DYNAMICAL SYSTEMS OF TYPE (m,n) AND THEIR C*-ALGEBRAS

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Given positive integers n and m, we consider dynamical systems in which n copies of a topological space is homeomorphic to m copies of that same space. The universal such system is shown to arise naturally from the study of a C*-algebra we denote by $\mathcal{O}_{m,n}$, which in turn is obtained as a quotient of the well known Leavitt C*-algebra $L_{m,n}$, a process meant to transform the generating set of partial isometries of $L_{m,n}$ into a tame set. Describing $\mathcal{O}_{m,n}$ as the crossed-product of the universal (m,n)–dynamical system by a partial action of the free group $\mathbb{F}_{m+n}$, we show that $\mathcal{O}_{m,n}$ is not exact when n and m are both greater than or equal to 2, but the corresponding reduced crossed-product, denoted $\mathcal{O}^r_{m,n}$, is shown to be exact and non-nuclear. Still under the assumption that $m,n\geq 2$, we prove that the partial action of $\mathbb{F}_{m+n}$ is topologically free and that $\mathcal{O}^r_{m,n}$ satisfies property (SP) (small projections). We also show that $\mathcal{O}^r_{m,n}$ admits no finite dimensional representations. The techniques developed to treat this system include several new results pertaining to the theory of Fell bundles over discrete groups.

1. Introduction.

The well known one-sided shift on n symbols is a dynamical system in which the configuration space is homeomorphic to n copies of itself. In this paper we study systems in which n copies of a topological space Y is homeomorphic to m copies of it.

Precisely, this means that one is given a pair (X,Y) of compact Hausdorff topological spaces such that

$$X=\bigcup_{i=1}^{n} H_i=\bigcup_{j=1}^{m} V_j,$$

where the H_i are pairwise disjoint clopen subsets of X, each of which is homeomorphic to Y via given homeomorphisms $h_i : Y \to H_i$, and the V_i are pairwise disjoint clopen subsets of X, each of which is homeomorphic to Y via given homeomorphisms $v_i : Y \to V_i$.

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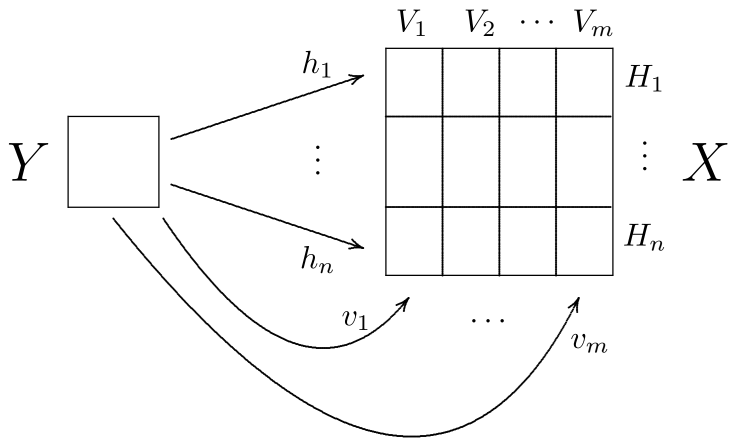


Diagram (1.1)

To the quadruple $\big(X, Y, \{h_i\}_{i=1}^n, \{v_j\}_{j=1}^m\big)$ we give the name of an (m,n)–dynamical system. When n or m are 1, this essentially reduces to the shift, but when $m, n \geq 2$, a very different behavior takes place.

The origin of the ideas developed in the present paper can be traced back to the seminal work of Cuntz and Krieger [**CK**], where a dynamical interpretation of the Cuntz-Krieger C*-algebras is given. In particular, the Cuntz algebra $\mathcal{O}_n$ corresponds to the full shift on n symbols. Since we are using an "external" model for this dynamical system, the C*-algebra $\mathcal{O}_{1,n}$ that we attach to the $(1,n)$-dynamical system is isomorphic to the algebra $M_2(\mathcal{O}_n)$.

From a purely algebraic perspective, a motivation to study such systems comes from the study of certain rings constructed by Leavitt [**L**] with the specific goal of having the free module of rank n be isomorphic to the free module of rank m. We refer the reader to [**AA**], [**AMP**], [**AG1**], [**H**] for various interpretations and generalizations of the algebras constructed by Leavitt to the setting of graph algebras.

A similar idea lies behind the investigations conducted by Brown [**B**] and McClanahan [**M1**], [**M2**], [**M3**], on the C*-algebras $U^{\mathrm{nc}}_{m,n}$. These are the C*-algebras generated by the entries of a universal unitary matrix of size $m \times n$. It has been observed in [**AG2**] that there are isomorphisms

$$L_{m,n} \cong M_{m+1}(U^{\mathrm{nc}}_{m,n}) \cong M_{n+1}(U^{\mathrm{nc}}_{m,n}),$$

where $L_{m,n}$ is the universal C*-algebra generated by partial isometries

$$s_1, \dots, s_n,\ t_1, \dots, t_m,$$

sharing the same source projection, and such that the sum of the range projections of the s_i, as well as that of the t_j, add up to the complement of the common source projection. Incidentally $L_{m,n}$ may also be constructed as a separated graph C*-algebra [**AG2**].

The partial isometries generating this algebra have a somewhat stubborn algebraic behavior, not least because their final projections fail to commute. Sidestepping this very delicate issue we choose to mod out all of the nontrivial commutators and, after performing this perhaps rather drastic transformation, we are left with a C*-algebra which we denote by $\mathcal{O}_{m,n}$, and which is consequently generated by a *tame* (see definition (2.2) below) set of partial isometries.

We then take advantage of the existing literature on C*-algebras generated by tame sets of partial isometries [**ELQ, E3, EL**] to describe $\mathcal{O}_{m,n}$ as the crossed product associated to a partial action θ^u of the free group $\mathbb{F}_{m+n}$ on a compact space Ω^u. In symbols

$$\mathcal{O}_{m,n} \simeq C(\Omega^u) \rtimes_{\theta^u} \mathbb{F}_{m+n}.$$

It is perhaps no coincidence that the above partial action of $\mathbb{F}_{m+n}$ is given by an (m, n)–dynamical system, as defined above, which is in fact the universal one (3.8).

While our description of the universal (m, n)–dynamical system Ω^u as a subset of the power set of $\mathbb{F}_{m+n}$ is satisfactory for some purposes, its tree-like structure may not make it easy to be studied from some points of view. We therefore present an alternative version of it in terms of functions defined on a certain space of finite paths (4.1). With this description at hand we are able to show that the partial action of $\mathbb{F}_{m+n}$ on Ω^u is topologically free (4.6). When $3 \leq m+n$ we show that every nonzero hereditary subalgebra of $\mathcal{O}^r_{m,n}$ contains a nonzero projection belonging to $C(\Omega^u)$.

We then initiate a systematic study of $\mathcal{O}_{m,n}$, begining with the fundamental questions of nuclearity and exactness (see [**BO**] for an extensive study of these important properties of C*-algebras).

When either $n = 1$, or $m = 1$, these algebras are Morita–Rieffel equivalent to Cuntz algebras, so we concentrate on the case in which n and m are greater than or equal to 2. Under this condition we prove that $\mathcal{O}_{m,n}$ is not nuclear, and not even exact (7.2). However, when we pass to its *reduced* version, namely the reduced crossed product [**M4**]

$$\mathcal{O}^r_{m,n} = C(\Omega^u) \rtimes^r_{\theta^u} \mathbb{F}_{m+n}$$

we find that $\mathcal{O}^r_{m,n}$ is exact, although still not nuclear.

Since the crossed product by a partial action may be defined as the cross-sectional C*-algebra of the semidirect product Fell bundle, we dedicate a significant amount of attention to these and in fact many of our statements about $\mathcal{O}_{m,n}$ or $\mathcal{O}^r_{m,n}$ come straight from corresponding results we prove for general Fell bundles.

If $\mathscr{B}$ is a Fell bundle over a discrete exact group whose unit fiber is an exact C*-algebra, we prove in (5.2) that the reduced cross-sectional C*-algebra $C^*_r(\mathscr{B})$ is exact. From this it follows that the reduced crossed product of an exact C*-algebra by a partial action of an exact group is exact, and hence that $\mathcal{O}^r_{m,n}$ is exact.

Being $\mathcal{O}_{m,n}$ a *full* crossed product, we are led to study full cross-sectional C*-algebras of Fell bundles. The well known fact [**BO**: 10.2.8] that the maximal tensor product of the reduced group C*-algebra by itself contains the full group C*-algebra is generalized in (6.2), where we prove that if $\mathscr{B}$ is a Fell bundle over the group G, then the full cross-sectional C*-algebra $C^*(\mathscr{B})$ is a subalgebra of $C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(G)$. As an immediate consequence we deduce that, if $C_r^*(\mathscr{B})$ is nuclear, then the full and reduced cross-sectional C*-algebras of $\mathscr{B}$ agree (6.4).

As another Corollary of (6.2) we prove that, if H is a subgroup of G, then the full cross-sectional C*-algebra of the bundle restricted to H embeds in the C*-algebra of the whole bundle (6.3). This result turns out to be crucial in our proof that, in a partial action, every residually finite-dimensional isotropy group is amenable when the full cross-sectional algebra is exact (7.1).

When $m, n \geq 2$, we show that there are non-amenable (7.2) isotropy groups in the universal (m, n)–dynamical system, so exactness of $\mathcal{O}_{m,n}$ is ruled out by (7.1).

We also consider the question of existence of finite dimensional representations of $\mathcal{O}_{m,n}$ and of $\mathcal{O}^r_{m,n}$. A trivial argument (8.1) proves that, when $n \neq m$, neither $\mathcal{O}_{m,n}$ nor $\mathcal{O}^r_{m,n}$ admit finite dimensional representations.

The case $m = n$ is however a lot more subtle. While it is easy to produce many finite dimensional representations of $\mathcal{O}_{m,n}$, we have not been able to decide whether or not there are enough of these to separate points. In other words we have not been able to decide whether $\mathcal{O}_{m,n}$ is residually finite.

With respect to $\mathcal{O}^r_{m,n}$, we settle the question in (9.5), proving that $\mathcal{O}^r_{m,n}$ admits no finite dimensional representation for all $m, n \geq 2$.

2. The Leavitt C*-algebra.

Throughout this paper we fix positive integers n and m, with $m \leq n$.

2.1. Definition. The *Leavitt C*-algebra of type* (m, n) is the universal unital C*-algebra $L_{m,n}$ generated by partial isometries $s_1, \ldots, s_n,\ t_1, \ldots, t_m$ satisfying the relations

$$\left.\begin{aligned} & s_i^* s_{i'} = 0, \ \text{for } i \neq i', \\ & t_j^* t_{j'} = 0, \ \text{for } j \neq j', \\ & s_i^* s_i = t_j^* t_j =: w, \\ & \sum_{i=1}^n s_i s_i^* = \sum_{j=1}^m t_j t_j^* =: v, \\ & vw = 0, \quad v + w = 1. \end{aligned}\right\} (\mathcal{R})$$

By choosing a specific representation, it is not difficult to see that $s_1 s_1^*$ does not commute with $t_1 t_1^*$ when $m, n \geq 2$, and hence that $s_1^* t_1$ is not a partial isometry (see e.g. [**E3**: 5.3]). This is in contrast with many well known examples of C*-algebras generated by sets of partial isometries which are almost always *tame* according to the following:

2.2. Definition. A set U of partial isometries in a C*-algebra is said to be *tame* if every element of $\langle U \cup U^* \rangle$ (meaning the multiplicative semigroup generated by $U \cup U^*$) is a partial isometry.

See [**E3**: 5.4] for equivalent conditions characterizing tame sets of partial isometries.

The standard partial isometries generating the Cuntz–Krieger algebras form a tame set [**E2**: 5.2], as do the corresponding ones for graph C*-algebras, higher rank graph C*-algebras and many others.

Rather than attempt to face the wild set of partial isometries in $L_{m,n}$ (incidentally a task not everyone shies away from [**AG2**]), we will force it to become tame by considering a quotient of $L_{m,n}$. In what follows we will denote by U the subset of partial isometries in $L_{m,n}$ that is most relevant to us, namely

$$U = \{s_1, \ldots, s_n,\ t_1, \ldots, t_m\}.$$

2.3. Definition. We will let $\mathcal{O}_{m,n}$ be the quotient of $L_{m,n}$ by the closed two-sided ideal generated by all elements of the form

$$xx^*x - x,$$

as x runs in $\langle U \cup U^* \rangle$. We will denote the images of the s_i and the t_j in $\mathcal{O}_{m,n}$ by $\underline{s}_i$ and $\underline{t}_j$, respectively.

It is therefore evident that

$$\{\underline{s}_1, \ldots, \underline{s}_n,\ \underline{t}_1, \ldots, \underline{t}_m\}$$

is a tame set of partial isometries. In fact it is not hard to prove that $\mathcal{O}_{m,n}$ is the universal unital C*-algebra generated by a tame set of partial isometries satisfying relations $(\mathcal{R})$.

Let $\mathbb{F}_{m+n}$ denote the free group generated by a set with $m+n$ elements, say

$$\{a_1, \ldots, a_n,\ b_1, \ldots, b_m\}.$$

Using [**E3**: 5.4] we conclude that there exists a (necessarily unique) semi-saturated [**E3**: 5.3] partial representation

$$\sigma : \mathbb{F}_{m+n} \to \mathcal{O}_{m,n}$$

such that $\sigma(a_i) = \underline{s}_i$, and $\sigma(b_j) = \underline{t}_j$ (when stating conditions such as these, which are supposed to hold for every $i = 1, \ldots, n$, and every $j = 1, \ldots, m$, we will omit making explicit reference of the range of variation of i and j, which should always be understood as being 1–n, and 1–m, as above).

Another universal property enjoyed by $\mathcal{O}_{m,n}$ is described next.

2.4. Proposition. *Let ρ be a semi-saturated partial representation of $\mathbb{F}_{m+n}$ in a unital C*-algebra B such that the elements $s'_i := \rho(a_i)$ and $t'_j := \rho(b_j)$ satisfy relations $(\mathcal{R})$. Then there exists a unique unital *-homomorphism $\varphi : \mathcal{O}_{m,n} \to B$ such that $\rho = \varphi \circ \sigma$.*

Proof. Since ρ is a partial representation, one has that the s'_i and the t'_j are partial isometries. By universality of $L_{m,n}$ one concludes that there exists a unital *-homomorphism $\psi : L_{m,n} \to B$, such that $\psi(s_i) = s'_i$, and $\psi(t_j) = t'_j$.

