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ON THE POWER PSEUDO-VARIETY \mathbf{PCS}

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Abstract: The pseudovariety \mathbf{PCS} which is generated by all power semigroups of finite completely simple semigroups is characterized in various ways. For example, the equalities

$$\mathbf{PCS} = \mathbf{J} \textcircled{\&} \mathbf{CS} = \mathbf{BG} \textcircled{\&} \mathbf{RB}$$

are established. This resolves a problem raised by Kad'ourek and leads to several transparent algorithms for deciding membership in \mathbf{PCS} .

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In the study of pseudovarieties of finite semigroups, the *power operator* \mathbf{P} that assigns to every pseudovariety \mathbf{V} the pseudovariety \mathbf{PV} generated by all power semigroups of the members of \mathbf{V} has attracted considerable attention – see Chapter 11 of the monograph [1] and the survey article [2] by Almeida. For example, the recognition of the pseudovariety \mathbf{PG} generated by all power groups (that is, power semigroups of groups) as the pseudovariety \mathbf{BG} of all block groups, often stated as the famous equality

$$(\#) \quad \mathbf{PG} = \mathbf{BG},$$

is a celebrated result in finite semigroup theory.

The proof of the equality $(\#)$ was originally obtained by Henckell and Rhodes [9] as a consequence of a deep result of Ash [5]. The reader interested in the historical background, in further developments and in generalizations of the above-mentioned equality is referred to Section 4.17 of the monograph [14] by Rhodes and Steinberg and to the introduction of Kad'ourek's paper [10]. The equality $(\#)$ in particular implies the decidability of membership in \mathbf{PG} which does not follow from the definition of the power operator. Indeed, the operator \mathbf{P} does not preserve decidability of membership [6].

It seems to be natural to replace “groups” with “completely simple semigroups” and to ask for a suitable description of the power pseudovariety **PCS**. This question occurs in the list of open problems in [1]. The first solution to this was found by Steinberg [15] who proved the equality

$$\mathbf{PCS} = \mathbf{BG} * \mathbf{RZ}$$

which implied the decidability of membership in **PCS**. However, neither a transparent structural description of the members of **PCS** nor an efficient algorithm for the membership problem in **PCS** were presented. The latter fact was considered by Kad'ourek to be not completely satisfactory; he therefore proposed further research on that topic: in [10] he studied finite semigroups S all of whose subsemigroups of the form aSb (for $a, b \in S$) are block groups. It is easy to see that these exactly comprise the Mal'cev product $\mathbf{BG} \textcircled{m} \mathbf{RB}$, see Corollary 3 below. Kad'ourek proved that this class can be represented also as the Mal'cev product $\mathbf{J} \textcircled{m} \mathbf{CS}$. Moreover, he related **PCS** with the latter pseudovariety by proving the inclusion $\mathbf{PCS} \subseteq \mathbf{J} \textcircled{m} \mathbf{CS}$. The question whether that inclusion is proper or not has been left open. It is the intention of the present paper to resolve this problem. In fact, we shall prove more, namely we shall establish the equalities

$$\mathbf{PCS} = \mathbf{J} \textcircled{m} \mathbf{CS} = \mathbf{BG} \textcircled{m} \mathbf{RB} = \mathbf{ER} \textcircled{m} \mathbf{RZ} \cap \mathbf{EL} \textcircled{m} \mathbf{LZ} = \mathbf{BG} * \mathbf{RZ}.$$

The paper is organized as follows. In Section 1 we shall mention all preliminaries needed while in Section 2 we shall formulate and prove the main result. This will be done semantically as well as syntactically; the members of **PCS** will thus be characterized in various ways.

1. Preliminaries

1.1. Definitions and notation. The reader is assumed to be familiar with central facts of semigroup theory, in particular with the theory of pseudovarieties of finite semigroups, including pseudoidentities and Tilson's derived semigroupoid theorem; sources are the monographs [1], [14] as well as Tilson's seminal paper [16]. All semigroups considered in this paper are finite.

We start by introducing some notation. Given a semigroup S and an element $a \in S$, then $L(a) = Sa \cup \{a\}$ is the principal left ideal of S generated by a , and likewise, $R(a) = aS \cup \{a\}$ is the principal right ideal of S generated by a . Further, we denote by $L(a)^\rho$ the semigroup of all inner right translations (acting on the right) $\rho_s: L(a) \rightarrow L(a)$, $x \mapsto xs$ for $s \in L(a)$; then $\rho^{L(a)}: L(a) \twoheadrightarrow L(a)^\rho$, $s \mapsto \rho_s$ is a (not necessarily injective) homomorphism, called the *right regular representation* of $L(a)$.

Dually, ${}^\lambda R(a)$ denotes the semigroup of all inner left translations (acting on the left) $\lambda_s: R(a) \rightarrow R(a)$, $x \mapsto sx$ for $s \in R(a)$; then $\lambda^{R(a)}: R(a) \rightarrow {}^\lambda R(a)$, $s \mapsto \lambda_s$ is a (not necessarily injective) homomorphism, called the *left regular representation* of $R(a)$.

Let S and T be two semigroups, both generated by the same set A . Then the subsemigroup of the direct product $S \times T$ generated by the set $\{(a, a) \mid a \in A\}$, **considered as a relation** from S to T , is the *canonical relational morphism* $S \rightarrow T$ with respect to A .

