.0352951	0.0643397	0.0906895	0.1374189	0.1399317	0.1437876	0.1578388	0.1538846
$\hat{\hat{Y}}_{3ERSS_a} = \bar{y}_{[a]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.0343591	0.0376966	0.046057	0.0689711	0.0985591	0.2839814	0.2827303	0.2785348	0.2435255	0.2236065
	ERSS	0.0178802	0.0247913	0.0346036	0.0654427	0.0937825	0.1419155	0.1458068	0.141693	0.1573677	0.1596848
$\hat{\hat{Y}}_{4ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+S_X}{\bar{x}_{(a)}+S_X}\right]$	*SRS*	0.0431915	0.0477002	0.0535503	0.0718311	0.094948	0.2536793	0.2470929	0.2436818	0.2271933	0.2072925
	ERSS	0.0227204	0.0286125	0.0369096	0.0703841	0.09294	0.1271421	0.1330668	0.1314235	0.1430808	0.1503416

Table 2: *MSE values of different estimators using SRS and ERSS for* $m = 4, r = 4$.

Estimator	ρ_{XY}	0.99	0.9	0.8	0.7	0.5	−0.99	−0.9	−0.8	−0.7	−0.5
$\hat{\hat{Y}}_{1ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{(a)}+\rho_{XY}}\right]$	*SRS*	0.021717	0.0235508	0.0249642	0.0366157	0.0464009	0.1658218	0.1563325	0.152332	0.1354153	0.1171365
	ERSS	0.0107867	0.0147103	0.0190514	0.0347934	0.0460406	0.0783965	0.0774434	0.0807905	0.0812439	0.0815075
$\hat{\hat{Y}}_{2ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+C_X}{\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.0174917	0.0190349	0.0221608	0.0349446	0.045639	0.1358885	0.1438294	0.1313778	0.1232827	0.1116966
	ERSS	0.0092618	0.0131617	0.01722	0.0320217	0.0465724	0.0681436	0.0708875	0.0721552	0.0742076	0.0752605
$\hat{\hat{Y}}_{3ERSS_a} = \bar{y}_{[a]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.0168429	0.0192322	0.0216478	0.0353861	0.0478808	0.1430786	0.1383039	0.1367204	0.116574	0.1148881
	ERSS	0.0089404	0.0123172	0.0166099	0.0348346	0.0480005	0.0686419	0.0688538	0.0731911	0.0790872	0.0751499
$\hat{\hat{Y}}_{4ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+S_X}{\bar{x}_{(a)}+S_X}\right]$	*SRS*	0.0218861	0.0244523	0.0257768	0.0381466	0.0447497	0.123449	0.1257248	0.12019	0.1146243	0.1009705
	ERSS	0.0110367	0.0144961	0.0186961	0.0342148	0.0485601	0.0637671	0.0646209	0.0668673	0.0728451	0.0727519

Table 3: MSE values of different estimators using SRS and ERSS for $m = 6, r = 2$.

Estimator	ρ_{XY}	0.99	0.9	0.8	0.7	0.5	−0.99	−0.9	−0.8	−0.7	−0.5
$\hat{\hat{Y}}_{1ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{(a)}+\rho_{XY}}\right]$	*SRS*	0.0276638	0.0310981	0.0339876	0.0492321	0.0623409	0.2198379	0.2056067	0.2040597	0.1750035	0.15697
	ERSS	0.0125093	0.0179924	0.0232678	0.0444828	0.0625907	0.0909455	0.0915863	0.0956375	0.0963579	0.1004085
$\hat{\hat{Y}}_{2ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+C_X}{\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.0228799	0.028244	0.0311799	0.0452084	0.0644545	0.1936849	0.1822071	0.1833652	0.1721255	0.1467526
	ERSS	0.0102854	0.0154112	0.0217249	0.0437198	0.0611709	0.0788966	0.0813828	0.08313	0.0966679	0.098809
$\hat{\hat{Y}}_{3ERSS_a} = \bar{y}_{[a]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.023543	0.0266576	0.0307184	0.0468753	0.0627374	0.1894285	0.187493	0.1834673	0.1657038	0.148337
	ERSS	0.0101325	0.0154559	0.0211064	0.0454989	0.0635411	0.0769606	0.0848768	0.08612	0.0936018	0.0972394
$\hat{\hat{Y}}_{4ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+S_X}{\bar{x}_{(a)}+S_X}\right]$	*SRS*	0.0284588	0.0308181	0.0338651	0.048739	0.0646897	0.1614895	0.1726636	0.1640717	0.1508197	0.1369065
	ERSS	0.0128794	0.0173838	0.0238402	0.0439633	0.0633808	0.0704374	0.0764649	0.0776193	0.0874932	0.0983925

Table 4: MSE values of different estimators using SRS and ERSS for $m = 6, r = 4$.