Observe that if x is in $\langle U \cup U^* \rangle$, then $\psi(x)$ lies in the multiplicative semigroup generated by the s'_i, the t'_j, and their adjoints. Employing [**E3**: 5.4] we have that $\psi(x)$ is a partial isometry and hence that $\psi(xx^*x-x)=0$. This implies that ψ vanishes on the ideal referred to in (2.3) and hence that it factors through $\mathcal{O}_{m,n}$ providing a *-homomorphism $\varphi:\mathcal{O}_{m,n}\to B$, such that $\varphi(\underline{s}_i)=s'_i$, and $\varphi(\underline{t}_j)=t'_j$. Therefore

$$\varphi(\sigma(a_i))=\varphi(\underline{s}_i)=s'_i=\rho(a_i),$$

and similarly $\varphi(\sigma(b_j))=\rho(b_j)$. In other words, $\varphi\circ\sigma$ coincides with ρ on the generators of $\mathbb{F}_{m+n}$. Since both σ and ρ are semi-saturated, we now conclude that $\varphi\circ\sigma=\rho$ on the whole of $\mathbb{F}_{m+n}$. □

So $\mathcal{O}_{m,n}$ is the universal unital C*-algebra for partial representations of $\mathbb{F}_{m+n}$ subject to the relations $(\mathcal{R})$, according to [**ELQ**: 4.3], and hence we may apply [**ELQ**: 4.4] to deduce that there exists a certain partial dynamical system $(\Omega^u,\mathbb{F}_{m+n},\theta^u)$ and a *-isomorphism

$$\Psi:\mathcal{O}_{m,n}\to C(\Omega^u)\rtimes_{\theta^u}\mathbb{F}_{m+n}. \tag{2.5}$$

The choice of notation, specifically the use of the superscript "u", is motivated by *universal* properties to be described below. Before giving further details on the above result let us introduce a variation of $\mathcal{O}_{m,n}$.

2.6. Definition. For every pair of positive integers (m,n) we shall let $\mathcal{O}^r_{m,n}$ denote the corresponding *reduced* crossed product

$$\mathcal{O}^r_{m,n}=C(\Omega^u)\rtimes^r_{\theta^u}\mathbb{F}_{m+n}.$$

For the convenience of the reader we will now give a brief description of Ω^u and of the partial action θ^u. We refer the reader to [**ELQ**: Section 4] for further details.

The first step is to write the relations defining our algebra in terms of the *final projections*

$$e(g):=\sigma(g)\sigma(g^{-1}),$$

for $g\in\mathbb{F}_{m+n}$. Once this is done we arrive at

$$\left.\begin{aligned}
&e(a_i)e(a_{i'})=0,\ \text{for } i\neq i',\\
&e(b_j)e(b_{j'})=0,\ \text{for } j\neq j',\\
&e(a_i^{-1})=e(b_j^{-1})=:w,\\
&\sum_{i=1}^n e(a_i)=\sum_{j=1}^m e(b_j)=:v,\\
&vw=0,\quad v+w=1.
\end{aligned}\right\}(\mathcal{R}')$$

Observe that, since the $e(g)$ are projections, all of the above relations expressing orthogonality, that is, those having a zero as the right-hand-side, follow from "$v+w=1$".

If we are to apply the theory of [**ELQ**: Section 4] to our algebra, we need to add another relation to $(\mathcal{R}')$ in order to account for the fact that the partial representations involved in (2.4) are required to be semi-saturated. Although the definition of semi-saturatedness, namely

$$|hk| = |h| + |k| \ \Rightarrow\ \sigma(hk) = \sigma(h)\sigma(k),$$

is not expressed in terms of the $e(g)$, we may use [**E2**: 5.4] to replace it with the equivalent form

$$|hk| = |h| + |k| \ \Rightarrow\ e(hk) \le e(h).$$

The next step is to translate each of the above relations in terms of equations on $\{0,1\}^{\mathbb{F}_{m+n}}$. For this we will find it convenient to identify this product space with the power set $\mathscr{P}(\mathbb{F}_{m+n})$ in the usual way.

According to [**ELQ**: Section 4] and [**EL**: Section 2] the translation process consists in replacing each occurrence of a final projection $e(g)$ in the above relations with the scalar valued function 1_g defined by

$$1_g : \xi \in \{0,1\}^{\mathbb{F}_{m+n}} \mapsto [g \in \xi].$$

Here we use brackets to denote Boolean value and we see the truth values "1" and "0" as complex numbers. Therefore 1_g is nothing but the characteristic function of the set

$$\big\{\xi \in \{0,1\}^{\mathbb{F}_{m+n}} : g \in \xi\big\}.$$

The description of Ω^u given in [**ELQ**: 4.1] therefore becomes: a necessary and sufficient condition for a given $\xi \in \{0,1\}^{\mathbb{F}_{m+n}}$ to belong to Ω^u is that $1 \in \xi$, and that

$$\left.\begin{array}{l}
(1_{hk}1_h - 1_{hk})\,(g^{-1}\xi) = 0, \ \text{whenever } |hk| = |h| + |k|,\\
\big(1_{a_i}1_{a_{i'}}\big)\,(g^{-1}\xi) = 0, \ \text{for } i \ne i',\\
\big(1_{b_j}1_{b_{j'}}\big)(g^{-1}\xi) = 0, \ \text{for } j \ne j',\\
1_{a_i^{-1}}(g^{-1}\xi) = 1_{b_j^{-1}}(g^{-1}\xi) =: w(g^{-1}\xi),\\
\displaystyle\sum_{i=1}^{n} 1_{a_i}(g^{-1}\xi) = \sum_{j=1}^{m} 1_{b_j}(g^{-1}\xi) =: v(g^{-1}\xi),\\
(vw)\,(g^{-1}\xi) = 0, \quad (v+w)\,(g^{-1}\xi) = 1,
\end{array}\right\}\ (\mathcal{R}'')$$

for every $g \in \xi$.

For example, to account for the second equation above, it is required that

$$0 = (1_{a_i}1_{a_{i'}})(g^{-1}\xi) = 1_{a_i}(g^{-1}\xi)\ 1_{a_{i'}}(g^{-1}\xi) = \big[a_i \in g^{-1}\xi\big]\big[a_{i'} \in g^{-1}\xi\big] =$$

$$= \big[ga_i \in \xi\big]\big[ga_{i'} \in \xi\big] = \big[ga_i \in \xi \ \wedge\ ga_{i'} \in \xi\big].$$

This may be interpreted as saying that for every $g \in \xi$, not more than one element of the form ga_i belongs to ξ.

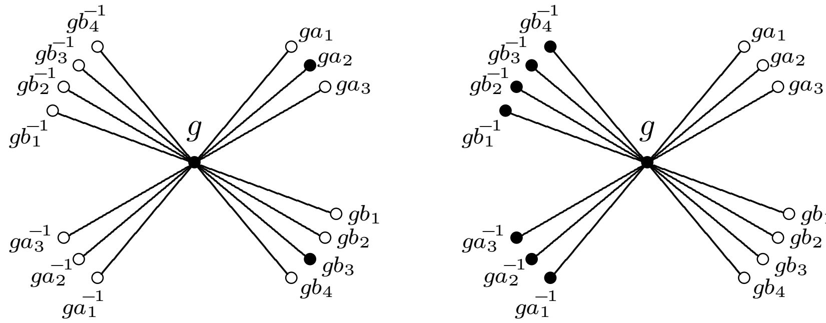
As another example, recall from that [**EL**: 4.5] that, in order for ξ to satisfy the conditions related to the first equation in $(\mathcal{R}'')$, it is required that ξ be convex [**EL**: 4.4].

The reader may now check that the elements of Ω^u are precisely those $\xi \subseteq \mathbb{F}_{m+n}$ such that

(a) $1 \in \xi$,

(b) ξ is convex,

(c) for any $g \in \xi$, one and only one of the conditions below are satisfied:



Pattern (c_1) Pattern (c_2)

(c_1) there exists a unique $i \leq n$ and a unique $j \leq m$, such that ga_i and gb_j lie in ξ, and for every i and j, none of ga_i^{-1} or gb_j^{-1} lie in ξ,

(c_2) for every i and j, none of ga_i or gb_j lie in ξ, and for every i and j, all of ga_i^{-1} and gb_j^{-1} lie in ξ.

Having completed the description of Ω^u, the partial action of $\mathbb{F}_{m+n}$ is now easy to describe: for each $g \in \mathbb{F}_{m+n}$ we put

$$\Omega_g^u = \{\xi \in \Omega^u : g \in \xi\},$$

and we let

$$\theta_g^u : \Omega_{g^{-1}}^u \to \Omega_g^u,$$

be given by $\theta_g^u(\xi) = g\xi = \{gh : h \in \xi\}$.

In possession of the proper notation we may now also describe the isomorphism Ψ mentioned in (2.5). It is characterized by the fact that

$$\Psi(\sigma(g)) = 1_g^u \delta_g, \quad \forall\, g \in \mathbb{F}_{m+n}, \tag{2.7}$$

where 1_g^u refers to the characteristic function of the clopen set $\Omega_g^u \subseteq \Omega^u$.

In what follows we will concentrate ourselves in studying the above partial action of $\mathbb{F}_{m+n}$ as well as the structure of $\mathcal{O}_{m,n}$ based on its crossed product description.

3. Dynamical systems of type (m, n).

In this section we will study pairs of compact Hausdorff topological spaces (X, Y) such that

$$X = \bigcup_{i=1}^{n} H_i = \bigcup_{j=1}^{m} V_j,$$

where the H_i are pairwise disjoint clopen subsets of X, each of which is homeomorphic to Y via given homeomorphisms $h_i : Y \to H_i$. Likewise we will assume that the V_i are pairwise disjoint clopen subsets of X, each of which is homeomorphic to Y via given homeomorphisms $v_i : Y \to V_i$. See diagram (1.1).

3.1. Definition. We will refer to the quadruple $\big(X, Y, \{h_i\}_{i=1}^{n}, \{v_j\}_{j=1}^{m}\big)$ as an (m, n)–*dynamical system*.

As an example, consider the situation in which Y^u is the subset of Ω^u consisting of all the ξ relative to which the configuration at $g = 1$ follows pattern (c_2). Equivalently

$$Y^u = \big\{\xi \in \Omega^u : a_i^{-1}, b_j^{-1} \in \xi, \text{ for all } i \text{ and } j\big\}.$$

Let X^u be the complement of Y^u relative to Ω^u, and put

$$h_i^u : \xi \in Y^u \mapsto a_i\xi \in X^u, \quad \text{and} \qquad v_j^u : \xi \in Y^u \mapsto b_j\xi \in X^u.$$

We leave it for the reader to verify that this provides an example of an (m, n)–dynamical system.

3.2. Definition. The system $\big(X^u, Y^u, \{h_i^u\}_{i=1}^{n}, \{v_j^u\}_{j=1}^{m}\big)$ described above will be referred to as the *standard* (m, n)*–dynamical system*.

It is our next immediate goal to prove that the standard (m, n)–dynamical system possesses a universal property. We thus fix, throughout, an arbitrary (m, n)–dynamical system

$$\big(X, Y, \{h_i\}_{i=1}^{n}, \{v_j\}_{j=1}^{m}\big).$$

Our goal will be to prove that there exists a unique map

$$\gamma : X \mathbin{\dot\cup} Y \to \Omega^u$$

such that $\gamma(Y) \subseteq Y^u$, $\gamma(X) \subseteq X^u$, $\gamma \circ h_i = h_i^u \circ \gamma$, and $\gamma \circ v_j = v_j^u \circ \gamma$.

We shall initially construct a partial action of $\mathbb{F}_{m+n}$ on the topological disjoint union

$$\Omega := X \mathbin{\dot\cup} Y.$$

For this consider the inverse semigroup $\mathcal{I}(\Omega)$ formed by all homeomorphisms between clopen subsets of Ω. Evidently the h_i and the v_j are elements of $\mathcal{I}(\Omega)$. Next consider the unique map

$$\theta : \mathbb{F}_{m+n} \to \mathcal{I}(\Omega)$$

such that

$$\theta(a_i^{\pm1}) = h_i^{\pm1}, \quad \theta(b_j^{\pm1}) = v_j^{\pm1},$$

and such that for each $g \in \mathbb{F}_{m+n}$, written in reduced form[1]

$$g = x_1 x_2 \dots x_p,$$

one has that

$$\theta(g) = \theta(x_1)\theta(x_2)\dots\theta(x_p).$$

3.3. Proposition. *θ is a partial action of $\mathbb{F}_{m+n}$ on Ω.*

Proof. It is not hard to prove this fact from scratch. Alternatively one may deduce it from known results as follows: using [**DP**: 1.1], one may faithfully represent $\mathcal{I}(\Omega)$ as an inverse semigroup of partial isometries on a Hilbert space. Applying [**E3**: 5.4] we then conclude that there exists a unique semi-saturated partial representation of $\mathbb{F}_{m+n}$ in $\mathcal{I}(\Omega)$, sending the a_i to h_i, and the b_j to v_j. Evidently this partial representation coincides with θ, and hence we conclude that θ is a partial representation. Therefore, for every $g, h \in \mathbb{F}_{m+n}$ one has that

$$\theta_g\theta_h = \theta_g\theta_h\theta_{h^{-1}}\theta_h = \theta_{gh}\theta_{h^{-1}}\theta_h = \theta_{gh}\theta_h^{-1}\theta_h \subseteq \theta_{gh},$$

meaning that θ_{gh} is an extension of $\theta_g\theta_h$, a property that characterizes partial actions. □

We may then form the crossed product $C(\Omega) \rtimes_\theta \mathbb{F}_{m+n}$. Given $g \in \mathbb{F}_{m+n}$, denote by Ω_g the range of θ_g. Since θ_g lies in $\mathcal{I}(\Omega)$, we have that its range is clopen. So the characteristic function of Ω_g, which we shall denote by 1_g, is a continuous function on Ω.

3.4. Proposition. *The map*

$$\rho : g \in \mathbb{F}_{m+n} \mapsto 1_g\delta_g \in C(\Omega) \rtimes_\theta \mathbb{F}_{m+n},$$

is a semi-saturated partial representation, and moreover the elements

$$s_i' := \rho(a_i), \quad \text{and} \quad t_j' := \rho(b_j)$$

satisfy relations $(\mathcal{R})$.

Proof. Given $g, h \in \mathbb{F}_{m+n}$ we have

$$\rho(g)\rho(h) = (1_g\delta_g)(1_h\delta_h) = \theta_g\big(\theta_{g^{-1}}(1_g)1_h\big)\delta_{gh} = \theta_g\big(1_{g^{-1}}1_h\big)\delta_{gh} = 1_g1_{gh}\delta_{gh}. \tag{3.4.1}$$

Therefore

$$\rho(g)\rho(h)\rho(h^{-1}) = (1_g1_{gh}\delta_{gh})(1_{h^{-1}}\delta_{h^{-1}}) = \theta_{gh}\big(\theta_{(gh)^{-1}}(1_g1_{gh})1_{h^{-1}}\big)\delta_g =$$

$$= \theta_{gh}\big(1_{h^{-1}}1_{(gh)^{-1}}1_{h^{-1}}\big)\delta_g = \theta_{gh}\big(1_{(gh)^{-1}}1_{h^{-1}}\big)\delta_g.$$

[1] That is, each x_k is either $a_i^{\pm1}$ or $b_j^{\pm1}$, and $x_{k+1} \neq x_k^{-1}$.

On the other hand

$$\rho(gh)\rho(h^{-1}) = (1_{gh}\delta_{gh})(1_{h^{-1}}\delta_{h^{-1}}) = \theta_{gh}\big(\theta_{(gh)^{-1}}(1_{gh})1_{h^{-1}}\big)\delta_g =$$

$$= \theta_{gh}\big(1_{(gh)^{-1}}1_{h^{-1}}\big)\delta_g,$$

which coincides with the above and hence proves that $\rho(g)\rho(h)\rho(h^{-1}) = \rho(gh)\rho(h^{-1})$. We leave it for the reader to prove that $\rho(g^{-1}) = \rho(g)^*$, after which the verification that ρ is a partial representation will be concluded.

