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Asymptotically optimal filtering in linear systems with fractional Brownian noises⋆

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Abstract

In this paper, the filtering problem is revisited in the basic Gaussian homogeneous linear system driven by fractional Brownian motions. We exhibit a simple approximate filter which is asymptotically optimal in the sense that, when the observation time tends to infinity, the variance of the corresponding filtering error converges to the same limit as for the exact optimal filter.

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1 Introduction

Several contributions have been already reported around filtering problems concerning models where the driving processes are fractional Brownian motions (fBm's for short) : see Kleptsyna *et al.* (2000) for a rather general approach and further references. The specific case of a homogeneous linear system has been investigated in Kleptsyna and Le Breton (2002) where explicit closed form equations are derived both for the optimal

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filter and the variance of the filtering error. Moreover, therein it is shown that this filter is asymptotically stable in the sense that the variance of the filtering error converges to a finite limit as the observation time tends to infinity. Here our aim is to exhibit a simple approximate filter which has the same asymptotic behaviour as the optimal one. Let us fix this more precisely.

As in Kleptsyna and Le Breton (2002), we deal with real-valued processes $X = (X_t, t \geq 0)$ and $Y = (Y_t, t \geq 0)$, representing the signal and the observation respectively, governed by the following homogeneous linear system of stochastic differential equations interpreted as integral equations :

$$\begin{cases} dX_t &= \theta X_t dt + dV_t^H\,,\ t \geq 0\,,\ X_0 = 0\,, \\ dY_t &= \mu X_t dt + dW_t^H\,,\ t \geq 0\,,\ Y_0 = 0\,. \end{cases} \tag{1.1}$$

Here $V^H = (V_t^H, t \geq 0)$ and $W^H = (W_t^H, t \geq 0)$ are independent normalized fBm's with the same Hurst parameter H in $[\frac{1}{2}, 1)$ and the coefficients θ and $\mu \neq 0$ are fixed real constants. The system (1.1) has a uniquely defined solution process (X, Y) which is Gaussian. Supposing that only Y is observed but one wishes to know X, the classical problem of filtering the signal X at time t from the observation of Y up to time t occurs. The solution to this problem is the conditional distribution of X_t given $\{Y_s\,, 0 \leq s \leq t\}$, which of course is Gaussian. Then, it is completely determined by the conditional mean $\pi_t(X) = \mathbb{E}(X_t/\{Y_s\,, 0 \leq s \leq t\})$, which we shall call the *exact optimal filter*, and the variance $\gamma_{XX}(t) = \mathbb{E}(X_t - \pi_t(X))^2$ of the filtering error. In Kleptsyna and Le Breton (2002), a system of Volterra type integral equations for these characteristics is provided and the following stability property of the filter is also shown :

$$\lim_{t \to +\infty} \gamma_{XX}(t) = \gamma_H\,,$$

where the constant γ_H is given by

$$\gamma_H = \frac{\Gamma(2H+1)}{2(\theta^2+\mu^2)^H}\Big[1 + \frac{\sqrt{\theta^2+\mu^2}+\theta}{\sqrt{\theta^2+\mu^2}-\theta}\sin \pi H\Big]\,. \tag{1.2}$$

In the classical case $H = \frac{1}{2}$ where the noises are standard Brownian motions, the system of filtering equations reduces to the well-known Kalman-Bucy system (see, *e.g.*, Davis (1977) and Liptser and Shiryaev (1978)) and the asymptotic variance of the filtering error is $\gamma_{\frac{1}{2}} = \mu^{-2}[\sqrt{\theta^2+\mu^2}+\theta]$. In that case, substituting the constant $\gamma_{\frac{1}{2}}$ for the function $\gamma_{XX}(t)$ in the Kalman-Bucy system, one gets the simpler filtering equation

$$d\pi_t^*(X) = -\sqrt{\theta^2+\mu^2}\,\pi_t^*(X)dt + \mu\gamma_{\frac{1}{2}}\,dY_t\,;\ \pi_0^*(X) = 0\,, \tag{1.3}$$

which generates the filter

$$\pi_t^*(X) = \mu\gamma_{\frac{1}{2}} \int_0^t \mathrm{e}^{-\sqrt{\theta^2+\mu^2}(t-s)} dY_s\,. \tag{1.4}$$

It turns out that $\pi_t^*(X)$ is an *asymptotically optimal filter* in the sense that the variance $\mathbb{E}(X_t - \pi_t^*(X))^2$ of the corresponding filtering error converges to $\gamma_{\frac{1}{2}}$ as t goes to infinity. Observe that actually, in this case, the asymptotic optimality in filtering is achieved in the class of filters which can be represented as $\int_0^t \phi(t-s)dY_s$. In the present paper, we show that this still holds for $H > \frac{1}{2}$ and we identify in this class a filter for which the variance of the filtering error converges to γ_H.

The paper is organized as follows. At first in Section 2, we fix some notations and preliminaries; in particular we associate to the problem under study an equivalent deterministic control problem. Then, our main result is stated and proved in Section 3 by exploiting the solution of this auxiliary problem which belongs to a family of infinite time horizon deterministic control problems which are investigated in Section 4.

