EQUIVALENCE RELATIONS IN SET THEORY, COMPUTATION THEORY, MODEL THEORY AND COMPLEXITY THEORY

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One of Harvey's most influential articles is his joint work with Lee Stanley [8] in which he introduces a notion of *Borel reducibility* between isomorphism relations on the countable models of a theory in infinitary logic. Through the work of many researchers, this theory later blossomed into a rich field devoted to the more general study of Borel reducibility between Borel and analytic equivalence relations (and quasi-orders). For a look at some of this work see [11, 12, 17, 19, 23, 26, 27, 30].

The aim of the present article is to illustrate how a similar idea has recently been used to good effect in four new contexts: *effective* descriptive set theory, computation theory, model theory and complexity theory. This work has deepened research in these fields, produced a number of unexpected results and raised a host of interesting new open problems.

Section 1. Effective Descriptive Set Theory

We begin with a brief description of the classical, non-effective setting, before turning to the more recent work [6] in the effective context. The principal objects of study in the classical theory are analytic (Σ^1_1 with parameters) equivalence relations on Polish spaces (think of the reals). Such equivalence relations are compared using *Borel reducibility* in the following way:

E_0 *is Borel reducible to* E_1 iff there is a Borel function $f\colon X_0 \to X_1$ such that

$$xE_0y \text{ iff } f(x)E_1f(y).$$

E_0 and E_1 are *Borel bireducible* if each Borel reduces to the other. Then $\mathcal{B}$ denotes the resulting set of degrees, ordered under Borel reducibility. When discussing Borel reducibility we sometimes identify an equivalence relation with its degree. Work of Silver [37] and of Harrington-Kechris-Louveau [16] identifies an interesting initial segment of $\mathcal{B}$:

The author wishes to thank the John Templeton Foundation for its support through the project *Myriad Aspects of Infinity*, Project ID# 13152, and the Austrian Science Fund (FWF) for its support through research project P 20835-N13.

Theorem 1. *$\mathcal{B}$ has the initial segment*

$$1 < 2 < \cdots < \omega < id < E_0,$$

where:

n = Borel equivalence relations with exactly n classes;
ω = Borel equivalence relations with exactly $\aleph_0$ classes;
id is $({}^\omega\omega, =)$ (equality on reals);
E_0 is the equivalence relation xE_0y iff $x(n) = y(n)$ for all but finitely many n. In fact, any Borel equivalence relation is Borel equivalent to one of the above or lies strictly above E_0 under Borel reducibility.

The question for the effective theory is: What happens if we replace "Borel" by "effectively Borel"? In what follows we simply write "Hyp" for "effectively Borel" (= lightface Δ^1_1). We define:

If E and F are Hyp equivalence relations on the reals, then E is *Hyp reducible to F*, written $E \leq_H F$, iff For some Hyp function f, xEy iff $f(x)Ff(y)$

$\leq_H$ is reflexive and transitive. We write $E \equiv_H F$ for $E \leq_H F$ and $F \leq_H E$.

So the new object of study is $\mathcal{H}$, the degrees of Hyp equivalence relations on the reals under Hyp reducibility.

There are some surprises! Again we have degrees

$$1 < 2 < \cdots < \omega < \mathrm{id} < E_0,$$

defined as follows:

n is represented by xE^ny iff $x(0) = y(0) < n-1$ or $x(0), y(0) \geq n-1$;
ω is represented by $xE^\omega y$ iff $x(0) = y(0)$;
id, E_0 are as before: $x\mathrm{id}y$ iff $x = y$, xE_0y iff $x(n) = y(n)$ for all but finitely many n.

Proposition 2. *There are Hyp equivalence relations strictly between* 1 *and* 2*!*

Here is why: Let E be a Hyp equivalence relation. Recall that the $\mathcal{H}$-degree n is represented by the equivalence relation E^n where:

$$xE^ny \text{ iff } x(0) = y(0) < n-1 \text{ or } x(0), y(0) \geq n-1.$$

Fact 1. E^n is Hyp reducible to E iff at least n distinct E-equivalence classes contain Hyp reals.