Pseudovarieties of semigroups are usually denoted by bold-face letters **V**, **W**, etc. Let **RZ**, **LZ**, **RB**, **G**, **CS**, and **J** be, respectively, the pseudovarieties of all right zero semigroups, left zero semigroups, rectangular bands, groups, completely simple semigroups, and \mathcal{J} -trivial semigroups. Of central importance will be the pseudovariety **BG** of all *block groups*. It consists of all semigroups all of whose regular elements have a unique inverse. Equivalently, **BG** is comprised of all semigroups which do not contain non-trivial right zero and left zero subsemigroups. It is well known that **BG** is defined by the single pseudoidentity $(x^\omega y^\omega)^\omega = (y^\omega x^\omega)^\omega$. In addition, we shall need the pseudovariety **ER** which consists of all semigroups whose idempotent generated subsemigroups are \mathcal{B} -trivial. A semigroup then belongs to **ER** if and only if it does not contain a non-trivial right zero subsemigroup. It is well known that **ER** is defined by the single pseudoidentity $(x^\omega y^\omega)^\omega = (x^\omega y^\omega)^\omega x^\omega$. At last, we shall need yet the pseudovariety **EL** which is defined dually, that is, it consists of all semigroups containing no non-trivial left zero subsemigroup. Likewise, **EL** is defined by the single pseudoidentity $(y^\omega x^\omega)^\omega = x^\omega (y^\omega x^\omega)^\omega$. For the latter three pseudovarieties the equality **BG** = **ER** \cap **EL** holds. All pseudovarieties considered in this paper are pseudovarieties of **semigroups**.

For a semigroup S , let $\mathfrak{P}(S)$ be the set of all non-empty subsets of S ; endowed with set-wise multiplication, the set $\mathfrak{P}(S)$ itself becomes a semigroup. For a homomorphism $\alpha: S \rightarrow T$ between semigroups S and T , the induced mapping $\mathfrak{P}(S) \rightarrow \mathfrak{P}(T)$, $X \mapsto \{x\alpha \mid x \in X\}$ is a homomorphism, the *induced homomorphism*. For a pseudovariety **V** denote by **P'V** the pseudovariety generated by all power semigroups $\mathfrak{P}(S)$ with $S \in \mathbf{V}$. Likewise, the full powerset $\mathfrak{P}(S) \cup \{\emptyset\}$ of a semigroup S is also a semigroup under set-wise multiplication. For a pseudovariety **V** let **PV** be the pseudovariety generated by all full power semigroups $\mathfrak{P}(S) \cup \{\emptyset\}$ with $S \in \mathbf{V}$. It is well known that **P'V** and **PV** coincide if and only if **V** contains a nontrivial monoid, see Lemma 5.1 in [2]. In all cases considered in this paper, the equality **P'V** = **PV** holds.

Finally, a semigroup S equipped with a partial order \leq is an *ordered semigroup* if $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. It is well known and easy to see that an ordered monoid M satisfying $a \leq 1$ for all $a \in M$ is \mathcal{J} -trivial [12].

1.2. Pseudovarieties of the forms $\mathbf{V} \textcircled{\cap} \mathbf{RZ}$, $\mathbf{V} \textcircled{\cap} \mathbf{LZ}$, $\mathbf{V} \textcircled{\cap} \mathbf{RB}$, and $\mathbf{V} * \mathbf{RZ}$. The results in this subsection are likely to be known by experts, but the author is not aware of a proper reference. For arbitrary pseudovarieties \mathbf{V} and \mathbf{W} , the *Mal'cev product* $\mathbf{V} \textcircled{\cap} \mathbf{W}$ consists of all semigroups S for which there exists a semigroup $T \in \mathbf{W}$ and a relational morphism $\phi: S \rightarrow T$ such that, for each idempotent $e \in T$, the inverse image of e under ϕ , that is, $e\phi^{-1} = \{s \in S \mid (s, e) \in \phi\}$ (which is a subsemigroup of S) belongs to \mathbf{V} . Suppose that \mathbf{W} is locally finite; then necessary and sufficient for a semigroup S to be contained in $\mathbf{V} \textcircled{\cap} \mathbf{W}$ is that the former condition holds for the canonical relational morphism $\phi: S \rightarrow F_{\mathbf{W}}(A)$ (with respect to A) where A is **some** generating set of S and $F_{\mathbf{W}}(A)$ is the free semigroup in \mathbf{W} generated by A . This allows to give quite transparent descriptions of the members of the Mal'cev products of the forms $\mathbf{V} \textcircled{\cap} \mathbf{RZ}$, $\mathbf{V} \textcircled{\cap} \mathbf{LZ}$ and $\mathbf{V} \textcircled{\cap} \mathbf{RB}$, for an arbitrary pseudovariety \mathbf{V} .

Lemma 1. *A semigroup S belongs to $\mathbf{V} \textcircled{\cap} \mathbf{RZ}$ if and only if each principal left ideal of S belongs to \mathbf{V} ; dually, S belongs to $\mathbf{V} \textcircled{\cap} \mathbf{LZ}$ if and only if each principal right ideal of S belongs to \mathbf{V} .*

Proof: Let S be a semigroup and let $RZ(S)$ be the (free) right zero semigroup on the set S . Thus, $RZ(S)$ is just the set S endowed with right zero multiplication. Let ϕ be the canonical relational morphism $S \rightarrow RZ(S)$ (with respect to S). Then, for each $a \in RZ(S)$ ($= S$) we have $a\phi^{-1} = L(a)$ whence, according to the discussion before the statement of the lemma, $S \in \mathbf{V} \textcircled{\cap} \mathbf{RZ}$ if and only if $L(a) \in \mathbf{V}$ for each $a \in S$. The dual case is proved analogously. \square

Lemma 2. *A semigroup S belongs to $\mathbf{V} \textcircled{\cap} \mathbf{RB}$ if and only if, for all $a, b \in S$, the subsemigroups $\{ab\} \cup aSb$ and $\{a, a^2\} \cup aSa$ belong to \mathbf{V} .*

Proof: Let S be a semigroup and let $RB(S)$ be the free rectangular band on the set S . That is, $RB(S)$ is the set $S \times S$ endowed with multiplication $(a, b) \cdot (c, d) = (a, d)$. Let further $\phi: S \rightarrow RB(S)$ be the canonical relational morphism with respect to S ; that is, ϕ is the subsemigroup of $S \times RB(S)$ generated by all pairs $(r, (r, r))$, $r \in S$. Then $S \in \mathbf{V} \textcircled{\cap} \mathbf{RB}$ if and only if $(a, b)\phi^{-1} \in \mathbf{V}$ for all $a, b \in S$. Now it remains to observe that $(a, b)\phi^{-1} = \{ab\} \cup aSb$ for $a \neq b$ and $(a, a)\phi^{-1} = \{a, a^2\} \cup aSa$. \square