2 Preliminaries

Fractional Brownian motion. Here, for some $H \in [\frac{1}{2}, 1)$, $B^H = (B_t^H, t \geq 0)$ is a normalized fractional Brownian motion with Hurst parameter H. This means that B^H is a Gaussian process with continuous paths such that $B_0^H = 0$, $\mathbb{E}B_t^H = 0$ and

$$\mathbb{E}B_s^H B_t^H = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], \quad s, t \geq 0. \tag{2.1}$$

Of course the fBm reduces to the standard Brownian motion when $H = \frac{1}{2}$. For $H \neq \frac{1}{2}$, the fBm is outside the world of semimartingales but a theory of stochastic integration with respect to fBm has been developed (see, *e.g.*, Decreusefond and Üstünel (1999) or Duncan *et al.* (2000)). Actually the case of deterministic integrands, which is sufficient for the purpose of the present paper, is easy to handle (see, *e.g.*, Norros *et al.* (1999)). In particular, for a stochastic integral

$$S_t = \int_0^t g(t-s)dB_s^H, \tag{2.2}$$

we can evaluate

$$\mathbb{E}S_t^2 = \begin{cases} \int_0^t g^2(s)ds \quad \text{if } H = \frac{1}{2}, \\ H(2H-1)\int_0^t \int_0^t g(s)g(r)\,|s-r|^{2H-2}\,dsdr \quad \text{if } H \in (\frac{1}{2}, 1). \end{cases}$$

In the second case, exploiting the representation

$$|s-r|^{2H-2} = \frac{1}{B(H-\frac{1}{2}, 2-2H)} \int_{s\vee r}^{+\infty} (\tau - s)^{H-\frac{3}{2}}(\tau - r)^{H-\frac{3}{2}} d\tau,$$

where $B(.,.)$ denotes the Beta function, it is easy to check that we can rewrite

$$\mathbb{E}S_t^2 = \frac{H(2H-1)}{B(H-\frac{1}{2}, 2-2H)} \int_0^{+\infty} \{\int_0^{s\wedge t} g(r)(s-r)^{H-\frac{3}{2}}dr\}^2 ds.$$

Therefore, we have also for all $H \in [\frac{1}{2}, 1)$

$$\lim_{t\to+\infty} \mathbb{E} S_t^2 = \frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \int_0^{+\infty} \widetilde{g}^2(s)ds\,, \tag{2.3}$$

where

$$\widetilde{g}(s) = \frac{d}{ds} \int_0^s g(r)(s - r)^{H-\frac{1}{2}} dr\,, \tag{2.4}$$

and Γ is the Gamma function. Actually, the connection (2.4) can be inverted by

$$g(s) = \frac{1}{B(H + \frac{1}{2}, \frac{3}{2} - H)} \frac{d}{ds} \int_0^s \widetilde{g}(r)(s - r)^{\frac{1}{2}-H} dr\,. \tag{2.5}$$

Filtering errors. As announced in Section 1, in the system (1.1), we shall concentrate on filters which take the form

$$\pi_t^\phi(X) = \int_0^t \phi(t - s)dY_s\,.$$

From the first equation in (1.1), we have

$$X_t = \mathrm{e}^{\theta t} \int_0^t \mathrm{e}^{-\theta s} dV_s^H\,,$$

and, taking into account the second one, we get

$$\pi_t^\phi(X) = \mu \int_0^t \phi(t - s)\mathrm{e}^{\theta s}\{\int_0^s \mathrm{e}^{-\theta u} dV_u^H\}ds + \int_0^t \phi(t - s)dW_s^H\,.$$

Hence, it comes that

$$\pi_t^\phi(X) = \mu \int_0^t \{\int_u^t \phi(t - s)\mathrm{e}^{\theta s} ds\}\mathrm{e}^{-\theta u} dV_u^H + \int_0^t \phi(t - s)dW_s^H\,,$$

or

$$\pi_t^\phi(X) = \mu \int_0^t \{\int_0^{t-u} \phi(w)\mathrm{e}^{-\theta w} dw\}\mathrm{e}^{\theta(t-u)} dV_u^H + \int_0^t \phi(t - s)dW_s^H\,.$$

Finally, the filtering error corresponding to the filter $\pi_t^\phi(X)$ can be written as

$$X_t - \pi_t^\phi(X) = \int_0^t \mathrm{e}^{\theta(t-s)}\{1 - \mu \int_0^{t-s} \phi(w)\mathrm{e}^{-\theta w} dw\}dV_s^H - \int_0^t \phi(t - s)dW_s^H\,,$$

or equivalently

$$X_t - \pi_t^\phi(X) = \int_0^t Z^\phi(t - s)dV_s^H - \int_0^t \phi(t - s)dW_s^H\,, \tag{2.6}$$