Addressing semi-saturatedness, let $g, h \in \mathbb{F}_{m+n}$ be such that $|gh| = |g| + |h|$. This means that the reduced form of gh is precisely the concatenation of the reduced forms of g and h, and hence we see that $\theta_{gh} = \theta_g \circ \theta_h$. In particular this implies that these two partial homeomorphisms have the same range. Therefore

$$ran(\theta_g \circ \theta_h) = \theta_g(\Omega_{g^{-1}} \cap \Omega_h) = \Omega_g \cap \Omega_{gh}$$

coincides with the range of θ_{gh}, which is Ω_{gh}. Having concluded that $\Omega_g \cap \Omega_{gh} = \Omega_{gh}$, we deduce that

$$1_g 1_{gh} = 1_{gh}.$$

Employing (3.4.1) we then deduce that

$$\rho(g)\rho(h) = 1_{gh}\delta_{gh} = \rho(gh),$$

proving that ρ is semi-saturated.

Finally we leave it for the reader to prove that ${s'_i}^* s'_i$ and ${t'_j}^* t'_j$ coincide with the characteristic function of Y, that $s'_i {s'_i}^*$ is the characteristic function of H_i (the range of h_i) and that $t'_j {t'_j}^*$ is the characteristic function of V_j (the range of v_j). The checking of relations $(\mathcal{R})$ now becomes straightforward. □

We may of course apply the above result for the standard (m, n)–dynamical system (see (3.2)), and hence there is a semi-saturated partial representation

$$\rho^u : \mathbb{F}_{m+n} \to C(\Omega^u) \rtimes_{\theta^u} \mathbb{F}_{m+n}, \tag{3.5}$$

given by $\rho^u(g) = 1^u_g\delta_g$, for every g in $\mathbb{F}_{m+n}$. With this notation (2.7) simply says that $\Psi \circ \sigma = \rho^u$.

As another consequence of (3.4) and (2.4) we have that there exists a *-homomorphism

$$\Phi : \mathcal{O}_{m,n} \to C(\Omega) \rtimes_\theta \mathbb{F}_{m+n},$$

such that $\rho = \Phi \circ \sigma$.

Wrapping up our previous results we obtain the commutative diagram:

$$\begin{tikzcd} & C(\Omega^u) \rtimes \mathbb{F}_{m+n} \\ \mathbb{F}_{m+n} \arrow[ur, "\rho^u"] \arrow[r, "\sigma"] \arrow[dr, "\rho"'] & \mathcal{O}_{m,n} \arrow[u, "\Psi"'] \arrow[d, "\Phi"] \\ & C(\Omega) \rtimes_\theta \mathbb{F}_{m+n} \end{tikzcd}$$

Observing that the correspondence

$$f \in C(\Omega) \mapsto f\delta_1 \in C(\Omega) \rtimes_\theta \mathbb{F}_{m+n}$$

is an embedding, we will henceforth identify $C(\Omega)$ with its image within $C(\Omega) \rtimes_\theta \mathbb{F}_{m+n}$ without further notice, and similarly for $C(\Omega^u)$.

3.6. Proposition. *If Γ is defined as the composition $\Gamma := \Phi\Psi^{-1}$, then $\Gamma\big(C(\Omega^u)\big) \subseteq C(\Omega)$.*

Proof. Since we will be dealing with two different dynamical systems here we will insist in the convention (already used above) that 1_g denotes the characteristic function of Ω_g, reserving 1_g^u for the characteristic function of Ω_g^u. For each $g \in \mathbb{F}_{m+n}$ we have that

$$\rho^u(g)\rho^u(g^{-1}) = (1_g^u\delta_g)(1_{g^{-1}}^u\delta_{g^{-1}}) = 1_g^u\delta_1 = 1_g^u,$$

and similarly $\rho(g)\rho(g^{-1}) = 1_g$. Since $\Gamma \circ \rho^u = \rho$, we deduce that

$$\Gamma(1_g^u) = 1_g. \tag{3.6.1}$$

It is easy to see that the set $\{1_g^u : g \in \mathbb{F}_{m+n}\}$ separates points of Ω^u, and hence by the Stone-Weierstrass Theorem, the closed *-subalgebra it generates coincides with $C(\Omega^u)$. So the result follows from (3.6.1). □

As a consequence of the last result we see that there exists a unique continuous map

$$\gamma : \Omega \to \Omega^u, \tag{3.7}$$

such that $\Gamma(f) = f \circ \gamma$, for every $f \in C(\Omega^u)$.

3.8. Theorem. *The standard (m,n)–dynamical system is universal in the following sense: given any (m,n)–dynamical system*

$$\big(X, Y, \{h_i\}_{i=1}^n, \{v_j\}_{j=1}^m\big),$$

there exists a unique continuous map

$$\gamma : \Omega = X \dot\cup Y \to \Omega^u,$$

such that

(i) $\gamma(Y) \subseteq Y^u$,
(ii) $\gamma(X) \subseteq X^u$,
(iii) $\gamma \circ h_i = h_i^u \circ \gamma$,
(iv) $\gamma \circ v_j = v_j^u \circ \gamma$.

Proof. Regarding existence we will prove that the map γ constructed in (3.7) satisfies the above properties. Notice that $1_{a_1^{-1}}$ is the characteristic function of the domain of $\theta(a_1)$ $(= h_1)$, namely Y. Similarly $1^u_{a_1^{-1}}$ is the characteristic function of Y^u. Applying (3.6.1) to $g = a_1^{-1}$ we get $\Gamma(1_{Y^u}) = 1_Y$, or equivalently

$$1_{Y^u} \circ \gamma = 1_Y.$$

For $x \in \Omega$ this says that $x \in Y$ iff $\gamma(x) \in Y^u$, thus proving both (i) and (ii).

Given $g \in \mathbb{F}_{m+n}$, and $f \in C_0(\Omega^u_g)$, one may prove by direct computation that

$$\rho^u(g^{-1}) f \rho^u(g) = f \circ \theta^u_g,$$

and similarly for $f \in C_0(\Omega_g)$. So

$$f \circ \theta^u_g \circ \gamma = \Gamma\big(f \circ \theta^u_g\big) = \Gamma\big(\rho^u(g^{-1}) f \rho^u(g)\big) =$$

$$= \rho(g^{-1})\Gamma(f)\rho(g) = \Gamma(f) \circ \theta_g = f \circ \gamma \circ \theta_g.$$

Since f is arbitrary it follows that $\theta^u_g \circ \gamma = \gamma \circ \theta_g$. Point (iii) then follows by plugging $g = a_i$, while (iv) follows with $g = b_j$.

Addressing the uniqueness of γ, suppose one is given another map

$$\gamma' : \Omega \to \Omega^u$$

satisfying (i-iv). Then it is clear that γ' is covariant for the corresponding partial actions of $\mathbb{F}_{m+n}$ on Ω and Ω^u. Letting

$$\pi : f \in C(\Omega^u) \mapsto f \circ \gamma' \in C(\Omega),$$

one may easily prove that the pair (π, ρ) is a covariant representation of the partial dynamical system $(\Omega^u, \theta^u, \mathbb{F}_{m+n})$ in $C(\Omega) \rtimes_\theta \mathbb{F}_{m+n}$. Using [**ELQ**: 1.3] we conclude that there exits a *-homomorphism

$$\pi \times \rho : C(\Omega^u) \rtimes_{\theta^u} \mathbb{F}_{m+n} \to C(\Omega) \rtimes_\theta \mathbb{F}_{m+n},$$

such that $(\pi \times \rho)(f) = f \circ \gamma'$, for every $f \in C(\Omega^u)$, and such that

$$(\pi \times \rho) \circ \rho^u = \rho. \tag{3.8.1}$$

Since the range of σ generates $\mathcal{O}_{m,n}$, and since Ψ is an isomorphism, we deduce that the range of ρ^u generates $C(\Omega^u) \rtimes_{\theta^u} \mathbb{F}_{m+n}$. We then conclude from (3.8.1) that

$$(\pi \times \rho) \circ \rho^u = \Gamma \circ \rho^u,$$

and hence that $\pi \times \rho = \Gamma$, which in turn implies that $\gamma' = \gamma$. ∎

We shall next discuss the existence of fixed points in the universal (m,n)–dynamical system.

3.9. Proposition. *If $n \geq m \geq 2$, then there is a point y in Y^u such that*

$$(v_1^u)^{-1}h_1^u(y) = y = (v_2^u)^{-1}h_2^u(y).$$

Proof. In order to prove the statement it is enough to show that there exists some (m,n)–dynamical system $\big(X, Y, \{h_i\}_{i=1}^n, \{v_j\}_{j=1}^m\big)$, and a point $y \in Y$ such that

$$v_1^{-1}h_1(y) = y = v_2^{-1}h_2(y).$$

By (3.8), the image of y in Y^u under γ will clearly satisfy the required conditions.

We shall introduce another convenient variable by putting

$$p := n - m + 1.$$

Let $Y = \{1, 2, \ldots, p\}^{\mathbb{N}}$, with the product topology, and let X be given as the disjoint union of m copies of Y. To be precise,

$$X = \{1, 2, \ldots, m\} \times Y.$$

For every $i = 1, \ldots, m$, we define

$$h_i : y \in Y \mapsto (i, y) \in X,$$

and let us now define the v_j via a process that is not as symmetric as above. For $j \leq m-1$, we put

$$v_j : y \in Y \mapsto (j, y) \in X, \tag{3.9.1}$$

so that $v_j = h_j$, for all j's considered so far. In order to define the remaining v_j's, namely for j of the form

$$j = m - 1 + k, \text{ with } k = 1, \ldots, p,$$

we let

$$v_{m-1+k}(y) = (m, ky), \quad \forall\, y \in Y, \tag{3.9.2}$$

where "ky" refers to the infinite sequence in $\{1, 2, \ldots, p\}^{\mathbb{N}}$ obtained by preceding k to y. The easy task of checking that the above does indeed gives an (m,n)–dynamical system is left for the reader.

We claim that the point $y = (1, 1, 1, 1, \ldots)$ satisfies the required conditions. On the one hand we have the elementary calculation

$$v_1^{-1}h_1(y) = v_1^{-1}(1, y) = y,$$

where we are using the hypothesis that $m \geq 2$, to guarantee that the definition of v_1 is given by (3.9.1) rather than by (3.9.2).

If $m \geq 3$, the same easy computation above yields $v_2^{-1}h_2(y) = y$, and the proof would be complete, so let us assume that $m = 2$. Under this condition notice that $2 = m - 1 + k$, with $k = 1$, so v_2 is defined by (3.9.2), and hence

$$v_2(y) = (2, ky) = \big(2, k(1, 1, 1 \ldots)\big) = \big(2, (1, 1, 1 \ldots)\big) = h_2(y),$$

whence $v_2^{-1}h_2(y) = y$, and the claim is proven.

As already mentioned, $\gamma(y)$ is then the element of Y^u satisfying the requirements. $\square$

4. Configurations and functions.

The purpose of this section is to give a description of the space Y^u of configurations of pattern (c_2) at 1. This will be done in terms of certain functions, which we are now going to describe.

Set $Z_0 := \{a_1,\dots,a_n\}$, $Z_1 := \{b_1,\dots,b_m\}$ and $E = Z_0 \sqcup Z_1$. We will denote the elements of E^r as words $\alpha = e_1e_2\cdots e_r$ in the alphabet E. Set $E^+ := \bigsqcup_{r=1}^{\infty} E^r$. For $\alpha = e_1e_2\cdots e_r \in E^+$ define the color of α as $c(\alpha) = 1-i$, if $e_r \in Z_i$. We consider the compact Hausdorff space

$$Z := \prod_{\alpha\in E^+} Z_{c(\alpha)},$$

where each Z_i is given the discrete topology and Z is endowed with the product topology. Elements of Z will be interpreted as functions $f: E^+ \to E$ such that $f(\alpha) \in Z_{c(\alpha)}$ for all $\alpha \in E^+$.

Let D be the subspace of Z consisting of the functions f such that the following properties (*) and (**) hold for all $\alpha \in E^+ \sqcup \{\cdot\}$, all $e \in E$ and all $\beta \in E^+$:

(*) $f(\alpha e f(\alpha e)) = e$.
(**) $f(\alpha e f(\alpha e)\beta) = f(\alpha\beta)$.

Observe that, for $e \in E$ and $\alpha \in E^+ \sqcup \{\cdot\}$, we have $e \in Z_i \iff f(\alpha e) \in Z_{1-i}$. It is easy to show that D is a closed subspace of Z, and thus D is a compact Hausdorff space with the induced topology.

Our aim in this section is to show the following result:

4.1. Theorem. *There is a canonical homeomorphism $D \cong Y^u$.*

To show this we need some preliminaries.

4.2. Definition. A *partial E-function* is a family $(\Omega_1, f_1), (\Omega_2, f_2), \dots, (\Omega_r, f_r)$, for some $r \geq 1$, satisfying the following relations:

(1) $\Omega_1 = E$, and $f_1 : E \to E$ is a function such that $f_1(e) \in Z_{c(e)}$ for all $e \in E$.
(2) For each $i = 1,\dots,r$,

$$\Omega_i = \{x_1x_2\cdots x_i \in E^i \mid x_{j+1} \neq f_j(x_1x_2\cdots x_j) \text{ for } j = 1,\dots,i-1\},$$

and $f_i : \Omega_i \to E$ is a function such that $f_i(\alpha) \in Z_{c(\alpha)}$ for all $\alpha \in \Omega_i$.

An *E-function* is an infinite sequence $(\Omega_1, f_1), (\Omega_2, f_2), \dots$, satisfying the above conditions for all indices.

It is quite clear that any partial E-function can be extended (in many ways) to an E-function.

4.3. Lemma. *Given an E-function $(\Omega_1, f_1), (\Omega_2, f_2), \dots$, there is a unique function $f \in D$ such that $f(\alpha) = f_i(\alpha)$ for $\alpha \in \Omega_i$ and all $i \in \mathbb{N}$. Therefore D can be identified with the space of all E-functions. Moreover a basis for the topology of D is provided by the partial E-functions by the rule:*

$$\mathfrak{f} = ((\Omega_1, f_1), (\Omega_2, f_2), \dots, (\Omega_r, f_r)) \longmapsto U_{\mathfrak{f}},$$

where $U_{\mathfrak{f}} = \{f \in D \mid f$ extends $\mathfrak{f}\}$.

Proof. Let $(\Omega_1, f_1), (\Omega_2, f_2), \dots,$ be an E-function. We have to construct an extension of it to a function $f : E^+ \to E$ such that $f \in D$. It will be clear from the construction that f is unique.

Note that $f(ef(e))$ must be equal to e for $e \in E$ by condition (*). This, together with the extension property determines completely f on $E^{\le 2}$. Assume that f has been defined on E^{r-1} for some $r \ge 3$. Then we define f on E^r as follows: First $f(\alpha) = f_r(\alpha)$ if $\alpha \in \Omega_r$. If $\alpha = x_1x_2 \cdots x_r \notin \Omega_r$, there are various possibilities, that we are going to consider:

If $x_2 = f(x_1)$, then we set

$$f(x_1 f(x_1) x_3 \cdots x_r) = f(x_3 \cdots x_r).$$

Observe that this is forced by condition (**).