Proof. Suppose that E^n Hyp reduces to E via the Hyp function f. Each of the n equivalence classes of E^n contains a Hyp real; let $x_0, \ldots, x_{n-1}$ be Hyp, pairwise E^n-inequivalent reals. Then the reals $f(x_i)$, $i < n$, are Hyp, pairwise E-inequivalent reals. Conversely, if $y_0, \ldots, y_{n-1}$ are Hyp, pairwise E-inequivalent reals then send the E^n-equivalence class of x_i to the real y_i; this is a Hyp reduction of E^n to E. □

Fact 2. E is Hyp reducible to E^2 iff E has at most 2 equivalence classes.

Proof. If E is Hyp reducible to E^2, then E has at most 2 equivalence classes because E^2 has only 2 equivalence classes. Conversely, suppose that the equivalence classes of E are A_0 and A_1. We may assume that A_0 has a Hyp element x. Then A_0 is Hyp as it consists of those reals E-equivalent to x and A_1 is Hyp as it consists of those reals not E-equivalent to x. Now we can reduce E to E^2 by choosing E^2-inequivalent Hyp reals y_0, y_1 and sending the elements of A_0 to y_0 and the elements of A_1 to y_1. □

So to get a Hyp equivalence relation between 1 and 2 we need only find one with two equivalence classes but with all Hyp reals in just one class. The existence of such an equivalence relation follows from a classical fact from Hyp theory (see [35], page 52, Theorem 1.1):

Fact 3. There are nonempty Hyp sets of reals which contain no Hyp element.

Proof. Let A be the set of non-Hyp reals. Then A is Σ^1_1 and therefore the projection of a Π^0_1 subset P of Reals $\times$ Reals. P is nonempty. A Hyp real $h = (h_0, h_1)$ in P would give a Hyp real h_0 in A, contradiction. □

Now we ask a harder question: Are there incomparable degrees between 1 and 2? To answer this we prove:

Theorem 3. *(*[6]*) There exist Hyp sets of reals A, B such that for* no *Hyp function F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.*

Given this Theorem, define E_A to be the equivalence relation with equivalence classes A and $\sim A$ (the complement of A); define E_B similarly. Note that the sets A, B contain no Hyp reals, else there would be a constant Hyp function F mapping one of them into the other. So a Hyp reduction of E_A to E_B would have to send the elements of $\sim A$ (which contains Hyp reals) to elements of $\sim B$, and therefore the elements of A to elements of B, contradicting the Theorem. Similarly there is no Hyp reduction of E_B to E_A.

Proof Sketch of Theorem 3. First we quote a result of Harrington [15] (also see [33], Theorem XIII.3.5). For reals a, b and a recursive ordinal α we say that a is *α-below* b iff a is recursive in the α-jump of b.

Fact. For any recursive ordinal α there are Π^0_1 singletons a, b such that a is not α-below b and b is not α-below a.

Now using Barwise Compactness, find a nonstandard ω-model M of ZF$^-$ with standard ordinal ω_1^{CK} in which are there are Π^0_1 singletons a, b such that for all recursive α, a is not α-below b and b is not α-below a (i.e., a and b are Hyp incomparable.) Let a, b be the unique solutions in M to the Π^0_1 formulas φ_0, φ_1, respectively. The desired sets A, B are $\{x \mid \varphi_0(x)\}$ and $\{x \mid \varphi_1(x)\}$. If F were a Hyp function mapping A into B, then it would send the element a of A to an element $F(a)$ of $B \cap M$; but then $F(a)$ must equal b and therefore b is Hyp in a, contradicting the choice of a, b. □

Now fix A, B as in the Theorem. Using them we can get incomparable Hyp equivalence relations between n and $n+1$ for any finite n, by considering E_A, E_B where the equivalence classes of E_A are A together with a split of $\sim A$ (the complement of A) into n classes, each of which contains a Hyp real (similarly for E_B).