Lemma 1 and Lemma 2 are very special instances of the ‘basis theorem’ for Malcev products by Pin and Weil, see Theorem 4.1 in [13]. As mentioned in the introduction, Kad’ourek considered in [10] semigroups S all of whose subsemigroups of the form aSb with $a, b \in S$ are block groups; he called such semigroups *aggregates of block groups*. Whether or not a semigroup is a block group depends entirely on its set of idempotents. It is further clear that there are no idempotents in $(\{ab\} \cup aSb) \setminus aSb$ nor in $(\{a, a^2\} \cup aSa) \setminus aSa$. It follows that $\{ab\} \cup aSb$ is a block group if and only if so is aSb and likewise $\{a, a^2\} \cup aSa$ is a block group if and only if so is aSa , for arbitrary $a, b \in S$. Consequently, the pseudovariety $\mathbf{BG} \textcircled{\cap} \mathbf{RB}$ is comprised exactly of all aggregates of block groups. This observation is implicitly contained in [10] but (unfortunately) is not mentioned explicitly. In any case, we may formulate a criterion for membership in $\mathbf{BG} \textcircled{\cap} \mathbf{RB}$.

Corollary 3. *A semigroup S belongs to $\mathbf{BG} \textcircled{\cap} \mathbf{RB}$ if and only if, for each choice of elements $a, b \in S$, the semigroup aSb is a block group.*

Next we present a similar description of the members of the semidirect product pseudovariety $\mathbf{V} * \mathbf{RZ}$, but only for the case when \mathbf{V} is local. Here we use the derived semigroupoid theorem [16], [14].

Proposition 4. *Let \mathbf{V} be a local pseudovariety; a semigroup S belongs to $\mathbf{V} * \mathbf{RZ}$ if and only if the image $L(a)^\rho$ of each principal left ideal $L(a)$ of S under the right regular representation $\rho^{L(a)}$ belongs to \mathbf{V} .*

Proof: Let S be a semigroup, let $RZ(S)$ be the (free) right zero semigroup on the set S and let $\phi: S \rightarrow RZ(S)$ be the canonical relational morphism (with respect to S). Then, according to the derived semigroupoid theorem, the semigroup S belongs to $\mathbf{V} * \mathbf{RZ}$ if and only if the derived semigroupoid D_ϕ of ϕ divides a member of \mathbf{V} . (For a proof that the consideration of the canonical relational morphism $S \rightarrow RZ(S)$ suffices here, see Proposition 3.5 in combination with Theorem 3.3 in [4].) Since \mathbf{V} is local, this happens if (and only if) each local semigroup of D_ϕ is in \mathbf{V} . The set of objects of the derived semigroupoid D_ϕ is $RZ(S) \cup \{1\} = S \cup \{1\}$ where 1 represents a new identity adjoined to S . However, the local semigroup at the object 1 is empty. So consider any other object a , say. Then, as in the proof of Lemma 1, we have $a\phi^{-1} = L(a)$, the principal left ideal of S generated by a . According to the construction of the derived semigroupoid D_ϕ , the local semigroup at the given object a may be identified with the semigroup of all mappings $L(a) \rightarrow L(a)$ induced by multiplication on the right by elements of $L(a)$ which semigroup by definition coincides with $L(a)^\rho$. \square

Finally, we recall the well-known equalities

$$\mathbf{CS} = \mathbf{G} \textcircled{\mathbf{m}} \mathbf{RB} \text{ and } \mathbf{CS} = \mathbf{G} * \mathbf{RZ}.$$

The former equality is a consequence of Lemma 2 while the latter is a consequence of Proposition 4.

1.3. Pseudoidentity bases of pseudovarieties of the form $\mathbf{V} * \mathbf{RZ}$.

Proposition 4 provides us with a method to obtain a basis of pseudoidentities of $\mathbf{V} * \mathbf{RZ}$ from a basis of \mathbf{V} if \mathbf{V} is a local pseudovariety.

Corollary 5. *Let \mathbf{V} be a local pseudovariety and let Σ be a basis of pseudoidentities of \mathbf{V} . Then a basis of $\mathbf{V} * \mathbf{RZ}$ is given by the set of all pseudoidentities of the form*

$$x\pi(y_1x, \dots, y_nx) = x\sigma(y_1x, \dots, y_nx)$$

where the pseudoidentity $\pi(x_1, \dots, x_n) = \sigma(x_1, \dots, x_n)$ is a member of Σ and, for each i , y_i can take the value x_i or the empty value, and x is a variable not contained in $\{x_1, \dots, x_n\}$.

Proof: Suppose that the semigroup S belongs to $\mathbf{V} * \mathbf{RZ}$ and let $\pi = \sigma$ be a pseudoidentity from Σ . Choose any $a \in S$ and any $b_1, \dots, b_n \in S \cup \{1\}$ where 1 represents a new identity adjoined to S . Then b_1a, \dots, b_na and therefore also $\pi(b_1a, \dots, b_na)$ and $\sigma(b_1a, \dots, b_na)$ are well defined elements of the principal left ideal $L(a)$. Since $L(a)^\rho$ belongs to \mathbf{V} by Proposition 4 and since \mathbf{V} satisfies the pseudoidentity $\pi(x_1, \dots, x_n) = \sigma(x_1, \dots, x_n)$, substituting ρ_{b_ia} for x_i , for each i , we obtain

$$\pi(\rho_{b_1a}, \dots, \rho_{b_na}) = \sigma(\rho_{b_1a}, \dots, \rho_{b_na})$$

and therefore also

$$\rho_{\pi(b_1a, \dots, b_na)} = \rho_{\sigma(b_1a, \dots, b_na)}.$$

This means that multiplication by $\pi(b_1a, \dots, b_na)$ and by $\sigma(b_1a, \dots, b_na)$ on the right induces the same transformation $L(a) \rightarrow L(a)$. In particular, the elements $a\pi(b_1a, \dots, b_na)$ and $a\sigma(b_1a, \dots, b_na)$ coincide. Since $a \in S$ and $b_1, \dots, b_n \in S \cup \{1\}$ have been arbitrarily chosen, S satisfies each pseudoidentity of the (above-mentioned) form

$$x\pi(y_1x, \dots, y_nx) = x\sigma(y_1x, \dots, y_nx).$$

Let conversely S be a semigroup which satisfies all pseudoidentities of the form $x\pi(y_1x, \dots, y_nx) = x\sigma(y_1x, \dots, y_nx)$ described above. Then, reading the preceding paragraph backward, we see that the image $L(a)^\rho$ of each principal left ideal $L(a)$ of S under $\rho^{L(a)}$ satisfies all pseudoidentities from Σ . This means that $L(a)^\rho$ belongs to \mathbf{V} and so S belongs to $\mathbf{V} * \mathbf{RZ}$ by Proposition 4. \square

