ABSOLUTE CONVERGENCE OF FOURIER INTEGRALS

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Abstract. In this survey, results on the representation of a function as an absolutely convergent Fourier integral are collected, classified and discussed. Certain applications are also given.

Contents

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1. Introduction

Chronologically, one should apparently start with the celebrated paper by S.N. Bernstein of 1914 (see, e.g., [172] or [4]). The following main results of it read as follows.

Each periodic function from the $\mathrm{Lip}\,\alpha$ class with $\alpha > \frac{1}{2}$ is expanded in the absolutely convergent Fourier series, and an example of a function from $\mathrm{Lip}\,\frac{1}{2}$ (by using the Legendre symbols) is built whose Fourier series is not absolutely convergent. More precisely, the next sufficient condition holds true:

$$\int\limits_0^1 \frac{\omega(f,t)}{t^{3/2}}\,dt < \infty,$$

where $\omega(f,t)$ is the modulus of continuity of the function f.

Later on, A. Zygmund discovered a bit different test: If, in addition, *f is of bounded variation*, sufficient condition is

$$\int\limits_0^1 \frac{\sqrt{\omega(f,t)}}{t}\,dt < \infty.$$

Both conditions are sharp on the whole class (see [172], [4], [65]). In the multivariate case of the n-dimensional Euclidean space $\mathbb{R}^n$ an analog of Bernstein's condition reads, roughly speaking, as follows: the smoothness in the L_2 metrics should be greater than $\frac{n}{2}$ (see, e.g., [130, Ch.VII, Cor.1.9]).

However, there are important problems of a similar type, non-periodic by their nature. And here our story starts. In the 20-s–40-s of the 20-th century, besides natural attempts to find non-periodic analogs of the aforementioned results, first of all by E. Titchmarsh, a variety of results on the absolute convergence of Fourier integrals appeared one after another as part and parcel of different theories. Among those were the moment problem, Tauberian theorems, probability, differential equations, etc. Later on a need in such results sprung up in other topics. We will discuss some of them in more detail. As frequently happens, the subject in question has become an area to study interesting in its own. It is apparently difficult to say who was "Columbus" on that continent (or, say, "Armstrong" on that satellite) but naming the spaces in question by N. Wiener

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is by no means accidental. He definitely opened many roads to the territory of absolutely convergent Fourier integrals (and series as well) and determined for many years the main ways of their study. It was certainly Wiener's famous 1/f-theorem after which the interest essentially moved towards the ideology where the space (algebra) of absolutely convergent Fourier integrals was the principal object rather than separate functions.

A simple basic reason has initiated many results and conditions, mostly sufficient but sometimes also necessary. The Fourier transform of an integrable function is uniformly continuous and vanishing at infinity, that is, belongs to $C_0(\mathbb{R}^n)$. A natural question is whether every C_0 function is the Fourier transform of an integrable function. It is well known that the answer is NO. A simple counter-example on $\mathbb{R}$ is delivered by the function

$$f(t) = \begin{cases} \frac{t}{\ln 2}, & |t| \le 1, \\ \frac{1}{\ln(1+t)}, & t > 1, \\ -\frac{1}{\ln(1-t)}, & t < -1; \end{cases}$$

for more general approach, see [130, Ch.I, 4.1]. However, it is important to know whether the given C_0 function (or a class of such functions) is the Fourier transform of an integrable function or, more generally and under weaker assumptions, the Fourier transform of a measure.

The class of functions representable as Fourier integrals of absolutely integrable functions, although often denoted by $A(\mathbb{R}^n)$, in numerous publications is also denoted as $W_0(\mathbb{R}^n)$ and called Wiener ring or Wiener algebra, both in generalizations, see, for instance, [40], and applications in operator theory. In this survey we follow the latter tradition.

We deal with the Wiener algebra in three forms:

$$W_0 = W_0(\mathbb{R}^n) := \left\{ f(x) : \ f(x) = \int\limits_{\mathbb{R}^n} e^{-ixy} g(y)\, dy, \ g \in L_1(\mathbb{R}^n) \right\} \tag{1.1}$$

with $\|f\|_{W_0} = \|g\|_{L_1}$,

$$W_1 = W_1(\mathbb{R}^n) := \{ f(x) : \ f(x) = c + f_0, \ f_0 \in W_0(\mathbb{R}^n), \ c \in \mathbb{C} \} \tag{1.2}$$

with $\|f\|_{W_1} = |c| + \|f_0\|_{W_0}$ and

$$W = W(\mathbb{R}^n) := \left\{ f(x) : \ f(x) = \int\limits_{\mathbb{R}^n} e^{-ixy} d\mu(y), \mathrm{var}\mu < \infty \right\} \tag{1.3}$$

with $\|f\|_W = \mathrm{var}\mu$, so that W reduces to W_0 in the case of absolutely continuous measures, and to the space of almost-periodic functions with absolutely convergent Fourier series if restricted to discrete measures. In fact, absolute convergence of Fourier series is an important particular case of the general approach when the measure in the definition of algebra W is discrete and is concentrated on an arithmetic progression, for instance, at the lattice points.

There is a great lot of investigations devoted to the well known problem of the representation of a function as an absolutely convergent Fourier integral. The role and importance of this topic can be detected by examining the Fourier multipliers theory in parallel. The intimate relations of the two topics can be described in dimension one as follows.

Let $m\colon \mathbb{R}\to\mathbb{C}$ be a bounded measurable function ($m\in L_\infty(\mathbb{R})$). As is recently proved (see [79] and (1.5)), $m\in M_p$, $1\le p\le\infty$, can be considered as almost everywhere continuous. Define on $L_2(\mathbb{R})\cap L_p(\mathbb{R})$ a linear operator Λ by means of the following identity for the Fourier transforms of functions $f\in L_2(\mathbb{R})\cap L_p(\mathbb{R})$:

$$\widehat{\Lambda f}(x)=m(x)\widehat{f}(x). \tag{1.4}$$

If a constant $D>0$ exists such that for each $f\in L_2(\mathbb{R})\cap L_p(\mathbb{R})$ there holds

$$\|\Lambda f\|_{L_p(\mathbb{R})}\le D\|f\|_{L_p(\mathbb{R})},$$

then Λ is called a Fourier multiplier taking $L_p(\mathbb{R})$ into $L_p(\mathbb{R})$. Fourier multipliers in $L_p(\mathbb{R}^n)$ when $n\ge 2$ are defined in a similar manner. This is written as $m\in M_p$ and $\|m\|_{M_p}=\|\Lambda\|_{L_p\to L_p}$. Obviously, $M_2=L_\infty(\mathbb{R})$ and $\|m\|_{M_2}=\|m\|_{L_\infty}$. The case of the space M_p when $p\in(1,+\infty)$ has been studied by M. Riesz, J. Marcinkiewicz, S.G. Mikhlin, L. Hörmander, E.M. Stein, Ch. Fefferman, P.I. Lizorkin, and many others (see, e.g., [129, Ch.IV], [36, Ch.8], or [60, Ch.5]). A key sufficient condition for belonging to M_p, $1<p<+\infty$, is the boundedness of total variations of m on the dyadic intervals $[2^k,2^{k+1}]$ and $[-2^{k+1},-2^k]$, $k\in\mathbb{Z}$. In the case related to our subject, in contrast to the mentioned results on Fourier multipliers which ensure the boundedness of the operator (1.4) in $L_p(\mathbb{R})$ for $1<p<\infty$, the absolute integrability of the function $\widehat{m}(x)$ allows one to cover also the cases $p=1$ and $p=\infty$. In all of these cases

$$M_\infty(\mathbb{R}^n)=M_1(\mathbb{R}^n)\subset M_p(\mathbb{R}^n),\qquad 1<p<+\infty. \tag{1.5}$$

When m is a continuous function one has $m\in M_\infty$ if and only if m is the Fourier transform of a finite Borel (complex-valued) measure μ:

$$m(x)=\int\limits_{\mathbb{R}^n} e^{-ixy}d\mu(y). \tag{1.6}$$

By this, $\|m\|_{M_\infty}=\|m\|_{M_1}=\mathrm{var}\mu$ (the total variation of the measure). The multiplier itself is a convolution of f and this measure. In [79] the reader will also find a convenient list of many useful properties of multipliers.

Needless to say that Fourier multipliers have many important applications. For instance, one of the ways to define Sobolev spaces of fractional order is to introduce them as $L_p\cap I^\alpha(L_p)$ or $L_p\cap B^\alpha(L_p)$, where I^α, B^α are the Riesz and Bessel operators of fractional integration. In view of the well known properties of the Fourier transforms of the kernels of these integral operators, to justify the

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embedding $L_p \cap B^\alpha(L_p) \subset L_p \cap I^\alpha(L_p)$, we arrive at the problem to show that the function $\frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}}$ is a Fourier multiplier; see (8.2) for this example.

For the application of multipliers to deriving criteria of comparison of differential operators in dimension one, see [157]. Thus, for instance, let P and Q be algebraic polynomials, with $\deg Q < \deg P = r$, $D = \frac{d}{dx}$. In order that

$$\|Q(D)f\|_{L_q(\mathbb{R})} \le C\|P(D)f\|_{L_p(\mathbb{R})}$$

for all $f \in W_p^r(\mathbb{R})$ with a constant C independent of f, it is necessary and sufficient that $q \ge p \ge 1$, and for $(p,q) \ne (1,\infty)$

$$\sup_{x\in\mathbb{R}} \frac{|Q(ix)|}{|P(ix)|} < \infty.$$

And if $p = 1$ and $q = \infty$, one has to replace in the denominator $|P(ix)|$ by $|P(ix| + |P'(ix)|$ in this criterion.

In the case of several variables the classical Gagliardo-Nirenberg inequality is well known (see, e.g., [15, Ch.III, §15]):

$$\sum_{|j|=r} \|D^j f\|_{L_p(\mathbb{R}^n)} \le C\|f\|_{L_{p_1}(\mathbb{R}^n)}^{1-\theta} \left(\sum_{|j|=l} \|D^j f\|_{L_{p_2}(\mathbb{R}^n)} \right)^\theta,$$

where $1 \le p_1, p_2 \le \infty$, $0 \le r < l$, $\frac{r}{l} \le \theta \le 1$, and

$$\frac{n}{p} - r = (1-\theta)\frac{n}{p_1} + \theta\left(\frac{n}{p_2} - l\right),$$

except certain special values of the indicated parameters.

Many examples of multipliers are dispersed in the body of the paper. Note that various specific multipliers attracted much attention in the 50-70s (see, e.g., [39] and [129, Ch.4, 7.4], and references therein). One of them is

$$m(x) := m_{\alpha,\beta}(x) = \theta(x)\frac{e^{i|x|^\alpha}}{|x|^\beta}, \tag{1.7}$$

where θ is a C^∞ function on $\mathbb{R}^n$, which vanishes near zero, and equals 1 outside a bounded set, and $\alpha, \beta > 0$. This multiplier served as a model case while proving general theorems. It is known that for $n \ge 2$,

$$\text{if} \quad \frac{\beta}{\alpha} > \frac{n}{2}, \quad \text{then} \quad m \in M_1 \quad (\text{or} \quad m \in M_\infty); \tag{1.8}$$

and

$$\text{if} \quad \frac{\beta}{\alpha} \le \frac{n}{2}, \quad \text{then} \quad m \notin M_1 \quad (\text{or} \quad m \notin M_\infty). \tag{1.9}$$

The first assertion holds true for $n = 1$ as well, while the second one only when $\alpha \ne 1$; however, the case $\alpha = n = 1$ is obvious: $m \in M_\infty$ for any $\beta > 0$. The instance $\frac{\beta}{\alpha} = \frac{n}{2}$ is also considered in [39]; it is proved there that in this case m is a multiplier from L_1 to a Lorentz space; see a detailed analysis of this multiplier and relations to differential and pseudo-differential operators in [101], [102].

Multipliers are used in embedding theorems, differential equations, operator theory, etc. We emphasize applications of multipliers in the study of the convergence of various summability methods of multiple Fourier series. Further we shall concentrate on the representation of a function in the form (1.6). When the measure μ in such a representation is absolutely continuous with respect to the Lebesgue measure, the representation (say, of a multiplier function μ) is that of the form of an absolutely convergent Fourier integral. The study of such a representability goes back to the results of N. Wiener [164] and [165] in the 30-s, in particular to the famous $\frac{1}{f}$-Wiener theorem. Those results gave a strong impulse to the general theory of commutative normed rings developed by I.M. Gel'fand, A.A. Raikov and G.E. Shilov [47], [48] and were also an essential tool in applications to Wiener-Hopf equations and operator theory in general, see, e.g., I. Gohberg and M. Krein [52] and I. Gohberg and I.A. Fel'dman [51].

The main goal of this work is to survey (in accordance with history, bibliography, approaches and methods) the results on the latter subject of representability of functions by Fourier integrals of functions in $L_1(\mathbb{R}^n)$. For the reader's convenience, we divide the survey into sections corresponding to the types of the used terms (in terms of integrability of derivatives, integrability of fractional derivatives, radial functions, and so on). However, placing a test into that or another section is often quite relative. Note that we here give results for estimates of $\|\widehat{f}\|_1$ even if the cited paper contains an estimate for $\|\widehat{f}\|_q,\ q \geq 1$.

Let us outline the structure of the paper. Immediately after this introduction we present in Section 2 a list of symbols and notations that are systematically used in the text. The next Section 3 is of the same nature, here definitions and general notions are collected. Some of them are well known but still are worth recalling, the other are less known but maybe deserve more attention. Naturally, those appearing only once, "locally", are given in the appropriate place.

We begin to present the results in the following Section 4, some general statements first, necessary conditions afterwards, and conclude the section with certain criteria. The latter have mostly not found applications but maybe this publication will stimulate further search for those.

We then single out a special Section 5 for the one-dimensional case to show the genesis of many further results and generalizations. The one-dimensional tests obtained mainly from twenties to forties of the XX-th century generated three different, although contiguous, trends in for tests of the absolute integrability:

1) in terms of finite differences (goes back to S.N. Bernstein and E. Titchmarsh),
2) in terms of integrability of derivatives (goes back to A. Beurling),
3) in BV-terms (goes back to A. Beurling and B. Sz.-Nagy).

We also mention the technique of localization (decomposition theorems), which goes back to L. Hörmander.

The above does not mean that no more one-dimensional results exist, nor will be presented in the next sections. However, we indeed will mostly concentrate on multidimensional assertions. For example, in Section 6 many results from the previous section, in which conditions on the derivatives ensure integrability, are generalized to several dimensions by means of involving partial derivatives, Laplacians, and other differential operators.

Section 7 returns us to what is said in the very beginning of the paper. In this section various generalizations of Bernstein's theorem and of Zygmund's theorem are given separately in two subsections. Extensions of Zygmund's theorem are quite recent and probably do not cover all the possibilities to give sufficient conditions in terms of belonging to two different smoothness spaces.

In the next Section 8 the case of radial and quasi-radial functions is considered. As often happens, unity of both one-dimensional and multidimensional properties lead to interesting phenomena and very special conditions.

Back to conditions in terms of the derivatives, one may see that some of these results are not sharp enough. Involving fractional derivatives in Section 9 allows one to overcome this obstacle. The price for the sharpness is well known: such conditions are sometimes more difficult to be verified.

The lengthy title of Section 10 expresses the type of the considered conditions. Beginning with more or less old results, we then present absolutely recent conditions in terms of a simultaneous behavior of the function and its derivative(s). It is worth mentioning that the diversity of such results is impressive, many of them are not published just because it is still difficult to choose which are "true" and which look the same but are not applicable. Undoubtedly, numerous conditions of such type as well as their applications will appear in close future.

In Section 11 various results on the positive definiteness are given in connection with our topic. Many of these results are quite recent, some open problems are still waiting for being solved.

In Section 12 classes of functions are studied for which the integrability or even asymptotics of the Fourier transform is of the same nature as that for convex function. These results are applied to various problems of summability of one- and multidimensional Fourier series.

The survey is concluded with a section containing a table of the Fourier transforms of certain integrable functions and a table of various positive definite functions.

The number of items in the list of references may impress but we have to say that the list is by no means comprehensive. We have referred only to the works we consider to be important in that or another sense or to those which consist further information.

Nowadays, as many other ideas born in classical analysis, the notion of Wiener algebras is flourishing in Functional Analysis, where Wiener algebras of operators in Banach spaces are studied. The property defining such algebras is that if any element in this algebra is invertible (probably one-sidedly), then its inverse is also

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in the algebra. Such algebras are widely studied last decades and have numerous applications. We do not touch at all this topic, but refer to the recent paper [3], where the reader can find many other references.

A bit more on the motivation and prospects. The authors have contributed much in this topic, and very often felt that the known results were too incoherent. This resulted in an idea that one source should exist where basic results were collected and provided with general outlook. Such an attempt had been taken by one of the authors in [121]. However, it turned out that, on the one hand, that publication was (and still is) almost inaccessible and, on the other hand, certain publications had not been included. The latter means not only the absence of certain items in the list of reference but also the absence of certain related subjects, first of all those from Sections 11 and 12. The last but not least, new results have recently appeared (see first of all Section 10). All these have naturally led the authors to unite their efforts in a new attempt to survey all possible results as the components of one big puzzle. The authors hope that such a general overview, the picture as a whole and more detailed elucidation of the most important fragments, will increase the interest to this topic. One of the expected outcomes is as follows: from certain sufficient conditions for $m \in M_1 = M_\infty$ and the obvious one for M_2 (see above) sufficient conditions may apparently be derived for $m \in M_p$, $1 < p < +\infty$, by means of an appropriate interpolation theorem.

2. Notation

We begin with some general notation systematically used throughout the text without further indication.

$\dot{\mathbb{R}}^n$ is the compactification of the Euclidean space $\mathbb{R}^n$ by the unique infinite point,
$x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$,
$xy = x_1y_1 + \cdots + x_ny_n, \qquad |x| = \sqrt{x_1^2 + \cdots + x_n^2}$,
$\mathbb{R}^n_+$ is the n-tant of points $x \in \mathbb{R}^n$ with all $x_j \geq 0$;
when $n = 1$ we use this and similar notation with the superscript omitted;
by $\mathbb{C}$ we denote the set of complex numbers;
for a set $E \subset \mathbb{R}^n$ by $|E|$ we denote its Lebesgue measure;
$\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ centered at the origin, by σ and $x' = \frac{x}{|x|}$ we denote points on $\mathbb{S}^{n-1}$, $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$;
$e_i = (\underbrace{0, ...0, 1,}_{i} 0, ..., 0)$;
the standard notation $\mathbb{Z}$ stands for all integers on $\mathbb{R}$; while $\mathbb{N}$ are all the positive integers;
for multi-indices with non-negative integer components we use the notation $j = (j_1, \dots, j_n)$ and $|j| = j_1 + \cdots + j_n$ will stand for the length of the multi-index;

$$j \leq j^0 \text{ for } j^0 = (j_1^0, ..., j_n^0) \iff j_i \leq j_i^0, \ i = 1, ..., n;$$

$\{0,1\}^n$ is the set of multi-indices with components 0 and 1; vectors in $\{0,1\}^n$ will be denoted as χ, η, ζ; the vectors $0 = (0, ..., 0)$ and $1 = (1, ..., 1)$ are the elements of $\{0,1\}^n$ - we hope that no confusion will arise through a misunderstanding whether 0 and 1 are vectors or scalars;

$x^j = x_1^{j_1} \cdots x_n^{j_n}, \quad D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}; \quad D^\chi f(x) = \left(\prod_{j:\chi_j=1} \frac{\partial}{\partial x_j} \right) f(x);$

$D_{\rm rad} f(x) = \frac{\partial}{\partial |x|} f(|x|x') = x' \cdot {\rm grad} f(x), \quad \Delta f(x) = \frac{\partial^2 f}{\partial^2 x_1} + \cdots + \frac{\partial^2 f}{\partial^2 x_n};$

$\mathbb{R}_+ = [0, \infty), \qquad \overline{\mathbb{R}_+} = [0, \infty];$

$F \circ f = F[f(\cdot)];$

$[\gamma]$ and $\{\gamma\}$ are the integral and fractional parts, respectively, of a real number γ;

$Ff(\xi) = \widehat{f}(\xi) = \int\limits_{\mathbb{R}^n} e^{-i\xi x} f(x)\, dx, \quad F^{-1}f(x) = \check{f}(x) = \int\limits_{\mathbb{R}^n} e^{i\xi x} f(\xi)\, \frac{d\xi}{(2\pi)^n};$

$C_0(\mathbb{R}^n) = \{f \in C(\dot{\mathbb{R}}^n) \colon f(\infty) = \lim\limits_{|x| \to \infty} f(x) = 0\};$

$C_0^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions with compact support;

$\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of test functions rapidly decreasing at infinity;

$\Psi = \Psi(\mathbb{R}^n)$ is the subspace of functions in $\mathcal{S}$ which vanish at the origin together with all their derivatives;

$\Phi = \Phi(\mathbb{R}^n) = \widehat{\Psi}$ is the Lizorkin space of functions in $\mathcal{S}$ which are orthogonal to the polynomials;

$L_{\overline{p}}(\mathbb{R}^n) = \{f : \|f\|_{\overline{p}} : \| \cdots \| \, \|f\|_{p_k}^{(k)} \|_{p_{k-1}}^{(k-1)} \cdots \|_{p_1}^{1}\}$ is the space with mixed norm, $\overline{p} = (p_1, p_2, \ldots, p_n)$;

$B_{p,q}^\alpha = B_{p,q}^\alpha(\mathbb{R}^n)$ denotes the Besov space. One of of the possible (equivalent) definitions of this space is as follows ([93]). Let χ_Ω be the indicator function of a domain $\Omega \in \mathbb{R}^n$, and

$$\Gamma_m = \{x \in \mathbb{R}^n : \; 2^{m-1} < \max_{i=1,\ldots,n} |x_i| < 2^m\}, \quad m = 0, \pm 1, \pm 2 \ldots$$

For a function $f \in L_p$, we set $f_m(\xi) = F^{-1}[\chi_{\Gamma_m} Ff](\xi)$, the Fourier transform is understood in the sense of $\mathcal{S}'$-distributions. Then the norms in the Besov space is defined by

$$\|f\|_{B_{p,q}^\alpha} = \|f\|_{L_p} + \Big(\sum_{m=-\infty}^{\infty} 2^{m\alpha q} \|f_m\|_p^q \Big)^{1/q}$$

where $\alpha > 0$, $1 \le p \le \infty$ and $1 \le q \le \infty$ (with the usual modification for $q = \infty$).