where the function Z^ϕ is defined from ϕ, $Z^\phi = Z$ say, by

$$Z(\tau) = e^{\theta\tau}\{1 - \mu \int_0^\tau \phi(w)e^{-\theta w} dw\} .$$

Notice that Z is governed by the differential equation

$$\dot{Z}(\tau) = \theta Z(\tau) - \mu\phi(\tau) ; \; Z(0) = 1 . \tag{2.7}$$

Asymptotic variance of filtering errors. Now, starting from (2.6), according to the identities (2.2)-(2.4) with (Z, V^H) and (ϕ, W^H) in place of (g, B^H) and due to the independence of V^H and W^H, we get that the asymptotic variance of the filtering error corresponding to the filter $\pi_t^\phi(X)$, *i.e.*,

$$\lim_{t\to+\infty} \mathbb{E}(X_t - \pi_t^\phi(X))^2 = J(\phi) , \tag{2.8}$$

is given by

$$J(\phi) = \frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \int_0^{+\infty} \{\widetilde{Z}^2(s) + \widetilde{\phi}^2(s)\} ds , \tag{2.9}$$

where, for Z linked to ϕ by (2.7),

$$\widetilde{Z}(s) = \frac{d}{ds} \int_0^s Z(r)(s - r)^{H-\frac{1}{2}} dr ; \quad \widetilde{\phi}(s) = \frac{d}{ds} \int_0^s \phi(r)(s - r)^{H-\frac{1}{2}} dr . \tag{2.10}$$

Actually, it is readily seen from (2.7) and (2.10) that the dynamics which links $\widetilde{Z}$ to $\widetilde{\phi}$ is nothing but

$$\widetilde{Z}(t) = \theta \int_0^t \widetilde{Z}(s) ds - \mu \int_0^t \widetilde{\phi}(s) ds + t^{H-\frac{1}{2}} . \tag{2.11}$$

Notice that of course if $H = \frac{1}{2}$, and hence $\widetilde{\phi} \equiv \phi$ and $\widetilde{Z} \equiv Z$, equation (2.11) is nothing but equation (2.7) written in integral form and if $H > \frac{1}{2}$, then (2.11) can be rewritten as

$$\dot{\widetilde{Z}}(t) = \theta\widetilde{Z}(s) - \mu\widetilde{\phi}(s) ds + (H - \frac{1}{2}) t^{H-\frac{3}{2}} ; \quad \widetilde{Z}(0) = 0 .$$

Due to the limiting property (2.8), our guess is that in order to define an asymptotically optimal filter $\pi_t^*(X)$, one may take $\pi_t^*(X) = \pi_t^{\phi^*}(X)$ where the function ϕ^* corresponds through (2.5) to an optimal control $\widetilde{\phi}^*$ in the control problem :

$$\min_{\widetilde{\phi}} \widetilde{J}(\widetilde{\phi}) \quad \text{subject to (2.11)} , \tag{2.12}$$

with the performance criterion $\widetilde{J}(\widetilde{\phi}) = J(\phi)$ defined by (2.9).

The concerned infinite time horizon deterministic control problem (2.12) belongs to the class of control problems which are solved in Section 4. Their solutions make us able to formulate and prove our main result.

3 Asymptotically optimal filtering

At first, let us discuss the case when $H = \frac{1}{2}$. Here, in the control problem studied in Section 4, we must take $x = 1$, $K \equiv 0$, $a = \theta$, $b = -\mu$ and $q = r = 1$. Hence, applying Theorem 4.1 (see also the particular case 4.1), it comes that the optimal control in (2.12) is

$$\phi^*(t) = \mu\gamma_{\frac{1}{2}} \, \mathrm{e}^{-\sqrt{\theta^2+\mu^2}t} ,$$

where

$$\gamma_{\frac{1}{2}} = \frac{\sqrt{\theta^2+\mu^2}+\theta}{\mu^2} ,$$

is the value of the optimal cost. This means nothing but that, as claimed in Section 1, an asymptotically optimal filter is $\pi_t^*(X) = \pi_t^{\phi^*}(X)$ given by (1.4).

Now, we turn to the case $H \in (\frac{1}{2}, 1)$ where we can prove the following statement which provides also an asymptotically optimal filter :

Theorem 3.1 *Define the function V^* by*

$$V^*(t) = \frac{H-\frac{1}{2}}{B(H+\frac{1}{2}, \frac{3}{2}-H)} \int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}t\tau} \frac{\tau^{H-\frac{1}{2}}}{\tau+1} d\tau , \quad t > 0 . \tag{3.1}$$

Let the pair of functions (ϕ^*, Z^*) be defined by

$$\begin{cases} \phi^*(t) &= \dfrac{\theta+\sqrt{\theta^2+\mu^2}}{\mu}[Z^*(t)+V^*(t)] , \\ \dot{Z}^*(t) &= \theta Z^*(t) - \mu\phi^*(t) ; \quad Z^*(0) = 1 . \end{cases} \tag{3.2}$$

Then the filter

$$\pi_t^*(X) = \int_0^t \phi^*(t-s)dY_s ,$$

is asymptotically optimal, i.e.,

$$\lim_{t\to+\infty} \mathbf{E}(X_t - \pi_t^*(X))^2 = \gamma_H ,$$

where γ_H is given by (1.2).