Corollary 5 is known and has been formulated as Theorem 2.2 in [8]; the well-informed reader will recognize it as a very special case (of a proved instance) of the Almeida-Weil basis theorem [4, Corollary 5.4]. For more information on the basis theorem we refer the interested reader to Section 7 in [3] and to Section 3.7 in [14].

2. The main result

2.1. Semantic characterization of the members of PCS. The main result of the paper can be formulated as follows.

Theorem 6. *The following equalities of pseudovarieties hold:*

$$\mathbf{PCS} = \mathbf{J} \textcircled{\cap} \mathbf{CS} = \mathbf{BG} \textcircled{\cap} \mathbf{RB} = \mathbf{ER} \textcircled{\cap} \mathbf{RZ} \cap \mathbf{EL} \textcircled{\cap} \mathbf{LZ} = \mathbf{BG} * \mathbf{RZ}.$$

Proof: We shall verify the inclusions

$$\mathbf{PCS} \subseteq \mathbf{J} \textcircled{\cap} \mathbf{CS} \subseteq \mathbf{BG} \textcircled{\cap} \mathbf{RB} \subseteq \mathbf{ER} \textcircled{\cap} \mathbf{RZ} \cap \mathbf{EL} \textcircled{\cap} \mathbf{LZ} \subseteq \mathbf{BG} * \mathbf{RZ} \subseteq \mathbf{PCS}.$$

1. The inclusion $\mathbf{PCS} \subseteq \mathbf{J} \textcircled{\cap} \mathbf{CS}$ has been proved by Kad'ourek [10, Theorem 4.2]. Our proof uses the same relational morphism but otherwise is different and shorter. Since $\mathbf{P'CS} = \mathbf{PCS}$ it is sufficient to show that $\mathfrak{P}(S)$ belongs to $\mathbf{J} \textcircled{\cap} \mathbf{CS}$ for each $S \in \mathbf{CS}$.

So, let S be a completely simple semigroup. Then the quotient set S/\mathcal{R} forms a left zero semigroup under set-wise multiplication and the quotient mapping $R: S \rightarrow S/\mathcal{R}, a \mapsto R_a$ is a homomorphism. Likewise, the quotient set S/\mathcal{L} is a right zero semigroup and the quotient mapping $L: S \rightarrow S/\mathcal{L}, a \mapsto L_a$ is a homomorphism. Denote by \mathfrak{R} and \mathfrak{L} the respective induced homomorphisms $\mathfrak{R}: \mathfrak{P}(S) \rightarrow \mathfrak{P}(S/\mathcal{R})$ and $\mathfrak{L}: \mathfrak{P}(S) \rightarrow \mathfrak{P}(S/\mathcal{L})$. Note that $\mathfrak{P}(S/\mathcal{R})$ is a left zero semigroup while $\mathfrak{P}(S/\mathcal{L})$ is a right zero semigroup. Moreover, for every $X \in \mathfrak{P}(S)$ we have

$$X\mathfrak{R} = \{U \in S/\mathcal{R} \mid X \cap U \neq \emptyset\} \text{ and } X\mathfrak{L} = \{V \in S/\mathcal{L} \mid X \cap V \neq \emptyset\}.$$

Consider the direct product

$$T := \mathfrak{P}(S/\mathcal{R}) \times S \times \mathfrak{P}(S/\mathcal{L})$$

which is again a completely simple semigroup. For every $X \in \mathfrak{P}(S)$ set

$$X\Phi := \{(X\mathfrak{R}, x, X\mathfrak{L}) \mid x \in X\} = \{X\mathfrak{R}\} \times X \times \{X\mathfrak{L}\}.$$

Then Φ is a relational morphism $\mathfrak{P}(S) \rightarrow T$. Let $(\mathcal{A}, e, \mathcal{B})$ be an idempotent of T with $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$ non-empty; note that e is an idempotent of S . We intend to show that the semigroup $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$ is \mathcal{J} -trivial. In order to do so, it is sufficient to verify that the set $\text{Reg}((\mathcal{A}, e, \mathcal{B})\Phi^{-1})$ of all regular elements is contained in a \mathcal{J} -trivial subsemigroup of $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$.

Then each regular \mathcal{J} -class of $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$ is trivial whence the semigroup $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$ is itself \mathcal{J} -trivial.

Set

$$S(\mathcal{A}, \mathcal{B}) := \bigcup_{U \in \mathcal{A}, V \in \mathcal{B}} U \cap V$$

and let I be the (completely simple) subsemigroup generated by all idempotents of $S(\mathcal{A}, \mathcal{B})$. Then I is an idempotent of $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$. Let

$$\mathfrak{M}_I := \{X \in (\mathcal{A}, e, \mathcal{B})\Phi^{-1} \mid IX = XI = X\}$$

be the local submonoid of $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$ with respect to I . We claim that $X \supseteq I$ for each $X \in \mathfrak{M}_I$. Indeed, let $X \in \mathfrak{M}_I$ and, for an arbitrary idempotent g of I let I_g be the \mathcal{H} -class of I containing g . First observe that $I_e = eI_e \subseteq XI = X$ since $e \in X$. Next, let f be the unique idempotent such that $e \mathcal{R} f \mathcal{L} g$; then $I_g = gI_e f \subseteq IXI = X$. So, X contains each \mathcal{H} -class of I and thus contains I itself. Therefore, the monoid \mathfrak{M}_I is ordered by reverse inclusion \supseteq with I the greatest element, and so \mathfrak{M}_I is \mathcal{J} -trivial [12]. It suffices now to show that $\text{Reg}((\mathcal{A}, e, \mathcal{B})\Phi^{-1})$ is contained in \mathfrak{M}_I .