We will denote absolute constants by C, not the same in different occurrences. When we have to emphasize that the constant depends on certain parameters, say $a, b, ...$, we will denote it by $C(a, b, ...)$.

3. Definitions and preliminaries

Denote by $AC_{\rm loc}(\mathbb{R}_+)$ and $BV_{\rm loc}(\mathbb{R}_+)$ classes of functions that are locally absolutely continuous and locally of bounded variation, respectively.

3.1. Spaces of functions with bounded variation.

In the following definition we follow [29], [141], [144].

Definition 3.1. We say that a function $g\colon [0,\infty)\to\mathbb{C}$ is in the space BV^{ω}_{k+1}, where $k=0,1,2,\dots$ and $\omega\ge 0$, if it is bounded on $[0,\infty)$ and
1) $g^{(i)}\in AC_{\rm loc}([0,\infty)),\ i=0,1,\dots,k-1$;
2) $\lim\limits_{r\to\infty} g^{(i)}(r)=0,\ i=0,1,\dots,k-1$;
3) $\lim\limits_{r\to\infty} g^{(k)}$ locally has bounded variation and $A(g) := \int\limits_0^\infty r^{k+\omega}|dg^{(k)}(r)| < \infty$.
Equipped with the norm

$$\|f\|_{BV_{k+1}} := |g(\infty)| + A(g)$$

this is a Banach space. In the case $\omega=0$ we will write $BV^0_{k+1} =: BV_{k+1}$.

We will also need a generalization $BV^{\omega}_{\alpha+1}$ (see [142]) of the space BV^{ω}_{k+1} for the case of fractional values of k. To this end, we recall the notion of the Cossar fractional derivative ([33], see also [119, p.163], [142, Section 3]).

Definition 3.2. Cossar fractional derivative of order $\alpha\in(0,1)$ of a function f defined on $\mathbb{R}_+$ or $\mathbb{R}$ is given by

$$f^{(\alpha)}(x) = -\frac{1}{\Gamma(1-\alpha)}\lim_{m\to\infty}\frac{d}{dx}\int\limits_x^m \frac{f(t)\,dt}{(t-x)^\alpha}. \tag{3.1}$$

To values $\alpha\ge 1$ this definition extends as $f^{(\alpha)}(x) = \left(\frac{d}{dx}\right)^{[\alpha]} f^{(\{\alpha\})}(x)$.

Construction (3.1) differing from the usual form of the Liouville fractional differentiation by the order of operations $\frac{d}{dx}$ and $\lim\limits_{m\to\infty}$, is applicable to functions with worse behavior at infinity.

For the following definition, we refer to [108], [127], [143]. This definition as well as the next one may be found in [106].

Definition 3.3. We say that $f\in BV^{\omega}_{\alpha+1} = BV^{\omega}_{\alpha+1}(\mathbb{R}_+)$, if $f\in C(\mathbb{R}_+)$, $f^{(\alpha-i)}\in AC_{\rm loc}([0,\infty),\ i=1,\dots,[\alpha]$, $f^{([\alpha])}\in BV_{\rm loc}(\mathbb{R}_+)$ and

$$\int\limits_0^\infty t^{\alpha+\omega}|df^{(\alpha)}| < \infty.$$

Definition 3.4. We say that $f\in BV^{\omega,\sigma}_{\alpha+1} = BV^{\omega,\sigma}_{\alpha+1}(\mathbb{R}_+)$, where $\alpha\ge 0$, $\omega\ge 0$, $\sigma\ge 0$, if $t^{-\sigma}f(t)\in BV^{\omega}_{\alpha+1}(\mathbb{R}_+)$.

Let now α^* be the greatest integer less than α.

Definition 3.5. We say that $g \in MV^{\omega}_{\alpha+1} = MV^{\omega}_{\alpha+1}(\mathbb{R}_+)$, with $\alpha > 0$ and $\omega \geq 0$, if $g \in C(0,\infty)$ and satisfies the following conditions:

$$g, g', ..., g^{(\alpha^*)} \quad \text{are locally absolutely continuous on} \quad (0,\infty); \tag{3.2}$$

$$\lim_{t\to\infty} g(t) = 0, \qquad \lim_{t\to\infty} t^{\alpha+\omega} g^{(\alpha)}(t) = 0; \tag{3.3}$$

$$\|g\|_{MV^{\omega}_{\alpha+1}} := \sup_{t>0} |t^{\omega} g(t)| + \int_0^{\infty} \left| d\Big(t^{\alpha+\omega} g^{(\alpha)}(t)\Big)\right| < \infty. \tag{3.4}$$

In the problems of integrability of the Fourier transform the following T-transform of a function $h(u)$ defined on $(0,\infty)$ is of importance

$$Th(t) = \int_{\to 0}^{t/2} \frac{h(t+s) - h(t-s)}{s}\, ds. \tag{3.5}$$

A bit of history is in order. In the discrete case (for sequences of the coefficients of trigonometric series) very similar transform can be found already in [64] and in a weaker form in [22]. The T-transform first appeared in [81], and in [43] it is called the Telyakovskii transform to designate that for obtaining results for the Fourier transform in [81] and [43] it is used to generalize one of the most general result for the integrability of trigonometric series (see, e.g., [135]). It is clear that the T-transform, being a truncated Hilbert transform, should have much in common with the latter; this is revealed and discussed in [81] and later on in, e.g., [43], [84], etc. For certain related notions, see also Section 12.

3.2. **The class $C^q(\mathbb{S}^{n-1}), q > 0$.** Let

$$f(\sigma) \sim \sum_{m=0}^{\infty} Y_m(f,\sigma)$$

be the Fourier-Laplace expansion of a function f defined on the unit sphere $\mathbb{S}^{n-1}$, into spherical harmonics $Y_m(f,\sigma)$, (see [129], [130], [104] for details of harmonic analysis on the sphere).

By $C^{\infty}(\mathbb{S}^{n-1})$ we denote the space of functions $f(\sigma), \sigma \in \mathbb{S}^{n-1}$ such that $f\left(\frac{x}{|x|}\right)$ is infinitely differentiable in the layer $\frac{1}{2} < |x| < 2$.

Definition 3.6. By $C^q(\mathbb{S}^{n-1}), q > 0$, we denote the closure of $C^{\infty}(\mathbb{S}^{n-1})$ with respect to the norm

$$\|f\|_{C^{\infty}(\mathbb{S}^{n-1})} := \|f\|_{C(\mathbb{S}^{n-1})} + \|\delta^{\frac{q}{2}} f\|_{C(\mathbb{S}^{n-1})},$$

where $\delta f = -|x|^2 \Delta f\left(\frac{x}{|x|}\right)$ is the Laplace-Beltrami operator so that

$$\delta^{\frac{q}{2}} f(\sigma) := \sum_{m\mu} [m(m+n-2)]^{\frac{q}{2}} Y_m(f,\sigma).$$

3.3. **Some classes of functions with smoothness in radial and angular variables.**

Definition 3.7. ([120]) We say that $f \in C^{N,\gamma}(\dot{\mathbb{R}}^1_+)$, where $N = 0,1,2,...$, $0 < \gamma < N$, if

1) $f(r)$ is continuously differentiable up to order N for all $r \in (0,\infty)$;

2) there exist the limits $\lim\limits_{r\to\infty} \frac{d^k}{dr^k} f\left(\frac{1}{r}\right)$, $k = 0,1,...,N$;

3) there exist the limits $\lim\limits_{r\to 0} f^{(k)}(r)$, $k = 0,1,...,[\gamma]$, while for $k = [\gamma]+1,...,N$ the estimates

$$|f^{(k)}(r)| \le C r^{\gamma-k}, \quad 0 < r \le 1, \tag{3.6}$$

hold, where in the case of integer γ and $k = \gamma$ one should write

$$|f^{(k)}(r)| \le C \ln \frac{2}{r}, \quad 0 < r \le 1. \tag{3.7}$$

In the sequel, when formulating some tests for functions to belong to $W(\mathbb{R}^n)$ in Section 10, we will use classes $C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\})$ of functions with $C^{N,\gamma}(\mathbb{R}_+)$-behavior in the radial variable and $C^\infty(\mathbb{S}^{n-1})$-behavior in the angular variables.

For $x \in \mathbb{R}^n$ we use the standard notation $x = r\sigma$, $r \in (0,\infty)$, $\sigma \in \mathbb{S}^{n-1}$.

The next two definitions can be found in [74].

Definition 3.8. We say that $f(r\sigma)$ belongs to $C^{N,\gamma}(\dot{\mathbb{R}}^1_+)$ in r *weakly uniformly* with respect to σ, if $f(r\sigma) \in C^{N,\gamma}(\mathbb{R}_+)$ in $r \in (0,\infty)$ for every $\sigma \in \mathbb{S}^{n-1}$ with the constants in (3.6) and (3.7) do not depend on σ. If also the values

$$\frac{\partial^k}{\partial r^k} f(r\sigma)\bigg|_{r=0}, \quad k = 0,1,...,[\gamma], \qquad \text{and} \qquad \frac{\partial^k}{\partial r^k} f\left(\frac{\sigma}{r}\right)\bigg|_{r=0}, \quad k = 0,1,...,N,$$

do not depend on σ, we say that $f(r\sigma)$ belongs to $C^{N,\gamma}(\dot{\mathbb{R}}^1_+)$ in r *uniformly* with respect to σ.

Definition 3.9. We say that f belongs to the class $C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\})$ (or to the class $C_q^{N,\gamma}(\dot{\mathbb{R}}^n)$), where $N = 0,1,2,...$, $q > 0$ and $0 < \gamma < N$, if

1) $f \in C(\dot{\mathbb{R}}^n)$,

2) $f(r\sigma)$ and $\delta^{\frac{q}{2}} f$ belong to$C^{N,\gamma}(\dot{\mathbb{R}}^1_+)$ in r weakly uniformly in σ (uniformly, resp.),

3) $\frac{\partial^N f(r\sigma)}{\partial r^N} \in C^q(\mathbb{S}^{n-1})$ for every $r > 0$.

Remark 3.10. The requirement for $\frac{\partial^k f(r\sigma)}{\partial r^k}\Big|_{r=0}$ not to depend on $\sigma \in \mathbb{S}^{n-1}$ in the definition of the class $C_q^{N,\gamma}(\dot{\mathbb{R}}^n)$ is rather restrictive in comparison with that of the class $C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\})$. For instance,

$$\frac{x_1}{1+|x|^2} \in C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\})$$

for all $N \geq 0$, $q \geq 0$, $0 \leq \gamma \leq N$, but

$$\frac{x_1}{1+|x|^2} \notin C_q^{N,\gamma}(\dot{\mathbb{R}}^n), \quad \text{if} \quad N > 0.$$

Examples of functions in $C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\})$ are given, for instance, by $f(x) = a(|x|) + b(|x|)u\left(\frac{x}{|x|}\right)$, where $u \in C^q(\mathbb{S}^{n-1})$, $a, b \in C_q^{N,\gamma}(\dot{\mathbb{R}}^1_+)$ and $b^{(k)}(0) = 0$, $k = 0, 1, ..., [\gamma]$.

We refer to [74] for some properties of the classes $C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\})$ and $C_q^{N,\gamma}(\dot{\mathbb{R}}^n)$.

3.4. **On α-distances in $\mathbb{R}^n$.** Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $A_t, t > 0$ be a diagonal matrix of the form $A_t = \text{diag}(t^{\alpha_1}, ..., t^{\alpha_n})$. A function $f(x)$, $x \in \mathbb{R}^n$, is called α-homogeneous of degree λ (see [15], [16]), if

$$f(A_t x) = t^\lambda f(x).$$

Definition 3.11. A function $\varrho(x)$ positive for $x \neq 0$ and α-homogeneous of degree 1, that is,

$$\varrho(A_t x) = t\varrho(x),$$

is called an α-*distance* in $\mathbb{R}^n$.

We refer to [15], [16] for properties of α-distances. The following are examples of α-distances in $\mathbb{R}^n$:
1) $\varrho(x) = \sum_{i=1}^n |x_i|^{\frac{1}{\alpha_i}}$,
2) the unique positive solution $\xi = \xi(x)$ of the equation $\sum_{i=1}^n \xi^{-2\alpha_i} x_i^2 = 1$.

4. Starting points

In this section we discuss apparently less practical results that reflect general properties of the Wiener algebra in all its three forms (1.1)-(1.2).

4.1. **Some general theorems.** The Wiener algebras $W_0(\mathbb{R}^n)$, $W_1(\mathbb{R}^n)$, and $W(\mathbb{R}^n)$ are obviously related by

$$W_0(\mathbb{R}^n) \subset W_1(\mathbb{R}^n) \subset W(\mathbb{R}^n). \tag{4.1}$$

For these classes, the following general statements are known.

Theorem 4.1. *$W_0(\mathbb{R}^n)$, $W_1(\mathbb{R}^n)$ and $W(\mathbb{R}^n)$ are Banach algebras with point-wise multiplication, and $W_0(\mathbb{R}^n)$ is an ideal in $W(\mathbb{R}^n)$.*

We remark that $W_0(\mathbb{R}^n)$ can be extended to $W_1(\mathbb{R}^n)$ by adding the unity element to the former.

The next Wiener-Lévy theorem is one of the basic results of the theory (see, e.g., [111, Ch.1, 1.3] or [158, 6.1.8 and further]).

Theorem 4.2. *Let $f \in W_0(\mathbb{R}^n)$, and let F be an analytic function on an open set which contains the range of f, with $F(0) = 0$. Then $F \circ f \in W_0(\mathbb{R}^n)$ too.*

We now give important corollaries of this result.

Corollary 4.3. *If $f \in W_0(\mathbb{R}^n)$ and the complex number ℓ is such that $f(x) \neq \ell$ for all $x \in \mathbb{R}^n$ and $\ell \neq 0$, then $\frac{f(x)}{f(x)-\ell} \in W_0(\mathbb{R}^n)$.*

In the theory of Fourier series there is well known the so-called Wiener's $1/f$-theorem; it also holds (see [47], [48]) for the Wiener algebra $W(\mathbb{R}^n)$ as stated in the following theorem important in applications, but is not valid in the general case, that is, for the algebra $W(\mathbb{R}^n)$; for sufficient conditions on μ to provide such a result, see, e.g., [166]. Note that Theorem 4.4 does not follow from Wiener-Levy Theorem 4.2.

Theorem 4.4. *If $f \in W_1(\mathbb{R}^n)$ and $f(x) \neq 0$ for all $x \in \dot{\mathbb{R}}^n$, then $\frac{1}{f} \in W_1(\mathbb{R}^n)$.*

We then proceed to Wiener's local theorem.

Theorem 4.5. *If a function f coincides with some function $\varphi_{x_0}(x) \in W_0(\mathbb{R}^n)$ (or $W_1(\mathbb{R}^n), W(\mathbb{R}^n)$) in a neighborhood of every point $x_0 \in \dot{\mathbb{R}}^n$, then $f \in W_0(\mathbb{R}^n)$ ($W_1(\mathbb{R}^n)$, $W(\mathbb{R}^n)$ respectively)*

Theorem 4.1 is elementary, for Theorems 4.2 and 4.5, see [110], and [48, §17, §35] in the context of the general theory of commutative normed rings. We emphasize the evident importance of these theorems in applications, see e.g. an application of Theorem 4.5 to some examples of functions in [118, p.21], [147, Th.7].

The following corollary of Theorem 4.5 is important in applications as well.

Corollary 4.6. *If a function $f(x)$ coincides with a function $g(x) \in W(\mathbb{R}^n)$ for $|x| < R$ and with a function $h(x) \in W_0(\mathbb{R}^n)$ for $|x| > r$, where $0 < r < R < \infty$, then $f \in W_0(\mathbb{R}^n)$.*

In applications, the following version of Wiener's Theorem 4.4 is also useful, see [95, p.102].

Theorem 4.7. *Let $f \in W_0(\mathbb{R}^n)$. If $f(x) \neq 0$ on closed bounded set $V \subset \mathbb{R}^n$, then $\frac{1}{f(x)}$ is extendable to a function in $W_0(\mathbb{R}^n)$, i.e. there exists a function $g \in W_0(\mathbb{R}^n)$ such that $f(x) \equiv g(x)$ on V.*

We mention that the difference between $W_0(\mathbb{R}^n)$ and $W(\mathbb{R}^n)$ is revealed only in the behavior of the function near infinity. By the Riemann-Lebesgue lemma, f vanishes at infinity if $f \in W_0(\mathbb{R}^n)$. We also note that if $f \in W(\mathbb{R}^n)$, $\lim f(x) = 0$

as $|x| \to \infty$ and f is of bounded total Vitali variation off a cube, then $f \in W_0(\mathbb{R}^n)$ ([147, Theorem 2]).

We remind the reader that the total Vitali variation of the function $\phi\colon E \to \mathbb{C}$, with $E \subset \mathbb{R}^n$, is defined as follows. If the boundary of E consists of a finite number of planes given by equations $x_i = c_i$, then

$$V(f) = \sup \sum |\Delta_h f(x)|, \qquad \Delta_h f(x) = \left(\prod_{i=1}^{n} \Delta_{h_i}\right) f(x),$$

where $h = (h_1, \cdots, h_n)$ and

$$\Delta_{h_i} f(x) = f(x + h_i e_i) - f(x - h_i e_i), \quad 1 \le i \le n. \tag{4.2}$$

Here Δ_h is the mixed difference with respect to the vertices of the parallelepiped $[x-h, x+h]$ and sup of the sums is taken while summing up over all the choices of such non-overlapping parallelepipeds in E. For smooth enough functions on E, one has

$$V(f) = \int_E \left| \frac{\partial^n f(x)}{\partial x_1 \cdots \partial x_n} \right| dx.$$

We note that in Marcinkiewicz's sufficient condition for M_p, $1 < p < \infty$, only the finiteness of total variations over all dyadic parallelepipeds with no intersections with coordinate hyper-planes is assumed (see, e.g., [129]).

For further presentation, concerning the terminology, we mention that the absolute integrability of the Fourier transform of a function f in a sense is the same as the belonging of f to $W_0(\mathbb{R}^n)$. Within the framework of the L_1-theory of Fourier transforms, it is well known that

$$f, \widehat{f} \in L_1(\mathbb{R}^n) \implies f \in W_0(\mathbb{R}^n).$$

The latter is true almost everywhere, of course; it becomes true everywhere under the assumption that f is continuous.

There exists an immense of publications on conditions for $\widehat{f} \in L_1(\mathbb{R}^n)$, $n = 1, 2, \ldots$ It is not our aim to survey them all, nor to include as many as possible of them in the list of references. On the contrary, just few of them are referred to, really relevant to our subject. On the other hand, some of the results being such are reformulated in this paper as the statements on belonging to W_0.

Within the scope of the distributions $f \in \mathcal{S}'(\mathbb{R}^n)$, however, the exact equivalence occurs:

If $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi = \widehat{f}$ is its Fourier transform in $\mathcal{S}'(\mathbb{R}^n)$: $(f, F^{-1}\omega) = (\varphi, \omega)$ for all $\omega \in \mathcal{S}(\Omega)$, then $f \in W_0(\mathbb{R}^n)$ if and only if $\widehat{f} \in L_1(\mathbb{R}^n)$.

Keeping in mind the treatment of Fourier transforms for distributions over test function spaces other than $\mathcal{S}(\mathbb{R}^n)$ (for instance, the Lizorkin space Φ), we introduce the following definition used in the subsequent lemma.

A test function space $Y = Y(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ is called complete, if the validity of the equality $(f, \omega) = 0$ for all $\omega \in Y$ for a locally integrable function $f \in Y'$ implies that $f(x) = 0$ a.e.

By $(\varphi, \omega), \omega \in \mathcal{S}, \varphi \in \mathcal{S}'$, we denote the bilinear form which corresponds to $\int_{\mathbb{R}^n} \varphi(x)\overline{\omega(x)}\, dx$ in case of regular distributions φ.

Lemma 4.8. *Let $f \in L_1^{loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$. If f has the Fourier transform in the sense of distributions over the test function space $X \subseteq \mathcal{S}'(\mathbb{R}^n)$:*

$$(2\pi)^n (f, F^{-1}\omega) = (\varphi, \omega) \quad \text{for all} \quad \omega \in X$$

where $\varphi \in L_1(\mathbb{R}^n)$ and the space $F^{-1}(X)$ is complete, then $f(x)$ coincides a.e. with a function in $W_0(\mathbb{R}^n)$.

The proof is direct.

4.2. **Necessary conditions.** Let us go on to necessary conditions for dimension one. Obviously, if $f \in W_0$, then for each $l \in \mathbb{R}$

$$\|\overline{f}\|_{W_0} = \|f(l\cdot)\|_{W_0} = \|f(\cdot + l)\|_{W_0} = \|e^{il(\cdot)} f(\cdot)\|_{W_0} = \|f\|_{W_0}. \tag{4.3}$$

Example 1. If $f \in W_0(\mathbb{R})$, then also $(e^{it} + 1)^\gamma f(t) \in W_0(\mathbb{R})$ for $\gamma > 0$.