Analogously, if $x_{j+1} \ne f(x_1 \cdots x_j)$ for $j = 1, \dots, i-1$ and $x_{i+1} = f(x_1x_2 \cdots x_i)$ for some $i < r-1$, define

$$f(x_1x_2 \cdots x_i f(x_1 \cdots x_i) x_{i+2} \cdots x_r) = f(x_1x_2 \cdots x_{i-1}x_{i+2} \cdots x_r).$$

Also we have here that this is forced by (**).

Finally if $x_{j+1} \ne f(x_1 \cdots x_j)$ for $j = 1, \dots, r-2$ and $x_r = f(x_1x_2 \cdots x_{r-1})$, define

$$f(x_1x_2 \cdots x_{r-1} f(x_1 \cdots x_{r-1})) = x_{r-1}.$$

Note that this is forced by (*).

We obtain a map $f : E^+ \to E$ such that $f(\alpha) \in Z_{c(\alpha)}$ for all $\alpha \in E^+$. We have to check conditions (*) and (**).

For (*), let $\alpha \in E \cup \{\cdot\}$ and $e \in E$. We will check that $f(\alpha e f(\alpha e)) = e$ by induction on $|\alpha|$. If $\alpha = \cdot$ then we have that $f(ef(e)) = e$ by construction. Suppose that the equality holds for words of length r and let α a word of length $r+1$. Write $\alpha = x_1x_2 \cdots x_{r+1}$. Assume that, for $1 \le i \le r$, we have that $x_{j+1} \ne f(x_1 \cdots x_j)$ for all $j = 1, \dots, i-1$, and that $x_{i+1} = f(x_1x_2 \cdots x_i)$. Then we have

$$\begin{aligned} f(\alpha e f(\alpha e)) &= f(x_1x_2 \cdots x_i f(x_1x_2 \cdots x_i) x_{i+2} \cdots x_{r+1} e f(\alpha e)) \\ &= f(x_1 \cdots x_{i-1} x_{i+2} \cdots x_{r+1} e f(\alpha e)) \\ &= f(x_1 \cdots x_{i-1} x_{i+2} \cdots x_{r+1} e f(x_1x_2 \cdots x_i f(x_1x_2 \cdots x_i) x_{i+2} \cdots x_{r+1} e)) \\ &= f(x_1 \cdots x_{i-1} x_{i+2} \cdots x_{r+1} e f(x_1x_2 \cdots x_{i-1} x_{i+2} \cdots x_{r+1} e)) \\ &= e \end{aligned}$$

where we have used the induction hypothesis in the last step.

Assume that $x_{j+1} \ne f(x_1 \cdots x_j)$ for all $j = 1, \dots, r$, and that $e = f(x_1 \cdots x_r x_{r+1})$. Then we have

$$\begin{aligned} f(\alpha e f(\alpha e)) &= f(x_1 \cdots x_r x_{r+1} f(x_1 \cdots x_r x_{r+1}) f(\alpha e)) \\ &= f(x_1 \cdots x_r f(\alpha e)) \\ &= f(x_1 \cdots x_r f(x_1x_2 \cdots x_r x_{r+1} f(x_1x_2 \cdots x_r x_{r+1}))) \\ &= f(x_1 \cdots x_r x_{r+1}) \\ &= e. \end{aligned}$$

Finally if $x_{j+1} \neq f(x_1 \cdots x_j)$ for all $j = 1, \cdots, r$ and $e \neq f(x_1 x_2 \cdots x_{r+1})$ then we have

$$f(x_1 \cdots x_{r+1} e f(x_1 \cdots x_{r+1} e)) = e$$

by the definition of f.

The checking of (**) is similar. We prove that $f(\alpha e f(\alpha e)\beta) = f(\alpha\beta)$ by induction on $|\alpha|$. If $\alpha = \cdot$ then $f(ef(e)\beta) = f(\beta)$ by definition of f. Suppose (**) holds when the length of α is $\leq r$ and set $\alpha = x_1 x_2 \cdots x_{r+1}$. Assume that, for $1 \leq i \leq r$, we have that $x_{j+1} \neq f(x_1 \cdots x_j)$ for all $j = 1, \ldots, i-1$, and that $x_{i+1} = f(x_1 x_2 \cdots x_i)$. Then we have

$$\begin{aligned} f(\alpha e f(\alpha e)\beta) &= f(x_1 x_2 \cdots x_i f(x_1 x_2 \cdots x_i) x_{i+2} \cdots x_{r+1} e f(\alpha e)\beta) \\ &= f(x_1 \cdots x_{i-1} x_{i+2} \cdots x_{r+1} e f(x_1 \cdots x_{i-1} x_{i+2} \cdots x_{r+1} e)\beta) \\ &= f(x_1 \cdots x_{i-1} x_{i+2} \cdots x_{r+1} \beta) \\ &= f(x_1 \cdots x_{r+1} \beta) \\ &= f(\alpha\beta) \end{aligned}$$

where we have used the induction hypothesis for the third equality.

Assume that $x_{j+1} \neq f(x_1 \cdots x_j)$ for all $j = 1, \ldots, r$, and that $e = f(x_1 \cdots x_r x_{r+1})$. Then we have that $f(\alpha e) = f(x_1 \cdots x_r x_{r+1} e) = x_{r+1}$ by definition of f, and so

$$\begin{aligned} f(\alpha e f(\alpha e)\beta) &= f(x_1 \cdots x_r x_{r+1} f(x_1 \cdots x_r x_{r+1}) f(\alpha e)\beta) \\ &= f(x_1 \cdots x_r f(\alpha e)\beta) \\ &= f(x_1 \cdots x_r x_{r+1}\beta) \\ &= f(\alpha\beta). \end{aligned}$$

Finally if $x_{j+1} \neq f(x_1 \cdots x_j)$ for all $j = 1, \cdots r$ and $e \neq f(x_1 x_2 \cdots x_{r+1})$ then we have

$$f(x_1 \cdots x_{r+1} e f(x_1 \cdots x_{r+1} e)\beta) = f(x_1 \cdots x_{r+1}\beta) = f(\alpha\beta)$$

by the definition of f.

Given the description of D as a subspace of $Z = \prod_{\alpha \in E^+} Z_{c(\alpha)}$, it is clear that the family $\{U_{\mathfrak{f}} \mid \mathfrak{f}$ is a partial $E-$ function$\}$ is a basis for the topology of D. $\square$

Proof of Theorem (4.1). We will define mutually inverse maps $\varphi : Y^u \to D$ and $\psi : D \to Y^u$.

Let $\xi \subseteq \mathbb{F}_{m+n}$ be a configuration of pattern (c_2) at 1. Then we have that $x^{-1} \in \xi$ for all $x \in E$. Now the configuration at x^{-1} must be of pattern (c_1), so that, for each $x \in E$ there is a unique $f_1(x) \in Z_{c(e)}$ such that $x^{-1} f(x) \in \xi$. This defines a partial E-function (E, f_1). For each $x_1 \in E$, the configuration at $x_1^{-1} f_1(x_1)$ must be of pattern (c_2), so all words of the form $x_1^{-1} f(x_1) x_2^{-1}$, with $x_2 \neq f_1(x_1)$ must be in ξ. In the next step we look at the configuration at vertices of the form $x_1^{-1} f(x_1) x_2^{-1}$, where $x_2 \neq f_1(x_1)$. Here the configuration must be of pattern $(c1)$, so there is a unique $f_2(x_1 x_2) \in Z_{c(x_2)} = Z_{c(x_1 x_2)}$ such that $x_1^{-1} f_1(x_1) x_2^{-1} f_2(x_1 x_2) \in \xi$. This gives us a partial E-function $(\Omega_1, f_1), (\Omega_2, f_2)$, where of course $\Omega_2 = \{x_1x_2 \in E^2 \mid x_2 \neq f_1(x_1)\}$. Proceeding in this way we obtain an E-function $\varphi(\xi) = ((\Omega_1, f_1), (\Omega_2, f_2), \ldots)$.

To define ψ we just need to revert the previous process. Given an E-function

$$\mathfrak{f} = ((\Omega_1, f_1), (\Omega_2, f_2), \ldots),$$

the configuration $\psi(\mathfrak{f})$ consists of 1 together with the elements of $\mathbb{F}_{m+n}$ of the form

$$x_1^{-1} f_1(x_1) x_2^{-1} f_2(x_1x_2) \cdots x_r^{-1}$$

and

$$x_1^{-1} f_1(x_1) x_2^{-1} f_2(x_1x_2) \cdots x_r^{-1} f_r(x_1x_2 \cdots x_r).$$

where $x_1x_2 \cdots x_r \in \Omega_r$, for $r \geq 1$. It is clear that $\psi(\mathfrak{f})$ is a configuration of pattern (c_2) at 1, and that φ and ψ are mutually inverse maps.

Since both D and Y^u are compact Hausdorff spaces, in order to show that φ is a homeomorphism it is enough to prove that ψ is an open map. Since the family $\{U_\mathfrak{f} \mid \mathfrak{f}$ is a partial E − function$\}$ is a basis for the topology of D by Lemma (4.3), it is enough to show that $\psi(U_\mathfrak{f})$ is an open subset of Y^u for every partial E-function $\mathfrak{f}$. Thus let $\mathfrak{f} = ((\Omega_1, f_1), (\Omega_2, f_2), \ldots, (\Omega_r, f_r))$ be a partial E-function, and consider the set

$$T := \{x_1^{-1} f_1(x_1) x_2^{-1} f_2(x_1x_2) \cdots x_r^{-1} f_r(x_1x_2 \cdots x_r) \mid x_1x_2 \cdots x_r \in \Omega_r\}.$$

By using the convexity of the elements of Y^u it is straightforward to show that

$$\psi(U_\mathfrak{f}) = \{\xi \in Y^u \mid g \in \xi \quad \forall g \in T\}.$$

Since T is a finite subset of $\mathbb{F}_{m+n}$, we conclude that $\psi(U_\mathfrak{f})$ is an open subset of Y^u. This concludes the proof of Theorem (4.1).

It will be useful to get a detailed description of the action θ^u of $\mathbb{F}_{m+n}$ on Y^u in terms of the picture of Y^u using E-functions (Theorem 4.1).

4.4. Lemma. *Let*

$$g = z_r^{-1} x_r z_{r-1}^{-1} x_{r-1} \cdots z_1^{-1} x_1$$

be a reduced word in $\mathbb{F}_{m+n}$, where $x_1, \ldots, x_r, z_1, \ldots, z_r \in E$. Then Dom$(\theta_g^u) = \emptyset$ unless $z_i \in Z_{c(x_i)}$ for all $i = 1, \ldots, r$. Assume that the latter condition holds. Then the domain of g is precisely the set of all E-functions $\mathfrak{f} = (f_1, f_2, \ldots)$ such that $f_i(x_1 \cdots x_i) = z_i$ for all $i = 1, \ldots, r$, and the range of g is the set of those E-functions $\mathfrak{h} = (h_1, h_2, \ldots,)$ such that $h_i(z_r z_{r-1} \cdots z_{r-i+1}) = x_{r-i+1}$ for all $i = 1, \ldots, r$. Moreover for $\mathfrak{f} \in$ Dom(θ_g^u) let $\mathfrak{h} = {}^g\mathfrak{f}$ denote the image of $\mathfrak{f}$ under the action of g. Then $\mathfrak{h} = ((\Omega_1, h_1), (\Omega_2, h_2), \ldots)$ with

$$h_{r+t}(z_r z_{r-1} \cdots z_1 y_1 y_2 \cdots y_t) = f_t(y_1 \cdots y_t) \quad \textit{if} \quad z_r \cdots z_1 y_1 \cdots y_t \in \Omega_{r+t}. \tag{4.4.1}$$

Moreover, for $i = 2, 3, \ldots, r$ and $z_r z_{r-1} \cdots z_i y_1 \cdots y_t \in \Omega_{r-i+1+t}$ with $z_{i-1} \neq y_1$,

$$h_{r-i+1+t}(z_r z_{r-1} \cdots z_i y_1 y_2 \cdots y_t) = f_{i-1+t}(x_1 \cdots x_{i-1} y_1 y_2 \cdots y_t), \tag{4.4.2}$$

and for $y_1 \cdots y_t \in \Omega_t$ *with* $y_1 \neq z_r$ *we have*

$$h_t(y_1 \cdots y_t) = f_{r+t}(x_1 x_2 \cdots x_r y_1 y_2 \cdots y_t) \,. \tag{4.4.3}$$

Proof. Suppose that $\mathrm{Dom}(\theta^u_g) \neq \emptyset$ and take $\xi \in \mathrm{Dom}(\theta^u_g)$, with corresponding E-function $\mathfrak{f}$. Since $1 \in \xi$ we get that $g \in \xi$ and by convexity we get that $z_r^{-1} x_r z_{r-1}^{-1} x_{r-1} \cdots z_i^{-1} x_i \in \xi$ for all $i = 1, \dots, r$. We thus obtain that $h_i(z_r z_{r-1} \cdots z_{r-i+1}) = x_{r-i+1}$ for all $i = 1, \dots, r$, where $\mathfrak{h} = {}^g\mathfrak{f}$. In particular it follows that $x_{r-i+1} \in Z_{c(z_{r-i+1})}$ for $i = 1, \dots, r$, which is equivalent to $z_i \in Z_{c(x_i)}$ for $i = 1, \dots, r$. Moreover since $g x_1^{-1} f_1(x_1) \in \xi$, we get $z_r^{-1} x_r \cdots x_2 z_1^{-1} f_1(x_1) \in \xi$. Since $z_1, f(x_1) \in Z_{c(x_1)}$ we get that $z_1 = f_1(x_1)$. Similarly we get that $f_i(x-1 \cdots x_i) = z_i$ for all $i = 1, \dots, r$.

Conversely, assume that $z_i \in Z_{c(x_i)}$ for all $i = 1, \dots, r$. Then there are infinitely many E-functions $\mathfrak{f} = (f_1, f_2, \dots)$ such that $f_i(x_1 x_2 \cdots x_i) = z_i$ for all $i = 1, \dots, r$. Let $\mathfrak{f}$ be one of these functions. Then it is easy to verify that ${}^g\mathfrak{f}$ is the E-function $\mathfrak{h} = (h_1, h_2, \dots)$ determined by $h_i(z_r z_{r-1} \cdots z_{r-i+1}) = x_{r-i+1}$ for $i = 1, \dots, r$ and by the rules (4.4.1), (4.4.2) and (4.4.3). $\square$

Recall the following definition from [**ELQ**].

4.5. Definition. Let θ be a partial action of a group G on a compact Hausdorff space X. The partial action θ is *topologically free* if for every $t \in G \setminus \{1\}$, the set $F_t := \{x \in U_{t^{-1}} \mid \theta_t(x) = x\}$ has empty interior.