For each $X \subseteq S$ denote by E_X the set of all idempotents of X . Let F be an idempotent of $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$; then F is a (completely simple) subsemigroup of $S(\mathcal{A}, \mathcal{B})$ with $F\mathfrak{R} = \mathcal{A}$ and $F\mathfrak{L} = \mathcal{B}$. Let $U \in \mathcal{A}$, $V \in \mathcal{B}$, $a \in F \cap U$, $b \in F \cap V$; then $(ab)^\omega \in F \cap U \cap V$. It follows that F contains all idempotents of $S(\mathcal{A}, \mathcal{B})$, hence $F \supseteq I$ and $E_F = E_I$. Then $F = F^2 \supseteq FI \supseteq FE_I = FE_F = F$ whence $FI = F$ and dually also $IF = F$. Moreover, for each regular element X of $(\mathcal{A}, e, \mathcal{B})\Phi^{-1}$ there exist idempotents F, G such that $FX = X = XG$; then $IX = IFX = FX = X$ and dually also $XI = X$. Consequently, $\text{Reg}((\mathcal{A}, e, \mathcal{B})\Phi^{-1})$ is contained in \mathfrak{M}_I , as required.

2. The inclusion $\mathbf{J} \textcircled{\text{m}} \mathbf{CS} \subseteq \mathbf{BG} \textcircled{\text{m}} \mathbf{RB}$ is a consequence of the equality $\mathbf{BG} = \mathbf{J} \textcircled{\text{m}} \mathbf{G}$ (see Section 7 in [11]) and of standard facts concerning the operation $\textcircled{\text{m}}$ (see Lemma 1.4 in [17]):

$$\mathbf{J} \textcircled{\text{m}} \mathbf{CS} = \mathbf{J} \textcircled{\text{m}} (\mathbf{G} \textcircled{\text{m}} \mathbf{RB}) \subseteq (\mathbf{J} \textcircled{\text{m}} \mathbf{G}) \textcircled{\text{m}} \mathbf{RB} = \mathbf{BG} \textcircled{\text{m}} \mathbf{RB}.$$

3. An indirect argument will be used to prove the inclusion

$$\mathbf{BG} \textcircled{\text{m}} \mathbf{RB} \subseteq \mathbf{ER} \textcircled{\text{m}} \mathbf{RZ} \cap \mathbf{EL} \textcircled{\text{m}} \mathbf{LZ}.$$

Indeed, suppose that $S \notin \mathbf{ER} \textcircled{\text{m}} \mathbf{RZ}$. Then, according to Lemma 1, there exists a principal left ideal $L(a)$ of S which does not belong to \mathbf{ER} . Consequently there exist distinct idempotents e, f in $L(a)$ forming a (non-trivial) right zero subsemigroup. This right zero subsemigroup then

is, in fact, contained in the left ideal Sa . However, since $e\{e, f\} = \{e, f\}$, the right zero subsemigroup $\{e, f\}$ is even contained in eSa , so that the latter subsemigroup of S cannot be a block group. Corollary 3 now implies that $S \notin \mathbf{BG} \textcircled{\text{m}} \mathbf{RB}$. By the dual argument, one also shows that $S \notin \mathbf{EL} \textcircled{\text{m}} \mathbf{LZ}$ entails that $S \notin \mathbf{BG} \textcircled{\text{m}} \mathbf{RB}$.

4. In order to prove the inclusion

$$\mathbf{ER} \textcircled{\text{m}} \mathbf{RZ} \cap \mathbf{EL} \textcircled{\text{m}} \mathbf{LZ} \subseteq \mathbf{BG} * \mathbf{RZ},$$

suppose that $S \in \mathbf{ER} \textcircled{\text{m}} \mathbf{RZ} \cap \mathbf{EL} \textcircled{\text{m}} \mathbf{LZ}$. According to Lemma 1 this means that each principal left ideal of S is in \mathbf{ER} and each principal right ideal of S is in \mathbf{EL} . It is well known and easy to see that \mathbf{BG} is local (by taking advantage of the *consolidation operation* [16], [14] – the reader may also consult the proof of Proposition 1 in [15]). Therefore we may now apply Proposition 4: in order to show that S belongs to $\mathbf{BG} * \mathbf{RZ}$, it is sufficient to verify that the image $L(a)^\rho$ of each principal left ideal $L(a)$ of S under $\rho^{L(a)}$ belongs to $\mathbf{ER} \cap \mathbf{EL}$ ($= \mathbf{BG}$). Since each principal left ideal $L(a)$ is in \mathbf{ER} so is its image $L(a)^\rho$. It remains to prove that $L(a)^\rho$ is in \mathbf{EL} . We need to show that each (at most) two-element left zero subsemigroup $\{\rho_{ca}, \rho_{da}\}$ of $L(a)^\rho$ is trivial; here ca, da are arbitrary elements of $L(a)$ (c and/or d may be the empty symbol). By assumption, the equalities

$$a(ca) = a(ca)^2, \quad a(da) = a(da)^2, \quad a(ca)(da) = a(ca), \quad a(da)(ca) = a(da)$$

hold. Shifting the brackets to the left we observe that the inner left translations $\lambda_{ac}, \lambda_{ad}: R(a) \rightarrow R(a)$ form a left zero subsemigroup of ${}^\lambda R(a)$, the image of $R(a)$ under $\lambda^{R(a)}$. Since $R(a)$ is in \mathbf{EL} so is ${}^\lambda R(a)$. Hence $\{\lambda_{ac}, \lambda_{ad}\}$ is a trivial semigroup and so $\lambda_{ac} = \lambda_{ad}$. It follows that $(ac)a = (ad)a$ and therefore also $a(ca) = a(da)$ hold. But then $\rho_{ca} = \rho_{da}$, that is, the left zero semigroup $\{\rho_{ca}, \rho_{da}\}$ is trivial.