We now consider Hyp equivalence relations with infinitely many equivalence classes. Recall the Silver and Harrington-Kechris-Louveau dichotomies:

Theorem 4. *(a) (Silver) A Borel equivalence relation is either Borel reducible to ω or Borel reduces id.*
(b) (Harrington-Kechris-Louveau) A Borel equivalence relation is either Borel reducible to id or Borel reduces E_0.

How effective are these results? Harrington's proof of (a) and the original proof of (b) show:

Theorem 5. *(a) A Hyp equivalence relation is either Hyp reducible to ω or Borel reduces id.*
(b) A Hyp equivalence relation is either Hyp reducible to id or Borel reduces E_0.

The sets A, B of Theorem 3 can be used to show that the Silver and Harrington-Kechris-Louveau dichotomies are *not* fully effective:

Theorem 6. *(*[6]*) (a) There are incomparable Hyp equivalence relations between ω and id.*
(b) There are incomparable Hyp equivalence relations between id and E_0.

Proof Sketch. (a) Consider the relations

$E_A(x, y)$ iff ($x \in A$ and $x = y$) or ($x, y \notin A$ and $x(0) = y(0)$)

E_B: The same, with A replaced by B.

Now E^ω Hyp reduces to E_A by $n \mapsto (n, 0, 0, ...)$. Also E_A Hyp reduces to id via the map $G(x) = x$ for $x \in A$, $G(x) = (x(0), 0, 0, ...)$ for $x \notin A$ (same for B)

There is no Hyp reduction of E_A to E_B: If F were such a reduction, then let C be $F^{-1}[\sim B]$. As $\sim B$ is Hyp, C is also Hyp and therefore $A \cap C$ is also Hyp. But

$A \cap C$ must be countable as F is a reduction. So if $A \cap C$ were nonempty it would have a Hyp element, contradicting the fact that A has no Hyp element. Therefore F maps A into B, which is impossible by the choice of A, B. By symmetry, there is no Hyp reduction of E_B to E_A.

(b) Now we define E_A on $\mathbb{R} \times \mathbb{R}$ by: $(x, y)E_A(x', y')$ iff $x = x'$ and either $x \notin A$ or ($x \in A$ and yE_0y'). E_B is the same, with A replaced by B.

We need two Facts (see [18], Lemma 2.49 and [24], Theorem 2.2.5 (a), respectively):

1. If $h \colon \mathbb{R} \to \mathbb{R}$ is Baire measurable and constant on E_0 classes, then h is constant on a comeager set.
2. If $B \subseteq \mathbb{R}^2$ is Hyp, then so is $\{x \mid \{y \mid (x, y) \in B\}$ is comeager$\}$.

Now suppose that F were a Hyp reduction of E_A to E_B. Let $\pi(x, y) = x$ for all x and define $h \colon \mathbb{R} \to \mathbb{R}$ by: $h(x) = z$ iff $\{y \mid \pi(F(x, y)) = z\}$ is comeager.

Using 1 and 2, h is a total Hyp function. We claim that $h[A] \subseteq B$, contradicting the choice of A, B: Assume $x \in A$. Then for comeager-many y, $\pi(F(x, y)) = h(x)$. So if $h(x) \notin B$ then F maps more than one E_A class into a single E_B class, contradiction. By symmetry there is no Hyp reduction of E_B to E_A. $\square$

The overall picture of the degrees of Hyp sets of reals under Hyp reducibility is the following: Call a degree *canonical* if it is one of $1 < 2 < \cdots < \omega < \mathrm{id} < E_0$. For any two canonical degrees $a < b$ there is a rich collection of degrees which are above a, below b and incomparable with all canonical degrees in between.

However at least one nice thing happens: If a degree is above n for each finite n, then it is also above ω.

Because this field is so new (like the others introduced in this paper), there remain many open questions. Here are several:

1. If a Hyp equivalence relation is Borel reducible to E_0, then must it also be Hyp reducible to E_0? (This is true for finite n, ω, id.)
2. Are there any nodes other than 1? I.e., is there a Hyp equivalence relation with more than one equivalence class which is comparable with all Hyp equivalence relations under Hyp reducibility?
3. Is there a minimal degree? Are there incomparables above each degree?