Let $f \in W_0(\mathbb{R})$ and $f = \widehat{g}$, where $g \in L_1(\mathbb{R})$. Then the trigonometrically conjugate function (the Hilbert transform) is

$$\widetilde{f}(x) = \mathcal{H}f(x) = \frac{1}{\pi} \int\limits_{\to 0}^{\to\infty} \frac{f(x+t) - f(x-t)}{t} dt$$

$$= \frac{2i}{\pi} \lim_{\substack{\varepsilon \to +0 \\ M \to +\infty}} \int\limits_{-\infty}^{+\infty} g(y) e^{ixy} dy \int\limits_{\varepsilon}^{M} \frac{\sin ty}{t} dt.$$

Since the absolute values of the integrals over $[\varepsilon, M]$ are bounded by an absolute constant, it is possible to pass to the limit under integral sign. This yields (see [91] and also [61])

$$\widetilde{f}(x) = i \int\limits_{-\infty}^{+\infty} g(y) e^{ixy} \text{sign } y dy, \qquad \|\widetilde{f}\|_{W_0} = \|f\|_{W_0}.$$

We mention that the improper integral in the definition of $\widetilde{f}$ converges everywhere (and uniformly in x), but not necessarily absolutely.

Now we proceed to a necessary condition for belonging to $W_0(\mathbb{R}^n)$, valid for both radial and non-radial functions of n variables and depends on dimension n. To formulate it, we introduce the radial part of the function f:

$$f_0(t) = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} f(t\sigma)\, d\sigma, \qquad t > 0,$$

We are now in a position to formulate a multidimensional result.

Theorem 4.9. *If $f \in W_0(\mathbb{R}^n)$, then the radial part f_0 satisfies the following conditions:*

1) for $0 \le s \le \frac{n-1}{2}$,

$$f_0^{(s)} \in C(0,\infty);$$

2) there holds

$$\lim_{t\to\infty} t^s f_0^{(s)}(t) = 0, \qquad 0 \le s \le \frac{n-1}{2};$$

and

$$\lim_{t\to 0+} t^s f_0^{(s)}(t) = 0, \qquad 0 < s \le \frac{n-1}{2};$$

where s is any number in the given interval, not only integer.

3) besides that, $Tf_0^{(\frac{n-1}{2})}(t)$ exists for any $t > 0$.

This theorem is obtained in [147, Th.3] for s integers, in a recent paper [85] the derivatives of fractional order s are allowed as well.

4.3. **Criteria.** There also exist a few criteria, that is, necessary and sufficient conditions for the belonging to $W_0(\mathbb{R}^n)$.

Let us start with F. Riesz's criterion of the absolute convergence of Fourier integrals (its counterpart for series can be found in [65]).

Theorem 4.10. *A function f belongs to $W_0(\mathbb{R}^n)$ if and only if it is representable as the convolution of two functions from $L_2(\mathbb{R}^n)$:*

$$f(x) = \int_{\mathbb{R}^n} f_1(y) f_2(x-y)\,dy, \qquad f_1, f_2 \in L_2(\mathbb{R}^n). \tag{4.4}$$

By this, $\|f\|_{W_0} \le \|f_1\|_{L_2} \|f_2\|_{L_2}$.

The proof is based on the unitary property (up to the norming) of the Fourier operator in $L_2(\mathbb{R}^n)$ (cf. [140, Th.66]). For an application of this criterion, see Theorem 6.3.

Certain criteria are generalizations of R. Salem's criterion for Fourier series [113]. We start with a very general I.J. Schoenberg's theorem [123].

Theorem 4.11. *For $f \in L_\infty(\mathbb{R})$ to coincide a.e. with a function from $W(\mathbb{R})$ so that $\mu \in BV(\mathbb{R})$, it is necessary and sufficient that there exists a constant $C > 0$ such that*

$$\left| \int_{\mathbb{R}} f(t)h(t)\,dt \right| \le C \sup_{x\in\mathbb{R}} \left| \int_{\mathbb{R}} h(t) e^{-ixt}\,dt \right|$$

for all $h \in L_1(\mathbb{R})$.

In the case where μ is an absolutely continuous function of bounded variation, from Theorem 4.11we obtain the following A.C. Berry's theorem [13], where

$$F_f(h) := \int\limits_{\mathbb{R}} f(t)h(t)\,dt.$$

Theorem 4.12. *The following conditions are necessary and sufficient in order that $f \in L_\infty(\mathbb{R})$ coincides a.e. with a function from $W_0(\mathbb{R})$:*

a) There exists a constant $C > 0$ such that $|F_f(h)| \le C\|\widehat{h}\|_\infty$ for all $h \in L_1(\mathbb{R})$.

b) For every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|F_f(h)| \le \epsilon\|\widehat{h}\|_\infty$ whenever both h and $\widehat{h}$ belong to $L_1(\mathbb{R})$ and $\|\widehat{h}\|_{L_1(\mathbb{R})} \le \delta\|\widehat{h}\|_\infty$.

We also mention further generalizations of R. Salem's results in [112] and [37].

There is one more criterion (approximative) from which in [158, 6.4.3], for example, known sufficient conditions with different smoothness in various variables are derived. It is similar to that for absolute convergence of orthogonal Fourier series due to S.B. Stechkin (1955) (see, e.g., [4] or [65, Ch.II.3]). Assuming that $f \in L_2(\mathbb{R}^n)$, we denote, for $\sigma \ge 0$, by

$$a_\sigma(f)_2 = \inf\{\|f - g\|_2;\ g:\ |\mathrm{supp}\,\widehat{g}| \le \sigma\}$$

the best approximation to f in $L_2(\mathbb{R}^n)$ by functions with spectrum in a set of measure not greater than σ.

Theorem 4.13. *If $f \in L_2(\mathbb{R}^n)$ and $p \in (0, 2)$, then $\widehat{f} \in L_p(\mathbb{R}^n)$ if and only if*

$$\int\limits_0^\infty \sigma^{-p/2}[a_\sigma(f)_2]^p d\sigma < \infty.$$

A couple more criteria, hardly applicable in concrete situations, can be found in [103].

5. Some tests in the one-dimensional case

Sufficient conditions for a function $f\colon \mathbb{R} \to \mathbb{C}$ to belong to $W_0(\mathbb{R})$ are known comparatively long since. Probably, the first essential result of such a nature is Titchmarsh's theorem below. This theorem for $p = 2$ is, in a sense, a continual version of similar Bernstein's theorem known in the theory of Fourier series. In fact, absolute convergence of Fourier series is an important particular case of the general approach when the measure in the definition of algebra W is discrete and is concentrated on an arithmetic progression, for instance, at the lattice points. Chronologically, it was an initial point of the whole subject being started with the celebrated paper by S.N. Bernstein of 1914 (see, e.g., [172] or [4]). The following was proved in that paper:

Each periodic function from the $\operatorname{Lip}\alpha$ class with $\alpha > \frac{1}{2}$ is expanded in the absolutely convergent Fourier series, and an example of a function from $\operatorname{Lip}\frac{1}{2}$ (by using the Legendre symbols) is built whose Fourier series is not absolutely convergent. More precisely, the next sufficient condition holds true:

$$\int_0^1 \frac{\omega(f,t)}{t^{3/2}}\,dt < \infty,$$

where $\omega(f,t)$ is the modulus of continuity of the function f with step t.

Later on, A. Zygmund discovered a bit different condition: If, in addition, f *is of bounded variation*, sufficient condition is

$$\int_0^1 \frac{\sqrt{\omega(f,t)}}{t}\,dt < \infty.$$

Both conditions are sharp on the whole class (see [172], [4], [65]). In the multivariate case an analog of Bernstein's condition roughly speaking reads as follows: the smoothness in the L_2 metrics should be greater than $\frac{n}{2}$ (see, e.g., [130, Ch.VII, Cor.1.9]). As follows from Wiener's theorems below, a compactly supported function with the support in $[-a,a]^n$ belongs to $W_0(\mathbb{R}^n)$ if and only if being extended from the cube $[-a-1,a+1]^n$ periodically, for example with period $2a+2$ in each of the n variables it expands in the absolutely convergent Fourier series.

The mentioned Titchmarsh's result first appeared in 1927 (see [139]).

Theorem 5.1. *Let $f \in L_p(\mathbb{R}) \cap C_0(\mathbb{R}), 1 < p \le 2$. If*

$$\int_{\mathbb{R}} |f(x+h) - f(x)|^p\,dx \le Ch^{\alpha p} \quad \text{for} \quad 0 < h \le 1,$$

where $\frac{1}{p} < \alpha < 1$, then $f \in W_0(\mathbb{R}^n)$.

The proof of Theorem 5.1 can also be found in Titchmarsh's book [140].

Corollary 5.2. *If $f(x)$ satisfies the Hölder condition of order $\lambda > \frac{1}{2}$ on $\dot{\mathbb{R}}$, that is,*

$$|f(x) - f(y)| \le C\frac{|x-y|^\lambda}{(1+|x|)^\lambda(1+|y|)^\lambda}, \qquad \lambda > \frac{1}{2},$$

and $f(\infty) = 0$, then $f \in W_0(\mathbb{R})$.

Let us describe conditions closely related to that. If for $f \in C_0(\mathbb{R})$ both f and $\widehat{f}$ are integrable, then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{f}(x)e^{itx}dx, \qquad \|f\|_{W_0} = \frac{1}{2\pi}\|\widehat{f}\|_{L_1};$$

the same statement holds true for $f\colon \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$.

Most of the tests in terms of continuity moduli are usually called Bernstein type theorems, see such tests for functions of many variables in Subsection 7.1. Zygmund type results are discussed in Subsection 7.2.

The following basic results of 1938 are due to A. Beurling [17, p.5].

Theorem 5.3. *If* $f \in AC_{loc}$ *and* $f, f' \in L_2(\mathbb{R})$*, then* $f \in W_0(\mathbb{R}^n)$ *and*

$$\|f\|_{W_0} \le \sqrt{\pi \|f\|_{L_2} \|f'\|_{L_2}}. \tag{5.1}$$

Theorem 5.4. *If* $f \in C_0(\mathbb{R})$*, is even and has the derivative* f' *such that* $B := \int\limits_0^\infty t|df'(t)| < \infty$*, then* $f \in W_0(\mathbb{R}^n)$ *and* $\|f\|_{W_0} \le B$*.*

The reader may find the proof of Theorem 5.3 in [28, p.250], and of Theorem 5.4 in [145] (cf. Lemma 12.2).

Typical examples covered by Theorem 5.3 may be functions with a logarithmic-type decay at infinity:

$$f(x) = (1 + \ln(1 + |x|^\lambda))^{-\alpha}, \quad f(x) = (1 + \ln\ln(3 + |x|^\lambda))^{-\alpha} \qquad \text{etc}$$

where $\lambda, \alpha > 0$.

In addition to Theorem 5.3, the following test with a sharper assumption at infinity is due to T. Carleman [30, p.67]:

Theorem 5.5. *If* $f \in AC_{loc}$*,* $f \in L_1(\mathbb{R})$ *and* $f' \in L_2(\mathbb{R})$*, then* $f \in W_0(\mathbb{R}^n)$*.*

The following is B. Sz.-Nagy's generalization of Theorem 5.3 (1948, [134, p.67]).

Theorem 5.6. *Let* $f \in C_0(\mathbb{R})$ *be even,* $f \in AC_{loc}$*, and let* f' *locally be of bounded variation everywhere except for the points* $x = a_i, i = 1, \dots, s,\ 0 = a_0 < a_1 < \dots < a_s$*. If the integrals*

$$\int\limits_0^\varepsilon x|df'(x)|, \int\limits_N^\infty x|df'(x)| \quad \text{and} \quad \int\limits_{a_i-\varepsilon}^{a_i+\varepsilon} |x - a_i| \ln \frac{1}{|x - a_i|} |df'(x)|, \ i = 1, \dots, s$$

converge for some $0 < \varepsilon < N < \infty$*, then* $f \in W_0(\mathbb{R})$*.*

The proof of Theorem 5.6 in the case $s = 0$ may be found in the paper [9] as well as in the book [28, p.251]; for the odd case, see, e.g., the next Theorem 5.7.

In fact, in Theorems 5.3, 5.5 and 5.6 one can assume that $f \in AC_{\rm loc}$ rather than to suppose the absolute continuity, see [145] (cf. also Lemma 12.2 below).

The following theorem, which may be considered as development and sharpening of Beurling's Theorem 5.4, involves the spaces $BV_{\alpha+1}$ and $BV_{\alpha+1}^{\sigma;\sigma}$ defined in terms of fractional differentiation, see Definitions 3.3–3.4. Though the tests in terms of fractional derivatives are considered separately in Section 9, it seems reasonable to formulate the following one here (see [106, Th.6.3] and [144]).

By

$$f_{\pm}(t) = \frac{f(t) \pm f(-t)}{2}$$

we denote, respectively, the even and odd components of $f(t)$.

Theorem 5.7. *Let $f \in C_0(\mathbb{R})$, $f_+ \in BV_{\alpha+1}$ and $f_- \in BV_{\alpha+1}^{\sigma;\sigma}$ for some $\alpha, \sigma > 0$. Then $f \in W_0(\mathbb{R})$.*

The following corollary is useful in application to concrete examples of functions.

Corollary 5.8. *Let $f \in C_0(\mathbb{R})$ be twice continuously differentiable on $\mathbb{R}$ and*

1) $\int\limits_0^\infty x|f''(x) + f''(-x)|\,dx < \infty$,

2) there exists a $\sigma > 0$ such that $\int\limits_0^\infty x^{1+\sigma} \left| \frac{d^2}{dx^2} \frac{f(x)-f(-x)}{x^\sigma} \right| dx < \infty$,

then $f \in W_0(\mathbb{R})$.

A typical example is

$$f(t) = \frac{A}{[\ln\ln(e + |t|^\lambda)]^\alpha} + \frac{B\mathrm{sign}\, t\ \mu(|t|)}{[\ln(1 + |t|)]^\beta},$$

where $\alpha > 0 . \beta > 1, \lambda > 0$, A and B are constants and $\mu(r)$ is a C^∞-step function equal to 1 at infinity and vanishing in a neighborhood of the origin.

6. Tests in terms of integrability of derivatives

The following direct generalization of Beurling's Theorem 5.3 to the multidimensional case was obtained in 1970 ([95, pp. 99–100]; without estimate of type (5.1)).

Theorem 6.1. *Let*

$$f, \Delta^m f \in L_2(\mathbb{R}^n),$$

where $m > \frac{n}{4}$ $(m = 1, 2, \ldots)$. Then $f \in W_0(\mathbb{R}^n)$.

A simple argument in the next theorem gives even more in terms of fractional powers $(-\Delta)^{\frac{\alpha}{2}}$ of the minus Laplace operator. It should be properly defined in the case $\alpha \neq 2, 4, 6, \ldots$ We treat it in the distributional sense, which in fact means that the relation

$$\widehat{(-\Delta)^{\frac{\alpha}{2}} f}(y) = |y|^\alpha \widehat{f}(y)$$

serves as a definition of $(-\Delta)^{\frac{\alpha}{2}} f$, or more precisely

$$\big((-\Delta)^{\frac{\alpha}{2}} f, \omega\big) := (2\pi)^{-n} (Ff, |\cdot|^\alpha F^{-1}\omega).$$

The latter definition is not applicable within the scope of the Schwartz test function space, since the operation $(-\Delta)^{\frac{\alpha}{2}}$ does not preserve the Schwartz test function space. However, the Lizorkin test function space Φ of Schwartz functions orthogonal to all polynomials serves well for this goal (we refer to Chapters 2,3 of [118] for details on the distributional interpretation of $(-\Delta)^{\frac{\alpha}{2}}f$).

In the next theorem and everywhere in the sequel, in particular in Section 9, we treat $(-\Delta)^{\frac{\alpha}{2}}f$ in the described distributional sense.

Theorem 6.2. *Let* $f \in L_2(\mathbb{R}^n)$ *and* $(-\Delta)^{\frac{\alpha}{2}}f \in L_2(\mathbb{R}^n), \alpha > \frac{n}{2}$. *Then* $f \in W_0(\mathbb{R}^n)$.

Proof. Since

$$\widehat{f}(y) = (\widehat{f}(y)(1 + |y|^\alpha))\,\frac{1}{1 + |y|^\alpha}$$

and $\frac{1}{1+|y|^\alpha} \in L_2(\mathbb{R}^n)$, by the definition of $(-\Delta)^{\frac{\alpha}{2}}f$, we can represent the function f as the convolution of the two L_2-functions. Then the statement follows from Theorem 4.10. □

Other differential operators can be used in the same way, say elliptic, while applying embedding theorems allows one to digress on the function classes defined via moduli of continuity of partial derivatives.

The following version of such a generalization has, roughly speaking, a similar restriction with respect to the order of the used derivatives, but is finer due to its anisotropic nature (see [137] of 1961 and [98] of 1974). In [158], this test is derived from the criterion from Theorem 4.13.

Theorem 6.3. *Let* β_i *be positive integers such that* $\sum_{i=1}^n \frac{1}{\beta_i} < 2$. *If* $f \in \mathcal{S}'(\mathbb{R}^n)$ *satisfies the conditions*

$$f \in L_2(\mathbb{R}^n) \qquad \text{and} \qquad D_i^{\beta_i} := \frac{\partial^{\beta_i} f}{\partial x^{\beta_i}} \in L_2(\mathbb{R}^n),\ i = 1, \dots, n,$$

then $f \in W_0(\mathbb{R}^n)$ *and*

$$\|f\|_{W_0} \le C\|f\|_{L_2} + C\sum_{i=1}^n \|D_i^{\beta_i} f\|_{L_2}.$$

For even functions f, the following statement holds [141], which is a partial generalization of Theorem 5.4 to the multidimensional case.

Theorem 6.4. *Let* $f \in C_0(\mathbb{R}^n)$ *be even in every variable and have locally integrable derivatives* $D^j f, j = (j_1, \dots, j_n)$ *with* $j_i = 0, 1, 2$ *for all* $i = 1, 2, \dots, n$, *such that* $\lim_{x\to\infty} D^j f(x) = 0$. *If*

$$\int_0^\infty \cdots \int_0^\infty |D^j f(x)| \prod_{j_i \ne 0} x_i^{j_i - 1} dx_i < \infty$$

uniformly in x_i *when* $j_i = 0$, *then* $f\left(|x_1|^{\lambda_1}, \dots, |x_n|^{\lambda_n}\right) \in W_0(\mathbb{R}^n)$ *provided that* $\lambda_i > 0,\ i = 1, \dots, n$.

Theorem 6.4 covers even functions with rather slow decay at infinity and possible singularities of derivatives at the coordinate hyper-planes. A typical example ([141]) is

$$f(x) = \frac{1}{[1 + \ln(1 + |x|^{\lambda_1})]^n}.$$

A generalization of Beurling's inequality to the case of many variables is presented in the next theorem of 1975 in [24, p.18].

Theorem 6.5. *Let* $f \in C_0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash \{0\})$ *and*

$$D^j f \in L_2(\mathbb{R}^n) \quad \text{for all } j \quad \text{with} \quad 0 \le |j| \le N \quad \text{for some} \quad N > \frac{n}{2}.$$

Then

$$\|f\|_{W_0} \le C \|f\|_{L_2}^{1-\frac{n}{2N}} \Bigg(\sum_{|j|=N} \|D^j f\|_{L_2} \Bigg)^{\frac{n}{2N}}. \tag{6.1}$$

Estimates of type (6.1) are sometimes called *Carlson-Beurling inequalities*; see, e.g., [5], cf. also Theorem 9.1.

The next theorem ([99]) is essentially of the same type as Theorem 6.3 is. The use of the mixed derivatives allows to lessen the orders β_j of the involved derivatives with respect to separate variables. We first need the following definition.

Definition 6.6. A set A of multi-indices j is called *acceptable*, if

$$\frac{1}{\sum\limits_{j \in A} |x^j|} \in L_2(\mathbb{R}^n).$$

The following are examples of acceptable sets:

i) $A = \{j : \; |j| = m\}, \quad m > \frac{n}{2}$,

ii) $A = \{0,1\}^n$,

iii) $A = \{j_i : \beta_i e_i, \; i = 1, \ldots, n\}$, where $\beta_i > 0$ and $\sum_{i=1}^n \frac{1}{\beta_i} < 2$.

Theorem 6.7. *Let* A *be an acceptable set of multi-indices containing* 0. *If* $f \in \mathcal{S}'(\mathbb{R}^n)$ *has mixed* $\mathcal{S}'$*-distributional derivatives* $D^j f \in L_2(\mathbb{R}^n)$ *for all* $j \in A$, *then* $f \in W_0(\mathbb{R}^n.)$

A test corresponding to the acceptable set $\{0,1\}^n$ but using L_p-norms is contained in the following theorem of 1977 from [114] (see also [118, Lemma 1.22]).

Theorem 6.8. *If* $f \in L_1(\mathbb{R}^n)$ *and* f *has mixed* $\mathcal{S}'$*-distributional derivatives* $D^j f \in L_p(\mathbb{R}^n)$ *for all* $j \in \{0,1\}^n, j \ne 0$, *where* $1 < p \le 2$, *then* $f \in W_0(\mathbb{R}^n)$ *and*

$$\|f\|_{W_0(\mathbb{R}^n)} \le C \|f\|_{L_1} + C \sum_{j \in \{0,1\}^n, j \ne 0} \|D^j f\|_{L_p}.$$

An extension of Theorem 6.8 to the case of fractional derivatives does exist, see Theorem 9.8. The other version of Theorem 6.8 is as follows [34].

Theorem 6.9. *Let f have mixed $\mathcal{S}'$-distributional derivatives $D^j f \in L_p(\mathbb{R}^n)$ for all $j \in \{0,1\}^n$, where $1 < p \leq 2$, and let $\tau = (\tau_1, ..., \tau_n)$, $\tau_i > 0$, $i = 1, 2, \dots, n$. Then there exists a constant $C > 0$ not depending on f and τ such that*

$$\|f\|_{W_0(\mathbb{R}^n)} \leq C \sum_{j \in \{0,1\}^n} \tau^{\frac{1}{\mathbf{p}} - j} \|D^j f\|_{L_p}, \tag{6.2}$$

where $\frac{1}{\mathbf{p}} = \left(\frac{1}{p}, \dots, \frac{1}{p}\right)$.

