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A recursion formula for expected negative and positive powers of the central Wishart distribution

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Abstract

We use Haff's fundamental identity to express the expectation of S^p in lower-order terms, where S follows the central Wishart distribution.

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1 The recursion formula (positive powers)

Let $S \sim W_m(\Omega, n)$. Hence S follows the Central Wishart distribution with scale matrix $\Omega > 0$ and n degrees of freedom. We use Haff's Fundamental Identity (FI):

$$\mathcal{E} F_1 \Omega^{-1} F_2 = 2 \mathcal{E} F_1 \nabla F_2 + 2 (\mathcal{E} F_2' \nabla F_1')' + (n - m - 1) \mathcal{E} F_1 S^{-1} F_2,$$

with $F_1 = F_1(S)$ a differentiable matrix function of S and $n > m + 1$. Further ∇ is a matrix of differential operators with typical element

$$d_{ij} = \frac{1}{2} (1 + \delta_{ij}) \frac{\partial}{\partial s_{ij}} \quad (i, j = 1, \dots, m),$$

where δ_{ij} is a Kronecker delta: $\delta_{ii} = 1$ and $\delta_{ij} = 0$ ($i \neq j$).

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All matrices are square of dimension m . A very useful property is: when $dF = P'(dS)Q$ then $2\nabla F = PQ + (\text{tr } P)Q$, where dF is the differential of F . For these properties see Neudecker (2001). Further \mathcal{E} is the expectation operator.

We choose $F_1 = I_m$ and $F_2 = S^p$, ($p > 0$). By Haff's FI we get:

$$\mathcal{E}\Omega^{-1}S^p = 2\mathcal{E}\nabla S^p + (n - m - 1)\mathcal{E}S^{p-1},$$

because $\nabla I = 0$. For definition and computation of $\nabla(\cdot)$ see also Neudecker (2000).

Clearly

$$dS^p = \sum_{k=0}^{p-1} S^k (dS) S^{p-k-1}$$

where $S^0 = I_m$.

Hence

$$2\nabla S^p = (m + p)S^{p-1} + \sum_{k=1}^{p-1} (\text{tr } S^k) S^{p-k-1}.$$

Then

$$\Omega^{-1}\mathcal{E}S^p = (n + p - 1)\mathcal{E}S^{p-1} + \mathcal{E}\sum_{k=1}^{p-1} (\text{tr } S^k) S^{p-k-1}$$

or equivalently

$$\mathcal{E}S^p = (n + p - 1)\Omega\mathcal{E}S^{p-1} + \Omega\mathcal{E}\sum_{k=1}^{p-1} (\text{tr } S^k) S^{p-k-1}.$$

2 Discussion and application (positive powers)

The second recursion formula is not really suited for finding $\mathcal{E}S^p$, because this requires knowledge of $\mathcal{E}(\text{tr } S^k)S^{p-k-1}$ for $1 \leq k \leq p - 1$. This expression has to be found by other methods. See Neudecker (2001). The formula is, however, suited for establishing Loewner orderings. Relevant is that $\Omega\mathcal{E}S^p = \mathcal{E}S^p\Omega > 0$ (positive definite). This follows from the first recursion formula, which yields the identity $\Omega^{-1}\mathcal{E}S^p = \mathcal{E}S^p\Omega^{-1}$ and further the positive definiteness of the two expressions. Pre- and postmultiplication of the identity by matrix Ω yields: $\Omega\mathcal{E}S^p = \mathcal{E}S^p\Omega$. As Ω and $\mathcal{E}S^p$ are two commuting positive definite matrices, they are simultaneously diagonalizable by an orthogonal similarity transformation. Hence $T'\Omega T = \Lambda$ and $T'\mathcal{E}S^p T = M$, say, and consequently

$\Omega \mathcal{E} S^p = T \Lambda T' T M T' = T \Lambda M T' > 0$ (positive definite), as both Λ and M are diagonal positive definite. See e.g. Hadley (1972, ex. 7-14).