Proof. For $H \in (\frac{1}{2}, 1)$, in the control problem studied in Section 4, we must take $x = 0$, $K(t) = (H-\frac{1}{2})t^{H-\frac{3}{2}}$, $a = \theta$, $b = -\mu$ and

$$q = r = \frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} .$$

Hence, applying Theorem 4.1 (see also the particular case 4.2), we get that the following pair $(\widetilde{\phi}^*, \widetilde{Z}^*)$ is optimal in the control problem (2.12) :

$$\begin{cases} \widetilde{\phi}^*(t) &= \dfrac{\theta+\sqrt{\theta^2+\mu^2}}{\mu}[\widetilde{Z}^*(t)+\widetilde{V}^*(t)] , \\ \dot{\widetilde{Z}}^*(t) &= \theta\widetilde{Z}^*(t) - \mu\widetilde{\phi}^*(t) + (H-\frac{1}{2})t^{H-\frac{3}{2}} ; \quad \widetilde{Z}^*(0) = 0 , \end{cases}$$

where

$$\widetilde{V}^*(t) = (H - \frac{1}{2}) \int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}r}(t+r)^{H-\frac{3}{2}}dr\,. \tag{3.3}$$

Moreover, it is easy to check that the optimal cost in (2.12) is $\widetilde{J}(\widetilde{\phi}^*) = \gamma_{H}$ where γ_{H} is given by (1.2). Hence, it is clear that to define an asymptotically optimal filter by $\pi_t^*(X) = \pi_t^{\phi^*}(X)$ we can take the second component ϕ^* of the triple (V^*, ϕ^*, Z^*) which corresponds through (2.5) to the triple $(\widetilde{V}^*, \widetilde{\phi}^*, \widetilde{Z}^*)$. It is easy to check that ϕ^* is defined by (3.2) where V^* corresponds through (2.5) to $\widetilde{V}^*$ and so, finally, we have just to identify V^*. From (3.3), we compute

$$\begin{aligned}
\int_0^t (t-s)^{\frac{1}{2}-H}\widetilde{V}^*(s)ds &= (H-\frac{1}{2})\int_0^t (t-s)^{\frac{1}{2}-H}\{\int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}r}(s+r)^{H-\frac{3}{2}}dr\}ds \\
&= (H-\frac{1}{2})\int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}r}\{\int_0^t (t-s)^{\frac{1}{2}-H}(s+r)^{H-\frac{3}{2}}ds\}dr \\
&= (H-\frac{1}{2})\int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}r}\{\int_0^{\frac{t}{t+r}} v^{\frac{1}{2}-H}(1-v)^{H-\frac{3}{2}}dv\}dr\,.
\end{aligned}$$

Observing that actually

$$\frac{d}{dt}\int_0^{\frac{t}{t+r}} v^{\frac{1}{2}-H}(1-v)^{H-\frac{3}{2}}dv = \frac{t^{\frac{1}{2}-H}r^{H-\frac{1}{2}}}{t+r}\,,$$

it follows that

$$\begin{aligned}
\int_0^t (t-s)^{\frac{1}{2}-H}\widetilde{V}^*(s)ds &= (H-\frac{1}{2})\int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}r}\{\int_0^t \frac{u^{\frac{1}{2}-H}r^{H-\frac{1}{2}}}{u+r}du\}dr \\
&= (H-\frac{1}{2})\int_0^t\{\int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}r}\frac{u^{\frac{1}{2}-H}r^{H-\frac{1}{2}}}{u+r}dr\}du \\
&= (H-\frac{1}{2})\int_0^t\{\int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}u\tau}\frac{\tau^{H-\frac{1}{2}}}{\tau+1}d\tau\}du\,.
\end{aligned}$$

From (2.5), we see that this means exactly that V^* is given by (3.1). □

Remark 3.1 (*a*) Observe that from (3.1) we have also

$$\dot{V}^*(t) = -\sqrt{\theta^2+\mu^2}\frac{H-\frac{1}{2}}{B(H+\frac{1}{2},\frac{3}{2}-H)}\int_0^{+\infty} \mathrm{e}^{-\sqrt{\theta^2+\mu^2}t\tau}\frac{\tau^{H+\frac{1}{2}}}{\tau+1}d\tau\,, \quad t>0\,.$$

Then, splitting the integral into two terms corresponding to the decomposition of $\tau^{H+\frac{1}{2}}$ as the difference $[\tau^{H+\frac{1}{2}} + \tau^{H-\frac{1}{2}}] - \tau^{H-\frac{1}{2}}$, one may easily check that V^* is actually the solution of the differential equation

$$\dot{V}^*(t) = \sqrt{\theta^2+\mu^2}V^*(t) - \beta_{H}(H-\frac{1}{2})t^{-\frac{1}{2}-H}\,; \quad \lim_{t\to+\infty} V^*(t) = 0\,,$$

where

$$\beta_{H} = \frac{(\theta^2+\mu^2)^{\frac{1-2H}{4}}}{\Gamma(\frac{3}{2}-H)}\,.$$