There is also a jump operation, which is in need of further study.

Section 2. Computation Theory

We now turn to equivalence relations not on the reals but on the natural numbers, where computation theory play a central role. As seen in the last section, Hyp-reducibility for Hyp equivalence relations on the real numbers has a rich structure; however the analogous theory in the context of the natural numbers is trivial:

Proposition 7. *(*[4]*, Section 2.2, Fact 2.10.2) Any Hyp equivalence relation on the natural numbers is Hyp reducible to the equality relation on ω.*

Therefore the central objects of interest in our study of equivalence relations on the natural nubmers are not the Hyp equivalence relations but instead the Σ^1_1 equivalence relations. Indeed, in the classical theory of Borel reducibility one considers not only the Borel equivalence relations but more generally analytic (Σ^1_1 with parameters) equivalence relations which are not Borel; indeed these appeared already in [8]:

Let T be any theory in first-order logic (or any sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$). Then the isomorphism relation on the countable models of T is an analytic equivalence relation which need not be Borel.

There are analytic equivalence relations which are not Borel reducible to such an isomorphism relation; an example is E_1, the equivalence relation on $\mathbb{R}^\omega$ defined by:

$$\vec{x}E_1\vec{y} \text{ iff } \vec{x}(n) = \vec{y}(n) \text{ for almost all } n.$$

Note that E_1 is even Hyp.

A motivating question for our study is the following:

Question. Is every Σ^1_1 equivalence relation on the natural numbers reducible to isomorphism on a Hyp class of *computable* structures?

Of course we can identify a computable structure with a natural number which serves as an index for it. The reducibility we use is: $E_0 \leq_H E_1$ iff there is a Hyp function $f\colon \mathcal{N} \to \mathcal{N}$ such that mE_0n iff $f(m)E_1f(n)$. (We say that E_0 is *Hyp reducible* to E_1.)

Theorem 8. *(*[5]*) Every Σ^1_1 equivalence relation on $\mathcal{N}$ is Hyp reducible to isomorphism on computable trees.*

This answers the above Question positively.

Proof Sketch: Let E be a Σ^1_1 equivalence relation on $\mathcal{N}$ and choose a computable $f\colon \mathcal{N}^2 \to$ *Computable Trees* such that $\sim mEn$ iff $f(m,n)$ is well-founded.

Now associate to pairs m,n computable trees $T(m,n)$ so that:

$T(m,n)$ is isomorphic to $T(n,m)$;

mEn implies that $T(m,n)$ is isomorphic to the "canonical" non-well-founded computable tree;

$\sim mEn$ implies that $T(m,n)$ is isomorphic to the "canonical" computable tree of rank α, where α is least so that $f(m',n')$ has rank at most α for all $m' \in [m]_E$, $n' \in [n]_E$.

Now to each n associate the tree T_n gotten by gluing together the $T(n,i)$, $i \in \omega$. If mEn, then T_m is isomorphic to T_n as they are obtained by gluing together isomorphic trees. And if $\sim mEn$ then T_m, T_n are not isomorphic as they are obtained by gluing together trees which on some component are non-isomorphic. □

It can be shown that the isomorphism relation on computable trees (and therefore any Σ^1_1 equivalence relation on $\mathcal{N}$) Hyp-reduces to the isomorphism relation on each of the following Hyp classes:

1. Computable graphs
2. Computable torsion-free Abelian groups
3. Computable Abelian p-groups for a fixed prime p
4. Computable Boolean Algebras
5. Computable linear orders
6. Computable fields

These results came as a surprise, because in the classical setting, the analogue of 2 is an open problem and the analogue of 3 is false!