Estimator	ρ_{XY}	0.99	0.9	0.8	0.7	0.5	−0.99	−0.9	−0.8	−0.7	−0.5
$\hat{\hat{Y}}_{1ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{(a)}+\rho_{XY}}\right]$	*SRS*	0.0133039	0.0148865	0.0170387	0.0239208	0.0312614	0.1091832	0.1102434	0.0999666	0.090003	0.0764261
	ERSS	0.0065831	0.0088927	0.0114613	0.0226373	0.0310097	0.0456315	0.0447123	0.0472675	0.0501376	0.0510662
$\hat{\hat{Y}}_{2ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+C_X}{\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.0117771	0.0131748	0.0154717	0.0228518	0.030882	0.0919903	0.0920865	0.0866318	0.0833786	0.0718192
	ERSS	0.0052139	0.007885	0.0110488	0.0216899	0.0321166	0.0385758	0.0400184	0.0407774	0.0457131	0.0481164
$\hat{\hat{Y}}_{3ERSS_a} = \bar{y}_{[a]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{(a)}+C_X}\right]$	*SRS*	0.011078	0.013142	0.0149048	0.0243094	0.0308484	0.0915886	0.0917993	0.0891019	0.079593	0.0754453
	ERSS	0.0048665	0.0075851	0.0106771	0.0222789	0.0308866	0.0390286	0.0414351	0.0413122	0.0467898	0.0487322
$\hat{\hat{Y}}_{4ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+S_X}{\bar{x}_{(a)}+S_X}\right]$	*SRS*	0.013897	0.0148891	0.0174452	0.0236295	0.0314632	0.0877026	0.0829768	0.0808907	0.0719681	0.0665056
	ERSS	0.0062653	0.0092167	0.0120893	0.022158	0.0312568	0.0361718	0.0369842	0.0391167	0.0410935	0.0468384

Table 5: *MSE values of different estimators using SRS and ERSS for $m=5, r=2$.*

Estimator	ρ_{XY}	0.99	0.9	0.8	0.7	0.5	-0.99	-0.9	-0.8	-0.7	-0.5
$\hat{\hat{Y}}_{1ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{(c)}+\rho_{XY}}\right]$	*SRS*	0.0337842	0.0364239	0.0406543	0.0606875	0.078297	0.262396	0.2657483	0.255815	0.2125039	0.1911728
	ERSS	0.0156276	0.0217249	0.028424	0.0537139	0.0739129	0.1054132	0.1133406	0.1088354	0.1215593	0.1208256
$\hat{\hat{Y}}_{2ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+C_X}{\bar{x}_{(c)}+C_X}\right]$	*SRS*	0.0272919	0.0313738	0.0354903	0.0559322	0.0752243	0.2202212	0.2207612	0.2131319	0.1939198	0.1723402
	ERSS	0.013084	0.0188902	0.0268646	0.0524263	0.0737738	0.091501	0.0921889	0.0989449	0.1129062	0.1136797
$\hat{\hat{Y}}_{3ERSS_c} = \bar{y}_{[c]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{(c)}+C_X}\right]$	*SRS*	0.0277326	0.0304337	0.0352675	0.0577364	0.0741742	0.2386541	0.2225659	0.21779	0.1927738	0.182693
	ERSS	0.0123854	0.0193276	0.0258538	0.0518648	0.0761968	0.0972475	0.0981524	0.1052062	0.1178463	0.1166842
$\hat{\hat{Y}}_{4ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+S_X}{\bar{x}_{(c)}+S_X}\right]$	*SRS*	0.0353522	0.0371511	0.0436183	0.0577978	0.0757769	0.2082045	0.2042378	0.2015726	0.1789549	0.1646477
	ERSS	0.0147976	0.0206443	0.0283466	0.053068	0.0742967	0.0853865	0.0904359	0.0950412	0.1041661	0.1122718