Since $\int_0^{+\infty} V^*(t)dt = 1$, it means also that that V^* is the solution of the integral equation

$$V^*(t) = \int_0^t \sqrt{\theta^2+\mu^2}V^*(s)ds + \beta_H t^{\frac{1}{2}-H} - 1\,. \tag{3.4}$$

(b) Let us emphasize that, similarly to the case $H = \frac{1}{2}$ where the filter $\pi_t^*(X)$ can be generated by the approximate Kalman-Bucy algorithm (1.3), a recursive scheme can be also provided for the asymptotically optimal filter in the case $H \in (\frac{1}{2}, 1)$. At first, we observe that due to the first equation in (3.2) we can write

$$\pi_t^*(X) = \mu\gamma_{\frac{1}{2}}[\mathcal{Z}_t^* + \mathcal{V}_t^*]\,, \tag{3.5}$$

where

$$\mathcal{Z}_t^* = \int_0^t Z^*(t-s)dY_s\,; \quad \mathcal{V}_t^* = \int_0^t V^*(t-s)dY_s\,.$$

Since the function Z^* is differentiable, we have

$$\mathcal{Z}_t^* = Z^*(0)Y_t + \int_0^t \{\int_0^s \dot{Z}^*(s-r)dY_r\}ds\,.$$

Hence, due to the second equation in (3.2), the process $\mathcal{Z}^*$ is generated from Y by the equation

$$\mathcal{Z}_t^* = \theta\int_0^t \mathcal{Z}_s^* ds - \mu\int_0^t \pi_s^*(X)ds + Y_t\,, \tag{3.6}$$

Now, using equation (3.4), we can write

$$\mathcal{V}_t^* = \int_0^t \psi(t-s)dY_s + \beta_H\int_0^t (t-s)^{\frac{1}{2}-H}dY_s\,,$$

where the function ψ satisfies

$$\dot{\psi}(t) = \sqrt{\theta^2+\mu^2}V^*(t)\,; \quad \psi(0) = -1\,.$$

Consequently, we get that

$$\begin{aligned}\int_0^t \psi(t-s)dY_s &= \psi(0)Y_t + \int_0^t\{\int_0^s \dot{\psi}(s-r)dY_r\}ds \\ &= \sqrt{\theta^2+\mu^2}\int_0^t \mathcal{V}_s^* ds - Y_t\,.\end{aligned}$$

Finally, the following equation holds for $\mathcal{V}_t^*$:

$$\mathcal{V}_t^* = \sqrt{\theta^2+\mu^2}\int_0^t \mathcal{V}_s^* ds + \int_0^t [\beta_H(t-s)^{\frac{1}{2}-H} - 1]dY_s\,. \tag{3.7}$$

The system (3.5)-(3.7) provides a closed-form recursion which generates the filter $\pi_t^*(X)$ from the observation process Y. It is readily seen that when $H = \frac{1}{2}$, and hence $\mathcal{V}^* \equiv 0$ and $\pi_t^*(X) = \mu\gamma_{\frac{1}{2}}\mathcal{Z}_t^*$, this system reduces to the single equation

$$\pi_t^*(X) = -\sqrt{\theta^2 + \mu^2}\int_0^t \pi_s^*(X)ds + \mu\gamma_{\frac{1}{2}} Y_t\,,$$

which is nothing but equation (1.3). (c) Suppose that $H > \frac{1}{2}$ but one does as if the noises were standard Brownian motions and hence uses the filter generated by the approximate Kalman-Bucy algorithm (1.3), *i.e.*, the filter

$$\widetilde{\pi}_t(X) = \mu\gamma_{\frac{1}{2}}\int_0^t \mathrm{e}^{-\sqrt{\theta^2+\mu^2}(t-s)}dY_s\,.$$

Then it can be checked that the corresponding asymptotic variance of the filtering error $\lim_{t\to+\infty}\mathbb{E}(X_t - \widetilde{\pi}_t(X))^2$ is the constant

$$\widetilde{\gamma}_H = \frac{\Gamma(2H+1)}{(\theta^2+\mu^2)^{H-\frac{1}{2}}}\gamma_{\frac{1}{2}}\,.$$

Moreover the consequent loss of performance with respect to the asymptotically optimal filter can be evaluated by

$$\widetilde{\gamma}_H - \gamma_H = \frac{\Gamma(2H+1)}{2(\theta^2+\mu^2)^H}\mu^2\gamma_{\frac{1}{2}}^2(1-\sin\pi H)\,.$$

Let us observe that, for fixed parameters θ and μ, the asymptotic relative efficiency

$$\frac{\gamma_H}{\widetilde{\gamma}_H} = \frac{1+\mu^2\gamma_{\frac{1}{2}}^2\sin\pi H}{1+\mu^2\gamma_{\frac{1}{2}}^2}\,,$$

of $\widetilde{\pi}_t(X)$ decreases as H increases in $(\frac{1}{2}, 1)$.