Fokina and I show in [4] that the global structure of Σ^1_1 equivalence relations on $\mathcal{N}$ under Hyp reducibility is very rich: it embeds the partial order of Σ^1_1 sets under Hyp many-one reducibility. But it is not known if there is a single isomorphism relation on computable structures which is neither Hyp nor complete under Hyp-reducibility! However we do have:

Theorem 9. *([4]) Every Σ^1_1 equivalence relation is Hyp bireducible to a bi-embeddability relation on computable structures.*

The proof is based on the analagous result in the non-effective setting:

Theorem 10. *([11]) Every analytic equivalence relation on the reals is Borel bireducible to a bi-embeddability relation on countable structures.*

I should also mention that there has been considerable prior work on *computably enumerable* equivalence relations, of which provable equivalence is a natural example. For those interesting results we refer to [13] and the references therein.

Section 3. Model Theory

It is natural to expect that insights into the model-theoretic properties of a first-order theory could be derived from the descriptive set-theoretic behaviour of the isomorphism relation on its countable models under Borel reducibility. This idea was pursued by Laskowski [29], Marker [31] and in depth by Koerwien [28]. But the conclusion was rather negative: theories can be complicated model-theoretically and simple descriptive set-theoretically (an example is dense linear orderings), or vice-versa (an example is described in [28]).

A solution to this difficulty emerged through the study of isomorphism on a theory's *uncountable* models. The work of [10] (see Chapter V, Theorem 64) shows, for example, that a theory is classifiable and shallow in Shelah's model-theoretic sense exactly if the isomorphism relation on its models of size κ (for an appropriate choice of regular uncountable cardinal κ) is "Borel" in a generalised sense.

Naturally, a prerequisite for this study is the development of a suitable descriptive set theory of the uncountable, which has turned out to be a fascinating area of independent interest. Armed with such a theory it becomes possible to bring in the methods of model-theoretic stability theory to uncover deep connections between the model theory and descriptive set theory of first-order theories.

I begin with the uncountable descriptive set theory. It is favourable to choose κ to be uncountable and such that $\kappa^{<\kappa} = \kappa$. The *Generalised Baire Space* κ^κ is the space of all functions $f\colon kappa \to \kappa$ topologised with basic open sets of the form $N_s = \{f \mid s \subseteq f\}$, s an element of $\kappa^{<\kappa}$. In this context the *Borel* sets are obtained by closing the open sets under the operations of complementation and unions of size at most κ. The Σ^1_1 sets are the projections of Borel sets, the Π^1_1 sets are the complements of the Σ^1_1 sets and the Δ^1_1 sets are those which are both Σ^1_1 and Π^1_1. Borel sets are Δ^1_1 but the converse is false. As usual, a set is *nowhere dense* if its closure contains no nonempty open set; a set is *meager* if it is the union of κ-many nowhere dense sets. The Baire Category Theorem holds in the sense that the intersection of κ-many open dense sets is dense. A set has the *Baire Property (BP)* if its symmetric difference with some open set is meager. Borel sets have the BP. A *perfect set* is the range of a continuous injection from 2^κ (the Generalised Cantor Space) into κ^κ. A set has the *Perfect Set Property (PSP)* iff it either has size at most κ or contains a perfect subset.

Theorem 11. *(see* [10]*) (a) It is consistent that all Δ^1_1 sets have the BP.*
(b) For any stationary subset S of κ, the filter CUB(S), the closed unbounded filter restricted to S, is a Σ^1_1 set without the BP.
(c) In L, CUB(S) for stationary S is not Δ^1_1, but there are nevertheless Δ^1_1 sets without the BP and without the PSP.

(d) It is consistent relative to an inaccessible cardinal that all Σ^1_1 sets have the PSP (and the use of an inaccessible is necessary).

Remark. Part (a) was proved independently by Lücke-Schlicht, in the case $S = \kappa$ part (b) is due to Halko-Shelah and part (d) was proved independently by Schlicht.