To formulate the next two tests, we denote by $L_{p,m}(\mathbb{R}^n)$ the Sobolev space with the norm

$$\|f\|_{p,m} := \left\{ \int\limits_{\mathbb{R}^n} \left[\sum_{|j| \leq m} |D^j f(x)|^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}$$

and by $\mathcal{L}_{p,m}(\mathbb{R}^n)$ the space of functions $f \in \mathcal{S}'(\mathbb{R}^n)$ which have distributional derivatives of order m such that the semi-norm

$$\|f\|'_{p,m} := \left\{ \int\limits_{\mathbb{R}^n} \left[\sum_{|j| = m} |D^j f(x)|^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}$$

is finite.

Theorem 6.10. *Let $f \in \mathcal{L}_{p,n-1} \cap \mathcal{L}_{p,n}$ with $1 < p < \frac{2n}{2n-1}$. Then*

$$f(x) = P_{n-2}(x) + f_0(x),$$

where $f_0 \in W_0(\mathbb{R}^n)$ and $P_{n-2}(x)$ is a polynomial of degree not exceeding $n-2$ and

$$\|f_0\|_{W_0} \leq 2^{\frac{n-1}{n}} C(p) \left(\alpha \|f\|'_{p,n} + \beta \|f\|'_{p,n-1}\right),$$

where $\alpha = \left(\frac{2}{p-1}\right)^{\frac{n}{p}}$, $\beta = \left(\frac{2n}{n-n(p-1)}\right)^{\frac{n}{p}}$ and $C(p)$ is the constant from the Hausdorff-Young $p \to p'$-inequality for the Fourier transform.

Theorem 6.11. *Let $f \in L^{q,n} \cap \mathcal{L}_{p,n} \cap \mathcal{L}_{p,n-1}$, where $1 < p < \frac{2n}{2n-1}$ and $p \leq q < \infty$. Then $f \in W_0(\mathbb{R}^n)$ and*

$$\|f\|_{W_0} \leq C \left(\|f\|_{q,n} + \|f\|'_{p,n} + \|f\|'_{p,n-1}\right).$$

These tests are referred to [132] and [133], respectively, as well as the following typical examples:

$$f(x) = \frac{1}{ix_1 + x_2^2 + x_3^2 + 1}, \qquad n = 3,$$

$$f(x) = \frac{1}{ix_1 + x_2^2 x_3^2 + (x_2 - x_3)^2 + x_4^2 + 1}, \qquad n = 4.$$

They satisfy the assumptions of Theorems 6.10 and 6.11, but are not covered by Theorem 6.1.

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We conclude this section by tests formulated in the decomposition terms which go back to the known Hörmander multiplier theorem. First we need the following definition (see [99] for it and for the next test).

Definition 6.12. A set A of multi-indices is called *nice* if it is acceptable and the inclusion $j \in A$ implies $k \in A$ for every $k \leq j$.

The sets A in ii) and iii) in Definition 6.6 and the set $A = \{j : |j| \leq m\}$, $m > \frac{n}{2}$, are examples of nice sets.

Theorem 6.13. *Let A be a nice set of multi-indices and $f \in L\infty(\mathbb{R}^n)$. If f has $\mathcal{S}'$-distributional derivatives $D^j f(x)$ of all orders $j \in A$, satisfying one of the conditions*

$$\sum_{i=-\infty}^{\infty} \sum_{j \in A} \left\{ 2^{-in} \int_{E_i} \left| |x|^{|j|} D^j f(x) \right|^2 dx \right\}^{\frac{1}{2}} < \infty, \tag{6.3}$$

where $E_i = \{x : 2^{i-1} < |x| < 2^{i+1}\}$,

$$\sum_{|k| < \infty} \sum_{j \in A} \left\{ \frac{1}{|Q_k|} \int_{Q_k} \left| |x|^{|j|} D^j f(x) \right|^2 dx \right\}^{\frac{1}{2}} < \infty, \tag{6.4}$$

where $Q_k = \{x : 2^{k_i - 1} < |x_i| < 2^{k_i + 1}\}$, $k - (k_1, \ldots, k_n)$, then $f \in W_0(\mathbb{R}^n)$.

Let us give a recent simple sufficient condition [7, Th.1] (see also [89] and [91]).

Theorem 6.14. *If $f \in C_0(\mathbb{R}^n)$,*

$$\lim_{|x_j| \to \infty} D^{\chi} f(x) = 0,$$

for every vector χ, $0 \leq \chi \leq 1$, and for some $\varepsilon \in (0,1)$

$$\left| \frac{\partial^n f(x)}{\partial x_1 \cdots \partial x_n} \right| \leq \frac{C}{\prod_{i=1}^{n} |x_i|^{1-\varepsilon} (1 + |x_j|)^{2\varepsilon}},$$

then $\|f\|_{W_0} \leq C$, where $C = C(\varepsilon) < \infty$.

7. Tests in terms of finite differences

In this section we consider generalizations to the real axis and Euclidean spaces the afore-mentioned Bernstein and Zygmund's theorems on the absolute convergence of Fourier series.

7.1. **Bernstein type theorems.** By

$$\Delta_h^\ell f(x) = \sum_{k=0}^{\ell} (-)^k \binom{\ell}{k} f(x-kh)$$

we denote the non-centered finite difference of order ℓ with a vector step h.

Historically, the first extension of Bernstein's theorem to double Fourier series is apparently obtained in [32]. Close to our topic, for functions of many variables one of the first essential results of Bernstein type was proved in 1966 ([45] for $n = 2$ and [108, p.11] for all $n \geq 1$). It is given, to all appearance, by the following theorem.

Theorem 7.1. *Let $f \in C_0(\mathbb{R}^n)$ and satisfy the condition*

$$\int_0^\infty \sup_{0<|h|<t} \|\Delta_h^\ell f\|_2 \frac{dt}{t^{1+\frac{n}{2}}} < \infty \quad \textit{with} \quad \ell > \frac{n}{2}. \tag{7.1}$$

Then f is representable as the sum of the Fourier transform of a function in $L_1(\mathbb{R}^n)$ and a polynomial of degree $\leq \ell - 1$.

Conditions

$$\|\Delta_h^\ell f\|_2 \leq C|h|^{\frac{n}{2}+\varepsilon} \quad \text{for} \quad |h| \leq 1$$

and

$$\|\Delta_h^\ell f\|_2 \leq C|h|^{\frac{n}{2}-\varepsilon} \quad \text{for} \quad |h| \geq 1, \quad \varepsilon > 0$$

of such a type (sufficient for (7.1)) were also noted by E. Stein [127, p.331]. Note that Theorem 7.1 was formulated in [108] in equivalent interpolation terms. Observe also that the statement of this theorem implies the embedding

$$B_{2,1}^{\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow W_0(\mathbb{R}^n) \tag{7.2}$$

of the isotropic Besov space $B_{2,1}^{\frac{n}{2}}(\mathbb{R}^n)$ into $W_0(\mathbb{R}^n)$, see, for instance, [15], [24], [107] for Besov spaces. A generalization

$$B_{p,1}^{\frac{n}{p}}(\mathbb{R}^n) \hookrightarrow W_0(\mathbb{R}^n), \quad 1 < p \leq 2, \tag{7.3}$$

also holds; more general statements of such a kind are known in anisotropic terms, with mixed finite differences, see for instance Theorem 7.7.

We start with the following L_2-type result ([56]; [14]), where

$$\Delta_{t_1}^1 \cdots \Delta_{t_n}^1 f(x) = \prod_{i=1}^n (I - \tau_{t_i}^i) f(x), \qquad \tau_{t_i}^i f(x) = f(x - t_i e_i)$$

stands for the mixed finite difference of a function f of vector order $(1, 1, \ldots, 1)$.

Theorem 7.2. *Let* $f \in C_0(\mathbb{R}^n)$ *and*

$$K := \Big(\int_0^\infty \cdots \int_0^\infty \|\Delta^1_{t_1} \cdots \Delta^1_{t_n} f\|_2 \prod_{i=1}^n t_i^{-\frac{3}{2}} dt_i \Big)^{\frac{1}{2}} < \infty.$$

Then $f \in W_0(\mathbb{R}^n)$ *and* $\|f\|_{W_0} \le CK$, *where* C *is independent of* f.

As formulated, Theorem 7.2 was proved in [14]. In [56, p.32] its proof was given under assumptions which need to be made more precise: an information about the behavior of functions at infinity was lost, see [14] and the following Theorem 7.3 in this connection.

The next condition (Lemma 4 in [147]) is very handy for proving various results.

Theorem 7.3. *Let* $f \in C_0(\mathbb{R}^n)$.

a) If

$$\sum_{s_1=-\infty}^\infty \cdots \sum_{s_n=-\infty}^\infty 2^{\frac{1}{2}\sum_{i=1}^n s_i} \|\Delta_{\frac{\pi}{2^{s_1}},\cdots,\frac{\pi}{2^{s_n}}}(f)\|_{L_2(\mathbb{R}^n)} < \infty,$$

then $f \in W_0(\mathbb{R}^n)$.

b) If $f = \widehat{g}$, *with* $g \in L_1(\mathbb{R}^n)$, *and for* $|u_j| \ge |v_j|$ *when* $\operatorname{sign} u_j = \operatorname{sign} v_j$ *for all* $1 \le j \le n$ *there holds* $|g(u)| \le |g(v)|$, *then the series in a) converges.*

Let us comment on this result. The convergence of the series condition in **a)** is of Bernstein type and controls $\|f\|_{W_0(\mathbb{R}^n)}$, as the proof of Lemma 4 in [147] establishes. Concerning **a)**, see also [109]. As follows from [57], conditions in Theorem 7.2 and Theorem 7.3 a) are equivalent. However, it is worth mentioning that in [147] **b)** is proved too. Not only this is a necessary condition for a certain subclass, but application of this assertion to the extension of Beurling's theorem is given in Section 12 (see Theorem 12.3 below).

The next theorem makes use of the following norm in the decomposition terms

$$\|f\|'_{B^\alpha_{2,1}} := \|f\|_{L_2} + \sum_{k=-\infty}^\infty 2^{k\alpha} \Big\{ \int_{\Gamma_k} |\widehat{f}(x)|^2 dx \Big\}^{\frac{1}{2}}, \qquad \alpha > 0,$$

where $\Gamma_k = \{x \in \mathbb{R}^n : 2^{k-1} < \max_i |x_i| < 2^k\}$.

Theorem 7.4. *Let* $f \in C_0(\mathbb{R}^n$ *and* $\|f\|'_{B^\alpha_{2,1}} < \infty$. *Then* $f \in W_0(\mathbb{R}^n)$ *and* $\|f\|_{W_0} \le C\|f\|'_{B^\alpha_{2,1}}$.

As shown in [93], Theorem 7.4 proven in [92] is a corollary of the following more general statement.

Theorem 7.5. *Let* $f \in L_2(\mathbb{R}^n)$ *and*

$$\|f\|_{S^{(1/2)}_{2,1,b}} := \sum_m 2^{\frac{|m|}{2}} \Big\{ \int_{\Pi_m} |\widehat{f}(x)|^2 dx \Big\}^{\frac{1}{2}} < \infty,$$

where $m = (m_1, \dots, m_n), |m| = m_1 + \dots + m_n,\ m_i = 0, \pm 1, \pm 2, \dots$ *and* $\Pi_m = \{x \in \mathbb{R}^n : 2^{m_i - 1} < |x_i| < 2^{m_i},\ i = 1, \dots, n\}$. *Then* $f \in W_0(\mathbb{R}^n)$ *and* $\|f\|_{W_0} \le C(\|f\|_{L_2} + \|f\|_{S^{(1/2)}_{2,1,b}})$.

The condition $f \in L_2(\mathbb{R}^n)$ in Theorem 7.5 is superfluous. The following version of this theorem from [93] holds true.

Theorem 7.6. *Let* $\|f\|_{S^{(1/2)}_{2,1,b}} < \infty$. *If there exists a sequence* $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ *converging to* f *in* $\mathcal{S}'$ *and with respect to the semi-norm* $\|f\|_{S^{(1/2)}_{2,1,b}}$. *Then* $f \in W_0(\mathbb{R}^n)$.

Theorem 7.5 yields also a statement more general than that in Theorem 7.2, in terms of the anisotropic Besov space $B^\alpha_{2,1}$, where $\alpha = (\alpha_1, ..., \alpha_n)$ and $\frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_n} = 2$, see [93].

An extension of Theorem 7.2 to the case of arbitrary mixed differences and mixed norms is also known [25]. Let $E_n = \{1, 2, \dots, n\}$, $e = \{j_1, \dots, j_N\}$, $1 \le N \le n$, an arbitrary set in E_n, $\mathbf{p} = (p_1, \dots, p_n)$ and $\frac{1}{\mathbf{p}} = \left(\frac{1}{p_1}, \dots, \frac{1}{p_n}\right)$.

Theorem 7.7. *Let*

$$\|f\||_{SB^{\frac{1}{\mathbf{p}}}_{\mathbf{p},1}} := \|f\|_{L_{\mathbf{p}}} + \sum_{e \subset E_n} \int\limits_{R(e)} \sup_{\substack{|h_j| < t_j \\ j \in e}} \|\Delta_h^{m^e}\|_{L_p} \prod_{j \in e} t_j^{-1-\frac{1}{p_j}} dt_j,$$

where $R(e) = \{t = (t_{j_1}, \dots, t_{j_N}) : 0 < t_{j_i} \le 1, j \in e\}$ *and* $m^e = (m^e_1, \dots, m^e_n)$, *where*

$$m^e_j = \begin{cases} m_j, & \text{if } j \in e, \\ 0, & \text{if } j \in E_n \backslash e, \end{cases}$$

If $\|f\||_{SB^{\frac{1}{\mathbf{p}}}_{\mathbf{p},1}} < \infty$, *then* $f \in W_0(\mathbb{R}^n)$ *and* $\|f\|_{W_0} \le C\|f\||_{SB^{\frac{1}{\mathbf{p}}}_{\mathbf{p},1}}$.

7.2. **Zygmund type theorems.** We can characterize the difference between the two types of results as follows: in the Bernstein type theorems the local uniform smoothness should be greater than $\frac{n}{2}$, while in the Zygmund type theorems greater than $\frac{n-1}{2}$ (see an example in Theorem 12.14). Zygmund's theorem on the absolute convergence of Fourier series, though has attracted less attention than Bernstein's theorem, has led to a considerable amount of generalizations. Let us refer to a couple of them, apparently more interesting than some others: [105], [137] (see also [138]).

The next theorem is definitely of Zygmund type (see [147] or [158, 6.4.5]).

Theorem 7.8. *Let* $q = [\frac{n-1}{2}]$ *(integral part), and let* $f \in C^q(\mathbb{R}^n)$ *be compactly supported. If, besides, the partial derivative* $D_{q,j}(f) = \frac{\partial^q f}{\partial x_j^q}$ *has, as a function of* x_j, *a bounded, with respect to the other variables, number of points separating the intervals of convexity for each* $1 \le j \le n$, *then the condition*

$$\int\limits_0^1 \omega(t) t^{q-(n+1)/2} dt < \infty,$$

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where $\omega(t) = \max_j \omega(D_{q,j}; t)$, *yields* $f \in W_0(\mathbb{R}^n)$ *(or* $\widehat{f} \in L_1(\mathbb{R}^n)$*).*

Example 2. If P is a nontrivial and nonnegative on $[0, R]$, $R > 0$, algebraic polynomial and $P(R) = 0$, then the function

$$\varphi(x) = \begin{cases} P^\delta(|x|), & |x| \le R, \\ 0, & |x| > R, \end{cases}$$

belongs to $W_0(\mathbb{R}^n)$ if and only if $\delta > \frac{n-1}{2}$.

In fact, Theorem 9.3 and some others may also be related to the results of the considered type, but we prefer to present them in accordance with other features they possess (use of fractional derivative, say).

Let us give a look at the initial Zygmund's result from a different side. Bernstein's type results are given in terms of the high enough smoothness of a function. Zygmund's theorem assumes lower smoothness provided the given function is of bounded variation. However, being of bounded variation means that the function just belongs to a different smoothness space, in the classical case the Lip1 space in the L_1 metrics. Therefore, Zygmund type theorems may be treated as a collection of results in which the assertion on the integrability of the Fourier transform (or, alternatively, Fourier series) is deduced from the belonging of the given function to two appropriate smoothness spaces. This approach was apparently first initiated in [80] and then developed in [87]. Here we reformulate certain results (there are many others, some even more general) from the latter paper as one statement on $W_0(\mathbb{R}^n)$.

Let us consider some spaces we have not dealt with yet.

Let $J^\alpha = (I - \Delta)^{-\alpha/2}$, where I is the identity operator and Δ is the Laplace operator, be the Bessel potential. The potential space $L^\alpha_{\overline{p}}(\mathbb{R}^n)$ is defined to be the image of $L_{\overline{p}}$ under the operator J^α:

$$L^\alpha_{\overline{p}}(\mathbb{R}^n) = \{f : f = J^\alpha\varphi, \quad \varphi \in L_{\overline{p}}, \quad \alpha > 0, \quad 1 \le p_i \le \infty\}. \tag{7.4}$$

We remind that given a real positive number α, the number α^* is the largest integer smaller than α. Let $\Lambda(\alpha, \bar{p}) = \Lambda(\alpha, \bar{p})(\mathbb{R}^n)$ be the class of functions $f \in L^{\alpha^*}_{\overline{p}}$ such that

$$\|\Delta^2_h J^{-\alpha^*} f\|_{L_{\overline{p}}} = O(|h|^{\alpha - \alpha^* t}). \tag{7.5}$$

Let $Lip(\alpha, \overline{p}) = Lip(\alpha, \overline{p}, \mathbb{R}^n)$ be the class of functions f such that for $|j| < \alpha^*$ we have $D^j f \in L_{\overline{p}}$, while for $|j| = \alpha^*$,

$$\|\Delta_h D^j f\|_{L_{\overline{p}}} = O(|h|^{\alpha-\alpha^*}) \qquad \text{if} \quad \alpha - \alpha^* < 1,$$
$$\|\Delta^2_h D^j f\|_{L_{\overline{p}}} = O(|h|) \qquad \text{if} \quad \alpha - \alpha^* = 1.$$

Theorem 7.9. *Let*

a) *either* $f \in L^\alpha_{\overline{p}} \cap L^\beta_{\bar{q}}$, *where* $1 < p_i < \infty$, $1 < q_i < \infty$;

b) or $f \in Lip(\alpha, \bar{p}) \cap Lip(\beta, \bar{q})$*; or* $f \in \Lambda(\alpha, \bar{p}) \cap \Lambda(\beta, \bar{q})$*), where* $1 \le p_i \le \infty$, $1 \le q_i \le \infty$*, with* $\theta/p_i + (1-\theta)/q_i = 1/2$*,* $i = 1, \dots n$ *for some* $0 < \theta < 1$*. If* $\theta\alpha + (1-\theta)\beta > n/2$ *then* $f \in W_0(\mathbb{R}^n)$*.*

The obtained result for Besov spaces reads in a different manner.

Theorem 7.10. *Suppose* $f \in B^{\alpha}_{p,1} \cap B^{\beta}_{q,1}$ *provided* $1 < p, q < \infty$, $\theta/p + (1-\theta)/q = 1/2$, $0 < \theta < 1$*. If* $\theta\alpha + (1-\theta)\beta = n/2$ *then* $f \in W_0(\mathbb{R}^n)$*.*

8. Tests for radial and quasi-radial functions

It is well known that

$$\frac{1}{(1+|x|^2)^{\frac{\alpha}{2}}} \in W_0(\mathbb{R}^n), \qquad \alpha > 0, \tag{8.1}$$

(see, e.g., [41], [119, p.396], [28]) and

$$\frac{|x|^{\alpha}}{(1+|x|^2)^{\frac{\alpha}{2}}} \in W(\mathbb{R}^n), \qquad \alpha > 0, \tag{8.2}$$

(see [129]); the inclusion $\frac{1}{1+|x|^{\alpha}} \in W_0(\mathbb{R}^n)$ was proved in [41]. It is also known that

$$\frac{|x|^{\alpha-\delta}}{1+|x|^{\alpha}}, \ \frac{|x|^{\alpha-\delta}}{(1+|x|^2)^{\frac{\alpha}{2}}} \in W_0(\mathbb{R}^n), \quad \alpha > 0, \tag{8.3}$$

for $0 < \delta \le \alpha$ (see e.g. [114], or [118], p.21). Known are various tests aiming specially at radial functions. Some of them, for example Theorems 8.2 and 8.7 purposefully extend the model cases (8.1)-(8.3), others have a more general nature.

To present a result from [53], we remind the reader the following definition which goes back to S. Bernstein [11]-[12].

Definition 8.1. A non-negative function is called *almost decreasing* on the interval (a, b), $-\infty \le a < b \le \infty$, if there exists a positive constant $C(\le 1)$ such that $f(t_1) \ge C f(t_2)$ for all $a < t_1 < t_2 < \infty$.

Theorem 8.2. *Let* $\varphi(r)$ *be a positive function, continuously differentiable up to order* n *for* $r \in [1, \infty)$ *and satisfying the assumptions*
1) there exists an $\alpha_0 \in (0,1)$ *such that the function* $r^{\alpha_0}\varphi(r)$ *is almost decreasing on* $(0, \infty)$*;*
2) $|\varphi^{(k)}(r)| \le C r^{-k} \varphi(r)$, $k = 1, 2, \dots, n$*.*
Then

$$\frac{\varphi\left(\sqrt{1+|x|^2}\right)}{(1+|x|^2)^{\frac{\alpha}{2}}} \in W_0(\mathbb{R}^n)$$

for any $\alpha > -\alpha_0$*.*

The following test [141] is effective, within the framework of radial dependence, for a wide class of functions. It is given in terms of bounded variation of derivatives and generalizes Theorem 5.4 (see Definition 3.1 for the class BV_{k+1}).

Theorem 8.3. *If* $g \in BV_{k+1}$, *where* $k = \left[\frac{n+1}{2}\right]$, *then*

$$g(|x|^\lambda) \in W_0(\mathbb{R}^n)$$

for every $\lambda > 0$.