4.6. Proposition. *For* $m, n \geq 2$*, the action of* $\mathbb{F}_{m+n}$ *on* Ω^u *is topologically free.*

Proof. Let $g \in \mathbb{F}_{m+n} \setminus \{1\}$. Assume first that

$$g = z_r^{-1} x_r z_{r-1}^{-1} x_{r-1} \cdots z_1^{-1} x_1$$

is a reduced word with $x_1, \dots, x_r, z_1, \dots, z_r \in E$. Obviously we may suppose that the domain of g is non-empty, so that $z_i \in Z_{c(x_i)}$ for $i = 1, \dots, r$ by Lemma (4.4).

Note that the domain of g is contained in Y^u. By Theorem (4.1) we only have to show that for any partial E-function $\mathfrak{f}$ there is an extension $\mathfrak{f}'$ of $\mathfrak{f}$ such that ${}^g\mathfrak{f}' \neq \mathfrak{f}'$. Obviously we can assume that $\mathfrak{f} = ((\Omega_1, f_1), (\Omega_2, f_2), \dots, (\Omega_s, f_s))$, with $s > r$ and that $U_{\mathfrak{f}} \cap \mathrm{Dom}(\theta^u_g) \neq \emptyset$. Set $t = s - r$ and choose $y_1, \dots, y_t$ in E such that $z_r z_{r-1} \cdots z_1 y_1 \cdots y_t \in \Omega_s$. Select $y_{t+1} \in E$ such that $y_{t+1} \neq f_s(z_r \cdots z_1 y_1 \cdots y_t)$. Since $m, n \geq 2$, there exits $u \in Z_{c(y_{t+1})}$ such that $u \neq f_{t+1}(y_1 \cdots y_t y_{t+1})$. Define $f_{s+1} \colon \Omega_{s+1} \to E$ in such a way that $f_{s+1}(z_1 \cdots z_r y_1 \cdots y_t y_{t+1}) = u$, and arbitrarily on the other elements of Ω_{s+1} subject to the condition that $f_{s+1}(w_1 \cdots w_{s+1}) \in Z_{c(w_{s+1})}$. Then $((\Omega_1, f_1), \dots, (\Omega_s, f_s), (\Omega_{s+1}, f_{s+1}))$ is a partial E-function extending $\mathfrak{f}$. Extend this partial E-function to an E-function $\mathfrak{f}'$. If ${}^g\mathfrak{f}' = \mathfrak{f}'$ then equation (4.4.1) gives

$$f_{s+1}(z_r \cdots z_1 y_1 \cdots y_t y_{t+1}) = f_{t+1}(y_1 \cdots y_t y_{t+1}),$$

which contradicts our choice of $f_{s+1}(z_r \cdots z_1 y_1 \cdots y_t y_{t+1})$.

We conclude that $U_{\mathfrak{f}}$ has points which are not fixed points for g.

Now assume that

$$g = x_r z_{r-1}^{-1} x_{r-1} \cdots z_1^{-1} x_1 z_0^{-1}$$

is a reduced word in $\mathbb{F}_{m+n}$, with $x_1, \ldots, x_r, z_0, \ldots, z_{r-1} \in E$. Write

$$g' := x_r z_{r-1}^{-1} x_{r-1} \cdots z_1^{-1} x_1.$$

Assume that $g \cdot \xi = \xi$ for all $\xi \in V$, where V is an open subset of X. Then $(z_0^{-1} g') \cdot \xi' = \xi'$ for all $\xi' \in z_0^{-1} V$. By the first part of the proof we get $z_0^{-1} g' = 1$ and thus $g = g' z_0^{-1} = 1$, as desired. □

As an easy consequence we obtain:

4.7. Corollary. *If ρ is a representation of $\mathcal{O}_{m,n}^r$ whose restriction to $C(\Omega^u)$ is injective, then ρ itself is injective.*

Proof. Follows immediately from (4.6) and [**ELQ**: 2.6]. □

Recall the following definition.

4.8. Definition. A C^*-algebra satisfies property (SP) (for *small projections*) in case every nonzero hereditary C^*-subalgebra contains a nonzero projection. Equivalently, for every nonzero positive element a in A there is $x \in A$ such that $x^* a x$ is a nonzero projection.

4.9. Theorem. *For $3 \leq m + n$, the C^*-algebra $\mathcal{O}_{m,n}^r$ satisfies property* (SP). *More precisely, given a nonzero positive element c in $\mathcal{O}_{m,n}^r$, there is an element $x \in \mathcal{O}_{m,n}^r$ such that $x^* c x$ is a nonzero projection in $C(\Omega^u)$. In particular every nonzero ideal of $\mathcal{O}_{m,n}^r$ contains a nonzero projection of $C(\Omega^u)$.*

Proof. This is well-known for the Cuntz algebras $\mathcal{O}_n$ so we may assume that $m, n \geq 2$.

Let c be a nonzero positive element in $\mathcal{O}_{m,n}^r$. Since the canonical conditional expectation E_r is faithful, we may assume that $\|E_r(c)\| = 1$. By Proposition (4.6) the partial action of $\mathbb{F}_{m+n}$ on Ω^u is topologically free. Hence, it follows from [**ELQ**: Proposition 2.4] that, given $1/4 > \epsilon > 0$, there is an element $h \in C(\Omega^u)$ with $0 \leq h \leq 1$ such that

(1) $\|h E_r(c) h\| \geq \|E_r(c)\| - \epsilon$,

(2) $\|h E_r(c) h - h c h\| \leq \epsilon$.

By [**KR**: Lemma 2.2] there is a contraction d in $\mathcal{O}_{m,n}^r$ such that $d^*(hch)d = (hE_r(c)h - \epsilon)_+$, and so it follows that $(hd)^* c (hd)$ is a nonzero positive element in $C(\Omega^u)$. Since $C(\Omega^u)$ is an AF-algebra it has property (SP) so there is an element y in $C(\Omega^u)$ such that $y^*(hd)^* c (hd) y$ is a nonzero projection in $C(\Omega^u)$. Taking $x = hdy$, we get the result. □

5. Exactness of the reduced cross-sectional C*-algebra of a Fell bundle.

Recall from [**EL**: Section 2] that the full (resp. reduced) crossed product may be defined as the full (resp. reduced) cross sectional C*-algebra of the semidirect product Fell bundle [**E5**: 2.8]. For this reason we shall now pause to prove some key results on Fell bundles in support our study of $\mathcal{O}_{m,n}$.

We begin by discussing the notion of (minimal) tensor product of a C*-algebra by a Fell bundle. We refer the reader to [**FD**] for an extensive study of the theory of Fell bundles. We thank N. Brown for an interesting conversation from which some of the ideas pertaining to this tensor product arose.

5.1. Proposition. *Let A be a C^*-algebra and let $\mathscr{B} = \{B_g\}_{g\in G}$ be a Fell bundle over a discrete group G. Then*

(i) *There exists a unique collection of seminorms $\{\|\cdot\|_g\}_{g\in G}$ on the algebraic tensor products $A\odot B_g$, such that $\|\cdot\|_1$ is the spacial (minimal) C^*-norm on $A\odot B_1$, and the completions*
$$A\otimes B_g := \overline{A\odot B_g}^{\|\cdot\|_g}$$
become the fibers of a Fell bundle $\{A\otimes B_g\}_{g\in G}$, in which the multiplication and involution operations extend the following:
$$(a_1\otimes b_1, a_2\otimes b_2)\in \big(A\odot B_{g_1}\big)\times\big(A\odot B_{g_2}\big) \quad\longmapsto\quad a_1a_2\otimes b_1b_2\in A\odot B_{g_1g_2}$$
$$(a\otimes b)\in A\odot B_g \quad\longmapsto\quad a^*\otimes b^*\in A\odot B_{g^{-1}}.$$

(ii) *Denoting the resulting Fell bundle by $A\otimes\mathscr{B}$, there exists a (necessarily unique) $*$-isomorphism*
$$\varphi : A\otimes C^*_r(\mathscr{B}) \to C^*_r(A\otimes\mathscr{B}),$$
such that $\varphi(a\otimes b_g) = a\otimes b_g$, whenever $a\in A$, and $b\in B_g$, for any g (the last two tensor product signs should be given the appropriate and obvious meaning in each case).

Proof. In order to prove uniqueness, suppose that a collection of norms is given as above. Then, for every $g\in G$, and any $c\in A\odot B_g$, one has that
$$\|c\|_g^2 = \|c^*c\|_1.$$

Since $c^*c\in A\odot B_1$, and since the norm on $A\odot B_1$ is assumed to be the spacial norm, uniqueness immediately follows. As for existence, let
$$\pi : A\to B(H)$$
be a faithful representation of A on a Hilbert space H, and let
$$\rho : \bigcup_{g\in G} B_g \to B(K)$$
be a representation (in the sense of [**E2**: 2.2]) of $\mathscr{B}$ on a Hilbert space K, which is isometric on each B_g. Such a representation may be easily obtained by composing the natural inclusion maps $B_g\to C^*_r(\mathscr{B})$, which are isometric by [**E2**: 2.5], with any faithful representation of $C^*_r(\mathscr{B})$.

Consider the representations
$$\begin{aligned} \pi' = \pi\otimes 1 \quad &: \quad A \quad \to \quad B(H\otimes K)\\ \rho' = 1\otimes\rho \quad &: \quad \textstyle\bigcup_{g\in G} B_g \quad \to \quad B(H\otimes K), \end{aligned}$$

and let

$$C_g = \overline{\operatorname{span}}\big(\pi'(A)\rho'(B_g)\big).$$

It is then easy to see that $C_g C_h \subseteq C_{gh}$, and $C_g^* \subseteq C_{g^{-1}}$, for every $g, h \in G$. So we may think of $\mathscr{C} = \{C_g\}_{g\in G}$ as a Fell bundle over G, with operations borrowed from $B(H \otimes K)$.

For each g in G consider the seminorm $\|\cdot\|_g$ on $A \odot B_g$ obtained as the result of composing the maps

$$A \odot B_g \xrightarrow{\pi'\otimes\rho'} C_g \xrightarrow{\|\cdot\|} \mathbb{R}.$$

Evidently the completion of $A \odot B_g$ under this seminorm is isometrically isomorphic to C_g. By [**BO**: 3.3.1] we have that $\|\cdot\|_1$ is the spatial norm and the remaining conditions in (i) may now easily be verified.

In order to prove (ii) we consider two other representations of our objects, namely

$$\begin{array}{rcccl} \pi'' = \pi' \otimes 1 = \pi \otimes 1 \otimes 1 & : & A & \to & B\big(H \otimes K \otimes \ell_2(G)\big) \\ \rho'' = \rho' \otimes \lambda = 1 \otimes \rho \otimes \lambda & : & \bigcup_{g\in G} B_g & \to & B\big(H \otimes K \otimes \ell_2(G)\big), \end{array}$$

where λ is the regular representation of G, and for any given b_g in B_g, we put

$$\rho''(b_g) = 1 \otimes \rho(b_g) \otimes \lambda_g.$$

Observing that ρ'' is also isometric on each B_g, we see that the closed *-subalgebra of $B\big(H \otimes K \otimes \ell_2(G)\big)$ generated by the range of ρ'' is isomorphic to $C_r^*(\mathscr{B})$ by [**E2**: 3.7] (the faithful conditional expectation is just the restriction to the diagonal). Alternatively one may also deduce this from [**E4**: 3.4].

By [**BO**: 3.3.1] one then has that $A \otimes C_r^*(\mathscr{B})$ is isomorphic to the subalgebra of operators generated by $\pi''(A)\rho''(\mathscr{B})$. For further reference let us observe that the present model of $A \otimes C_r^*(\mathscr{B})$ within $B\big(H \otimes K \otimes \ell_2(G)\big)$ is therefore generated by the set

$$\{a \otimes (b_g \otimes \lambda_g) : a \in A,\ g \in G,\ b_g \in B_g\}.$$

Observe that, for each $g \in G$, the map

$$\sigma_g \ :\ x \in C_g \longmapsto x \otimes \lambda_g \in B\big(H \otimes K \otimes \ell_2(G)\big)$$

is an isometry and, collectively, they provide a representation of $\mathscr{C}$ in $B\big(H \otimes K \otimes \ell_2(G)\big)$.

By the same reasoning employed above, based on [**E2**: 3.7] or [**E4**: 3.4], we have that $C_r^*(\mathscr{C})$ is isomorphic to the closed *-subalgebra of $B\big(H \otimes K \otimes \ell_2(G)\big)$ generated by the union of the ranges of all the σ_g. Therefore our model of $C_r^*(\mathscr{C})$ within $B\big(H \otimes K \otimes \ell_2(G)\big)$ is generated by the set

$$\{(a \otimes b_g) \otimes \lambda_g : a \in A,\ g \in G,\ b_g \in B_g\}.$$

The models being identical, we conclude that the algebras $A \otimes C_r^*(\mathscr{B})$ and $C_r^*(A \otimes \mathscr{B})$ are naturally isomorphic. $\square$

5.2. Proposition. *Let $\mathscr{B} = \{B_g\}_{g\in G}$ be a Fell bundle over an exact discrete group G. If B_1 is an exact C^*-algebra, then so is $C^*_r(\mathscr{B})$.*

Proof. Let

$$0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} Q \to 0$$

be an exact sequence of C*-algebras. We need to prove that

$$0 \to J \otimes C^*_r(\mathscr{B}) \xrightarrow{\iota\otimes 1} A \otimes C^*_r(\mathscr{B}) \xrightarrow{\pi\otimes 1} Q \otimes C^*_r(\mathscr{B}) \to 0$$

is also exact. Employing the isomorphisms obtained in (5.1) we may instead prove the exactness of the sequence

$$0 \to C^*_r(J \otimes \mathscr{B}) \xrightarrow{\iota\otimes 1} C^*_r(A \otimes \mathscr{B}) \xrightarrow{\pi\otimes 1} C^*_r(Q \otimes \mathscr{B}) \to 0. \tag{5.2.1}$$

In naming the arrows in the above sequence we have committed a slight abuse of language since we should actually have employed the isomorphisms obtained in (5.1). Nevertheless, if the map we labeled $\pi\otimes 1$ in the last sequence above is applied to an element in $C^*_r(A\otimes\mathscr{B})$ of the form $a\otimes b_g$, with $b_g \in B_g$, the result will be $\pi(a)\otimes b_g$, so we feel our choice of notation is justified.

As it is well known, the only possibly controversial point relating to the exactness of (5.2.1) is whether or not the kernel of $\pi \otimes 1$, which we will refer to as K, is contained in the image of $\iota \otimes 1$. We will arrive at this conclusion by applying [**E4**: 5.3] to K. For this we need to recall from [**E2**: 3.5] that, for each g in G, there is a contractive linear map

$$F_g : C^*_r(\mathscr{B}) \to B_g$$

satisfying $F_g(\sum_h b_h) = b_g$, whenever $(b_h)_h$ is a finitely supported section of $\mathscr{B}$. Here we shall make use of these maps both for the Fell bundle $A \otimes \mathscr{B}$ and for $Q \otimes \mathscr{B}$, and we will denote them by F^A_g and F^Q_g, respectively.