I turn now to Borel reducibility. Suppose that X_0, X_1 are Borel subsets of κ^κ. Then $f\colon X_0 \to X_1$ is a *Borel function* iff $f^{-1}[Y]$ is Borel whenever Y is Borel. This implies that the graph of f is Borel, as (x, y) belongs to the graph of f iff for all $s \in \kappa^{<\kappa}$, either y does not belong to N_s or x belongs to $f^{-1}[N_s]$.

If E_0, E_1 are equivalence relations on Borel sets X_0, X_1 respectively, then we say that E_0 is *Borel reducible to* E_1, written $E_0 \leq_B E_1$, iff for some Borel $f\colon X_0 \to X_1$:

$$x_0 E_0 y_0 \text{ iff } f(x_0) E_1 f(x_1).$$

Now recall the following picture from the classical case:

$$1 <_B 2 <_B \cdots <_B \omega <_B \text{id} <_B E_0$$

forms an initial segment of the Borel equivalence relations under $\leq_B$ where n denotes an equivalence relation with n classes for $n \leq \omega$, id denotes equality on ω^ω and E_0 denotes equality modulo finite on ω^ω.

At κ we easily get the initial segment

$$1 <_B 2 <_B \cdots <_B \omega <_B \omega_1 <_B \cdots <_B \kappa$$

where for each nonzero cardinal $\lambda \leq \kappa$ we identify λ with the $\equiv_B$ class of Borel equivalence relations with exactly λ-many classes. What happens above these equivalence relations? We might hope for:

Silver Dichotomy The equivalence relation id (equality on κ^κ) is the strong successor of κ under $\leq_B$, i.e., if a Borel equivalence relation E has more than κ classes then id is Borel reducible to E.

Theorem 12. *(a) The Silver Dichotomy implies the PSP for Borel sets. Therefore it fails in L and its consistency requires at least an inaccessible cardinal.*
(b) The Silver Dichotomy is false with Borel replaced by Δ^1_1.

Is the Silver Dichotomy consistent? This question remains open.

We can also consider what happens above id. In the case $\kappa = \omega$ we have:

Classical Glimm-Effros Dichotomy $E_0 =$ (equality mod finite) is the strong successor of id, i.e., if a Borel equivalence relation E is not Borel reducible to id (i.e., E is not *smooth*) then E_0 Borel-reduces to E.

At κ, what shall we take E_0 to be? For infinite regular $\lambda \leq \kappa$, define $E_0^{<\lambda}$ = equality for subsets of κ modulo sets of size $< \lambda$.

Proposition 13. *For $\lambda < \kappa$, $E_0^{<\lambda}$ is Borel bireducible with id.*

So we can forget about $E_0^{<\lambda}$ for $\lambda < \kappa$ and set $E_0 = E_0^{<\kappa}$, equality modulo bounded sets.

As in the classical case we have:

Proposition 14. *$E_0 = E_0^{<\kappa}$ is* not *Borel reducible to id.*

There are other versions of E_0: For regular $\lambda < \kappa$ define E_λ^κ = equality modulo the ideal of λ-nonstationary sets. These equivalence relations are key for connecting model-theoretic stability with uncountable descriptive set theory.

How do the relations E_λ^κ compare to each other under Borel reducibility for different λ? For simplicity, consider the special case $\kappa = \omega_2$.

Theorem 15. *([10]) (a) It is consistent that $E_\omega^{\omega_2}$ and $E_{\omega_1}^{\omega_2}$ are incomparable under Borel reducibility. (b) Relative to a weak compact it is consistent that $E_\omega^{\omega_2}$ is Borel reducible to $E_{\omega_1}^{\omega_2}$.*

It is not known if it is consistent for $E_{\omega_1}^{\omega_2}$ to be Borel reducible to $E_\omega^{\omega_2}$.

What is the relationship between E_0 and E_λ^κ?

Theorem 16. *(a) The relations E_λ^κ do not Borel reduce to E_0, as E_0 is Borel and the E_λ^κ are not.*
(b) If $\kappa = \mu^+$ for some cardinal μ, then E_0 reduces to E_λ^κ, unless λ is the cofinality of μ.
(c) In L, the condition in (b) that λ not be the cofinality of μ can be dropped.