Typical examples ([141]): Theorem 8.3 covers the functions

$$\frac{1}{1+\ln(1+|x|^2)}, \quad \frac{1}{[1+\ln\ln(e+|x|^\lambda)]^\alpha} \quad \text{and} \quad \frac{\phi(|x|)}{\ln|x|}$$

where $\lambda > 0, \alpha > 0$ and $\phi(r)$ is a C^∞-step function equal to 1 for $0 \le r \le \frac{1}{e}$ and 0 for $r \ge \frac{2}{e}$.

A generalization of the last theorem to the case of fractional k is exhibited in Theorem 9.3.

The next test from [55] assumes that functions are sufficiently smooth and may have power-logarithmic type decay at infinity. We denote

$$\mathcal{D}^1 g(r) := -\frac{1}{r} g'(r), \quad \mathcal{D}^N = \mathcal{D}^1(\mathcal{D}^{N-1} g).$$

Definition 8.4. We say that a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is in the class M_N, if g is differentiable up to order N, g and $\mathcal{D}^N g$ are non increasing and tend to zero as $r \to \infty$.

Theorem 8.5. *Let* $f(x) = f_0(|x|)$, *where* $f_0 \in M_N$, $N > \frac{n-1}{2}$, *and let*

$$A_N := \int\limits_0^\infty r^{2N-1} |\mathcal{D}^N f_0(r)|\, dr < \infty.$$

Then $f \in W_0(\mathbb{R}^n)$ *and* $\|f\|_{W_0} \le C A_N$, *where* C *does not depend on* f.

The class $C^{N,\gamma}(\dot{\mathbb{R}}^1_+)$ used in the next test from [120] has been introduced in Definition 3.7.

Theorem 8.6. *Let* $f_0 \in C^{N,\gamma}(\dot{\mathbb{R}}^1_+), 0 < \gamma \le N$. *If* $f_0(\infty) = 0$ *and* $N > n$, *then* $f(x) = f_0(|x|) \in W_0(\mathbb{R}^n)$. *If*

$$\lim_{r\to\infty} \frac{d^k}{dr^k} f_0\left(\frac{1}{r}\right) = 0, \quad k = 0, 1, \ldots, [\alpha], \quad \textit{and} \quad N > n + \alpha,$$

then $|x|^\alpha f(x) \in W_0(\mathbb{R}^n)$, $\alpha \ge 0$.

Now we pass to the case of quasi-radial functions $f(x) = g(\varrho(x))$, where $\varrho(x)$ is the quasi-distance from Definition 3.11. The following theorem (see [34]) provides a direct generalization of inclusion (8.3).

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Theorem 8.7. *Let $\varrho_i(x)$ be α-distances in $\mathbb{R}^n$ and $\varrho_i \in C^N(\mathbb{R}^n\setminus\{0\})$, $i = 1, 2$, where $N = 1 + \left[\frac{n}{2}\right]$. Then*

$$\frac{[\varrho_1(x)]^\beta}{[1 + \varrho_2(x)]^\gamma} \in W_0(\mathbb{R}^n), \qquad \textit{provided} \quad \beta < \gamma.$$

The next theorem from the same paper contains a test for quasi-radial functions in terms of BV-spaces. It generalizes Theorem 8.3.

Theorem 8.8. *Let $\varrho(x)$ be an α-distance in $\mathbb{R}^n$ and $\varrho \in C^N(\mathbb{R}^n\setminus\{0\})$, where $N = 1 + \left[\frac{n}{2}\right]$, and $g \in BV_{N+1}$. Then $g[\varrho(x)] \in W_0(\mathbb{R}^n)$ and $\|g(\varrho(\cdot))\|_{W_0} \le C(\varrho)\|g\|_{BV_{N+1}}$, where $C(\varrho)$ depends on ϱ, but does not depend on g.*

In [34], the following decomposition type theorem (compare with Theorem 6.13) was proved. Denote

$$Q_k = \{x \in \mathbb{R}^n : \ 2^{k-1} < \varrho(x) < 2^k\}, \qquad k = 0, \pm 1, \pm 2, \dots$$

Theorem 8.9. *Let $\varrho(x)$ be an α-distance in $\mathbb{R}^n$ and $\varrho \in C^\infty(\mathbb{R}^n\setminus\{0\})$ and let $f \in \mathcal{S}'$ have distributional derivatives $D^j f \in L_p(Q_k)$ for all j and k, such that*

$$\sum_{k=-\infty}^{\infty} 2^{k\left(\alpha j - \frac{|\alpha|}{p}\right)} \|D^j f\|_{L_p(Q_k)} \le B_p < \infty.$$

Then $f \in W_0(\mathbb{R}^n)$ and $\|f\|_{W_0} \le CB_p$.

Corollary 8.10. *Let $\varrho(x)$ be the same as in Theorem 8.9. If*

$$\Bigg(\int\limits_{\varrho(x)<2} |D^j f(x)|^p \, dx \Bigg)^{\frac{1}{2}} + \sum_{k=1}^{\infty} 2^{-k\frac{|\alpha|}{p}} \|\varrho^{\alpha j} D^j f\|_{L_p(Q_k)} \le B_p < \infty,$$

then $f \in W_0(\mathbb{R}^n)$ and $\|f\|_{W_0} \le CB_p$.

Remark 8.11. As is shown in [34, p.288] (see also [35, Lemma 1]), Theorems 8.7-8.9 remain valid in the case of more general homogeneous type distances, defined by an arbitrary “dilation” matrix $A_t = e^{P \ln t}$ not necessarily diagonal, under the assumption that the eigenvalues of the matrix P have positive real parts. We refer also to [125] concerning the estimate

$$\|f[\varrho(\cdot)]\|_{W_0} \le C \int\limits_0^\infty \|\phi(\cdot) f(t\cdot)\|_{B^2_{\frac{n}{2},1}} \frac{dt}{t},$$

where ϕ is a bump function.

A radial function $f(x) = f_0(|x|)$ belongs to $W_0(\mathbb{R}^n)$ if and only if for $t \in \mathbb{R}_+$ and $\lambda = \frac{n}{2} - 1$

$$f_0(t) = \int\limits_0^\infty g_1(u) j_\lambda(ut) du, \quad g_1 \in L_1(\mathbb{R}_+). \tag{8.4}$$

Here $j_\lambda(t) = \frac{J_\lambda(t)}{t^\lambda}$, where J_λ is the Bessel function of order $\lambda > -1$ (see [130, Ch.IV, §3] or [1]; cf. Theorem 11.7 below). Note that j_λ is an entire function of exponential type, that is, bounded on $\mathbb{R}$ along with all its derivatives; moreover, the following inequality holds:

$$|j_\lambda(t)| \le j_\lambda(0) = \frac{1}{2^\lambda \Gamma(1+\lambda)}.$$

For $\lambda > -\frac{1}{2}$, we have

$$j_\lambda(t) = \frac{1}{2^\lambda \Gamma(\lambda+1/2)\Gamma(1/2)} \int_{-1}^{1} (1-u^2)^{\lambda-\frac{1}{2}} e^{iut} du, \quad j_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi}} \cos t.$$

Theorem 8.12. *Let $n \ge 3$ be odd. In order that $f(x) = f_0(|x|) \in W_0(\mathbb{R}^n)$, it is necessary and sufficient that for all integers $\nu \in \left[0, \frac{n-3}{2}\right]$*

$$\frac{d^\nu}{dt^\nu} \left[t^{n/2-1} f_0(\sqrt{t}) \right]_{t=0} = 0,$$

and $f_1(|x|)$, defined by

$$f_1(\sqrt{t}) = \sqrt{t} \frac{d^{n/2-1/2}}{dt^{n/2-1/2}} \left[t^{n/2-1/2} f_0(\sqrt{t}) \right],$$

for $t \ge 0$, belong to $W_0(\mathbb{R}^n)$.

For radial functions, this theorem from [158, 6.3.6] allows one to pass easily, at least when the function and its derivative(s) are of exponential behavior, from the one-dimensional case (say, Theorem 10.5) to estimates in spaces of any dimension without loss of accuracy.

The next asymptotic result was obtained in the present form in [88] (for preceding related results, see [151] and [82]; cf. also Sections 11 and 12). In fact, it is the most important partial case of a more general result. Moreover, it was established in that paper that $BV_{\alpha+1}^b$ is properly embedded in $MV_{\alpha+1}^b$. However, for the most of the results obtained earlier for $BV_{\alpha+1}^b$ their validity for the whole $MV_{\alpha+1}^b$ has never been checked.

Theorem 8.13. *Let $f_0 \in MV^0_{\frac{n+1}{2}}$, set $F_n(t) = t^{\frac{n-1}{2}} f_0^{(\frac{n-1}{2})}(t)$. Then there holds, for the radial extension $f(x) = f_0(|x|)$ of f_0,*

$$\widehat{f}(x) = \frac{C(n)}{|x|^{n-1}} \int_0^\infty F_n(t) \cos(|x|t - \pi n/2)\, dt + O\left(\frac{\Psi(|x|)}{|x|^{n-1}} \right), \tag{8.5}$$

where $\Psi(t)$ is a function integrable on $\mathbb{R}_+$.

We observe that the integral on the right-hand side of (8.5) is the one-dimensional Fourier transform, cosine or sine, according to whether n is even or odd.

Corollary 8.14. *Under the assumptions of Theorem 8.13, $f \in W_0(\mathbb{R}^n)$ if and only if $F_n \in W_0(\mathbb{R})$.*

For functions from the class in question, we can apply one-dimensional conditions considered above. For example, in [88] certain one-dimensional results in Section 12 are extended to radial functions $f(x) = f_0(|x|)$ with $f_0 \in MV_{\frac{n+1}{2}}$.

9. Tests with fractional derivatives

Let

$$\mathbb{D}^\alpha f := F^{-1}|\xi|^\alpha F f$$

be the Riesz fractional derivative, interpreted in the distributional sense as mentioned in the beginning of Section 6, for the direct realization of $\mathbb{D}^\alpha f$ we refer to [119] or [118].

We start with the one-dimensional case $n = 1$.

Theorem 9.1. *Let $f, \mathbb{D}^\alpha f \in L_2(\mathbb{R})$, where $\alpha > \frac{1}{2}$. Then*

$$\|f\|_{W_0} \le C\|f\|_{L_2}^{1-\frac{1}{2\alpha}} \|\mathbb{D}^\alpha f\|_{L_2}^{\frac{1}{2\alpha}} \tag{9.1}$$

Theorem 9.1 is in fact due to A. Beurling [17], 1938, who noted that fractional derivatives of order $\alpha > \frac{1}{2}$ may be used in (5.1). Note that inequality (9.1) may be derived from the inequality

$$\int\limits_0^\infty |f(x)|\, dx \le C\left(\int\limits_0^\infty |f^2(x)| dx\right)^{\frac{\mu}{2(1+\mu)}} \left(\int\limits_0^\infty x^{1+\mu}|f^2(x)| dx\right)^{\frac{1}{2(1+\mu)}} \tag{9.2}$$

which is a particular case of *Carlson-Bellman* inequality ([78, p.62]; see also [5], [62]). Indeed, under the choice $\mu = 2\alpha - 1$, inequality (9.2) implies (9.1) in the case $\frac{1}{2} < \alpha < \frac{3}{2}$. If $\alpha \ge \frac{3}{2}$, one may use the interpolation inequality (an inequality for the intermediate derivative)

$$\|\mathbb{D}^\beta f\|_{L_p} \le C\|\mathbb{D}^\alpha f\|_{L_p}^{\frac{\beta}{\alpha}} \|f\|_{L_p}^{1-\frac{\beta}{\alpha}}, \qquad 0 < \beta < \alpha, \quad 1 < p < \infty$$

for fractional derivatives, see [119], p. 313.

In the case of many variables we shall also deal with *the Riesz fractional differentiation*

$$\mathbb{D}^\alpha f = (-\Delta)^{\frac{\alpha}{2}} f = F^{-1}|\xi|^\alpha F f, \qquad \alpha > 0,$$

which is known to be expressed in terms of hyper-singular integrals, see [118] and [119]. The following statement was in fact proved in [95] p. 100.

Theorem 9.2. *Let $f, \mathbb{D}^\alpha f \in L_2(\mathbb{R}^n)$, where $\alpha > \frac{n}{2}$. Then*

$$\|f\|_{W_0} \le C\left(\|f\|_{L_2} + \|\mathbb{D}^\alpha f\|_{L_2}\right). \tag{9.3}$$

The following multiplicative Beurling type version

$$\|f\|_{W_0} \le C\|f\|_{L_2}^{\frac{n}{2\alpha}}\|\mathbb{D}^\alpha f\|_{L_2}^{1-\frac{n}{2\alpha}}$$

of (9.3) also holds (A.Karapetyants [66]).

The next test in terms of the spaces $BV_{\alpha+1}$ with fractional α (see Definition 3.3) is for radial functions ([144]; cf. also [142]).

Theorem 9.3. *Let $f_0 \in BV_{\alpha+1}(\mathbb{R}_+)$, where $\alpha > \frac{n-1}{2}$. Then $f(x) = f_0(|x|) \in W_0(\mathbb{R}^n)$.*

The following lemma ([142]) provides easy to check sufficient conditions for a function $f(t), t \in (0,\infty)$ to belong to the class $BV_{\alpha+1}$.

Lemma 9.4. *Let $f(t)$ have a compact support and be continuously differentiable up to order $[\alpha]+1$ for all $t > 0$. If the function $g(t) = f^{([\alpha]+1)}(t)$ satisfies the conditions*

a) $\int\limits_0^\infty t^\alpha |g(t)|\, dt < \infty$,

b) $\int\limits_0^\infty t^\alpha |g(t+h) - g(t)|\, dt \le Ch^\delta$,

where $\delta > \alpha - [\alpha]$, then $f \in BV_{\alpha+1}$, $\alpha > 0$.

Examples covered by Theorem 9.2 ([143, p.47]): consider the generalized Riesz means

$$\hat{r}_\alpha^\Phi(t) := \left(1 - \frac{\Phi(t)}{\Phi(t)_0}\right)_+^\alpha, \qquad \alpha > 0, 0 < t < \infty,$$

where $t_0 > 0$ and $x_+^\alpha = \begin{cases} x^\alpha, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$. If the function $\Phi(t), 0 < t \le t_0$ satisfies the conditions

i) $\Phi(t) \ge 0, \Phi(0) = 0, \Phi(t)$ is strictly increasing;

ii) $\Phi(t)$ is differentiable up to order $[\alpha]+2$ for $t \in (0, t_0)$ with $\Phi'(t)$ monotonously increasing;

iii) $t^k \left|\Phi^{(k)}(t)\right| \le \Phi'(t), \quad k = 0, 1, \ldots, [\alpha]+1$.

Then $\hat{r}_\alpha^\Phi(t) \in BV_{\alpha+1}$, so that $\hat{r}_\alpha^\Phi(|x|) \in W_0(\mathbb{R}^n)$.

We note the following modification ([143], p.8) of the above statement:

Let $H(x)$ be a positive function homogeneous of degree $m > 0$ and infinitely differentiable for $x \ne 0$ and let

$$\hat{r}_\alpha^H(x) := [1 - H(x)]_+^\alpha, \quad \alpha > 0.$$

Then $\hat{r}_\alpha^H(x) \in W_0(\mathbb{R}^n)$, if $\alpha > \frac{n-1}{2}$.

More generally, the following theorem holds [145].

Theorem 9.5. *Let $H(x)$ satisfy the above assumptions . If $f_0 \in BV_{\alpha+1}$ with $\alpha > \frac{n-1}{2}$ and $f_0(\infty) = 0$, then $f(x) = f_0[H(x)] \in W_0(\mathbb{R}^n)$.*

We shall also mention some results which use Bessel fractional differentiation and are given in decomposition terms. Let

$$\mathbb{D}_B^\gamma f = F^{-1}(1+|\xi|^2)^{\frac{\gamma}{2}} F f$$

be the operator of Bessel isotropic fractional differentiation. Let also $A_t = t^P$ be an arbitrary dilation type matrix, where it is supposed that the eigenvalues of the matrix P have positive real parts. the anisotropic distance $\varrho(x)$ is defined by the relations

$$\varrho(x) = \frac{1}{t}, \quad t: \ Qt^P x \cdot t^p x = 1, \ x \neq 0,$$

where $Q = \int\limits_0^\infty e^{-tPe^{-tP'}} \, dt$ and P' is the matrix adjoint to P.

The following theorem is valid [35].

Theorem 9.6. *Let $f \in \mathcal{S}$ and $f_m(x) = f\left(A'_{2m}x\right)$ and let $\Theta \in C_0^\infty(\mathbb{R}^n)$ be supported in the layer $\{x : \frac{1}{2} \leq \varrho(x) \leq 2\}$. If*

$$\mathbb{D}_B^\gamma f_m \in L_{q,loc}(\mathbb{R}^n \backslash \{0\}),$$

where $1 < q \leq 2$ and $\gamma > \frac{n}{q}$, and

$$\sum_{m=-\infty}^{\infty} \|\mathbb{D}_B^\gamma(\Theta f_m)\|_{L_q} < \infty,$$

then $f \in W_0(\mathbb{R}^n)$.

Let now $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a vector with positive components and let

$$\mathbb{D}_B^{\overline{\alpha}} f = F^{-1} \prod_{i=1}^n (1+\xi_i^2)^{\frac{\alpha_i}{2}} F f$$

be the operator of the anisotropic Bessel fractional differentiation. The paper [92] contains the following result.

Theorem 9.7. *Let $\eta(x) = \prod_{i=1}^n \eta_i(x_i)$, where $\eta_i \in C_0^\infty(\mathbb{R})$ and $\eta_i(t) \equiv 1$ for $1 \leq t \leq 2$ and $\eta_i(t) \equiv 0$ for $t \notin \left[\frac{1}{4}, 4\right]$. If*

$$\sum_{m=-\infty}^{\infty} \|\mathbb{D}_B^{\overline{\alpha}} \eta f_m\|_{L_2} < \infty,$$

where $f_m = f\left(2^m x\right)$ and $\alpha_i > \frac{1}{2}$, then $f \in W_0(\mathbb{R}^n)$.

Finally, in this section we formulate the following extension of Theorem 6.8 to the case of fractional derivatives [117]. They are treated in the distributional sense over the Lizorkin pace Φ of Schwartzian test functions orthogonal to all polynomials (see [118] and [119] for details about the space Φ and its dual Φ').

We denote $\overline{\alpha} \circ j = (\alpha_1 j_1, \ldots, \alpha_n j_n)$ and

$$D^{\overline{\alpha}\circ j} f = F^{-1}(-i\xi)^{\overline{\alpha}\circ j} F f.$$

Theorem 9.8. *Let $f \in L_1(\mathbb{R}^n)$ have mixed Φ'-distributional derivatives $D^{\overline{\alpha}\circ j} f \in L_{\overline{p}}(\mathbb{R}^n)$ for all $j \in \{0,1\}\backslash\{0\}$, where $\overline{p} = (p_1, \dots, p_n)$. If*

$$\frac{1}{\alpha_i} < p_i \le 2, \quad i = 1, \dots, n$$

then $f \in W_0(\mathbb{R}^n)$ and

$$\|f\|_{W_0} \le C\bigg(\|f\|_{L_1} + \sum_{j \in \{0,1\}\backslash\{0\}} \|D^{\overline{\alpha}\circ j} f\|_{L_{\overline{p}}} \bigg).$$

10. Tests for functions having derivatives with singularities near the origin and vanishing at infinity

In applications, tests for functions sufficiently smooth everywhere except the origin are of special interest. For instance, when a function behaves like a homogeneous one, the following test is adapted for it (see [94] or [95, p.102]).

Theorem 10.1. *Let $\varrho(x)$ be a homogeneous function in $\mathbb{R}^n$ of order $m > 0$, positive and infinitely differentiable for $x \ne 0$. Let a function $\varphi \in C^\infty(\mathbb{R}_+\backslash\{0\})$ satisfy the assumptions*
a) $|\varphi(t) - \varphi(0)| \le Ct^\alpha, \quad 0 < t < 1,$
b) $|\varphi^{(k)}| \le Ct^{\alpha-k}, \; 0 < t < 1, \qquad |\varphi^{(k)}| \le Ct^{-\beta-k}, \; t > 1,$ *for $k = 1, 2, \dots, N$, where $N > \frac{n}{2}$ and $\alpha > 0, \beta > 0$.*
Then $\varphi[\varrho(x)] \in W_0(\mathbb{R}^n)$.

The following two theorems due to J. Boman are given in [126]. They strengthen the above test.

Theorem 10.2. *Let $f \in C^N(\mathbb{R}^n), \; N = 1 + \left[\frac{n}{2}\right]$ and let there exists constants $C > 0$ and $\delta > 0$ such that*

$$|D^j f(x)| \le C|x|^{-\delta-|j|}, \quad x \in \mathbb{R}^n\backslash\{0\},$$

for all j with $|j| = 1, \dots, N$. Then $f \in W_0(\mathbb{R}^n)$.

Theorem 10.3. *Let $f \in C^N(\mathbb{R}^n\backslash\{0\}), \; N = 1 + \left[\frac{n}{2}\right]$, have compact support and let there exists constants $C > 0$ and $\delta > 0$ such that*

$$|D^j f(x)| \le C|x|^{\delta-|j|}, \quad x \in \mathbb{R}^n\backslash\{0\},$$

for all j with $|j| = 0, 1, \dots, N$. Then $f \in W_0(\mathbb{R}^n)$.

In some applications ([74]) there appears a need of tests which take into account different information , with respect to the radial and angular variables, the behavior of functions as $|x| \to 0$ or $|x| \to \infty$. The theorem below provides such a test in terms of the function classes $C_q^{N,\gamma}(\dot{\mathbb{R}}^n)$ and $C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\})$. Note that

$$C_q^{N,\gamma}(\dot{\mathbb{R}}^n) \subset C_q^{N,\gamma}(\mathbb{R}^n\backslash\{0\}),$$

but the usage of such wider classes $C_q^{N,\gamma}(\dot{\mathbb{R}}^n)$ and $C_q^{N,\gamma}(\mathbb{R}^n\setminus\{0\})$ yields a stronger restriction on the choice of the order of smoothness N. The next result can be found in [71], [72], and [74].