5. Finally, in order to verify the inclusion $\mathbf{BG} * \mathbf{RZ} \subseteq \mathbf{PCS}$, we may argue exactly as in [15], namely we may use the inclusion $\mathbf{PG} * \mathbf{RZ} \subseteq \mathbf{P}(\mathbf{G} * \mathbf{RZ})$ (proved in [7]) and the equalities $\mathbf{PG} = \mathbf{BG}$ and $\mathbf{CS} = \mathbf{G} * \mathbf{RZ}$. □

Just as in Theorem 4.2 of [10], the first step in the proof of Theorem 6 actually proves the inclusion $\mathbf{PQ} \subseteq \mathbf{J} \textcircled{\text{m}} \mathbf{Q}$ for **each** pseudovariety \mathbf{Q} of completely simple semigroups. The reason for this is that the left zero semigroup $\mathfrak{P}(S/\mathcal{R})$ is trivial if and only if so is S/\mathcal{R} and the right zero semigroup $\mathfrak{P}(S/\mathcal{L})$ is trivial if and only if so is S/\mathcal{L} . Consequently,

both power semigroups belong to the pseudovariety generated by S , and therefore also T belongs to the pseudovariety generated by S .

As already mentioned, two papers are devoted to the study of the pseudovariety \mathbf{PCS} , namely Steinberg's paper [15] and Kad'ourek's paper [10]. The main result in the former is the inclusion $\mathbf{PCS} \subseteq \mathbf{BG} * \mathbf{RZ}$ while the main results of the latter are the inclusion $\mathbf{PCS} \subseteq \mathbf{J} \textcircled{\text{m}} \mathbf{CS}$ and the equality $\mathbf{BG} \textcircled{\text{m}} \mathbf{RB} = \mathbf{J} \textcircled{\text{m}} \mathbf{CS}$. Whereas the inclusion of the latter paper forms an essential ingredient of our proof we get the inclusion of the former and the equality of the latter for free. It should be noted however, that we do make use of the celebrated equality $\mathbf{PG} = \mathbf{BG}$.

The last three items in the chain of equalities in Theorem 6 give rise to three (equivalent) structural characterizations of the members of \mathbf{PCS} . These characterizations in turn lead to transparent algorithms for testing membership in \mathbf{PCS} . Indeed, immediately from Corollary 3 we get: a semigroup S belongs to \mathbf{PCS} if and only if aSb is a block group for all $a, b \in S$ that is, if and only if it is an aggregate of block groups in the sense of Kad'ourek [10]. Next, expressing membership in $\mathbf{ER} \textcircled{\text{m}} \mathbf{RZ} \cap \mathbf{EL} \textcircled{\text{m}} \mathbf{LZ}$ in terms of Lemma 1 we obtain: a semigroup S belongs to \mathbf{PCS} if and only if each principal left ideal of S is in \mathbf{ER} and each principal right ideal of S is in \mathbf{EL} . Finally, applying Proposition 4 to $\mathbf{BG} * \mathbf{RZ}$ we see: a semigroup S belongs to \mathbf{PCS} if and only if the image $L(a)^\rho$ of each principal left ideal $L(a)$ of S under the right regular representation $\rho^{L(a)}$ is a block group. In the latter two cases, the one-sided ideals $L(a)$ and $R(a)$ may be replaced with the one-sided ideals Sa and aS , respectively.

The equality $\mathbf{PG} = \mathbf{BG}$ has two analogues in the completely simple case, namely, on the one hand Steinberg's equality

$$\mathbf{PCS} = \mathbf{P}(\mathbf{G} * \mathbf{RZ}) = \mathbf{BG} * \mathbf{RZ},$$

on the other hand the equality

$$\mathbf{PCS} = \mathbf{P}(\mathbf{G} \textcircled{\text{m}} \mathbf{RB}) = \mathbf{BG} \textcircled{\text{m}} \mathbf{RB}.$$

Finally, both equalities $\mathbf{PG} = \mathbf{J} * \mathbf{G}$ and $\mathbf{PG} = \mathbf{J} \textcircled{\text{m}} \mathbf{G}$ (see [11]) have their obvious analogues, namely

$$\mathbf{PCS} = \mathbf{J} * \mathbf{CS} \text{ and } \mathbf{PCS} = \mathbf{J} \textcircled{\text{m}} \mathbf{CS}.$$

The latter occurs in the statement of Theorem 6 and is important in its proof. In contrast, the author is not aware of a direct argument for the former: it follows merely from the equalities $\mathbf{PCS} = \mathbf{BG} * \mathbf{RZ}$, $\mathbf{BG} = \mathbf{J} * \mathbf{G}$, $\mathbf{CS} = \mathbf{G} * \mathbf{RZ}$ and the associativity of the operation $*$, see [10].

2.2. Syntactic characterization of the members of PCS. The last three items in the chain of equalities in Theorem 6 also give rise to three syntactic characterizations of the pseudovariety **PCS**. First, using the fact that **BG** is defined by the pseudoidentity $(x^\omega y^\omega)^\omega = (y^\omega x^\omega)^\omega$, it follows from Corollary 3 that **BG** \mathbb{m} **RB** is defined by the single pseudoidentity

$$((sxt)^\omega (syt)^\omega)^\omega = ((syt)^\omega (sxt)^\omega)^\omega,$$

see [10]. This statement can also be deduced from the already mentioned Pin-Weil basis theorem for Mal'cev products [13]. Further on, as mentioned in the preliminaries, the pseudoidentity $(y^\omega z^\omega)^\omega = (y^\omega z^\omega)^\omega y^\omega$ defines **ER**. Moreover, every idempotent in the principal left ideal $L(x)$ of a semigroup S can be written as $(ax)^\omega$ for some element a of S . In view of Lemma 1, it follows that the pseudovariety **ER** \mathbb{m} **RZ** is defined by the pseudoidentity

$$(\dagger) \quad ((ax)^\omega (bx)^\omega)^\omega = ((ax)^\omega (bx)^\omega)^\omega (ax)^\omega.$$

Dually, **EL** \mathbb{m} **LZ** is defined by the pseudoidentity

$$(\ddagger) \quad ((xb)^\omega (xa)^\omega)^\omega = (xa)^\omega ((xb)^\omega (xa)^\omega)^\omega.$$

(Again, both statements are immediate consequences of the above-mentioned basis theorem for Mal'cev products [13].) Consequently, the two pseudoidentities (\dagger) , (\ddagger) form a basis of **ER** \mathbb{m} **RZ** \cap **EL** \mathbb{m} **LZ**. Finally, as a consequence of Corollary 5 we obtain a basis of pseudoidentities of **BG** * **RZ** as follows.