The structure of the Δ_1^1 equivalence relations under Borel reducibility is (consistently) very rich:

Theorem 17. *Consistently, there is an injective, order-preserving embedding from $(\mathcal{P}(\kappa), \subseteq)$ into the partial order of Δ_1^1 equivalence relations under Borel reducibility.*

The above summarises the current state of knowledge regarding uncountable descriptive set theory. As has been mentioned, there remain many open questions, some of which we list at the end of this section.

Now we return to the connection between uncountable descriptive set theory and model theory. Let T be a countable, complete and first-order theory. Then T is *classifiable* iff there is a "structure theory" for its models. (Example: Algebraically closed fields (transcendence degree).) T is *unclassifiable* otherwise. (Example: Dense linear orderings.)

Shelah's Characterisation (Main Gap): T is classifiable iff T is superstable without the OTOP and without the DOP.

A classifiable T is *deep* iff it has the maximum number of models in all uncountable powers. (Example: Acyclic undirected graphs, every node has infinitely many neighbours.) T is *shallow* otherwise. (Remark: Actually, Shelah defined "deep" differently, in terms of rank. The fact that his definition is equivalent to the previous is one of the most profound results of his classification theory.)

Now for simplicity assume $\kappa = \lambda^+$ where λ is uncountable and regular and the GCH holds at λ. Isom^κ_T is the isomorphism relation on the models of T of size κ.

Theorem 18. *(*[10]*)*
(a) T is classifiable and shallow iff Isom^κ_T is Borel.
(b) T is classifiable iff for all regular $\mu < \kappa$, $E_{S^\kappa_\mu}$ is not Borel reducible to Isom^κ_T.
(c) In L, T is classifiable iff Isom^κ_T is Δ^1_1.

The proof uses Ehrenfeucht-Fraissé games. The Game $\mathrm{EF}^\kappa_t(\mathcal{A},\mathcal{B})$ is defined as follows, where $\mathcal{A}$, $\mathcal{B}$ are structures of size κ and t is a tree. Player I chooses size $< \kappa$ subsets of $A \cup B$ and nodes along an initial segment of a branch through t; player II builds a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$ which includes the sets that player I has chosen. Player II wins iff he survives until a cofinal branch is reached.

The tree t *captures Isom^κ_T* iff for all size κ models $\mathcal{A}$, $\mathcal{B}$ of T, $\mathcal{A} \simeq \mathcal{B}$ iff Player II has a winning strategy in $\mathrm{EF}^\kappa_t(\mathcal{A},\mathcal{B})$.

Now there are 4 cases:

Case 1: T is classifiable and shallow.

Then Shelah's work [36] shows that some well-founded tree captures Isom^κ_T. We use this to show that Isom^κ_T is Borel.

Case 2: T it classifiable and deep.

Then Shelah's work shows that no fixed well-founded tree captures Isom^κ_T. We use this to show that Isom^κ_T is not Borel.

Shelah's work also shows that $L_{\infty\kappa}$ equivalent models of T of size κ are isomorphic. This means that the tree $t = \omega$ (with a single infinite branch) captures Isom^κ_T. As the games $\mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B})$ are determined, this shows that Isom^κ_T is Δ^1_1.

We must also show: $E_{S^\kappa_\mu}$ (equality modulo the μ-nonstationary ideal) is not Borel reducible to Isom^κ_T for any regular $\mu < \kappa$. This is because (in this case) Isom^κ_T is absolutely Δ^1_1, whereas μ-stationarity is not.

Now we look at the unclassifiable cases. Recall: Classifiable means superstable without DOP and without OTOP.

Case 3: T is unstable, superstable with DOP or superstable with OTOP.

Work of Hyttinen-Shelah [20] and Hyttinen-Tuuri [21] shows that in this case no tree of size κ without branches of length κ captures Isom^κ_T. This can be used to show Isom^κ_T is not Δ^1_1.