Theorem 10.4. *I. Let $f \in C_q^{N,\gamma}(\mathbb{R}^n\setminus\{0\})$, where $N, q = 1, 2, \dots, N$ and $0 < \gamma < N$. If*

$$\lim_{|x|\to\infty} |x|^\alpha f(x) = 0, \quad \alpha \geq 0 \qquad \textit{and} \qquad N > n + \alpha, \quad q > \frac{3}{2}n,$$

then

$$|x|^\alpha f(x) \in W_0(\mathbb{R}^n). \tag{10.1}$$

II. Let $f \in C_q^{N,\gamma}(\dot{\mathbb{R}}^n)$, where $N, q = 1, 2, \dots, N$ and $0 < \gamma < N$. If

$$\lim_{|x|\to\infty} f(x) = 0, \quad \alpha \geq 0 \qquad \textit{and} \qquad N > \frac{n-1}{2} + \alpha, \quad q > \frac{3}{2}n,$$

then (10.1) *is valid for* $0 \leq \alpha < 1$.

The next two theorems as well as the corresponding corollaries are taken from [91] (see also [89]).

Theorem 10.5. *Let $f \in C_0(\mathbb{R})$, and let f be locally absolutely continuous on $\mathbb{R}$. Denote $f_0(t) = \sup_{|s|\geq|t|} |f(s)|$.*

a) Let f' be essentially bounded out of any neighborhood of zero and

$$f_1(t) = \operatorname{ess\,sup}_{|s|\geq|t|>0} |f'(s)|.$$

If, in addition,

$$A_1 = \int_0^1 f_1(t) \ln\frac{2}{t}\, dt < \infty \text{ and } A_{01} = \int_1^\infty \Big(\int_t^\infty f_0(s) f_1(s)\, ds\Big)^{\frac{1}{2}} \frac{dt}{t} < \infty,$$

then $f \in W_0(\mathbb{R})$, with $\|f\|_{W_0} \leq C(A_1 + A_{01})$.

b) Let f' be not necessarily locally bounded near infinity, $f_\infty(t) = \operatorname{ess\,sup}_{0<|s|\leq|t|} |f'(s)|$ and $f(t) = 0$ when $|t| \leq 2\pi$, with $f_\infty(4\pi) > 0$. If, in addition, there exists $\delta \in (0, 1)$ such that

$$A_\delta^{1+\delta} = \sup_{t\geq 2\pi} t\, f_0^\delta(t)\, f_\infty(t + 2\pi) < \infty,$$

then $f \in W_0(\mathbb{R})$, and $\|f\|_{W_0} \leq C(\delta) A_\delta (1 + A_\delta^{\frac{1}{\delta}} (f_\infty(4\pi))^{-\frac{1}{\delta}})$.

Conditions of this theorem differ from the known sufficient conditions in the way that near infinity a combined behavior of both the function and its derivative comes into play (see also the corollary below). Conditions for f_0 and f_1 in a) are not necessary in general, but are such for a certain subclass (cf. necessary conditions in Subsection 4.2). The condition $A_{01} < \infty$, holds, for example, if $(\ln t)^{2+\delta} f_0(t) f_1(t) \in L_1[1, \infty)$ for some $\delta > 0$, but not for $\delta = 0$ when the function in question may be not in $W_0(\mathbb{R})$.

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Corollary 10.6. *If* $A_1 < \infty$, $f(t) = O(|t|^{-\gamma_0})$ *for some* $\gamma_0 > 0$ *and* $f'(t) = O(|t|^{-\gamma_1})$ *for some* $\gamma_1 \in \mathbb{R}$ *as* $|t| \to \infty$, *with* $\gamma_0 + \gamma_1 > 1$, *then* $f \in W_0(\mathbb{R})$. *If* $\gamma_0 + \gamma_1 < 1$, *then there is a function* f *such that* $f(t) = O(|t|^{-\gamma_0})$ *and* $f'(t) = O(|t|^{-\gamma_1})$ *but* $f \notin W_0(\mathbb{R})$.

Let now $f\colon \mathbb{R}^n \to \mathbb{C}$ with $n \ge 2$. We will give a direct generalization of Theorem 10.5 to higher dimensions. In order to formulate it, we introduce certain notations.

Similarly to a) in the one-dimensional case, we set

$$f_\chi(x) = \sup_{\substack{|u_i|\ge|x_i|,\\ i:\chi_i=0}} \operatorname*{ess\,sup}_{\substack{|u_k|\ge|x_k|>0,\\ k:\chi_k=1}} |D^\chi f(u)|.$$

As for b), we set

$$F_\chi(x) = \sup_{\substack{|u_i|\ge|x_i|,\\ i:\chi_i=0}} \operatorname*{ess\,sup}_{\substack{|u_k|\le|x_k|>0,\\ i:\chi_i=1}} |D^\chi f(u)|.$$

Here and in what follows $\chi = (0, 0, \dots, 0)$ or $\chi = (1, 1, \dots, 1)$ means that only one type of majorizing is considered.

We denote by $\mathbb{R}_\eta$ the Euclidean space of dimension $|\eta| = \eta_1 + \cdots + \eta_n$ with respect to the variables x_i, where i-s are such that $\eta_i = 1$; correspondingly x_η is an element of this space.

Theorem 10.7. *Let* $f \in C_0(\mathbb{R}^n)$ *and let* f *and its partial derivatives* $D^\eta f$, *for all* η, $\eta \ne 1$, *be locally absolutely continuous on* $\mathbb{R}^n$ *in each variable.*

a) If all partial derivatives $D^\eta f$ *are essentially bounded out of any neighborhood of each coordinate hyperplane and*

$$A_\chi = \int_0^1 \dots \int_0^1 \prod_{k:\chi_k=1} \ln(2/x_k)\, dx_k$$

$$\int_1^\infty \dots \int_1^\infty \biggl(\int_{\prod_{i:\chi_i=0}[u_i,\infty)} f_\chi(x) f_1(x)\, dx_\eta \biggr)^{1/2} \prod_{i:\chi_i=0} \frac{du_i}{u_i} < \infty \tag{10.2}$$

for all $0 \le \chi < 1$, *then* $f \in W_0(\mathbb{R}^n)$.

b) Let $f(x) = 0$ *when* $|x_i| \le 2\pi$, $i = 1, 2, \dots, n$, *and* $F_\chi(4\pi, \dots, 4\pi) > 0$. *If there exists* $\delta \in (0, 1)$ *such that*

$$\sup_{x_i\ge 2\pi, i=1,\dots,n} (x_1 \cdots x_n)^{(1+\delta)^{n-1}} \prod_{0\le\chi\le1} F_\chi^{\frac{\delta^{n-|\chi|}}{2^{n-1}}}(x + 2\pi\chi) < \infty, \tag{10.3}$$

then $f \in W_0(\mathbb{R}^n)$.

In the multivariate case, a) and b) are the two extreme options. We omit possible cases of interplay of these options, since they are less transparent.

Slightly different results are obtained in [90]. As above, we first give a one-dimensional version and then its multivariate extension.

Theorem 10.8. *Let $f \in C_0(\mathbb{R})$, and let f be locally absolutely continuous on $\mathbb{R}$. Let for some $\delta \in (0,1]$*

$$\int_0^\infty \left[f_0^{1-\delta}(t)F^{1+\delta}(t) + f_0^{1+\delta}(t)F^{1-\delta}(t)\right]\, dt < \infty,$$

where

a) if $f' \in L_\infty$ off any neighborhood of zero, then

$$F(t) = \operatorname*{ess\,sup}_{|s|\ge|t|>0} |f'(s)|,$$

b) if f' is not bounded near infinity and belongs to L_∞ off any neighborhood of infinity, then $F(t) = f_\infty(t+2\pi)$, where $f_\infty(t) = \operatorname{ess\,sup}_{0<|s|\le|t|} |f'(s)|$.

Then $f \in W_0(\mathbb{R})$.

Denote $\Theta = (\chi, \eta, \zeta)$ and $/\Theta/ = \chi + \eta + \zeta$. Set for $/\Theta/ \le 1$

$$f_\Theta(x) = f_{\chi,\eta,\zeta}(x) = \sup_{\substack{|u_i|\ge|x_i|,\\ i:\chi_i=1}} \ \operatorname*{ess\,sup}_{\substack{0<|x_i|\le|u_i|,\\ i:\eta_i=1}} \ \operatorname*{ess\,sup}_{\substack{0<|u_k|\le|x_k|,\\ k:\zeta_k=1}} |D^{\eta+\zeta} f(u)|.$$

In the next result we deal with

$$D^{\eta+\zeta} f(x) = \Bigg(\prod_{i:\eta_i+\zeta_i=1} \frac{\partial}{\partial x_i}\Bigg) f(x).$$

Here $\eta_i+\zeta_i$ takes values either 0 or 1. Consider the main case $/\Theta/=1$. To each vector χ, $0 \le \chi \le 1$, certain η and ζ correspond that characterize the considered function. Thus, each majorant may be considered as depending only on χ and denoted by f_χ in the sense that only one triple $\Theta := \Theta(\chi)$ is taken for the given χ. By this the assumptions of the next theorem are formulated in terms of the function $\prod\limits_{0\le\chi\le1} f_\chi^{\delta_i(\chi)}$, where $i = 1, 2, \dots, 2^n$ and for some $\delta \in (0,1]$

$$\delta_i(\chi) = \frac{1}{2^{n-1}} \prod_{k=1}^n (1 + \omega_{ik}(\chi_k)\delta)$$

with $\omega_{ik}(\chi_k)$, taking values $+1$ or -1 by: $\{\omega_{ik}(1)\}_{k=1}^n$ differ for different i, and $\omega_{ik}(0) = -\omega_{ik}(1)$.

Theorem 10.9. *Let $f \in C_0(\mathbb{R}^n)$ and let f and its partial derivatives $D^\eta f$, $0 \le \eta < 1$, be locally absolutely continuous on $\mathbb{R}^n$ in each variable. If for some $\Theta(\chi)$ and $\delta_i(\chi)$ described above*

$$\int_{\mathbb{R}^n_+} \sum_{i=1}^{2^n} \prod_{0\le\chi\le1} f_\chi^{\delta_i(\chi)}(x + 2\pi\zeta)\, dx < \infty,$$

then $f \in W_0(\mathbb{R}^n)$.

Let us give more one- and multi-dimensional results of a similar nature (see [86] and [70]). These are generalizations of Theorem 5.5 (see also discussion after it).

Theorem 10.10. *Let $f \in C_0(\mathbb{R})$, and let f be locally absolutely continuous on $\mathbb{R}$.*

a) Let $f(t) \in L_p(\mathbb{R})$, $1 \le p < \infty$, and $f'(t) \in L_q(\mathbb{R})$, $1 < q < \infty$. If

$$\frac{1}{p} + \frac{1}{q} > 1,$$

then $f \in W_0(\mathbb{R})$.

b) If

$$\frac{1}{p} + \frac{1}{q} < 1,$$

then there exists a function f, $f(t) \in L_p(\mathbb{R})$ and $f'(t) \in L_q(\mathbb{R})$, with such p and q but $f \notin W_0(\mathbb{R})$.

The following multidimensional results were recently proved in [70].

Theorem 10.11. *Let $f \in C_0(\mathbb{R}^n)$ and let f and its partial derivatives $D^\eta f$, for all η, $\eta \ne 1$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^n$ in each variable. Let $f \in L_{p_0}(\mathbb{R}^n)$, $1 \le p_0 < \infty$, and let each partial derivative $D^\eta f$, $\eta \ne 0$, belong to $L_{p_\eta}(\mathbb{R}^n)$, where $1 < p_\eta < \infty$. If for all η, $\eta \ne 0$,*

$$\frac{1}{p_0} + \frac{1}{p_\eta} > 1, \tag{10.4}$$

then $f \in W_0(\mathbb{R}^n)$.

Remark 10.12. Condition (10.4) is sharp when $\eta = 1$, while for other η it is apparently not sharp.

We can also obtain a result in which all the derivatives interplay rather than the pairs p_0 and p_η.

Theorem 10.13. *Let $f \in C_0(\mathbb{R}^n)$ and let f and its partial derivatives $D^\eta f$, for all η, $\eta \ne 1$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^n$ in each variable. Let $f \in L_{p_0}(\mathbb{R}^n)$, $1 \le p_0 < \infty$, and let each partial derivative $D^\eta f$, $\eta \ne 0$, belong to $L_{p_\eta}(\mathbb{R}^n)$, where $1 < p_\eta < \infty$. If*

$$\sum_{0 \le \eta \le 1} \frac{1}{p_\eta} > 2^{n-1} \tag{10.5}$$

and

$$\sum_{\eta \ne 0} \frac{1}{p_\eta} \le 2^{n-1}, \tag{10.6}$$

then $f \in W_0(\mathbb{R}^n)$.

The next corollary gives a verifiable condition for $f \in W_0(\mathbb{R}^n)$ in terms of power growth near infinity.

Corollary 10.14. *If*

$$|D^\chi f(x)| \le C\frac{1}{(1+|x|)^{\gamma_\chi}}, \tag{10.7}$$

where $\gamma_\chi > 0$ *for all* χ, $0 \le \chi \le 1$, *and*

$$\sum_{0\le\chi\le 1} \gamma_\chi > n2^{n-1}, \tag{10.8}$$

then $f \in W_0(\mathbb{R}^n)$.

We mention that the multiplier function m given in (1.7), along with the positive and negative conditions (1.8) and (1.9), satisfies sufficient conditions of the above Theorems 10.7, 10.8, 10.11 and 10.13 as well as of Corollary 10.14 in the multivariate case. Of course, the corresponding one-dimensional versions allow one to treat the one-dimensional case. The sharpness of one-dimensional results can be shown by means of m; sometimes it works in a similar manner in several dimensions.

11. Positive definite functions

Positive definiteness is an important property of a function closely related to belonging to the considered algebras of functions. Indeed, each function from W_0 (as well as from W) is a linear combination of at most four positive definite functions.

Let E be a linear (vector) space over the field of real numbers. The function $f\colon E \to \mathbb{C}$ is said to be *positive definite* on E, if for any family $\{x_k\}_{k=1}^N$ of $x_k \in E$ for all k, and for any number set $\{\xi_k\}_{k=1}^N$, $\xi_k \in \mathbb{C}$ for all k,

$$\sum_{k,s=1}^{N} f(x_k - x_s)\xi_k\bar{\xi}_s \ge 0.$$

If E is a Hilbert space, than we can take, for example, $f_1(x) = e^{ixy}$, with any $y \in E$. As follows from Theorem 11.2 below, in which $E = \mathbb{R}^n$, an obvious necessary and sufficient condition for a function f from W_0 (or W) to be positive definite is $f(0) = \|f\|_W$.

Lemma 11.1. *Let f be an arbitrary positive definite function on E. Then for every $x \in E$ and $y \in E$*

a) $f(x) = \overline{f(-x)}$; *b)* $|f(x)| \le f(0)$;

c) $|f(x) - f(x+y)|^2 \le 2f(0)\,\mathrm{Re}(f(0) - f(y))$; *and*

d) $|f(x+y) - 2f(x) + f(x-y)| \le 2\mathrm{Re}(f(0) - f(y))$.

For the cases a) and b), see [97]; c) is due to M.G. Krein (see, e.g., [1]); the case d) in [158].

It follows either from c) or from d) that if

$$\underline{\lim}_{y\to 0}\frac{\mathrm{Re}(f(0)-f(y))}{|y|^2}=0,$$

then $f(x)\equiv f(0)$ for $x\in E$, while if there exists $y=x_0\neq 0$ such that $\mathrm{Re}f(x_0)=f(0)$, then $f(x+x_0)=f(x)$, that is, f is a periodic function.

By $W^+(\mathbb{R}^n)$ we denote the subset of $W(\mathbb{R}^n)$ consisting of functions positive definite on $E=\mathbb{R}^n$.

The following theorem is due to S. Bochner and A. Khintchin (see, e.g., [1], [21], [97], [158]).

Theorem 11.2. *Let f be a continuous function on $\mathbb{R}^n$. In order that $f\in W^+(\mathbb{R}^n)$, it is necessary and sufficient that there exists a finite positive measure μ on $\mathbb{R}^n$ such that for each $x\in\mathbb{R}^n$*

$$f(x)=\int\limits_{\mathbb{R}^n}e^{ixy}d\mu(y).$$

We remark that the sufficiency immediately follows from the above example of f_1.

In Probability Theory, the functions $f\in W^+(\mathbb{R})$, with $f(0)=1$, are called *characteristic* (see, e.g., [97] and the table below).

The above result yields the following statement from the L_1-theory of Fourier transforms, see, e.g., [130, Ch.I, Cor.1.26]. It concerns, in fact, positive definiteness.

The above results yield

Corollary 11.3. *Let $f\in L_1(\mathbb{R}^n)$. If $\widehat{f}(x)\geq 0$ and $f(x)$ is continuous at the point $x=0$, then $f\in W_0^+(\mathbb{R}^n)$.*

M.G. Krein investigated the problem of extension of a function from an interval to a positive definite function on $\mathbb{R}$, including the uniqueness of such extension; see, e.g., [75]. The following criterion for an entire function of exponential type to be positive definite is also due to M.G. Krein [76].

Theorem 11.4. *A function $f\colon\mathbb{R}\to\mathbb{C}$ satisfying $f(t)=0$ for $|t|\geq\sigma>0$ belongs to $W_0^+(\mathbb{R})$ if and only if there exists a function $g\in L_2(\mathbb{R})$, $g(t)=0$ for $|t|\geq\frac{\sigma}{2}$ such that for any $t\in\mathbb{R}$*

$$f(t)=\int\limits_{-\infty}^{+\infty}g(y)\overline{g(t+y)}\,dy.$$

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In what follows we find it convenient to identify $\mathbb{R}^n$ with $\ell_2^n := \{(x_1, \dots, x_n)\}$ defined as the subspace of the space

$$\ell_2 := \left\{ x = (x_1, \dots, x_n, \cdots),\ |x| := \sqrt{\sum_{i=1}^{\infty} x_n^2} < \infty \right\}$$

and denote

$$W^+(\ell_2) := \bigcap_{n=1}^{\infty} W^+(\mathbb{R}^n).$$

The function $f_2(|x|) = e^{-|x|^2}$ belongs to $W^+(\ell_2)$.

The next theorem is due to J. Marcinkiewicz (see, e.g., the corollary of Theorem 7.3.3 in [97]).

Theorem 11.5. *If P is a polynomial of degree not less than 3, then e^P is not a characteristic function.*

The proof of this result rests on the following lemma (see, e.g., [97]).

Lemma 11.6. *If $f \in W^+(\mathbb{R})$ and analytic in the disk $|z| < r$, then f is extendable into the strip $|\mathrm{Im} z| < r$, and is ridge-like in this strip:*

$$|f(z)| \le |f(i\mathrm{Im} z)|.$$

It remains to observe that if $P(z) = e^{i\alpha} z^m + ...$, with $\alpha \in [0, 2\pi)$ and $\deg P = m$, then the lemma yields for any $z = re^{i\varphi}$

$$\mathrm{Re}(e^{i\alpha} r^m e^{im\varphi} + \cdots) \le \mathrm{Re}\ [i^m (\mathrm{Im}\ re^{i\varphi})^m + \cdots].$$

Now, dividing by r^m and passing to the limit as $r \to \infty$, we get

$$\cos(m\varphi + \alpha) \le \mathrm{Re}(i^m \sin^m \varphi) \le |\sin \varphi|^m.$$

However, this inequality is invalid when $m \ge 3$ and $\varphi = \frac{\pi - \alpha}{m}$.

In what follows we shall write $f_0 \in rad\, W^+(\mathbb{R}^n)$ in order to emphasize that the positive definiteness is considered for the radial extension $f(x) = f_0(|x|)$. It is obvious that $W^+(\mathbb{R}^n) \subset W^+(\mathbb{R}^{n+1})$; however, for the subset of radial functions $f(x) = f_0(|x|)$ we have $rad\, W^+(\mathbb{R}^{n+1}) \subset rad\, W^+(\mathbb{R}^n)$.

Theorem 11.7. *In order that $f_0 \in rad\, W^+(\mathbb{R}^n)$, it is necessary and sufficient that there exists a finite positive measure μ_0 on $[0, +\infty)$ such that for $\lambda = n/2 - 1$ and $t \ge 0$*

$$f_0(t) = \int_0^{\infty} j_\lambda(tu)\, d\mu_0(u),$$

where, as above, $j_\lambda(t) = \frac{J_\lambda(t)}{t^\lambda}$, with J_λ being the Bessel function of order $\lambda > -1$.

For this result, see [124], [1] or [158, 6.3.4]. Necessary conditions when λ increases are given in [148]. Note that the entire function j_λ is an elementary function only when $\lambda + \frac{3}{2} \in \mathbb{N}$.

Example 3. If $2\lambda \notin Z_+$, then $j_\lambda \in rad\, W^+(\mathbb{R}^{[2\lambda]+2}) \setminus rad\, W^+(\mathbb{R}^{[2\lambda]+3})$.

Theorem 11.8. *Let n be odd and ≥ 3. In order that $f_0 \in rad\, W^+(\mathbb{R}^n)$, it is necessary and sufficient that for $0 \leq k \leq n/2 - 3/2$*

$$\frac{d^k}{dt^k}\left[t^{n/2-1} f_0(\sqrt{t})\right]_{t=0} = 0,$$

and $f_1(t)$, defined for $t \geq 0$ as

$$f_1(\sqrt{t}) = \sqrt{t}\frac{d^{n/2-1/2}}{dt^{m/2-1/2}}\left[t^{n/2-1/2} f_0(\sqrt{t})\right],$$

belong to $rad\, W^+(\mathbb{R})$.