4 About optimal control problems

Given a function $K = (K(t), t \geq 0)$ and constants a and b, we consider the state dynamics

$$\dot{\mathcal{X}}_t = a\mathcal{X}_t + b\mathcal{U}_t + K(t)\,,\ t\geq 0\,;\quad \mathcal{X}_0 = x\,, \tag{4.1}$$

where the control $\mathcal{U} = (\mathcal{U}_t, t \geq 0)$ can be chosen in order to drive the state $\mathcal{X} = (\mathcal{X}_t, t \geq 0)$. Let $\mathcal{A}$ be the class of measurable functions $\mathcal{U}$, called admissible controls, such that the corresponding differential equation (4.1) has a unique solution $\mathcal{X}$. Given constants $q > 0$ and $r > 0$, we define the performance criterion $\mathcal{J}$ by

$$\mathcal{J}(\mathcal{U}) = \int_0^{+\infty}[q\mathcal{X}_t^2 + r\mathcal{U}_t^2]dt\,. \tag{4.2}$$

The following statement gives the solution of the infinite time horizon deterministic control problem corresponding to (4.1)-(4.2).

Theorem 4.1 *Define the constants*

$$\rho = \frac{r}{b^2}[a + \delta]\,; \quad \delta = \sqrt{a^2 + \frac{b^2}{r}q}\,. \tag{4.3}$$

Assume that $\lim_{t\to+\infty} K(t) = 0$ *and also, setting*

$$\mathcal{V}_K(t) = \int_0^{+\infty} e^{-\delta r} K(t + r)dr\,,\ t \geq 0\,, \tag{4.4}$$

that the function $\mathcal{V}_K$ *is well-defined. Let the pair* $(\mathcal{U}^*, \mathcal{X}^*)$ *be governed by*

$$\begin{cases} \mathcal{U}_t^* &= -\frac{b}{r}\rho[\mathcal{X}_t^* + \mathcal{V}_K(t)]\,, \\ \dot{\mathcal{X}}_t^* &= a\mathcal{X}_t^* + b\mathcal{U}_t^* + K(t)\,; \quad \mathcal{X}_0^* = x\,, \end{cases} \tag{4.5}$$

Then, for $\mathcal{J}$ *defined by (4.2), the pair* $(\mathcal{U}^*, \mathcal{X}^*)$ *is optimal in the control problem*

$$\min_{\mathcal{U}\in\mathcal{A}} \mathcal{J}(\mathcal{U}) \quad \text{subject to (4.1)}\,.$$

Moreover, the value of the optimal cost is

$$\mathcal{J}(\mathcal{U}^*) = \rho[x + \mathcal{V}_K(0)]^2 + q\int_0^{+\infty} \mathcal{V}_K^2(s)ds\,. \tag{4.6}$$

Proof. Suppose that there exists a pair $(\mathcal{X}^*, p^*)$ which satisfies the Hamiltonian system

$$\begin{aligned} \dot{\mathcal{X}}_t^* &= a\mathcal{X}_t^* - \frac{b^2}{r}p_t^* + K(t)\,; \quad \mathcal{X}_0^* = x\,, \\ \dot{p}_t^* &= -q\mathcal{X}_t^* - ap_t^*\,; \quad \lim_{t\to+\infty} p_t^* = 0\,, \end{aligned} \tag{4.7}$$

Hence of course $\mathcal{X}^*$ is nothing but the state dynamics corresponding through (4.1) to the control $\mathcal{U}^*$ defined by $\mathcal{U}_t^* = -(b/r)p_t^*$. Let us show that for an arbitrary control $\mathcal{U} \in \mathcal{A}$ the inequality $\mathcal{J}(\mathcal{U}) \geq \mathcal{J}(\mathcal{U}^*)$ holds. Of course it is true when $\mathcal{J}(\mathcal{U}) = +\infty$ and so we concentrate on the case when $\mathcal{J}(\mathcal{U}) < +\infty$ which in particular means that $\lim_{t\to+\infty} \mathcal{X}_t = 0$ for the corresponding state dynamics $\mathcal{X}$. Defining for $T > 0$

$$\mathcal{J}_T(\mathcal{U}) = \int_0^T [q\mathcal{X}_t^2 + r\mathcal{U}_t^2]dt\,, \tag{4.8}$$

we evaluate

$$\mathcal{J}_T(\mathcal{U}) = \mathcal{J}_T(\mathcal{U}^*) + \int_0^T \{q[\mathcal{X}_t^2 - (\mathcal{X}_t^*)^2] + r[\mathcal{U}_t^2 - (\mathcal{U}_t^*)^2]\}dt\,.$$

Using the equality $y^2 - (y^*)^2 = (y - y^*)^2 + 2y^*(y - y^*)$ and exploiting the property $\mathcal{U}_t^* = -(b/r)p_t^*$, it is readily seen that

$$\mathcal{J}_T(\mathcal{U}) = \mathcal{J}_T(\mathcal{U}^*) + \Delta_1(T) + 2\Delta_2(T)\,, \tag{4.9}$$

where

$$\begin{aligned} \Delta_1(T) &= \int_0^T \{q[\mathcal{X}_t - \mathcal{X}_t^*]^2 + r[\mathcal{U}_t - \mathcal{U}_t^*]^2\}dt\,, \\ \Delta_2(T) &= \int_0^T \{q\mathcal{X}_t^*[\mathcal{X}_t - \mathcal{X}_t^*] - bp_t^*[\mathcal{U}_t - \mathcal{U}_t^*]\}dt\,. \end{aligned}$$