Corollary 7. *The pseudovariety **BG** * **RZ** is defined by the pseudoidentity*

$$(\star) \quad x((yx)^\omega (zx)^\omega)^\omega = x((zx)^\omega (yx)^\omega)^\omega.$$

Proof: It has been noticed in the preliminaries that the single pseudoidentity $(y^\omega z^\omega)^\omega = (z^\omega y^\omega)^\omega$ forms a basis of **BG**. As already mentioned in part 4 of the proof of Theorem 6, the pseudovariety **BG** is local. Thus we may apply to the pseudovariety **BG** * **RZ** the rule formulated in Corollary 5. In this way, we obtain four pseudoidentities, each of the form

$$x((ax)^\omega (bx)^\omega)^\omega = x((bx)^\omega (ax)^\omega)^\omega$$

where a as well as b can be a variable or the empty symbol. If a , say, represents the empty symbol then $(ax)^\omega$ becomes x^ω which is the same as $(xx)^\omega$. So, in the case that a and/or b represent the empty symbol, the corresponding pseudoidentity can be obtained from the one in which a as well as b represent variables, simply by substituting the variable a

and/or b with the variable x . Therefore the pseudoidentity (\star) defines $\mathbf{BG} * \mathbf{RZ}$. \square

We arrive at the main theorem in this section.

Theorem 8. *Each of the pseudoidentity systems (i)–(iii) forms a basis of pseudoidentities of \mathbf{PCS} :*

- (i) $((sxt)^\omega(syt)^\omega)^\omega = ((syt)^\omega(sxt)^\omega)^\omega,$
- (ii) $((ax)^\omega(bx)^\omega)^\omega = ((ax)^\omega(bx)^\omega)^\omega(ax)^\omega \quad \text{and}$
 $((xb)^\omega(xa)^\omega)^\omega = (xa)^\omega((xb)^\omega(xa)^\omega)^\omega,$
- (iii) $x((ax)^\omega(bx)^\omega)^\omega = x((bx)^\omega(ax)^\omega)^\omega.$

It is interesting to see directly, that is, using only syntactic arguments, that the systems of pseudoidentities (i)–(iii) are equivalent. In the following, we shall present such a syntactic derivation.

In order to prove that (i) implies (ii) we first note that by substituting ts for x in (i) and using $(stst)^\omega = (st)^\omega$ we get

$$(i') \quad ((st)^\omega(syt)^\omega)^\omega = ((syt)^\omega(st)^\omega)^\omega.$$

Now, using the equalities $y^\omega(y^\omega z^\omega)^\omega = (y^\omega z^\omega)^\omega = (y^\omega z^\omega)^\omega z^\omega$ we obtain

$$\begin{aligned} ((ax)^\omega(bx)^\omega)^\omega &= (((ax)^\omega(bx)^\omega)^\omega)^\omega \\ &= ((ax)^\omega((ax)^\omega(bx)^\omega)^\omega)^\omega \\ &= ((ax)^\omega(amx)^\omega)^\omega && \text{for } m = x(ax)^\omega{}^{-1}(bx)^\omega{}^{-1}b \\ &= ((amx)^\omega(ax)^\omega)^\omega && \text{by (i')} \\ &= (((ax)^\omega(bx)^\omega)^\omega(ax)^\omega)^\omega \end{aligned}$$

and

$$\begin{aligned} (ii) \quad ((ax)^\omega(bx)^\omega)^\omega &= (((ax)^\omega(bx)^\omega)^\omega(ax)^\omega)^\omega \\ &= (((ax)^\omega(bx)^\omega)^\omega(ax)^\omega)^\omega(ax)^\omega \\ &= ((ax)^\omega(bx)^\omega)^\omega(ax)^\omega && \text{by (ii)}. \end{aligned}$$

The second pseudoidentity of (ii) is derived analogously (by reading (i') from right to left, that is, the two sides of that pseudoidentity are swapped with each other).

Let us consider next the pseudoidentities (ii) and denote them by (ii1) and (ii2), respectively. In order to verify that (ii) implies (iii), we multiply (ii1) by x on the left to obtain:

$$\begin{aligned} x \cdot ((ax)^\omega (bx)^\omega)^\omega &= x \cdot ((ax)^\omega (bx)^\omega)^\omega (ax)^\omega && \text{by (ii1)} \\ &= (xa)^\omega ((xb)^\omega (xa)^\omega)^\omega \cdot x && \text{shifting brackets} \\ &= ((xb)^\omega (xa)^\omega)^\omega \cdot x && \text{by (ii2)} \\ &= x \cdot ((bx)^\omega (ax)^\omega)^\omega && \text{shifting brackets.} \end{aligned}$$

Finally, we intend to derive (i) from (iii). For this purpose, we note that shifting the brackets in (iii) we get an equivalent pseudoidentity, namely:

$$(iii^s) \quad ((xa)^\omega (xb)^\omega)^\omega x = ((xb)^\omega (xa)^\omega)^\omega x.$$