In this statement, $rad\, W^+$ can be replaced with $radW$; the latter is understood to be similarly restricted to the radial setting. For $W_0(\mathbb{R}^n)$, a similar statement is contained in Theorem 8.12.

Let us turn to completely monotone functions on $[0, +\infty)$. A function $f \in C[0, +\infty) \cap C^\infty(0, +\infty)$ satisfying the condition $(-1)^k f^{(k)}(x) \geq 0$ for all $x > 0$ and for all $k \in \mathbb{Z}_+$ is called a *completely monotone* function. Observe that being nonnegative, decreasing, concave, etc., such a function is of very regular behavior (is a restriction (trace) of a function analytic in the half-plane $\mathrm{Re}\, z > 0$).

For the next Bernstein's theorem, see, e.g., [161, p.161].

Theorem 11.9. *For a function f to be completely monotone on $[0, +\infty)$, it is necessary and sufficient that a finite positive measure on $[0, +\infty)$ exist such that for all $t \in [0, +\infty)$ there holds $f(t) = \int\limits_0^\infty e^{-ut} d\mu(u)$.*

As follows from the definition, the function $f(t^\alpha)$, with $\alpha \in [0, 1)$, is completely monotone along with $f(t)$. On the other hand, the theorem yields the ridge property of complete monotone functions:

$$|f(z)| \leq f(\mathrm{Re}\, z), \qquad Rez \geq 0.$$

We now describe, following I.J. Schoenberg, the set $rad\, W^+(\ell_2)$ of continuous functions from $W^+(\ell_2)$ depending only on a norm. We have $\mathbb{R}^n = \ell_2^n \subset \ell_2$ for all $n \in \mathbb{N}$, and hence $rad\, W^+(\ell_2) \subset \bigcap\limits_n W^+(\mathbb{R}^n)$. But $f_0(|x|) = \lim\limits_{n\to+\infty} f_0(|\mathrm{pr}_{\ell_2^n} x|_{\ell_2^n})$, which gives $W^+(\ell_2) = \bigcap\limits_n W^+(\mathbb{R}^n)$. The next result is due to I.J. Schoenberg (see, e.g., [1] or [158]).

Theorem 11.10. *In order that $f_0(|x|) \in W^+(\ell_2)$, it is necessary and sufficient that the function $t \mapsto f_0(\sqrt{t})$ be completely monotone on $[0, +\infty)$.*

Example 4. $e^{-|x|^\alpha} \in W^+(\ell_2)$ if and only if $\alpha \in (0, 2]$. The necessity of this condition follows from Lemma 11.1.

I.J. Schoenberg (1938) proved that $\exp(-|x|^\alpha_{l_p}) \in W^+(\ell_p)$ for all $p \in (0,2]$ and $\alpha \in [0,p]$. He posed the problem of conditions which ensure $\exp(-|x|^\alpha_{\ell_p^n}) \in W^+(\ell_p)$ for $p \in (2,+\infty]$. After numerous attempts of many mathematicians to solve this problem, it has surrendered just recently to V.P. Zastavnyi (see [167, 168]) and independently to A. Koldobskii [67, 68]. For the detailed bibliography, see [168] and [50].

Theorem 11.11. *Let $p \in (2,+\infty]$. If $n \geq 3$, then $e^{-|x|^\alpha_{l_p^n}} \in W^+(\ell_p)$ only for $\alpha = 0$, while for $n = 2$ only if $\alpha \in [0,1]$.*

In [169] (see also [158, Ch.6]) a more general result of V.P. Zastavnyi for normed spaces can be found; see also [100, p.210].

In the following theorems simple sufficient conditions for radial functions are given. A classical result due to Pólia reads as follows (see, e.g., [97] or [158]).

Theorem 11.12. *Each even, convex and monotone decreasing to zero function on $[0,\infty)$ belongs to $W_0^+(\mathbb{R})$.*

In fact, such functions belong even to the class $W_0^*(\mathbb{R})$, defined for any n by

$$W_0^*(\mathbb{R}^n) = \{f : f = \widehat{g},\ \operatorname{ess\,sup}_{|y|\geq t} |g(y)| \in L_1(\mathbb{R}_+)\}; \tag{11.1}$$

see [147] or [158]. By this, f may decrease arbitrarily slowly. What is really important, as Pólya observed, that $\widehat{f}(y) \geq 0$ for $y \in R \setminus 0$.

For the next result due to R. Askey, see [2] (or [158, 6.3.7]).

Theorem 11.13. *Let $n \in \mathbb{N}$, and $m = [n/2+1]$ (integral part). If $f_0 \in C[0,+\infty)$, $\lim_{t\to+\infty} f_0(t) = f_0(+\infty) \geq 0$, $f_0 \in C^{m-1}(0,+\infty)$, $(-1)^{m-1} f_0^{(m-1)}$ is concave on $(0,+\infty)$, and*

$$\lim_{t\to+0} t^m f_0^{(m)}(t) = \lim_{t\to+\infty} t^m f_0^{(m)}(t) = 0$$

(with, e.g., the right-hand derivative in mind), then $f_0 \in rad\, W_0^+(\mathbb{R}^n)$.

Example 5. The function $(1-|x|)_+^\alpha$ satisfies the assumptions of Theorem 11.13, with $n = 1$, if $\alpha \geq 1$. For $\alpha \in (0,1)$ this function does not belong to $W_0^+(\mathbb{R})$ by the necessary condition d) from Lemma 11.1 when $y = 1$ (condition c) gives $\alpha < \frac{1}{2}$).

In [149] and [153] (see also [148] and [158]), a problem of constructing compactly supported radial functions from $W^+(\mathbb{R}^n)$, each function of polynomial form in $|x|$ on the support, is considered. More precisely, $f_0(|x|) = p_m(|x|)$ for $|x| \leq 1$ and $f_0(x) = 0$ for $|x| \geq 1$. An important question is what the maximal smoothness of such "splines" of degree m is. We start with the case $n = 1$.

We now introduce A-splines by integrating a-splines even number of times, with a proper normalization. For $m = 1$, we set $A_1(x) = (1-|x|)_+$, while for

$m \geq 2$ we set, for all $x \in \mathbb{R}$,

$$A(x) = A_{3m-2}(x) = \frac{(-1)^{m-1}(3m-2)!2^{m-1}}{m!(2m-3)!(2m-3)!!} \int_1^{|x|} (|x|-u)^{2m-3} a_m(u)\, du,$$

where

$$a_m(\sqrt{t}) = \sqrt{t} \frac{d^{m-1}}{dt^{m-1}} \{t^{m-3/2}(1-\sqrt{t})_+^m\}.$$

Theorem 11.14. *For every $m \in \mathbb{N}$ we have $A_{3m-2} \in W_0^+(\mathbb{R}) \cap C^{2m-2}(\mathbb{R})$ with $A_{3m-2}(0) = 1$. Being an even function, it is a polynomial of degree $3m-2$ on $[0,1]$, and zero on $[1,\infty)$. These properties define the A-spline uniquely. Besides, it decreases strictly on $[0,1]$ and has one point of inflection if $m \geq 2$.*

Example 6. We have

$$A_4(x) = (1-|x|)_+^3(1+3|x|)$$

and

$$A_7(x) = (1-|x|)_+^5(1+5|x|+8x^2).$$

The next theorem of 1987 due to R.M. Trigub can be found in, e.g., [153], [171] and [158].

Theorem 11.15. *Let n be odd.*

a) For any $m \geq \frac{n+1}{2}$ there exists a function in $W_0^+(\mathbb{R}^n)$, of the form $p_m(|x|)$ for $|x| \leq 1$ and vanishing if $|x| \geq 1$, such that it belongs to $C^r(\mathbb{R}^n)$ with $r = 2[\frac{m}{3} - \frac{n+1}{6}]$ (integral part), and for no r greater than that indicated.

b) For each $r \in \mathbb{Z}_+$ there is a function in $W_0^+(\mathbb{R}^n)$ of the same form which belongs to $C^{2r}(\mathbb{R}^n)$ and is of degree $m = 3r + \frac{n+1}{2}$, and not smaller. This radial "spline" is unique provided that it is 1 at zero. This spline is of the form (radial basis function)

$$A_{0,n}(x) = \frac{2^{\frac{n-1}{2}}}{(n-3)!!} \int_0^1 (1-u^2)^{\frac{n-3}{2}} a_{\frac{n+1}{2}}(u|x|)\, du$$

and

$$A_{r,n}(x) = \gamma(r,n) \int_0^1 (1-u^2)^{\frac{n-3}{2}} du \int_1^{|x|u} (|x|u-t)^{2r-1} a_{r+\frac{n+1}{2}}(t)\, dt$$

for $r = 0$ and $r \geq 1$, respectively, where

$$\gamma(r,n) = (-1)^r \frac{2^{r+\frac{n-1}{2}}(3r+\frac{n+1}{2})!}{(n-3)!!(2r-1)!!(2r-1)!(r+\frac{n+1}{2})!}.$$

Example 7. The following is an example of a radial basis function:

$$A_{2,3}(x) = (1-|x|)_+^4(1+4|x|) \in W_0^+(\mathbb{R}^3) \cap C^2(\mathbb{R}^3).$$

To prove Theorems 11.14 and 11.15, the following criterion for a function to be characteristic was applied (see [150]).

Theorem 11.16. *Let f be a function continuous at zero. It is in $W^+(\mathbb{R})$ if and only if the following conditions are fulfilled:*

α) f is continuous and bounded on $\mathbb{R}$;

β) the improper integral $\int\limits_{\to 0}^{\to +\infty} \frac{f(t)-f(-t)}{t}\,dt$ converges;

γ) $\lim\limits_{T\to+\infty} \frac{1}{2T}\int\limits_{-T}^{T} f(t)\,dt \geq 0$;

δ) there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $x \neq 0$

$$(\operatorname{sign} x)^{k+1} \int\limits_{-\infty}^{\infty} \frac{f(t)}{(x+it)^{k+1}}\,dt \geq 0.$$

This criterion implies [158, 6.2.15].

It is worth noting that $A_{r,n}$ in Theorem 11.15 are the radial basis functions. More general cases of radial splines can be found in [160], [122], and [170].

It is of considerable interest to give a full description of the extremal points of a convex set of such "splines" of a given degree; see also [171].

We conclude with a couple of other applications of positive definite functions. The first one can be found in [158, 6.5.4], while the second one in [156].

Theorem 11.17. *Let $\varphi \in C(\mathbb{R}^n)$. In order that the series $\sum\limits_k \varphi(\varepsilon k)e^{ikx}$ be the Fourier series of a positive measure for every $\varepsilon > 0$, it is necessary and sufficient that $\varphi \in W^+(\mathbb{R}^n)$.*

Theorem 11.18. *Let $\tau_m(t) = \sum\limits_{k=-m}^{m} c_k e^{ikt}$ be a trigonometric polynomial of degree not greater than m.*

a) In order that the Bernstein inequality $\|\tau_m'\|_\infty \leq m\|\tau_m\|_\infty$ to hold, it is necessary and sufficient that the function $1-|t|$ when $|t| \leq 2$ and zero otherwise be extendable to a function from $W^+(\mathbb{R})$, and in a unique way with the period 4.

b) For any $\delta \in \mathbb{C}$ there holds $\|\widetilde{\tau}_m' + \delta\tau_m\|_\infty \leq (\frac{m}{2} + |\frac{m}{2} + \delta|)\|\tau_m\|_\infty$ (the norm in $C(\mathbb{T})$), and the inequality is sharp for any m and at least for any $\delta \in \mathbb{R}$.

For $\delta = 0$, inequality b) is established by Szegó.

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12. Convex and convex-type functions. Connection between the summability of Fourier series and absolute convergence of Fourier integrals

In this section, first of all, especially emphasized are convex and convex-type functions, and functions from intermediate classes V^* between that of functions of bounded variation and the class of functions that are linear combination of convex functions each. Our choice of the results for this section is very relative: various results from other sections may equally be included in this collection; mention, for example, Theorems 5.4, 5.7, 8.2, 9.3, and 9.5.

12.1. **Convexity and further.** As long ago as in 1928, A. Zygmund proved the following statement (see [65, Ch.II, §14]).

Theorem 12.1. *If an odd function f is convex on $\mathbb{R}_+$ and $f(0) = 0$, then $f \in W_0(\mathbb{R})$ locally (that is, can be extended from any interval in such a way that the extension will belong to $W_0(\mathbb{R})$) if and only if the integral*

$$\int_{\to 0} \frac{f(t)}{t}\, dt$$

converges.

Even functions convex on $\mathbb{R}_+$ are not subject to such integral condition (see Theorem 11.12). Without convexity, there may happen that a function after odd extension from $\mathbb{R}_+$ belongs to $W_0(\mathbb{R})$, while after even extension this is not the case (see [65, Ch.II, §15]). Roughly speaking, the difference is logarithmic (see, e.g., [64] or [65, Ch.VI, §2]).

G.E. Shilov in 1942 was the first who studied the asymptotics of the Fourier coefficients of an odd convex function (see, e.g., [4, Ch.IX, §6]). Theorems, both more precise and more general, will be given below (see Theorems 12.6 and 12.8).

Lemma 12.2. *Let $f \in C(\mathbb{R}_+)$ and $f' \in AC_{loc}(\mathbb{R}_+)$. In order that*

$$\int_0^\infty t|df'(t)| < \infty,$$

it is necessary and sufficient that f be the linear combination of at most four convex functions on $\mathbb{R}_+$.

By this, Theorem 5.4, for example, follows from Theorem 11.12.

We remind the reader that the class $W_0^*(\mathbb{R}^n)$ is defined by (11.1).

Theorem 12.3. *Let $f \in C_0(\mathbb{R}^n)$. If there is a $\phi \in W_0^*(\mathbb{R}^n)$ such that for some integer $r > n/2$, for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$*

$$|\Delta_h^r f(x)| \le |\Delta_h^r \phi(x)|,$$

then $f \in W_0(\mathbb{R}^n)$.

For $n = 1$ and $r = 1$, this theorem was proved by A. Beurling in [18]; far-going extensions are given in [147] (see also [158]). General properties of the algebra $W_0^*(\mathbb{R})$ are surveyed in [8] (see also [83]).

Theorem 12.4. *If a function $f\colon \mathbb{R}^n \to \mathbb{C}$ is even in each of the variables $x_1, x_2, \ldots, x_n$, belongs to AC_{loc} on every line parallel to any of the coordinate axes, $\lim\limits_{|x|\to\infty} D^\chi f(x) = 0$ for all $\chi \in \{0,1\}^n$ and*

$$\operatorname*{ess\,sup}_{|u_j|\ge x_j, 1\le j\le n} \left|\frac{\partial^n f(u)}{\partial u_1 \ldots \partial u_n}\right| \in L_1(\mathbb{R}^n_+),$$

then $f \in W_0(\mathbb{R}^n)$.

In particular, if f is radial, $f(x) = f_0(|x|)$, then the following conditions are sufficient for f to belong to $W_0(\mathbb{R}^n)$:

$$f_0 \in C_0(\mathbb{R}_+),\ f_0^{(n-1)} \in AC_{loc},\ \operatorname{ess\,sup}_{y\ge t} |y^{\nu-1} f_0^{(\nu)}(y)| \in L_1(\mathbb{R}_+), 1 \le \nu \le n.$$

In [147] (see also [158]), this theorem is derived from the previous one. It allows one to slightly strengthen Theorem 10.5 for even functions by replacing the condition $A_1 < \infty$ with the following:

$$\int_0^1 \operatorname*{ess\,sup}_{y\ge t} |f'(y)|\, dt < \infty.$$

Note that a similar replacement of the condition $A_{01} < \infty$ not always leads to the strengthening of Theorem 10.5.

Theorem 12.5. *Let for $q = [\frac{n-1}{2}]$ (integral part), $f_0 \in C^q[0,\pi]$, $f_0^{(q)}$ be convex on $[0,\pi]$, and $f_0(t) = 0$ when $t > \pi$. Then for the radial extension $f(x) = f_0(|x|)$, we have $f \in W_0(\mathbb{R}^n)$ (or $\widehat{f} \in L_1(\mathbb{R}^n)$) if and only if the integral*

$$\int_0^\pi \frac{f_0(\pi - t)}{t^{\frac{n+1}{2}}} dt$$

converges.

We now present a generalization and extension to the Fourier transform of the afore-mentioned result of G.E. Shilov (see, e.g., [158, 6.4.7 b)]). In order to formulate further generalizations, we define the class $V^*(\mathbb{R}_+)$ that consists of functions $f \in AC_{\rm loc}(\mathbb{R}_+)$ for which $\lim\limits_{t\to\infty} f(t) = 0$ and

$$\|f\|_{V^*} = \int_0^\infty \operatorname*{ess\,sup}_{y\ge t} |f'(y)|\, dt < \infty.$$

Theorem 12.6. *a) If f is convex on $[a,b]$ and vanishes off $[a,b]$, then for each $y \in \mathbb{R} \setminus [-2,2]$*

$$\widehat{f}(y) = \frac{i}{y}\left[f\left(b - \frac{d}{|y|}\right)e^{-iby} - f\left(a + \frac{d}{|y|}\right)e^{-iay}\right] + \theta F(|y|),$$

where $d = \min\{\frac{b-a}{2}, \pi\}$, F is decreasing on $[2,+\infty)$, and such that $\int\limits_2^\infty F(t)\,dt \le \frac{1}{d}V_a^b(f)$, while $|\theta| \le C$.

b) Let $f(t) = 0$ for $t < 0$, while on $[0,\infty)$ the function f is locally absolutely continuous, $\lim f(t) = 0$ as $t \to +\infty$, and $\|f\|_{V^*} < \infty$. Then for all $y \in \mathbb{R} \setminus \{0\}$

$$\widehat{f}(y) = -\frac{i}{y} f\left(\frac{\pi}{2|y|}\right) + F(y),$$

with $\|F\|_{L_1} \le C\|f\|_{V^*}$, where, unlike in a), F is, generally speaking, not monotone.

Corollary 12.7. *If a function $f \in V^*(\mathbb{R}_+)$, then $f \in W_0(\mathbb{R})$ if and only if*

$$\int\limits_0^\infty \frac{|f(t) - f(-t)|}{t}\,dt < \infty.$$

Let us go on to generalizations of these results. We just remind that by Tg we denote the T-transform of g.

Theorem 12.8. *Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be locally absolutely continuous, of bounded variation and $\lim\limits_{t\to\infty} f(t) = 0$. Let also $Tf' \in L_1(\mathbb{R}_+)$. Then the cosine Fourier transform of f*

$$\widehat{f_c}(x) = \int\limits_0^\infty f(t)\cos xt\,dt$$

is Lebesgue integrable on $\mathbb{R}_+$, with

$$\|\widehat{f_c}\|_{L_1(\mathbb{R}_+)} \le C\|f'\|_{L_1(\mathbb{R}_+)} + \|Tf'\|_{L_1(\mathbb{R}_+)},$$

and for the sine Fourier transform, we have, with $x > 0$,

$$\widehat{f_s}(x) = \int\limits_0^\infty f(t)\sin xt\,dt = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + F(x),$$

where

$$\|F\|_{L_1(\mathbb{R}_+)} \le C\|f'\|_{L_1(\mathbb{R}_+)} + \|Tf'\|_{L_1(\mathbb{R}_+)}.$$

Corollary 12.9. *Let a function f be odd and satisfy the assumptions of Theorem 12.8. Then $f \in W_0(\mathbb{R})$ if and only if*

$$\int_0^\infty \frac{|f(t)|}{t}\,dt < \infty.$$

These results remain to be true for important subspaces of the considered in Theorem 12.8 space, for instance, for the class $V^*(\mathbb{R}_+)$ (cf. Corollary 12.7). They are functions (Fourier transform) analogs of important sufficient sequence conditions for the integrability of trigonometric series (see, e.g., [136] and [42]) and can be found in [81] and in [49]. In fact, many of these subspaces have first appeared in [23]. For $1 < q < \infty$, set

$$\|g\|_{A_q} = \int_0^\infty \left(\frac{1}{u}\int_{u\le|t|\le 2u} |g(t)|^q dt\right)^{1/q} du.$$

The case $q = \infty$ (for the derivative) corresponds to $V^*(\mathbb{R}_+)$. In other words, belonging of g to one of the spaces A_q ensures the integrability of Tg. However, we can cast our eyes on this from another point of view. Indeed, routine calculations show that Tg is the Hilbert transform of the odd extension of g plus integrable value. If g is taken to be of compact support, a classical Zygmund $L\log L$ condition (see, e.g., [172]) ensures the integrability of the Hilbert transform. More precisely, the condition is the integrability of $g\log^+|g|$, where the $\log^+|g|$ notation means $\log|g|$ when $|g| > 1$ and 0 otherwise. As E.M. Stein has shown in [128], this condition is necessary on the intervals where the function is positive.

Let us now go on to the multidimensional case. We need additional notation, different from that in [81] and better, in our opinion. Let us denote by $T_i g(x)$ the T-transform of a function g of multivariate argument with respect to the i-th (single) variable:

$$T_i g(x) = \int_{x_i/2}^{3x_i/2} \frac{g(x)}{x_i - t}\,dt = \int_0^{x_i/2} \frac{g(x - te_i) - g(x + te_i)}{t}\,dt.$$

Analyzing the proof of Theorem 8 in [81], one can see that this theorem can be written in the following asymptotic form.