According to [**E4**: 5.2], to check that the ideal K in $C^*_r(A \otimes \mathscr{B})$ is *invariant* we must verify that $F^A_g(K) \subseteq K$, for each g in G. For this we consider the diagram

$$\begin{array}{ccc} C^*_r(A\otimes\mathscr{B}) & \xrightarrow{\pi\otimes 1} & C^*_r(Q\otimes\mathscr{B}) \\ F^A_g\Big\downarrow & & \Big\downarrow F^Q_g \\ A\otimes B_g & \xrightarrow{\pi\otimes 1} & Q\otimes B_g \end{array}$$

In order to check that this is commutative, let $x \in C^*_r(A \otimes \mathscr{B})$ have the form $x = a \otimes b_h$, where $a \in A$ and $b_h \in B_h$, for some $h \in G$. Employing Kronecker symbols we then have that

$$(\pi \otimes 1)F^A_g(x) = \delta_{gh}(\pi \otimes 1)(x) = \delta_{gh}\pi(a) \otimes b_h,$$

while

$$F^Q_g(\pi \otimes 1)(x) = F^Q_g(\pi(a) \otimes b_h) = \delta_{gh}\pi(a) \otimes b_h.$$

Since the set of elements x considered above clearly generates $C^*_r(A \otimes \mathscr{B})$, we see that the diagram is indeed commutative. If we now take an arbitrary element $x \in K$, we will have that

$$0 = F^Q_g(\pi \otimes 1)(x) = (\pi \otimes 1)F^A_g(x),$$

which implies that $F^A_g(x) \in K$, meaning that K is invariant under F^A_g.

Given that G is assumed to be exact, we may apply [**E4**: 5.3] to conclude that K is *induced*, meaning that it is generated, as an ideal, by its intersection with the unit fiber algebra, namely $K \cap (A \otimes B_1)$. The latter evidently coincides with the kernel of the restriction of $\pi \otimes 1$ to $A \otimes B_1$. However, since the image of $A \otimes B_1$ under $\pi \otimes 1$ is contained in $Q \otimes B_1$, we may view $K \cap (A \otimes B_1)$ as the kernel of the third map in the sequence

$$0 \to J \otimes B_1 \xrightarrow{\iota \otimes 1} A \otimes B_1 \xrightarrow{\pi \otimes 1} Q \otimes B_1 \to 0.$$

At this point we invoke our second main hypothesis, namely that B_1 is exact, to deduce that the sequence above is exact, and hence that $K \cap (A \otimes B_1) = J \otimes B_1$. Using angle brackets to denote generated ideals we then have that

$$K = \big\langle K \cap (A \otimes B_1) \big\rangle = \big\langle J \otimes B_1 \big\rangle \subseteq C^*_r(J \otimes \mathscr{B}),$$

which proves that (5.2.1) is exact in the middle. □

5.3. Corollary. *Given a partial action α of an exact discrete group G on an exact C*-algebra A, the reduced crossed product $A \rtimes^r_\alpha G$ is exact.*

Proof. It is enough to notice that $A \rtimes^r_\alpha G$ is the reduced cross-sectional C*-algebra of the semidirect product bundle, which is a Fell bundle over G, and has A as the unit fiber algebra. □

Recalling from (2.6) that $\mathcal{O}^r_{m,n}$ is the reduced crossed product of an abelian, hence exact, C*-algebra by the exact free group $\mathbb{F}_{m+n}$, we obtain:

5.4. Corollary. *For every positive integers n and m, one has that $\mathcal{O}^r_{m,n}$ is an exact C*-algebra.*

6. On full cross-sectional C*-algebras of Fell bundles.

We shall now prove some preparatory results in order to study $\mathcal{O}_{m,n}$ (rather than the reduced version $\mathcal{O}^r_{m,n}$). Our goal is to show that it is not an exact C*-algebra, for $m, n \geq 2$, from which it will follow that it indeed differs from its reduced counterpart.

Since $\mathcal{O}_{m,n}$ is the *full* crossed product $C(\Omega^u) \rtimes_{\theta^u} \mathbb{F}_{m+n}$, we will now concentrate on *full* cross-sectional algebras of Fell bundles. However we will start with a result about *reduced* cross-sectional algebras which will prove to be quite useful in the study of their full versions.

6.1. Proposition. *Let $\mathscr{B} = \{B_g\}_{g\in G}$ be a Fell bundle over a discrete group G and let H be a subgroup of G. Denote by $\mathscr{C} = \{C_h\}_{h\in H}$ the Fell bundle obtained by restricting $\mathscr{B}$ to H, meaning that $C_h = B_h$, for each $h \in H$, with norm, multiplication and involution borrowed from $\mathscr{B}$. Then:*

(i) *There exists a conditional expectation E on $C^*_r(\mathscr{B})$ whose range is isomorphic to $C^*_r(\mathscr{C})$.*

(ii) *If $C^*_r(\mathscr{B})$ is nuclear (resp. exact), then so is $C^*_r(\mathscr{C})$.*

Proof. Viewing each B_g as a subset of $C^*_r(\mathscr{B})$, as allowed by [**E2**: 2.5], let A be the closed linear span of $\bigcup_{h\in H} C_h$. The standard conditional expectation $E : C^*_r(\mathscr{B}) \to B_1$ given by [**E2**: 2.9] may be restricted to give a conditional expectation from A to $C_1 = B_1$, satisfying the hypothesis of [**E2**: 3.3]. Consequently there exists a surjective *-homomorphism

$$\lambda : A \to C^*_r(\mathscr{C}).$$

By [**E2**: 3.6] the kernel of λ is the set formed by the elements $a \in A$ such that $E(a^*a) = 0$. However, applying [**E2**: 2.12] to $C^*_r(\mathscr{B})$, one sees that only the zero element satisfies such an equation, which means that λ is injective and hence that A is isomorphic to $C^*_r(\mathscr{C})$.

We now claim that the map

$$E_H \ : \ \sum_{g\in G} b_g \ \in \ \bigoplus_{g\in G} B_g \ \mapsto \ \sum_{g\in H} b_g \ \in \ A$$

is continuous relative to the norm on its domain induced by $C^*_r(\mathscr{B})$. In order to see this recall that, strictly according to definition [**E2**: 2.3], $C^*_r(\mathscr{B})$ is the closed *-subalgebra of $\mathscr{L}\big(\ell_2(\mathscr{B})\big)$ (adjointable operators on the right Hilbert B_1–module $\ell_2(\mathscr{B})$) generated by the range of the left regular representation of $\mathscr{B}$.

Let ι be the natural inclusion of $\ell_2(\mathscr{C})$ into $\ell_2(\mathscr{B})$ and observe that its adjoint is the projection of the latter onto the former. Now consider the linear map

$$V : T \in \mathscr{L}\big(\ell_2(\mathscr{B})\big) \mapsto \iota^* T \iota \in \mathscr{L}\big(\ell_2(C)\big).$$

Viewing each B_g within $\mathscr{L}\big(\ell_2(\mathscr{B})\big)$, and each C_h within $\mathscr{L}\big(\ell_2(\mathscr{C})\big)$, by [**E2**: 2.2 & 2.5], one may easily show that for every $g \in G$, and every $b_g \in B_g$, one has that

$$V(b_g) = \begin{cases} b_g, & \text{if } g \in H, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, given any $\sum_{g\in G} b_g \in \bigoplus_{g\in G} B_g$, we have that

$$\Big\| \sum_{h\in H} b_h \Big\| = \Big\| V\Big(\sum_{g\in G} b_g \Big) \Big\| \leq \Big\| \sum_{g\in G} b_g \Big\|,$$

where the norm in the left hand side is computed in $\mathscr{L}\big(\ell_2(\mathscr{C})\big)$. However, due to the fact that A and $C^*_r(\mathscr{C})$ are isomorphic, the inequality above also holds if the norm in the left hand side is computed in A. This says that E_H is continuous, hence proving our claim.

One may then easily prove that the unique continuous extension of E_H to A is a conditional expectation, taking care of point (i).

Point (ii) now follows immediately from (i), since a closed *-subalgebra of an exact C*-algebra is exact [**BO**: Exercise 2.3.2], and the range of a conditional expectation on a nuclear C*-algebra is nuclear [**BO**: Exercise 2.3.1]. ☐

Recall that λ denotes the left regular representation of a group G in $C^*_r(G)$. Also, given a Fell bundle $\mathscr{B}$, we will let Λ be the regular representation of $\mathscr{B}$ in $C^*_r(\mathscr{B})$ [**E2**: 2.2].

Recall from [**FD**: VIII.16.12] that every representation π of $\mathscr{B}$ in a C*-algebra A extends to a *-homomorphism (also denoted π by abuse of language) from $C^*(\mathscr{B})$ to A.

We thank Eberhard Kirchberg for sharing with us a very interesting idea which, when applied to Fell bundles, yields the following curious result, mixing *reduced* cross-sectional C*-algebras and *maximal* tensor products to produce *full* cross-sectional C*-algebras. See also [**BO**: 10.2.8].

6.2. Theorem. *Let* $\Lambda \underset{\max}{\otimes} \lambda$ *be the representation of* $\mathscr{B}$ *in* $C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(G)$ *given by*

$$(\Lambda \underset{\max}{\otimes} \lambda) b_g = \Lambda(b_g) \otimes \lambda_g, \quad \forall\, g \in G, \quad \forall\, b_g \in B_g.$$

*Then the associated *-homomorphism*

$$\Lambda \underset{\max}{\otimes} \lambda : C^*(\mathscr{B}) \to C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(G)$$

is injective.

Proof. Choose a faithful representation $\pi : C^*(\mathscr{B}) \to B(H)$, where H is a Hilbert space, and consider the representation $\pi \otimes \lambda$ of $\mathscr{B}$ on $H \otimes \ell_2(G)$ given by

$$(\pi \otimes \lambda) b_g = \pi(b_g) \otimes \lambda_g, \quad \forall\, g \in G, \quad \forall\, b_g \in B_g.$$

This gives rise to the representation $\pi \otimes \lambda$ of $C^*(\mathscr{B})$ which factors through a representation

$$\pi_\lambda : C^*_r(\mathscr{B}) \to B\big(H \otimes \ell_2(G)\big),$$

by [**E4**: 3.4]. Let ρ be the *right* regular representation of G on $\ell_2(G)$, which in turn yields the representation $\tilde\rho$ of $C^*_r(G)$ on $H \otimes \ell_2(G)$ defined by

$$\tilde\rho = 1 \otimes \rho : C^*_r(G) \to B\big(H \otimes \ell_2(G)\big).$$

It is easy to see that the range of π_λ commutes with the range of $\tilde\rho$, so there exists a representation

$$\pi_\lambda \otimes \tilde\rho : C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(G) \to B\big(H \otimes \ell_2(G)\big),$$

such that

$$(\pi_\lambda \otimes \tilde\rho)(x \otimes y) = \pi_\lambda(x)\tilde\rho(y), \quad \forall\, x \in C^*_r(\mathscr{B}), \quad \forall\, y \in C^*_r(G).$$

Given any g in G, and any $b_g \in B_g$, observe that

$$(\pi_\lambda \otimes \tilde\rho)(\Lambda \underset{\max}{\otimes} \lambda) b_g = (\pi_\lambda \otimes \tilde\rho)\big(\Lambda(b_g) \otimes \lambda_g\big) = \pi_\lambda\big(\Lambda(b_g)\big)\ \tilde\rho(\lambda_g) =$$

$$= \big(\pi(b_g) \otimes \lambda_g\big)(1 \otimes \rho_g) = \pi(b_g) \otimes \lambda_g \rho_g.$$

Denoting by $\{\delta_g\}_{g\in G}$ the standard orthonormal basis of $\ell_2(G)$, pick any $\xi \in H$, and observe that the above operator, when applied to $\xi \otimes \delta_1$ produces

$$(\pi_\lambda \otimes \tilde\rho)(\Lambda \underset{\max}{\otimes} \lambda)b_g\big|_{\xi\otimes\delta_1} = \big(\pi(b_g) \otimes \lambda_g\rho_g\big)(\xi \otimes \delta_1) = \pi(b_g)\xi \otimes \delta_1.$$

By linearity, density and continuity we conclude that

$$(\pi_\lambda \otimes \tilde\rho)(\Lambda \underset{\max}{\otimes} \lambda)x\big|_{\xi\otimes\delta_1} = \pi(x)\xi \otimes \delta_1, \quad \forall\, x \in C^*(\mathscr{B}).$$

Therefore, assuming that $(\Lambda \underset{\max}{\otimes} \lambda)x = 0$, for some $x \in C^*(\mathscr{B})$, we deduce that $\pi(x)\xi = 0$, for all $\xi \in H$, and hence that $\pi(x) = 0$. Since π was supposed to be injective on $C^*(\mathscr{B})$, we deduce that $x = 0$. $\square$

The following is also based on an idea verbally communicated to us by Kirchberg.

6.3. Corollary. *Let $\mathscr{B}$ be a Fell bundle over a discrete group G and let H be a subgroup of G. Consider the Fell bundle $\mathscr{C} = \{C_h\}_{h\in H}$ obtained by restricting $\mathscr{B}$ to H, meaning that $C_h = B_h$, for each $h \in H$, with norm, multiplication and involution borrowed from $\mathscr{B}$. Then the natural map $\iota : C^*(\mathscr{C}) \to C^*(\mathscr{B})$ is injective.*

Proof. Recall from (6.1.i) that there exists a conditional expectation from $C^*_r(\mathscr{B})$ onto $C^*_r(\mathscr{C})$, as well as a conditional expectation from $C^*_r(G)$ to $C^*_r(H)$. Therefore by [**BO**: 3.6.6] one has that the natural maps below are injective:

$$C^*_r(\mathscr{C}) \underset{\max}{\otimes} C^*_r(H) \hookrightarrow C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(H) \hookrightarrow C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(G).$$

Consider the diagram

$$\begin{array}{ccc} C^*_r(\mathscr{C}) \underset{\max}{\otimes} C^*_r(H) & \hookrightarrow & C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(G) \\ \uparrow & & \uparrow \\ C^*(\mathscr{C}) & \xrightarrow{\iota} & C^*(\mathscr{B}) \end{array}$$

where the vertical arrows are the versions of $\Lambda \underset{\max}{\otimes} \lambda$ for $\mathscr{C}$ and $\mathscr{B}$, respectively. By checking on elements $c_h \in C_h$, it is elementary to prove that the above diagram commutes. Since all arrows, with the possible exception of ι, are known to be injective, we deduce that ι is injective as well. $\square$

The following is an interesting conclusion to be drawn from (6.2).

6.4. Theorem. *Let $\mathscr{B}$ be a Fell bundle over the discrete group G. If the reduced cross-sectional C^*-algebra $C^*_r(\mathscr{B})$ is nuclear, then the regular representation*

$$\Lambda : C^*(\mathscr{B}) \to C^*_r(\mathscr{B})$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} C^*(\mathscr{B}) & \xrightarrow{\Lambda \underset{\max}{\otimes} \lambda} & C^*_r(\mathscr{B}) \underset{\max}{\otimes} C^*_r(G) \\ \Lambda \big\downarrow & & \big\downarrow q \\ C^*_r(\mathscr{B}) & \xrightarrow{id \otimes \lambda} & C^*_r(\mathscr{B}) \otimes C^*_r(G) \end{array}$$

where q is the natural map from the maximal to the minimal tensor product. Assuming that $C^*_r(\mathscr{B})$ is nuclear we have that q is injective [**BO**: 3.6.12], and hence Λ is injective. $\square$

6.5. Remark. According to [**E2**: 4.1], the above result says that $\mathscr{B}$ is an *amenable* Fell bundle. However, as observed in the very last paragraph of [**E3**], we do not know whether this implies the *approximation property* for $\mathscr{B}$ [**E2**: 4.4]. Nevertheless, in view of [**BO**: 4.4.3], it is perhaps reasonable to believe that the approximation property could be deduced from the nuclearity of $C^*_r(\mathscr{B})$.