Having this in mind and assuming (iii), we obtain

$$\begin{aligned} ((sxt)^\omega (syt)^\omega)^\omega &= ((sxt)^\omega (syt)^\omega)^\omega ((sxt)^\omega (syt)^\omega)^\omega \\ &= \underline{lt}((\underline{sxt})^\omega (\underline{syt})^\omega)^\omega \\ &= \underline{lt}((\underline{syt})^\omega (\underline{sxt})^\omega)^\omega && \text{by (iii)} \\ &= ((sxt)^\omega (syt)^\omega)^\omega ((syt)^\omega (sxt)^\omega)^\omega \\ &= ((\underline{sxt})^\omega (\underline{syt})^\omega)^\omega \underline{sr} \\ &= ((\underline{syt})^\omega (\underline{sxt})^\omega)^\omega \underline{sr} && \text{by (iii}^s) \\ &= ((syt)^\omega (sxt)^\omega)^\omega ((syt)^\omega (sxt)^\omega)^\omega \\ &= ((syt)^\omega (sxt)^\omega)^\omega, \end{aligned}$$

where

$$l = ((sxt)^\omega (syt)^\omega)^{\omega-1} (sxt)^\omega (syt)^{\omega-1} sy$$

and

$$r = yt(syt)^{\omega-1} (sxt)^\omega ((syt)^\omega (sxt)^\omega)^{\omega-1},$$

which proves that (iii) implies (i). Altogether we have given an alternative, purely syntactic proof of the equalities

$$\mathbf{BG} \textcircled{m} \mathbf{RB} = \mathbf{ER} \textcircled{m} \mathbf{RZ} \cap \mathbf{EL} \textcircled{m} \mathbf{LZ} = \mathbf{BG} * \mathbf{RZ}.$$

What is more, we actually have proved the implications:

$$(i) \Rightarrow (i') \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).$$

From this it follows that (i') is yet another pseudoidentity defining **PCS**. Moreover, each of the pseudoidentity bases of Theorem 8 provides a new algorithm for deciding membership in **PCS**.

2.3. Another algorithmic problem. For two completely simple semigroups B and C , the direct product $\mathfrak{P}(B) \times \mathfrak{P}(C)$ embeds in $\mathfrak{P}(B \times C)$. Hence the pseudovariety **PCS** consists of all divisors of the power semigroups $\mathfrak{P}(C)$ of all completely simple semigroups C . For a given semigroup $S \in \mathbf{PCS}$ (that is, a semigroup S that has successfully passed a membership test for **PCS**), one may ask how to construct a completely simple semigroup C for which S divides $\mathfrak{P}(C)$. The analogous question for the members of **PG** had been a standing problem and was solved by Steinberg and the author [7]. Based on that we shall construct a solution for the present case.

So, let $S \in \mathbf{PCS}$, let $RZ(S)$ be the (free) right zero semigroup on S and let $\phi: S \rightarrow RZ(S)$ be the canonical relational morphism as in Lemma 1. The derived semigroupoid D_ϕ of ϕ has set of objects $RZ(S) \cup \{1\} = S \cup \{1\}$ where $1 \notin S$. The set $D_\phi(a, 1)$ of all morphisms $a \rightarrow 1$ is empty for each object a ; for $b \in S$, the set $D_\phi(1, b)$ can be identified with the set of all triples $(1, \gamma, b)$ where γ is a 'translation' $\{1\} \rightarrow L(b)$, $1 \mapsto 1g = g$ with $g \in L(b)$; finally, for two objects $a, b \in S$, the set $D_\phi(a, b)$ can be identified with the set of all triples (a, σ, b) where σ denotes the mapping $L(a) \rightarrow L(b)$, $x \mapsto xs$ for $s \in L(b)$. The composition of two composable morphisms is the obvious one:

$$(*) \quad (a, \sigma, b)(b, \tau, c) = (a, \sigma\tau, c).$$

Let $BG(S)$ be the *consolidated semigroup* of D_ϕ as in [16], [14] (see also below). Since $S \in \mathbf{BG} * \mathbf{RZ}$, the derived semigroupoid theorem implies that (i) the local semigroups of D_ϕ are block groups (and so $BG(S)$ itself is a block group since $BG(S)$ does not contain non-trivial left zero and right zero subsemigroups) and (ii) S divides the wreath product $BG(S) \wr RZ(S)$.

Next, from Theorem 4.5 in [7] we are aware how to construct from $BG(S)$ a group $G(S)$ such that $BG(S)$ divides the power group $\mathfrak{P}(G(S))$. Standard properties of the wreath product then imply that $BG(S) \wr RZ(S)$ and hence also S divide the wreath product $\mathfrak{P}(G(S)) \wr RZ(S)$. Finally, in Lemmas 2.1 and 2.2 in [7] a division from $\mathfrak{P}(G(S)) \wr RZ(S)$ to $\mathfrak{P}(G(S)) \wr RZ(S)$ is explicitly constructed. Consequently, the original semigroup S divides the power semigroup $\mathfrak{P}(G(S) \wr RZ(S))$ and $G(S) \wr RZ(S)$ is completely simple.

We summarize the procedure as follows: for $S \in \mathbf{PCS}$ let $BG(S)$ be the set consisting of an element 0 together with all triples (a, σ, b) for $a \in S \cup \{1\}$, $b \in S$ and σ being a mapping $\tilde{L}(a) \rightarrow L(b)$ of the form $x \mapsto xs$ for some $s \in L(b)$ (here $\tilde{L}(1) = \{1\}$ while $\tilde{L}(a) = L(a)$ if $a \neq 1$). Let $BG(S)$ be endowed with the multiplication $(*)$ whenever possible and let all other products be defined to be 0. Then $BG(S)$ is a block group; let $G(S)$ be a group for which $BG(S)$ divides the power group $\mathfrak{P}(G(S))$, as constructed in [7]. Then S divides the power semigroup $\mathfrak{P}(G(S)) \wr RZ(S)$ where $RZ(S)$ is the set S endowed with right zero multiplication, and $G(S) \wr RZ(S)$ is completely simple.

Personal remark of the author. The work on this paper started as a joint project with J. Kad'ourek (initiated by the author). Unfortunately, from the very beginning, this cooperation was overshadowed by dissent on various matters of the joint work. After some while the author felt that there was no longer the possibility to write the paper in a way that would be satisfactory for both authors and resigned from that cooperation. The outcome is now entirely according to the view of the author, and Kad'ourek did not accept co-authorship for it.

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