Theorem 12.10. *Let f be defined on $\mathbb{R}^n_+$; let all partial derivatives $D^\chi f(x)$, $\mathbf{0} \le \chi < \mathbf{1}$, be locally absolutely continuous with respect to any other variable, and f and all such $D^c hif(x)$ vanish at infinity as $x_1 + \cdots + x_n \to \infty$. Then for each $x_1, \ldots, x_n > 0$ and for any set of numbers $\{a_i : a_i = 0$ or $1\}$ we have*

$$\widehat{f}_a(x) = \int_{\mathbb{R}^n_+} f(u) \prod_{i=1}^n \cos(x_i u_j - \pi a_i/2)\,du_i$$

$$= (-1)^{n-1} f\Big(\frac{\pi}{2x_1}, \dots, \frac{\pi}{2x_n}\Big) \prod_{j=1}^{d} \frac{\sin(\pi a_i/2)}{x_i}$$

$$+ \sum_{0\le\chi\le 1} \prod_{i:\chi_i\ne 0} \frac{\sin(\pi a_i/2)}{x_i} \Psi_\chi(x),$$

where Ψ_χ are functions satisfying

$$\int\limits_{\mathbb{R}^n_+} |\Psi_\chi(x)|\,dx \le c \int\limits_{\mathbb{R}^n_+} \prod_{i:\chi_i\ne 0} \frac{\sin(\pi a_i/2)}{x_i} \Big| \prod_{\substack{k:\eta_k\ne 0,\\ 0\le\eta\le 1-\chi}} T_k D^{1-\chi} f(x)\Big|\,dx$$

for all possible η, $0 \le \eta \le 1-\chi$.

An earlier result in Theorem 12.3, b) and independently obtained results from [49] are interesting partial cases of Theorem 12.10.

Integrating the summands in Theorem 12.10, we obtain

Corollary 12.11. *Let f be as in Theorem 12.10. If all $a_j = 1$ and all the values of type* (**??**) *are finite, $\widehat{f}_a \in L_1(\mathbb{R}^n)$ if and only if*

$$\int\limits_{\mathbb{R}^n_+} |f(x)| \prod_{j=1}^{n} \frac{1}{x_j}\,dx < \infty.$$

12.2. **Summability of Fourier series.** Let us now consider a related problem of summability methods for Fourier series. More precisely, conditions for various types of summability will be given in terms of belonging of the function generating the summability method to W.

Thus, let $f \in L_1(\mathbb{T}^n)$, with $\mathbb{T} = [-\pi, \pi]$. If $f \in C(\mathbb{T}^n)$, we also suppose f to be 2π-periodic in each of the variables $x_1, x_2, \dots, x_n$). We write the trigonometric Fourier series of f as

$$f(x) \sim \sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{ikx}, \qquad \widehat{f}(k) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{T}^n} f(x) e^{-ikx} dx.$$

Kolmogorov's celebrated result [69] shows that this series may be divergent at each point.

The problem we are going to discuss reads as follows:

For which function $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ there holds

$$\lim_{\varepsilon\to+0} \sum_{k\in\mathbb{Z}^n} \varphi(\varepsilon k) \widehat{f}(k) e^{ikx} = f(x),$$

in norm or pointwise, for any function $f \in L_1(\mathbb{T}^n)$?

The reader can find a nice discussion of various aspects of this problem in, say, [130, Ch.I]. Finding the answer can be reduced to the boundedness of the norms

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of operators

$$\sup_{\varepsilon>0}\int_{\mathbb{T}^n}\left|\sum_{k\in\mathbb{Z}^n}\varphi(\varepsilon k)e^{ikx}\right|dx<\infty.$$

Many authors have studied this problem; a detailed survey is given in [84]. For compactly supported functions φ (polynomial summability methods) one of the most general conditions is the so-called Boas-Telyakovskii condition (see, e.g., [135] in dimension one; a general overview is given in [84]). This condition was a discrete prototype for the class described above via the T-transform. In particular, any function from V^* satisfies that condition.

Theorem 12.12. *Let a function $\varphi\colon \mathbb{R}^n\to\mathbb{C}$ be bounded and continuous a.e. For*

$$\Phi_\varepsilon(f)\sim\sum_{k\in\mathbb{Z}^n}\varphi(\varepsilon k)\widehat{f}(k)e^{ikx}$$

to converge to $f(x)$ as $\varepsilon\to 0$ for all $f\in L_1(\mathbb{T}^n)$ in the $L_1(\mathbb{T}^n)$ metric (or uniformly for all $f\in C(\mathbb{T}^n)$), it is necessary and sufficient that $\lim_{\varepsilon\to 0}\varphi(\varepsilon k)=1$ *for all $k\in\mathbb{Z}^n$ and φ, corrected by continuity at the points of discontinuity, belong to $W(\mathbb{R}^n)$.*

For this result, see [147] or [158].

It is said that x_0 is a *Lebesgue point* of $f\in L_1(\mathbb{T}^n)$ if

$$\lim_{h\to+0}\frac{1}{h^n}\int_{|u|\le h}|f(x_0+u)-f(x_0)|\,du=0.$$

By the Lebesgue theorem, almost all points of every integrable function f are Lebesgue points.

Theorem 12.13. *Let $\varphi\in W(\mathbb{R}^n)$. In order that*

$$\lim_{\varepsilon\to 0}\sum_{k\in\mathbb{Z}^n}\varphi(\varepsilon k)\widehat{f}(k)e^{ikx}=f(x)$$

for every $f\in L_1(\mathbb{T}^n)$ at any of its Lebesgue points, it is necessary and sufficient that $\varphi(0)=1$, and the measure μ in the representation of φ be absolutely continuous with respect to the Lebesgue measure, i.e., $d\mu(x)=g(x)\,dx$ with

$$\int_0^\infty t^{n-1}\operatorname*{ess\,sup}_{|x|\ge t}|g(x)|\,dt<\infty.$$

In this precise form the result is proved in [6] (see also [158, 8.1.3]).

Theorem 12.14. *The (Riesz type) means*

$$\sum_{|k|\le N}\left(1-\frac{|k|^\alpha}{N^\alpha}\right)^\beta\widehat{f}(k)e^{ikx}\to f(x)$$

as $N \to \infty$, on $L_1(\mathbb{T}^n)$ and on $C(\mathbb{T}^n)$ or at the Lebesgue points of $f \in L_1(\mathbb{T}^n)$, if and only if either $\alpha \neq 1$ and $\beta > \frac{n-1}{2}$ or $\alpha = 1$ and $\beta > 0$.

For $\alpha = 2$ and $\beta > 0$ these means are called the Bochner-Riesz means and were considered in the classical Bochner's paper [20]. For the general case, see, e.g., [158, 8.1, II].

Let us also consider a problem of the so-called strong summability introduced by Hardy and Littlewood (see [172] and [4]):

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left| f(t) - \sum_{k=-m}^{m} \widehat{f}(k) e^{ikt} \right| = 0.$$

Here the convergence is uniform on $\mathbb{T}$ for functions from $C(\mathbb{T})$ and in the $L_1(\mathbb{T})$ norm for $f \in L_1(\mathbb{T})$. Hardy and Littlewood showed that not always the convergence in the Lebesgue points takes place. However, J. Marcinkiewicz proved the almost everywhere convergence of these "strong means" for each $f \in L_1(\mathbb{T})$ (see [172, Vol.2, Ch.XIII, §8] or [4, Ch.VII, §8]). There are essential generalizations of strong summability on $\mathbb{T}^n$ in [44] and [77].

Let $\{\nu_m\}_{m=0}^{\infty}$ be a strictly increasing sequence of natural numbers. The next problem is which conditions provide

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left| f(t) - \sum_{k=-\nu_m}^{\nu_m} \widehat{f}(k) e^{ikt} \right| = 0 \tag{12.1}$$

for all $f \in C(\mathbb{T})$ and $t \in \mathbb{T}$. Studying this problem, R. Salem in 1955 proved that it suffices that the sequence $\{\nu_m\}_{m=0}^{\infty}$ be of power growth and necessary that a) in next theorem holds.

Theorem 12.15. *For (12.1) to hold,*

a) the condition $\log \nu_m = O(\sqrt{m})$ is necessary;

b) if the sequence ν_m is convex, then the condition $\log \nu_m = O(\sqrt{m})$ is sufficient.

Here b) was obtained independently by N.A. Zagorodnii and R.M. Trigub in 1979 (see, e.g., [158, 8.1, V]) and by L. Carleson (see [31] and also [96]).

Open is the next old problem: *For which sequences $\{\nu_m\}_{m=0}^{\infty}$ we have*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sum_{k=-\nu_m}^{\nu_m} \widehat{f}(k) e^{ikt} = f(t)$$

for every $f \in L_1(\mathbb{T})$ at all its Lebesgue points?

Theorems on Fourier multipliers in C and L_1 and their applications to problems of the theory of approximation of functions can be found in [28], [107], and [158].

Let us give one example.

Example 8. If $f \in C(\mathbb{T})$ and τ_n is a trigonometric polynomial specified by the conditions $\tau_n(x_k) = \frac{1}{2}(f(x_k)+f(x_{k+1}))$ for $|k| \le n$ and $x_k = \frac{2k\pi}{2n+1}$, then for certain absolute positive constants C_1 and C_2 there holds

$$C_1\omega(f;\frac{1}{n}) \le \|f(\cdot) - \tau_n(\cdot)\| \le C_2\omega(f;\frac{1}{n}),$$

where the norm is that in $C(\mathbb{T})$.

13. Tables

In this section we give two tables. The first one presents the Fourier transforms of various integrable functions, both in dimension one and in several dimensions. In the second one, some functions are given along with conditions on the involved parameters (if any) when these functions are positive definite.

13.1. **Functions with integrable Fourier transforms.** Note that the citation in the fourth column refers to one of the possible sources, not necessarily original.

The case $n = 1$.

N	**Function $f(\xi)$ in $W_0(\mathbb{R})$**	**Its Fourier transform $\widehat{f}(x)$**	**Reference**
1	$\frac{\sin\xi}{\xi}$	$\begin{cases} \pi, & \lvert x\rvert < 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	[59]
2	$\left(\frac{\sin\xi/2}{\xi/2}\right)^2$	$(1-\lvert x\rvert)_+$	[28]
3	$(1-\lvert\xi\rvert)_+$	$\frac{1}{2\pi}\left(\frac{\sin x/2}{x/2}\right)^2$	[28]
4	$(1-\xi^2)_+^\lambda$	$\frac{2^{\lambda-1}}{\pi}\Gamma(\lambda+1)\frac{J_{\lambda+\frac12}(\lvert x\rvert)}{\lvert x\rvert^{\lambda+\frac12}}$	[28]
5	$\left(\frac{\sin\xi/2}{\xi/2}\right)^4$	$\begin{cases} \frac23 - x^2 + \frac12\lvert x\rvert^3, & \lvert x\rvert \le 1 \\ \frac23\frac{(2-\lvert x\rvert)^3}{4}, & 1 < \lvert x\rvert \le 2 \\ 0, & \lvert x\rvert > 2 \end{cases}$	[28]
6	$\frac{1}{1+\xi^2}$	$\frac12 e^{-\lvert x\rvert}$	[28]
7	$e^{-\lvert\xi\rvert}$	$\frac{1}{\pi}\frac{1}{1+x^2}$	[28]
8	$e^{-\xi^2}$	$\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{4}}$	[28]
9	$\frac{1}{1+i\xi}$	$\begin{cases} e^-x, & x > 0 \\ 0, & x < 0 \end{cases}$	[28]
10	$\frac{1}{(\alpha-i\xi)^\lambda}$, $\mathrm{Re}\alpha > 0$, $\mathrm{Re}\lambda > 0$	$\begin{cases} \frac{1}{\Gamma(\lambda)}(-x)^{\lambda-1}e^{-\alpha x}, & x < 0 \\ 0, & x > 0 \end{cases}$	[38]
11	$\frac{1}{(\alpha+i\xi)^\lambda}$, $\mathrm{Re}\alpha > 0$, $\mathrm{Re}\lambda > 0$	$\begin{cases} \frac{1}{\Gamma(\lambda)}x^{\lambda-1}e^{\alpha x}, & x > 0 \\ 0, & x < 0 \end{cases}$	[38]
12	$\mathrm{Re}\frac{1}{(1-i\xi)^\alpha}$, $\alpha > 0$,	$\frac{\lvert x\rvert^{\alpha-1}e^{-\lvert x\rvert}}{2\Gamma(\alpha)}$	[28]
13	$\frac{(i\xi)^\alpha}{(1+i\xi)^\beta}$, $0 < \alpha < \beta < \infty$,		[46]
14	$\frac{\lvert\xi\rvert^\alpha}{e^{\lvert\xi\rvert}}$, $\alpha \ge 0$	$\frac{\Gamma(\alpha+1)}{\pi}\mathrm{Re}(1+ix)^{-1-\alpha}$	[28]
15	$\mathrm{Im}\frac{1}{(1-i\xi)^\alpha}$, $\alpha > 0$,	$i\frac{sign\, x\lvert x\rvert^{\alpha-1}e^{-\lvert x\rvert}}{2\Gamma(\alpha)}$	[28]
16	$sign\,\xi\lvert\xi\rvert^{\alpha-1}e^{-\lvert\xi\rvert}$, $\alpha > 0$	$\frac{\Gamma(1+\alpha)}{\pi i}\mathrm{Im}\frac{1}{(1-ix)^{\alpha+1}}$	[28]

17	$\frac{1}{(\alpha-\imath\xi)^{\mu}(\beta-i\xi)^{\lambda}}$, $\mathrm{Re}\alpha, \mathrm{Re}\beta, \mathrm{Re}(\mu+\lambda) > 0$,	$\begin{cases} \frac{e^{\alpha x}(-x)^{\mu+\lambda-1}}{\gamma(\mu+\lambda)} {}_1F_1(\lambda;\mu+\lambda;(\beta-\alpha)x), & x<0 \\ 0, & x>0 \end{cases}$	[38]
18	$\frac{1}{(\alpha+\imath\xi)^{\mu}(\beta+i\xi)^{\lambda}}$, $\mathrm{Re}\alpha, \mathrm{Re}\beta, \mathrm{Re}(\mu+\lambda) > 0$,	$\begin{cases} \frac{e^{-\alpha x}x^{\mu+\lambda-1}}{\gamma(\mu+\lambda)} {}_1F_1(\lambda;\mu+\lambda;(\alpha-\beta)x), & x>0 \\ 0, & x<0 \end{cases}$	[38]
19	$\frac{1}{1+\xi^2}\left(\frac{i-\xi}{i+\xi}\right)^n$ $n=1,3,5,\ldots$	$\begin{cases} \frac{(-1)^n x}{e^x} L^1_{n-1}(-2x), & x<0 \\ 0, & x>0 \end{cases}$ $L^{\alpha}_n(x)$ are the Laguerre polynomials	[38]
20	$\frac{\mathrm{ctg}(\pi\lambda+i\pi\xi)}{\lambda+i\xi}$ $-1<\mathrm{Re}\lambda<0$	$2e^{\lambda x} ln\|1-e^x\|$	[38]
21	$\frac{1}{(\lambda+i\xi)\sin(\pi\lambda+i\pi\xi)}$ $-1<\mathrm{Re}\lambda<0$	$2e^{\lambda x} ln\|1+e^x\|$	[38]
22	$H_n(\xi)=(-1)^n e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n}\left(e^{-\xi^2}\right)$ Hermite polynomials	$\frac{(-i)^n}{\sqrt{2\pi}} H_n(-x)$	[28]
23	$\frac{J_{n+\frac{1}{2}}(\xi)}{\sqrt{\xi}}$	$\begin{cases} i^n\sqrt{2\pi}P_n(-x), & \|x\|<1 \\ 0, & \|x\|>1 \end{cases}$ $P_n(x)$ are Legendre polynomials	[38]
24	$\frac{J_{\lambda+\frac{1}{2}}(\|\xi\|)}{\|\xi\|^{\lambda+\frac{1}{2}}}$ $\lambda>0$	$\frac{2^{-\lambda-\frac{1}{2}}}{\sqrt{\pi}\Gamma(1+\lambda)}(1-x^2)^{\lambda}_{+}$	[28]
25	$\begin{cases} (1-\xi^2)P^{(\lambda,\lambda)}_n(\xi), & \|\xi\|<1 \\ 0, & \|\xi\|\geq 1 \end{cases}$	$\frac{(-1)^n 2^{\lambda-\frac{1}{2}}}{n!\sqrt{\pi}} \frac{J_{n+\lambda+\frac{1}{2}}(x)}{x^{\lambda+\frac{1}{2}}}$ $P^{(\lambda,\lambda)}_n(x)$ are Jacobi polynomials	[38]
26	$\xi^n e^{i\xi} {}_1F_1(a;b;2i\xi)$ $\mathrm{Re}a>0,\ \mathrm{Re}(b-a)>0$	$\begin{cases} \frac{(-i)^n 2^{\mu}(1+x)^{-\lambda}}{B(a,b-a)(1-x)^{a+\mu}} P^{(-\lambda,b+\lambda)}_n(-x), & \|x\|<1 \\ 0, & \|x\|>1 \end{cases}$ $\lambda=n+1-a,\quad \mu=n+1-b$	[38]
27	$\frac{A}{\left[1+\ln\ln\left(1+\|\xi\|^{\lambda}\right)\right]^{\alpha}} + \frac{B\, sign\xi\mu(\|\xi\|)}{[1+\ln(1+\|\xi\|)]^{\beta}}$ $\alpha>0, \beta>1, \lambda>0$ $\mu\in C^{\infty}_0(\mathbb{R}_+)$, $\mu(r)=\begin{cases} 0,\ 0<r<1 \\ 1,\ r>2 \end{cases}$	see Theorem 5.7	[141] [144]

The case $n \geq 1$.

N	**A function $f(\xi)$ in $W_0(\mathbb{R}^n)$**	**Its Fourier transform $\widetilde{f}(x)$**	**Reference**
1	$e^{-\lvert\xi\rvert^2}$	$\pi^{-\frac{n}{2}} e^{-\frac{\lvert x\rvert^2}{2}}$	[132]
2	$e^{-\lvert\xi\rvert}$	$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+\lvert x\rvert^2)^{\frac{n+1}{2}}}$	[132]
3	$\frac{1}{(1+\lvert\xi\rvert^2)^{\frac{\alpha}{2}}},\ \alpha > 0$	$\frac{2^{\frac{2-n-\alpha}{2}}}{\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)} \frac{K_{\frac{n-\alpha}{2}}(\lvert x\rvert)}{\lvert x\rvert^{\frac{n-\alpha}{2}}}$	[132]
4	$\frac{K_\lambda(\lvert\xi\rvert)}{\lvert\xi\rvert^\lambda},\ \lambda < \frac{n}{2}$	$\frac{\Gamma\left(\frac{n}{2}-\lambda\right)}{2^{\lambda+1}\pi^{\frac{n}{2}}} \frac{1}{(1+\lvert x\rvert^2)^{\frac{n}{2}-\lambda}}$	[132]
5	$\frac{\lvert\xi\rvert^\alpha}{(1+\lvert\xi\rvert^2)^{\frac{\alpha}{2}}} - 1,\ \alpha > 0$		[116], p. 21
6	$\frac{\lvert\xi\rvert^{\alpha-\sigma}}{(1+\lvert\xi\rvert^2)^{\frac{\alpha}{2}}},\ 0 < \sigma \leq \alpha$	see Theorem 6.8	[45], [118]
7	$\frac{\lvert\xi\rvert^{\alpha-\sigma}}{(1+\lvert\xi\rvert)^{\alpha}},\ 0 < \sigma \leq \alpha$	see Theorem 6.8	[41],[45], [114], [118]
8	$\frac{1}{1+\ln(1+\lvert\xi\rvert^2)}$	see Theorem 8.3	[141]
9	$\frac{1}{[1+\ln\ln(1+\lvert\xi\rvert^\lambda)]^\alpha},\ \lambda > 0, \alpha > 0$	see Theorem 8.3	[141]
10	$\frac{\chi(\lvert\xi\rvert)}{\ln\lvert\xi\rvert}$ $\chi \in C_0^\infty(\mathbb{R}_+)$, $\begin{cases} \chi(r) \equiv 1, & 0 < r < \frac{1}{e} \\ \chi(r) \equiv 0, & r > \frac{2}{e} \end{cases}$	see Theorem 8.3	[141]
11	$\frac{1}{\left[1+\ln\left(1+\sum_{k=1}^n \lvert\xi_k\rvert^{\lambda_k}\right)\right]^n}$, $\lambda_k > 0, 1 \leq k \leq n$	see Theorem 6.4	[141]

13.2. **Positive definite functions.** Item 9 is due to I.J. Schoenberg; 6 is due to B.I. Golubov; 5 and 8 are due to V.P. Zastavnyi, the other items are due to R.M. Trigub.

N	**Function**	**Belongs to**	**Condition**
1	$(1-\lvert x\rvert)_+(1-\alpha x)$	$W_0^+(\mathbb{R})$	iff $\alpha \in [0,3]$
2	$(1-\lvert x\rvert)_+^2 + b(1-\lvert x\rvert)_+$	$W_0^+(\mathbb{R})$	iff $b \geq -\frac{2}{3}$
3	$(1-\lvert x\rvert)_+^2 + (1-\alpha x)$	$W_0^+(\mathbb{R})$	iff $\alpha \in [-\frac{1}{2}, 2]$
4	a_m before Theorem 11.14	$W_0^+(\mathbb{R})$	
5	$e^{-\alpha\lvert x\rvert}(1-\beta\lvert x\rvert)$	$W^+(\mathbb{R})$	iff $\lvert\beta\rvert \leq \alpha$
6	$(1-\lvert x\rvert)_+^\alpha$	$W_0^+(\mathbb{R}^n)$	iff $\alpha \geq \frac{n+1}{2}$
7	A_m before Theorem 11.14	$W_0^+(\mathbb{R})$	
8	$\mathrm{Re}\, e^{-z\lvert x\rvert^\gamma}$, $z \in \mathbb{C}$, $\gamma \in (0,1]$	$W_0^+(\mathbb{R}^n)$	iff $\lvert\arg z\rvert \leq \frac{\pi\gamma}{2n}$
9	$e^{-\lvert x\rvert^\alpha}$	$W^+(\ell_2)$	iff $\alpha \in (0,2]$
10	$\frac{1}{1+\lvert x\rvert^\alpha}$	$W^+(\ell_2)$	iff $\alpha \in (0,2]$
11	$\frac{1}{(\lvert x\rvert^2+a)^2+b^2}$	$W^+(\ell_2)$	iff $ib \in \mathbb{R}$, $a > \lvert b\rvert$

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