7. Isotropy groups for partial actions.

Given a partial action

$$\theta = \big\{\theta_g : X_{g^{-1}} \to X_g\big\}_{g\in G}$$

of a discrete group G on a locally compact Hausdorff topological space X, recall that the *isotropy subgroup* for a given point $x \in X$ is the subgroup of G defined by

$$G^x = \big\{g \in G : x \in X_{g^{-1}},\ \theta_g(x) = x\big\}.$$

7.1. Proposition. *Let X be a Hausdorff locally compact topological space, let G be a discrete group, and let θ be a partial action of G on X. Then:*

(i) *If the full crossed product $C_0(X) \rtimes_\theta G$ is exact, then for every x in X for which G^x is residually finite-dimensional* [**BO**: p. 96], *one has that G^x is amenable.*

(ii) *If the reduced crossed product $C_0(X) \rtimes^r_\theta G$ is nuclear, then the isotropy group of every point in X is amenable.*

(iii) *If the reduced crossed product $C_0(X) \rtimes^r_\theta G$ is exact, then the isotropy group of every point in X is exact.*

Proof. Given x in X, consider the restriction of θ to G^x, thus obtaining a partial action of G^x on X. Observing that the full crossed product is defined to be the full cross-sectional C*-algebra of the associated semidirect product Fell bundle, we deduce from (6.3) that $C_0(X) \rtimes_\theta G^x$ is isomorphic to a closed *-subalgebra of $C_0(X) \rtimes_\theta G$. By the assumption in (i) that the latter is exact, we deduce that $C_0(X) \rtimes_\theta G^x$ is also exact [**BO**: Exercise 2.3.2].

Consider the *-homomorphism

$$\pi : f \in C_0(X) \mapsto f(x) \cdot 1 \in C^*(G^x),$$

as well as the universal representation of G^x

$$u : G^x \to C^*(G^x).$$

Viewing $C^*(G^x)$ as an algebra of operators on some Hilbert space, it is easy to check that (π, u) is a covariant representation of the partial dynamical system $\big(C_0(X), G^x, \theta|_{G^x}\big)$, in the sense of [**ELQ**: 1.2]. Therefore, by [**ELQ**: 1.3] there exists a *-homomorphism

$$\pi \times u : C_0(X) \rtimes_\theta G^x \to C^*(G^x)$$

such that

$$(\pi \times u)(f\delta_h) = f(x)u_h,$$

for all h in G^x, and all f in $C_0(X_h)$. One moment of reflexion is enough to convince ourselves that $\pi \times u$ is surjective and hence that $C^*(G^x)$ is exact by [**BO**: 9.4.3].

Under the assumption that G^x is residually finite-dimensional we then deduce from [**BO**: 3.7.11] that G^x is amenable, completing the proof of (i).

We next consider the diagram

$$\begin{array}{ccc} C_0(X) \rtimes_\theta G^x & \xrightarrow{\pi\times u} & C^*(G^x) \\ \Lambda\big\downarrow & & \big\downarrow\Lambda^x \\ C_0(X) \rtimes_\theta^r G^x & \cdots\overset{\varphi}{\cdots}\cdots & C_r^*(G^x) \\ E\big\downarrow & & \big\downarrow\tau \\ C_0(X) & \xrightarrow{\chi^x} & \mathbb{C} \end{array} \tag{7.1.1}$$

where Λ is the *left regular representation* (see the paragraph following [**E2**: 2.3]), Λ^x is the version of Λ for the trivial one-dimensional Fell bundle over G^x, E is the standard conditional expectation [**E2**: 2.9], τ is the unique normalized trace on $C_r^*(G^x)$ such that $\tau(\lambda_h) = 0$, for all $h \neq 1$, and finally χ^x is the character on $C_0(X)$ given by point evaluation at x. Incidentally τ coincides with the standard conditional expectation in the context of the trivial bundle over G^x.

By checking on elements of the form $f\delta_h$, it is elementary to verify that the diagram commutes. We claim that $\pi \times u$ maps the kernel of Λ into the kernel of Λ^x. In order to see this, suppose that x lies in the kernel of Λ. Then by [**E2**: 3.6] we have that $E(x^*x) = 0$, so

$$0 = \chi^x\big(E\big(\Lambda(x^*x)\big)\big) = \tau\big(\Lambda^x\big((\pi \times u)(x^*x)\big)\big) = \tau\big(y^*y\big),$$

where $y = \Lambda^x\big((\pi \times u)x\big)$. Since τ is a faithful trace on $C_r^*(G^x)$ [**E2**: 2.12], we conclude that $y = 0$, which proves that $(\pi \times u)x$ belongs to the kernel of Λ^x, hence the claim.

As a consequence we see that there exists a *-homomorphism ψ filling the dots in (7.1.1) in a way as to preserve the commutativity of the diagram. Since Λ^x is surjective, ψ must also be surjective.

Assuming that $C_0(X) \rtimes_\theta^r G^x$ is nuclear (resp. exact), we now deduce that $C_r^*(G^x)$ shares this property. To conclude the proof it is now enough to recall that if $C_r^*(G^x)$ is nuclear then G^x is amenable [**BO**: 2.6.8], and that if $C_r^*(G^x)$ is an exact C*-algebra then G^x is an exact group [**BO**: 5.1.1]. $\square$

7.2. Theorem. *If $m, n \geq 2$, then $\mathcal{O}_{m,n}$ is not exact and hence it is not isomorphic to $\mathcal{O}^r_{m,n}$.*

Proof. Recall from (3.9) that there exists y in Y^u such that

$$(v^u_1)^{-1} h^u_1(y) = y = (v^u_2)^{-1} h^u_2(y).$$

This implies that $b_1^{-1} a_1$ and $b_2^{-1} a_2$ belong to $\mathbb{F}^y_{m+n}$, the isotropy group of y.

It is easy to see that the subgroup of $\mathbb{F}_{m+n}$ generated by these two elements is isomorphic to $\mathbb{F}_2$, so we conclude that $\mathbb{F}^y_{m+n}$ is not amenable.

It is well known that free groups are residually finite-dimensional [**C**: Corollary 22] and consequently the same applies to its subgroup $\mathbb{F}^y_{m+n}$. Using (7.1.i) one deduces that the full crossed product $C(\Omega^u) \rtimes_{\theta^u} \mathbb{F}_{m+n}$ cannot be exact, and hence the conclusion then follows from (2.5). ☐

8. Absence of finite dimensional representations.

The goal of this section is to prove that $\mathcal{O}^r_{m,n}$ does not admit any nonzero finite dimensional representation. In case $n \neq m$ the same is true even for the unreduced algebras and, since the proof of this fact is much simpler, we present it first.

8.1. Proposition. *If $n \neq m$ then $\mathcal{O}_{m,n}$ (and hence also $\mathcal{O}^r_{m,n}$) does not admit any non-trivial finite dimensional representation.*

Proof. Let $\rho : \mathcal{O}_{m,n} \to M_d(\mathbb{C})$ be a non-degenerate d-dimensional representation, with $d > 0$. Then, denoting by $\underline{v}$ and $\underline{w}$ the images of v and w (see the third and fourth relation in $(\mathcal{R})$), we have

$$\mathrm{tr}\big(\rho(\underline{v})\big) = \sum_{i=1}^{n} \mathrm{tr}\big(\rho(\underline{s}_i \underline{s}_i^*)\big) = \sum_{i=1}^{n} \mathrm{tr}\big(\rho(\underline{s}_i^* \underline{s}_i)\big) = n\, \mathrm{tr}\big(\rho(\underline{w})\big),$$

and similarly $\mathrm{tr}\big(\rho(\underline{v})\big) = m\, \mathrm{tr}\big(\rho(\underline{w})\big)$, so

$$n\, \mathrm{tr}\big(\rho(\underline{w})\big) = m\, \mathrm{tr}\big(\rho(\underline{w})\big).$$

Since $n \neq m$, this implies that $\mathrm{tr}\big(\rho(\underline{w})\big) = 0$, and hence also that $\mathrm{tr}\big(\rho(\underline{v})\big) = 0$. Therefore

$$d = \mathrm{tr}(1) = \mathrm{tr}(\rho(1)) = \mathrm{tr}\big(\rho(\underline{v} + \underline{w})\big) = 0,$$

a contradiction. ☐

From now on we will develop a series of auxiliary results in order to show the nonexistence of nonzero finite dimensional representations of $\mathcal{O}^r_{m,n}$ when $m = n$ (although our proof will not explicitly use that $m = n$, and hence it will serve as a proof for the general case). In what follows we will therefore assume that

$$\rho : \mathcal{O}^r_{m,n} \to M_d(\mathbb{C})$$

is non-degenerate d-dimensional representation and our task will be to arrive at a contradiction from it.

Restricting ρ to $C(\Omega^u)$ we get a finite dimensional representation of a commutative algebra which, as it is well known, is equivalent to a direct sum of characters. In other words, upon conjugating ρ by some unitary matrix, we may assume that there is a d-tuple $(\xi_1, \xi_2, \ldots, \xi_d)$ of elements of Ω^u such that

$$\rho(f) = \begin{pmatrix} f(\xi_1) & & & \\ & f(\xi_2) & & \\ & & \ddots & \\ & & & f(\xi_d) \end{pmatrix},$$

for every f in $C(\Omega^u)$.

8.2. Proposition. *The set $Z = \{\xi_1, \xi_2, \ldots, \xi_d\}$ is invariant under θ^u.*

Proof. We want to prove that for every g in $\mathbb{F}_{m+n}$, and every $\xi \in Z \cap \Omega^u_{g^{-1}}$, one has that $\theta^u_g(\xi)$ is in Z. Arguing by contradiction we assume that this is not so, that is, that we can find $\xi \in Z \cap \Omega^u_{g^{-1}}$ such that $\theta^u_g(\xi) \notin Z$. Observing that $\theta^u_g(\xi) \in \Omega^u_g$, we may pick an $f \in C_0(\Omega^u_g)$ such that $f(\theta^u_g(\xi))$ is nonzero, but such that f vanishes identically on Z. In particular this implies that $\rho(f) = 0$.

Using [**ELQ**: 1.4] we may write $\rho = \pi \times u$, where (π, u) is a covariant representation of the dynamical system $\big(C(\Omega^u), \mathbb{F}_{m+n}, \theta^u\big)$. Noticing that π is the restriction of ρ to $C(\Omega^u)$, we have

$$\rho\big(\theta^u_{g^{-1}}(f)\big) = \pi\big(\theta^u_{g^{-1}}(f)\big) = u_{g^{-1}}\pi(f)u_g = 0.$$

It follows that

$$0 = \theta^u_{g^{-1}}(f)\big|_\xi = f\big(\theta^u_g(\xi)\big) \neq 0,$$

a contradiction. □

8.3. Proposition. *If $m, n \geq 2$, then for every ξ in Z, the isotropy group $\mathbb{F}^\xi_{m+n}$, contains a subgroup isomorphic to $\mathbb{F}_2$.*

Proof. Assume first that $\xi \in Y^u$, that is, the configuration of ξ at the origin follows pattern (c_2). Then in particular $b_1^{-1} \in \xi$, and hence the configuration of ξ at b_1 must follow pattern (c_1). Therefore there exists a unique $i_1 \leq n$, such that $b_1^{-1}a_{i_1} \in \xi$. The configuration of ξ at $b_1^{-1}a_{i_1}$ must then follow pattern (c_2) so, in particular $b_1^{-1}a_{i_1}b_1^{-1} \in \xi$.

Continuing in this way we may construct an infinite sequence $i_1, i_2, \ldots$ such that

$$g_k := b_1^{-1}a_{i_1}b_1^{-1}a_{i_2}b_1^{-1} \ldots b_1^{-1}a_{i_k} \in \xi, \quad \forall\, k \in \mathbb{N}.$$

So $\xi \in \Omega^u_{g_k}$, and hence

$$\theta^u_{g_k^{-1}}(\xi) = g_k^{-1}\xi \in Z,$$

because Z is invariant under θ^u. Using the fact that Z is finite we conclude that there are positive integers $k < l$, such that

$$g_l^{-1}\xi = g_k^{-1}\xi,$$

so $g_k g_l^{-1}\xi = \xi$, and hence the element

$$x := g_k g_l^{-1}$$

lies in the isotropy group of ξ.

Let $\mathbb{F}_2$ be the free group on a set of two generators, say $\{c_1, c_2\}$, and consider the unique group homomorphism

$$\varphi : \mathbb{F}_{m+n} \to \mathbb{F}_2$$

such that

$$\varphi(a_i) = 1, \quad \forall\, i = 1, \ldots, n,$$

$$\varphi(b_1) = c_1, \quad \varphi(b_2) = c_2, \quad \varphi(b_j) = 1, \quad \forall\, j \geq 3.$$

It is then evident that $\varphi(g_k) = c_1^{-k}$, and hence that

$$\varphi(x) = \varphi(g_k g_l^{-1}) = c_1^{l-k},$$

where by assumption, $l - k > 0$.

Repeating the above argument with b_2 in place of b_1, we may find some y in the isotropy group of ξ such that $\varphi(y)$ is a positive power of b_2.

The subgroup of $\mathbb{F}^{\xi}_{m+n}$ generated by x and y is therefore a free group since its image within $\mathbb{F}_2$ via φ is certainly free.

This concludes the proof under the assumption that the configuration of ξ at the origin is (c_2), so let us suppose that the pattern is (c_1). Therefore there exists some i such that $a_i \in \xi$ and hence, again by invariance of Z, we have that $a_i^{-1}\xi \in Z$. Since $1 \in \xi$ we have that $a_i^{-1} \in a_i^{-1}\xi$, so the pattern of $a_i^{-1}\xi$ at the origin is necessarily (c_2).

By the case already studied there is a copy of $\mathbb{F}_2$ inside the isotropy group of $a_i^{-1}\xi$, but since

$$\mathbb{F}^{a_i^{-1}\xi}_{m+n} = a_i^{-1}(\mathbb{F}^{\xi}_{m+n})a_i,$$

the same holds for the isotropy group of ξ. $\square$

Since Z is invariant under θ^u we may restrict the latter to the former thus obtaining a partial action, say θ, of $\mathbb{F}_{m+n}$ on Z.

Given $\xi \in Z$, we will denote by 1_ξ the characteristic function of the singleton $\{\xi\}$, viewed as an element of $C(Z)$.