Table 6: *MSE values of different estimators using SRS and ERSS for $m=4, r=1$.*

Estimator	ρ_{XY}	0.99	0.95	0.90	0.80	0.75
$\hat{\hat{Y}}'_{1ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{[a]}+\rho_{XY}}\right]\left[\frac{\mu_Z+\rho_{YZ}}{\bar{z}_{(a)}+\rho_{YZ}}\right]$	*SRS*	0.0407724	0.0501944	0.0592333	0.0824846	0.0930625
	ERSS	0.0389418	0.0395594	0.0377836	0.0373521	0.0376105
$\hat{\hat{Y}}'_{2ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+C_X}{\bar{x}_{[a]}+C_X}\right]\left[\frac{\mu_Z+C_Z}{\bar{z}_{(a)}+C_Z}\right]$	*SRS*	0.0325959	0.0391727	0.0554632	0.0779341	0.0896016
	ERSS	0.0289404	0.029944	0.0299202	0.0294217	0.0303673
$\hat{\hat{Y}}'_{3ERSS_a} = \bar{y}_{[a]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{[a]}+C_X}\right]\left[\frac{\beta_{2Z}\mu_Z+C_Z}{\beta_{2Z}\bar{z}_{(a)}+C_Z}\right]$	*SRS*	0.0311288	0.0393582	0.0503898	0.0760209	0.0873974
	ERSS	0.0269807	0.0272861	0.0293202	0.0304836	0.0275289
$\hat{\hat{Y}}'_{4ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X+S_X}{\bar{x}_{[a]}+S_X}\right]\left[\frac{\mu_Z+S_Z}{\bar{z}_{(a)}+S_Z}\right]$	*SRS*	0.0446412	0.0502939	0.0623651	0.0846077	0.0935501
	ERSS	0.0386108	0.0411331	0.0394753	0.0393358	0.0392025

Table 7: *MSE values of different estimators using SRS and ERSS for $m = 5, r = 1$.*

Estimator	ρ_{XY}	0.99	0.95	0.90	0.80	0.75
$\hat{\bar{Y}}'_{1ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{[c]}+\rho_{XY}}\right]\left[\frac{\mu_Z+\rho_{YZ}}{\bar{z}_{(c)}+\rho_{YZ}}\right]$	*SRS*	0.0317161	0.0380387	0.0458682	0.0654852	0.0720379
	ERSS	0.0299674	0.028886	0.0288459	0.0299894	0.0306687
$\hat{\bar{Y}}'_{2ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+C_X}{\bar{x}_{[c]}+C_X}\right]\left[\frac{\mu_Z+C_Z}{\bar{z}_{(c)}+C_Z}\right]$	*SRS*	0.0242259	0.0321638	0.0398709	0.0599825	0.0709018
	ERSS	0.0221972	0.0233933	0.022709	0.0232737	0.0233012
$\hat{\bar{Y}}'_{3ERSS_c} = \bar{y}_{[c]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{[c]}+C_X}\right]\left[\frac{\beta_{2Z}\mu_Z+C_Z}{\beta_{2Z}\bar{z}_{(c)}+C_Z}\right]$	*SRS*	0.0238334	0.0311472	0.0413165	0.0602494	0.0688909
	ERSS	0.0214479	0.0212721	0.0222061	0.0205364	0.0219412
$\hat{\bar{Y}}'_{4ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+S_X}{\bar{x}_{[c]}+S_X}\right]\left[\frac{\mu_Z+S_Z}{\bar{z}_{(c)}+S_Z}\right]$	*SRS*	0.0330982	0.0391957	0.049966	0.069275	0.0761463
	ERSS	0.029793	0.0300217	0.0308689	0.0312941	0.0305486

Table 8: *MSE values of different estimators using SRS and ERSS for $m = 5, r = 2$.*

Estimator	ρ_{XY}	0.99	0.95	0.90	0.80	0.75
$\hat{\bar{Y}}'_{1ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+\rho_{XY}}{\bar{x}_{[c]}+\rho_{XY}}\right]\left[\frac{\mu_Z+\rho_{YZ}}{\bar{z}_{(c)}+\rho_{YZ}}\right]$	*SRS*	0.0159022	0.0182493	0.0223628	0.0317303	0.0351792
	ERSS	0.0149805	0.0146252	0.0147981	0.0150615	0.0149301
$\hat{\bar{Y}}'_{2ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+C_X}{\bar{x}_{[c]}+C_X}\right]\left[\frac{\mu_Z+C_Z}{\bar{z}_{(c)}+C_Z}\right]$	*SRS*	0.0115080	0.0154952	0.0197208	0.0313548	0.0351996
	ERSS	0.0110513	0.0109730	0.0110084	0.0111618	0.0111697
$\hat{\bar{Y}}'_{3ERSS_c} = \bar{y}_{[c]}\left[\frac{\beta_{2X}\mu_X+C_X}{\beta_{2X}\bar{x}_{[c]}+C_X}\right]\left[\frac{\beta_{2Z}\mu_Z+C_Z}{\beta_{2Z}\bar{z}_{(c)}+C_Z}\right]$	*SRS*	0.0109745	0.0143726	0.0195953	0.0311480	0.0356664
	ERSS	0.0107352	0.0106756	0.0103141	0.0105737	0.0103239
$\hat{\bar{Y}}'_{4ERSS_c} = \bar{y}_{[c]}\left[\frac{\mu_X+S_X}{\bar{x}_{[c]}+S_X}\right]\left[\frac{\mu_Z+S_Z}{\bar{z}_{(c)}+S_Z}\right]$	*SRS*	0.0160405	0.0187243	0.0234514	0.0316207	0.0367001
	ERSS	0.0146324	0.0153025	0.0146813	0.0143856	0.0153041