But, rewriting the quantity in the last integral as

$$(\mathcal{X}_t - \mathcal{X}_t^*)[q\mathcal{X}_t^* + ap_t^*] - p_t^*[a(\mathcal{X}_t - \mathcal{X}_t^*) + b(\mathcal{U}_t - \mathcal{U}_t^*)]\,,$$

and taking into account equations (4.1) and (4.7), we see that this integral can be written as

$$-\int_0^T (\mathcal{X}_t - \mathcal{X}_t^*)dp_t^* - \int_0^T p_t^* d(\mathcal{X}_t - \mathcal{X}_t^*)\,.$$

Therefore, integrating by parts, since $\mathcal{X}_0 - \mathcal{X}_0^* = 0$, it comes that

$$\Delta_2(T) = -p_T^*(\mathcal{X}_T - \mathcal{X}_T^*)\,.$$

Consequently, since $\Delta_1(T) \geq 0$, from (4.9) we get that

$$\mathcal{J}_T(\mathcal{U}) \geq \mathcal{J}_T(\mathcal{U}^*) - 2p_T^*(\mathcal{X}_T - \mathcal{X}_T^*)\,.$$

Hence, if $\lim_{T\to+\infty} \mathcal{X}_T^* = 0$, letting T tend to infinity in this inequality, due to the limiting conditions for p^* and $\mathcal{X}$, we obtain $\mathcal{J}(\mathcal{U}) \geq \mathcal{J}(\mathcal{U}^*)$.
Now, to show that the pair $(\mathcal{U}^*, \mathcal{X}^*)$ defined by (4.5) is optimal, it is sufficient to check that the pair $(\mathcal{X}^*, p^*)$, where $p_t^* = \rho[\mathcal{X}_t^* + \mathcal{V}_K(t)]$, satisfies the Hamiltonian system (4.7) and that also the limiting condition $\lim_{t\to+\infty} \mathcal{X}_t^* = 0$ holds. At first, it is easy to check that $(\mathcal{X}^*, p^*)$ satisfies the differential equations in (4.7). One can observe that the expression (4.4) for $\mathcal{V}_K$ can be rewritten as

$$\mathcal{V}_K(t) = \int_t^{+\infty} \mathrm{e}^{\delta(t-s)}K(s)ds\,,\ t \geq 0\,,$$

and since $\lim_{t\to+\infty} K(t) = 0$, actually $\mathcal{V}_K$ is nothing but the solution of the equation

$$\dot{\mathcal{V}}_K(t) = \delta\mathcal{V}_K(t) - K(t)\,; \quad \lim_{t\to+\infty} \mathcal{V}_K(t) = 0\,. \tag{4.10}$$

Now, since from the first equation in (4.7) we have

$$\dot{\mathcal{X}}_t^* = -\delta\mathcal{X}_t^* - \frac{b^2}{r}\rho\mathcal{V}_K(t) + K(t)\,; \quad \mathcal{X}_0^* = x\,, \tag{4.11}$$

due to $\lim_{t\to+\infty} K(t) = \lim_{t\to+\infty} \mathcal{V}_K(t) = 0$, it is clear that $\lim_{t\to+\infty} \mathcal{X}_t^* = 0$. Hence, we have also $\lim_{t\to+\infty} p_t^* = 0$.
Finally, we evaluate the optimal cost $\mathcal{J}(\mathcal{U}^*)$. At first, in order to compute the variation $p_T^*\mathcal{X}_T^* - p_0^*\mathcal{X}_0^*$, we express $p_t^*\dot{\mathcal{X}}_t^* + \dot{p}_t^*\mathcal{X}_t^*$ from (4.7). Then, for $\mathcal{J}_T$ defined by (4.8), it follows easily that

$$\mathcal{J}_T(\mathcal{U}^*) = p_0^* x - p_T^*\mathcal{X}_T^* + \int_0^T p_t^* K(t)dt\,.$$

Hence, since $p_t^* = \rho[\mathcal{X}_t^* + \mathcal{V}_K(t)]$, taking the limit for T tending to infinity, we get

$$\mathcal{J}(\mathcal{U}^*) = \rho[x + \mathcal{V}_K(0)]x + \rho \int_0^{+\infty} [\mathcal{X}_t^* + \mathcal{V}_K(t)]K(t)dt\,.$$

Proceeding similarly through the evaluation of the variation $\mathcal{V}_K(T)\mathcal{X}_T^* - \mathcal{V}_K(0)\mathcal{X}_0^*$ from (4.10)-(4.11), we obtain that

$$\int_0^{+\infty} \mathcal{X}_t^* K(t)dt = -\frac{b^2}{r}\rho \int_0^{+\infty} \mathcal{V}_K^2(t)dt + \int_0^{+\infty} K(t)\mathcal{V}_K(t)dt + \mathcal{V}_K(0)x\,.$$