We further conclude that $\Omega^{-1} \mathcal{E} S^p > (n+p-1) \mathcal{E} S^{p-1}$ or equivalently $(\mathcal{E} S)^{-1} \mathcal{E} S^p > n^{-1}(n+p-1) \mathcal{E} S^{p-1}$ as $\mathcal{E} S = n \Omega$ ($p \geq 2$). The first inequality follows from the second recursion formula. Repeated substitution finally yields the inequality

$$\mathcal{E} S^p > (\mathcal{E} S)^p$$

3 The recursion formula (negative powers)

With $F_1 = I_m$ and $F_2 = S^{-p}$, we get by Haff's FI

$$\Omega^{-1} \mathcal{E} S^{-p} = 2 \mathcal{E} \nabla S^{-p} + (n-m-1) \mathcal{E} S^{-(p+1)} \quad (1)$$

For $p = 1, 2, \dots, n-m-2$, we clearly have

$$dS^{-p} = - \sum_{k=1}^p S^{-k} (dS) S^{k-(p+1)}$$

Hence

$$\begin{aligned} 2 \nabla S^{-p} &= \sum_{k=1}^p S^{-(p+1)} - \sum_{k=1}^p (\text{tr} S^{-k}) S^{k-(p+1)} \\ &= -p S^{-(p+1)} - \sum_{k=1}^p (\text{tr} S^{-k}) S^{k-(p+1)} \end{aligned}$$

Insertion in (1) leads to the recursion formula

$$\Omega^{-1} \mathcal{E} S^{-p} = (n-m-p-1) \mathcal{E} S^{-(p+1)} - \sum_{k=1}^p \mathcal{E} (\text{tr} S^{-k}) S^{k-(p+1)} \quad (2)$$

or equivalently

$$\mathcal{E} S^{-p} = (n-m-p-1) \Omega \mathcal{E} S^{-(p+1)} - \Omega \sum_{k=1}^p \mathcal{E} (\text{tr} S^{-k}) S^{k-(p+1)} \quad (3)$$

4 Discussion and application (negative powers)

Because of the symmetry of the RHS of (2) we have

$$\Omega^{-1} \mathcal{E} S^{-p} = (\mathcal{E} S^{-p}) \Omega^{-1}. \quad (4)$$

We shall prove that $\Omega^{-1} \mathcal{E} S^{-p} > 0$ (positive definite).

Proof. From the commutativity property (4) we conclude that $\mathcal{E} S^{-p} = T \Lambda_p T'$ say and $\Omega^{-1} = T M T'$ where T is orthogonal and M and Λ_p are diagonal positive definite. Hence

$$\Omega^{-\frac{1}{2}} = T M^{\frac{1}{2}} T'$$

and

$$\begin{aligned} \Omega^{-\frac{1}{2}} (\mathcal{E} S^{-p}) \Omega^{-\frac{1}{2}} &= T M^{\frac{1}{2}} T' T \Lambda_p T' T M^{\frac{1}{2}} T' \\ &= T M^{\frac{1}{2}} \Lambda_p M^{\frac{1}{2}} T' = T M \Lambda_p T' = T M T' T \Lambda_p T' \\ &= \Omega^{-1} \mathcal{E} S^{-p} \end{aligned}$$

Hence

$$\Omega^{-1} \mathcal{E} S^{-p} > 0 \quad \text{as} \quad \Omega^{-\frac{1}{2}} (\mathcal{E} S^{-p}) \Omega^{-\frac{1}{2}} > 0 \quad \square$$

Because $\mathcal{E} S^{-(p+1)} > 0$ and $\sum_{k=1}^p \mathcal{E} (\text{tr } S^{-k}) S^{k-(p+1)} > 0$ we conclude from (2) that

$$\Omega^{-1} \mathcal{E} S^{-p} < (n - m - p - 1) \mathcal{E} S^{-(p+1)} \quad (5)$$

when $p < n - m - 1$. Or equivalently

$$(\mathcal{E} S^{-1}) \mathcal{E} S^{-p} < (n - m - 1)^{-1} (n - m - p - 1) \mathcal{E} S^{-(p+1)} < \mathcal{E} S^{-(p+1)} \quad (6)$$

as

$$\mathcal{E} S^{-1} = (n - m - 1)^{-1} \Omega^{-1}$$

and $(n - m - 1)^{-1} (n - m - p - 1) < 1$.

The inequality (6) ultimately leads by successive substitution to the inequality

$$\mathcal{E} S^{-p} > (\mathcal{E} S^{-1})^p \quad (p \geq 2) \quad (7)$$

As $\mathcal{E}S^{-1} = n(n-m-1)^{-1}(\mathcal{E}S)^{-1} > (\mathcal{E}S)^{-1}$, we can get the inequality

$$\mathcal{E}S^{-p} > (\mathcal{E}S^{-1})^p > (\mathcal{E}S)^{-p} \quad (8)$$

It is known that

$$\mathcal{E}S^p > (\mathcal{E}S)^p \quad p \geq 2 \quad (9)$$

See the end of section 2.

Combining (8) and (9) we finally have the inequality

$$\mathcal{E}S^{-p} > (\mathcal{E}S^{-1})^p > (\mathcal{E}S)^{-p} > (\mathcal{E}S^p)^{-1} \quad (10)$$

5 References

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