Population-I: Source: Murthy (1967).
$\log(Y)$: output of a factory, $\log(X)$: fixed capital.
$N = 80$, $m = 4$, 6, 8, $r = 1$, $\mu_Y = 8.480904$, $\mu_X = 6.750716$ and $\rho_{XY} = 0.9640175$.

Table 9: *Estimated MSE values.*

Estimator	*SRS* and *ERSS*	$m = 4$	$m = 6$	$m = 8$
$\hat{\hat{Y}}_{1ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X + \rho_{XY}}{\bar{x}_{(a)} + \rho_{XY}}\right]$	*SRS*	0.05314492	0.03622249	0.02676690
	ERSS	0.02439540	0.01409169	0.01021740
$\hat{\hat{Y}}_{2ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X + C_X}{\bar{x}_{(a)} + C_X}\right]$	*SRS*	0.07966297	0.05430063	0.04004554
	ERSS	0.03555703	0.01935204	0.01299920
$\hat{\hat{Y}}_{3ERSS_a} = \bar{y}_{[a]}\left[\frac{\beta_{2X}\mu_X + C_X}{\beta_{2X}\bar{x}_{(a)} + C_X}\right]$	*SRS*	0.08237797	0.05614814	0.04140083
	ERSS	0.03670370	0.01989512	0.01329300
$\hat{\hat{Y}}_{4ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X + S_X}{\bar{x}_{(a)} + S_X}\right]$	*SRS*	0.05842170	0.03982448	0.02941517
	ERSS	0.02660899	0.01513016	0.01075533

Population-II: Source: Murthy (1967).
$\log(Y)$: output of a factory, $\log(X)$: fixed capital and $\log(Z)$: number of workers.
$N = 80$, $m = 4, 6, 8$, $r = 1$, $\mu_Y = 8.480904$, $\mu_X = 6.750716$, $\mu_Z = 5.233816$,
$\rho_{XY} = 0.9640175$ and $\rho_{YZ} = 0.916134$.

Table 10: *Estimated MSE values.*

Estimator	*SRS* and *ERSS*	$m = 4$	$m = 6$	$m = 8$
$\hat{\hat{Y}}'_{1ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X + \rho_{XY}}{\bar{x}_{[a]} + \rho_{XY}}\right]\left[\frac{\mu_Z + \rho_{YZ}}{\bar{z}_{(a)} + \rho_{YZ}}\right]$	*SRS*	0.7599230	0.4833926	0.3632148
	ERSS	0.2603936	0.1286980	0.0919748
$\hat{\hat{Y}}'_{2ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X + C_X}{\bar{x}_{[a]} + C_X}\right]\left[\frac{\mu_Z + C_Z}{\bar{z}_{(a)} + C_Z}\right]$	*SRS*	1.0409530	0.6582852	0.4931993
	ERSS	0.3510492	0.1686980	0.1163822
$\hat{\hat{Y}}'_{3ERSS_a} = \bar{y}_{[a]}\left[\frac{\beta_{2X}\mu_X + C_X}{\beta_{2X}\bar{x}_{[a]} + C_X}\right]\left[\frac{\beta_{2Z}\mu_Z + C_Z}{\beta_{2Z}\bar{z}_{(a)} + C_Z}\right]$	*SRS*	1.0751500	0.6794242	0.5088861
	ERSS	0.3618654	0.1734536	0.1193290
$\hat{\hat{Y}}'_{4ERSS_a} = \bar{y}_{[a]}\left[\frac{\mu_X + S_X}{\bar{x}_{[a]} + S_X}\right]\left[\frac{\mu_Z + S_Z}{\bar{z}_{(a)} + S_Z}\right]$	*SRS*	0.7816216	0.4970571	0.3733052
	ERSS	0.2679331	0.1318287	0.0934524