Then, since from equation (4.10) we have $K(t) = \delta\mathcal{V}_K(t) - \dot{\mathcal{V}}_K(t)$, it follows that

$$\mathcal{J}(\mathcal{U}^*) = \rho[x^2 + 2\mathcal{V}_K(0)x] + \rho(\delta - a) \int_0^{+\infty} \mathcal{V}_K^2(t)dt - 2\rho \int_0^{+\infty} \mathcal{V}_K(t)\dot{\mathcal{V}}_K(t)dt\,.$$

But $\rho(\delta - a) = q$ and clearly the last integral equals $-\frac{1}{2}\mathcal{V}_K^2(0)$ and so the equality (4.6) holds. □

Remark 4.1 Actually, from (4.10), we observe that

$$\mathcal{V}_K(t)\dot{\mathcal{V}}_K(t) = \delta\mathcal{V}_K^2(t) - K(t)\mathcal{V}_K(t)\,,$$

and hence

$$\int_0^{+\infty} \mathcal{V}_K^2(t)dt = \frac{1}{\delta}\{\int_0^{+\infty} K(t)\mathcal{V}_K(t)dt - \frac{1}{2}\mathcal{V}_K^2(0)\}\,.$$

This allows to rewrite the value (4.6) of the optimal cost as

$$\mathcal{J}(\mathcal{U}^*) = \rho x[x + 2\mathcal{V}_K(0)] + \frac{\rho}{\delta}\{\frac{\delta + a}{2}\mathcal{V}_K^2(0) + (\delta - a) \int_0^{+\infty} K(t)\mathcal{V}_K(t)dt\}\,. \qquad (4.12)$$

Particular case 4.1 If we take $K \equiv 0$, and hence also $\mathcal{V}_K \equiv 0$, then the optimal pair $(\mathcal{U}^*, \mathcal{X}^*)$ is governed by

$$\begin{cases} \mathcal{U}_t^* &= -\frac{b}{r}\rho\mathcal{X}_t^*\,, \\ \dot{\mathcal{X}}_t^* &= a\mathcal{X}_t^* + b\mathcal{U}_t^*\,; \quad \mathcal{X}_0^* = x\,. \end{cases} \qquad (4.13)$$

Since $a - (b^2/r)\rho = -\delta$, this means that $\mathcal{X}_t^* = \mathrm{e}^{-\delta t}x$ and $\mathcal{U}_t^* = -(b/r)\rho\mathrm{e}^{-\delta t}x$. Substituting these expressions for $\mathcal{X}_t^*$ and $\mathcal{U}_t^*$ in the integral $\int_0^{+\infty}[q(\mathcal{X}_t^*)^2 + r(\mathcal{U}_t^*)^2]dt$, a direct computation gives the value $\mathcal{J}(\mathcal{U}^*) = \rho x^2$ of the optimal cost, which of course is nothing but what (4.6) says in the present case.

PARTICULAR CASE 4.2 If, for some $H \in (\frac{1}{2}, 1)$, we take $K(t) = (H - \frac{1}{2}) t^{H-\frac{3}{2}}$, then the optimal pair $(\mathcal{U}^*, \mathcal{X}^*)$ is governed by (4.5) with

$$\mathcal{V}_K(t) = (H - \frac{1}{2}) \int_0^{+\infty} \mathrm{e}^{-\delta r} (t+r)^{H-\frac{3}{2}} dr \,. \tag{4.14}$$

Moreover, the value of the optimal cost can be computed explicitly. Actually, here from (4.14), straightforward computations give that

$$\mathcal{V}_K(0) = \delta^{\frac{1}{2}-H} \Gamma(H + \frac{1}{2})\,; \quad \int_0^{+\infty} K(t) \mathcal{V}_K(t) dt = \delta^{1-2H} \frac{\Gamma(2H)\Gamma(H+\frac{1}{2})\Gamma(2-2H)}{2\Gamma(\frac{3}{2} - H)} \,.$$

Inserting this into the expression (4.12), one may finally get

$$\mathcal{J}(\mathcal{U}^*) = \rho x [x + \frac{2}{\delta^{H-\frac{1}{2}}} \Gamma(H + \frac{1}{2})] + \frac{q}{\delta^{2H}} \frac{\Gamma(2H)\Gamma(H+\frac{1}{2})\Gamma(2-2H)}{2\Gamma(\frac{3}{2}-H)} [1 + \frac{\delta + a}{\delta - a} \sin \pi H] \,.$$

5 Concluding comments

Linear Quadratic Gaussian (LQG) problems concerning dynamical systems governed by Brownian motions have well-known solutions which are now quite classical. When the driving processes are fBm's, the theory is not yet completed, specially from the asymptotical point of view. In this paper, concentrating on filtering, we have illustrated the actual solvability of the problems. Actually, the infinite time horizon stochastic control problems are also tractable and in forthcoming papers we shall report the results about the regulator problem both in the case of complete and incomplete observation, the last one mixing filtering and control.

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