8.4. Proposition. *For every $\xi \in Z$ there exists an embedding of $C^*_r(\mathbb{F}_2)$ in the reduced crossed product $C(Z) \rtimes^r_\theta \mathbb{F}_{m+n}$, such that the unit of the former is mapped to 1_ξ.*

Proof. Let G be any subgroup of $\mathbb{F}^{\xi}_{m+n}$. For each g in G, consider the element

$$u_g = 1_\xi \delta_g \in C(Z) \rtimes^r_\theta \mathbb{F}_{m+n}.$$

By direct computation one checks that $u_g u_h = u_{gh}$, and $u_{g^{-1}} = u_g^*$, for every g and h in G, and moreover that $u_1 = 1_\xi$. In other words, u is a unitary representation of G in the hereditary subalgebra of $C(Z) \rtimes^r_\theta \mathbb{F}_{m+n}$ generated by 1_ξ. Let

$$\varphi : C^*(G) \to C(Z) \rtimes^r_\theta \mathbb{F}_{m+n}$$

be the integrated form of u. Denoting by τ the canonical trace on $C^*(G)$, and by E the standard conditional expectation

$$E : C(Z) \rtimes_\theta^r \mathbb{F}_{m+n} \to C(Z),$$

one may easily prove that

$$E\big(\varphi(x)\big) = \tau(x)1_\xi, \quad \forall\, x \in C^*(G).$$

Since E is faithful, for every $x \in C^*(G)$ one has that

$$\varphi(x) = 0 \iff E(\varphi(x^*x)) = 0 \iff \tau(x^*x) = 0.$$

This said we see that the kernel of φ coincides with the kernel of the integrated form of the left regular representation, namely

$$\lambda : C^*(G) \to C_r^*(G).$$

Consequently φ factors through $C_r^*(G)$, providing a *-homomorphism

$$\tilde\varphi : C_r^*(G) \to C(Z) \rtimes_\theta^r \mathbb{F}_{m+n},$$

which is injective because of the above equality of null spaces. Clearly $\tilde\varphi(1) = 1_\xi$, as stated. To conclude the proof it is therefore enough to choose G to be the subgroup of $\mathbb{F}_{m+n}^\xi$ given by (8.3). $\square$

The next significant step in order to obtain a contradiction from the existence of ρ is to prove that it admits a factorization

$$\begin{tikzcd} \mathcal{O}_{m,n}^r \arrow[rr, "\rho"] \arrow[dr, "\varphi"'] & & M_d(\mathbb{C}) \\ & C(Z) \rtimes_\theta^r \mathbb{F}_{m+n} \arrow[ur, "\tilde\rho"'] & \end{tikzcd} \tag{8.5}$$

such that $\varphi(f) = f|_Z$, for all $f \in C(\Omega^u)$.

The poof of this factorization may perhaps be of independent interest, so we prove it in a more general context in the next section. Although it may not look a very deep result we have not been able to prove it in full generality, since we need to use the exactness of free groups.

9. Invariant ideals.

Let G be a discrete group and let α be a partial action of G on A. For each g in G, denote by A_g the range of α_g.

9.1. Definition. A closed two-sided ideal $K \trianglelefteq A$ is said to be α-invariant if

$$\alpha_g(K \cap A_{g^{-1}}) \subseteq K, \quad \forall\, g \in G.$$

Given such an ideal, let $B = A/K$, and denote the quotient map by

$$q : A \to B.$$

For each g in G, consider the closed two-sided ideal of B given by $B_g = q(A_g)$. Given any $b \in B_{g^{-1}}$, write $b = q(a)$, for some $a \in A_{g^{-1}}$, and define

$$\beta_g(a) := q\big(\alpha_g(a)\big).$$

It is then easy to see that β_g becomes a *-isomorphism from $B_{g^{-1}}$ to B_g, also known as a *partial automorphism* of B.

9.2. Proposition. *The collection of partial automorphisms $\{\beta_g\}_{g\in G}$ forms a partial action of G on B.*

Proof. If I and J are closed two-sided ideals of A, it is well known that every element $z \in I \cap J$ may be written as a product $z = xy$, with $x \in I$, and $y \in J$. In other words $I \cap J = IJ$. Therefore

$$q(I \cap J) = q(IJ) = q(I)q(J) = q(I) \cap q(J).$$

We then conclude that

$$\beta_g(B_{g^{-1}} \cap B_h) = \beta_g\big(q(A_{g^{-1}}) \cap q(A_h)\big) = \beta_g\big(q(A_{g^{-1}} \cap A_h)\big) = q\big(\alpha_g(A_{g^{-1}} \cap A_h)\big) =$$

$$= q\big(A_g \cap A_{gh}\big) = q(A_g) \cap q(A_{gh}) = B_g \cap B_{gh}.$$

We leave the verification of the remaining axioms ([**E1**], [**M4**], [**E5**]) to the reader. ☐

9.3. Proposition. *Under the above assumptions, there exists a unique surjective *-homomorphism*

$$\varphi : A \rtimes_\alpha^r G \to B \rtimes_\beta^r G,$$

such that $\varphi(a_g\delta_g) = q(a_g)\delta_g$, for all $g \in G$, and all $a_g \in A_g$.

Proof. Recalling that the reduced crossed product C*-algebra coincides with the reduced cross-sectional C*-algebra of the corresponding semidirect product bundle [**E5**: 2.8], denote by $\mathscr{A}$ and $\mathscr{B}$ the corresponding Fell bundles. Precisely $\mathscr{A} = \big\{A_g\delta_g\big\}_{g\in G}$, with multiplication

$$(a_g\delta_g, b_h\delta_h) \in A_g\delta_g \times A_h\delta_h \mapsto \alpha_g\big(\alpha_g^{-1}(a_g)b_h\big)\delta_{gh} \in A_{gh}\delta_{gh}$$

and involution

$$a_g\delta_g \in A_g\delta_g \mapsto \alpha_{g^{-1}}(a_g^*)\delta_{g^{-1}} \in A_{g^{-1}}\delta_{g^{-1}},$$

and likewise for $\mathcal{B}$. It is then easy to see that the correspondence

$$a_g\delta_g \in A_g\delta_g \mapsto q(a_g)\delta_g \in B_g\delta_g$$

defines a homomorphism in the category of Fell bundles and hence induces a *-homomorphism of full cross-sectional C*-algebras

$$\psi : C^*(\mathscr{A}) \to C^*(\mathscr{B}).$$

Denoting by

$$E : C^*(\mathscr{A}) \to A, \quad \text{and} \quad F : C^*(\mathscr{B}) \to B$$

the corresponding conditional expectations [**E2**: 2.9], one easily verifies that $F\psi = qE$. From this it follows that, for every element x in the kernel of the regular representation [**E2**: 2.2],

$$\Lambda_{\mathscr{A}} : C^*(\mathscr{A}) \to C^*_r(\mathscr{A}),$$

one has that

$$F\big(\psi(x^*x)\big) = q\big(E(x^*x)\big) = 0,$$

by [**E2**: 3.6]. Therefore, by [**E2**: 2.12], we see that $\psi(x^*x)$ lies in the kernel of the regular representation $\Lambda_{\mathscr{B}}$ relative to $\mathcal{B}$. We conclude that ψ factors through the quotient providing a map φ such that the diagram below is commutative.

$$\begin{array}{ccc} C^*(\mathscr{A}) & \xrightarrow{\psi} & C^*(\mathscr{B}) \\ \Lambda_{\mathscr{A}}\big\downarrow & & \big\downarrow\Lambda_{\mathscr{B}} \\ C^*_r(\mathscr{A}) & \xrightarrow{\varphi} & C^*_r(\mathscr{B}). \end{array}$$

Identifying reduced crossed products with their corresponding reduced cross-sectional algebras, the proof is complete. □

9.4. Proposition. *Let α be a partial action of a discrete exact group G on a C*-algebra A, and let ρ be a *-representation of $A \rtimes^r_\alpha G$ on a Hilbert space H. Letting K be the null-space of $\rho|_A$, then K is α-invariant, so we may speak of the* quotient partial action *β of (9.2), and of the map φ of (9.3). Under these conditions there exists a *-representation $\tilde\rho$ of $A/K \rtimes^r_\beta G$, such that the diagram*

$$\begin{array}{ccc} A \rtimes^r_\alpha G & \xrightarrow{\quad\rho\quad} & \mathscr{B}(H) \\ \varphi\searrow & & \nearrow\tilde\rho \\ & A/K \rtimes^r_\beta G & \end{array}$$

commutes.

Proof. Let J be the null space of ρ, so that $K = A \cap J$. Given any $g \in G$, and any $a \in K \cap A_{g^{-1}}$, observe that, identifying A with its image in $A \rtimes_\alpha^r G$, as usual, one has that

$$(b\delta_g)a(b\delta_g)^* = b\alpha_g(a)b^*, \quad \forall\, b \in A_g.$$

Applying ρ on both sides of the above equality, we conclude that $b\alpha_g(a)b^* \in K$. If we now let b run along an approximate identity for A_g, we conclude that $\alpha_g(a)$ lies in K, thus proving that K is α-invariant.

We next claim that

$$\mathrm{Ker}(\varphi) \subseteq \mathrm{Ker}(\rho). \tag{9.4.1}$$

With that goal in mind, let

$$E : A \rtimes_\alpha^r G \to A, \quad \text{and} \quad F : A/K \rtimes_\beta^r G \to A/K,$$

be the associated conditional expectations (unlike (9.3), here these are seen as maps on the *reduced* cross-sectional algebras). Given x in the kernel of φ, we have that

$$0 = F\big(\varphi(x^*x)\big) = q\big(E(x^*x)\big),$$

so we see that $E(x^*x)$ lies in $K \subseteq J$. Using [**E4**: 5.1] we deduce that x is in the ideal of $A \rtimes_\alpha^r G$ generated by K, and hence that x is in J. This proves (9.4.1) and, since φ is surjective, we have that ρ factors through φ, which means precisely that a map $\tilde{\rho}$ exists with the stated properties. ☐

Returning to the situation we left at the end of the previous section, recall that ρ is a non-degenerate d-dimensional representation of $\mathcal{O}_{m,n}^r$. Notice that

$$K := \mathrm{Ker}\big(\rho|_{C(\Omega^u)}\big) = \big\{f \in C(\Omega^u) : f(\xi_i) = 0,\ \forall i = 1, \ldots d\big\}.$$

The quotient of $C(\Omega^u)$ by K may then be naturally identified with $C(Z)$, and the quotient partial action given by (9.2) becomes the action induced by the restriction of θ^u to Z. Thus, when applied to our situation, the diagram in the statement of (9.4) becomes precisely (8.5).

The restriction of $\tilde{\rho}$ to the copy of $C_r^*(\mathbb{F}_2)$ provided by (8.4) will then be a (possibly degenerate) d-dimensional representation of the *simple infinite-dimensional* C*-algebra $C_r^*(\mathbb{F}_2)$. Such a representation must therefore be identically zero and hence, in particular,

$$\tilde{\rho}(1_\xi) = 0,$$

because, as seen above, 1_ξ lies in the copy of $C_r^*(\mathbb{F}_2)$ alluded to. Observing that the unit of $C(Z) \rtimes_\theta^r \mathbb{F}_{m+n}$ is given by

$$1 = \sum_{\xi \in Z} 1_\xi,$$

we deduce that $\tilde{\rho}(1) = 0$ and hence that $\tilde{\rho} = 0$. A glance at (8.5) then gives $\rho = 0$.

This proves the following main result:

9.5. Theorem. *$\mathcal{O}_{m,n}^r$ admits no nonzero finite dimensional representations.*

References

[**AA**] G. Aranda Pino and G. Abrams, "The Leavitt path algebra of a graph", *J. Algebra* **293** (2005), 319–334.

[**AG21**] P. Ara and K. R. Goodearl, "Leavitt path algebras of separated graphs", to appear in J. reine angew. Math.; arXiv:1004.4979v1 [math.RA] (2010).

[**AG2**] P. Ara and K. R. Goodearl, "C*-algebras of separated graphs", *J. Funct. Analysis* **261** (2011), 2540–2568.

[**AMP**] P. Ara, M. A. Moreno, E. Pardo, "Nonstable K-theory for graph algebras", *Algebr. Represent. Theory* **10** (2007), 157–178.

[**B**] L. G. Brown, "Ext of certain free product C*-algebras", *J. Operator Theory* **6** (1981), 135–141.

[**BO**] N. P. Brown and N. Ozawa, "C*-algebras and finite-dimensional approximations", Graduate Studies in Mathematics, 88, American Mathematical Society, 2008.

[**C**] D. E. Cohen, "Combinatorial group theory: a topological approach", London Mathematical Society Student Texts, 14. Cambridge University Press, 1989.

[**CK**] J. Cuntz and W. Krieger, "A class of C*-algebras and topological Markov chains", *Invent. Math.* **56** (1980), 251–268.

[**DP**] J. Duncan and A. L. T. Paterson, "C*-algebras of inverse semigroups", *Proc. Edinburgh Math. Soc. (2)* **28** (1985), no. 1, 41–58.

[**E1**] R. Exel, "Circle actions on C*-algebras, partial automorphisms and a generalized Pimsner–Voiculescu exact sequence", *J. Funct. Analysis,* **122** (1994), 361–401.

[**E2**] R. Exel, "Amenability for Fell bundles", *J. Reine Angew. Math.* **492** (1997), 41–73 [arXiv:funct-an/9604009].

[**E3**] R. Exel, "Partial representations and amenable Fell bundles over free groups", *Pacific J. Math.* **192** (2000), 39–63 [arXiv:funct-an/9706001].

[**E4**] R. Exel, "Exact groups, induced ideals, and Fell bundles[2]", http://arxiv.org/abs/math.OA/0012091 (2003).

[**EL**] R. Exel and M. Laca, "Cuntz–Krieger algebras for infinite matrices", *J. reine angew. Math.* **512** (1999), 119–172 [arXiv:funct-an/9712008].

[**ELQ**] R. Exel, M. Laca and John Quigg, "Partial dynamical systems and C*-algebras generated by partial isometries[3]", http://arxiv.org/abs/funct-an/9712007v1 (1997).

[**FD**] J. M. G. Fell and R. S. Doran, "Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vols. 1 & 2", Pure and Applied Mathematics, 126, Academic Press, 1988.

[**H**] R. Hazrat, "The graded structure of Leavitt path algebras", arXiv:1005.1900v4 [math.RA].

[**KR**] E. Kirchberg, M. Rørdam, "Infinite non-simple C^*-algebras: Absorbing the Cuntz algebra $\mathcal{O}_\infty$", *Advances Math.* **167** (2002), 195–264.

[**L**] W. G. Leavitt, "The module type of a ring", *Trans. Amer. Math. Soc.* **103** (1962), 113–130.

[**M1**] K. McClanahan, "C*-algebras generated by elements of a unitary matrix", *J. Funct. Anal.* **107** (1992), 439-457.

[**M2**] K. McClanahan, "K-theory and Ext-theory for rectangular unitary C*-algebras", *Rocky Mountain J. Math.* **23** (1993), 1063–1080.

[**M3**] K. McClanahan, "Simplicity of reduced amalgamated products of C*-algebras", *Canad. J. Math.* **46** (1994), 793–807.

[2] This paper was published as "Exact groups and Fell bundles", *Math. Ann.* **323** (2002), no. 2, 259–266. However, the referee required that the results pertaining to induced ideals be removed from the preprint version arguing that there were no applications of this concept. The reader will therefore have to consult the arxiv version, where the results we need may be found.

[3] This paper was published in *J. Operator Theory* **47** (2002), no. 1, 169–186. However, the referee required that the results pertaining to covariant representations be removed from the preprint version. The reader will therefore have to consult the arxiv version, where the results we need may be found.

[**M4**] K. McClanahan, "K-theory for partial crossed products by discrete groups", *J. Funct. Analysis* **130** (1995), 77–117.

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