But $E_{S^\kappa_\lambda} \leq_B \mathrm{Isom}^\kappa_T$ is harder. Following Shelah, there is a Borel map $S \mapsto \mathcal{A}(S)$ from subsets of κ to Ehrenfeucht-Mostowski models of T built on linear orders so that $\mathcal{A}(S_0) \simeq \mathcal{A}(S_1)$ iff $S_0 = S_1$ modulo the λ-nonstationary ideal.

Case 4: T is stable but not superstable.

This is the hardest case and requires some new model theory. In our joint paper [10], Hyttinen replaces Ehrenfeucht-Mostowski models built on linear orders with primary models built on trees of height $\omega + 1$ to show $E_{S^\kappa_\omega} \leq_B \mathrm{Isom}^\kappa_T$. (We don't know if $E_{S^\kappa_\lambda} \leq_B \mathrm{Isom}^\kappa_T$ or if Isom^κ_T could be Δ^1_1 in this case.)

Now we have all we need to prove the Theorem mentioned earlier:

(a) T is classifiable and shallow iff Isom^κ_T is Borel.

We mentioned that if T is classifiable and shallow then Isom^κ_T is Borel and if it is classifiable and deep it is not. If T is not classifiable, then some $E_{S^\kappa_\mu}$ Borel reduces to Isom^κ_T, so the latter cannot be Borel.

(b) T is classifiable iff for all regular $\mu < \kappa$, $E_{S^\kappa_\mu}$ is not Borel reducible to Isom^κ_T.

We mentioned that if T is not classifiable then $E_{S^\kappa_\mu}$ is Borel reducible to Isom^κ_T where μ is either λ or ω. We also mentioned that if T is classifiable and deep then no $E_{S^\kappa_\mu}$ is Borel reducible to Isom^κ_T, by an absoluteness argument. When T is classifiable and shallow there is no such reduction as Isom^κ_T is Borel.

(c) In L, T is classifiable iff Isom^κ_T is Δ^1_1.

We mentioned that if T is classifiable then Isom^κ_T is Δ^1_1, in ZFC. If T is not classifiable, then $E_{S^\kappa_\mu}$ Borel reduces to Isom^κ_T for some μ, and in L, $E_{S^\kappa_\mu}$ is not Δ^1_1.

This summarises the work in [10]. Some surprisingly basic and very interesting open questions remain in this new area. Below are some of them. Assume $\kappa^{<\kappa} = \kappa$, as before.

1. Under what conditions on an uncountable κ does Vaught's Conjecture hold in the following form: If an isomorphism relation on the models of size κ has more than κ classes, then id is Borel reducible to it?
2. Is the Silver Dichotomy for uncountable κ consistent?
3. Is it consistent for there to be Borel equivalence relations which are incomparable under Borel reducibility for an uncountable κ?
4. Is it consistent that $S^{\omega_2}_{\omega_1}$ Borel reduces to $S^{\omega_2}_{\omega}$?
5. We proved that the isomorphism relation of a theory T is Borel if and only if T is classifiable and shallow. Is there a connection between the depth of a shallow theory and the Borel degree of its isomorphism relation? Is one monotone in the other?
6. Can it be proved in ZFC that if T is stable unsuperstable then isomorphism for the size κ models of T (κ uncountable) is not Δ^1_1?
7. If $\kappa = \lambda^+$, λ regular and uncountable, then does equality modulo the λ-nonstationary ideal Borel reduce to isomorphism for the size κ models of T for all stable unsuperstable T?
8. Let DLO be the theory of dense linear orderings without end points and RG the theory of random graphs. Does the isomorphism relation of RG Borel reduce to that of DLO for an uncountable κ?

Section 4. Complexity Theory

We consider NP equivalence relations on finite strings. One motivation for this topic is the following: Borel reducibility allows us to compare isomorphism relations on Borel classes of countable structures. Is there an analogous reducibility for "nice" classes of *finite* structures?

The resulting theory of "strong isomorphism reductions" is introduced in [9] and studied systematically in [2]. We consider polynomial-time definable classes C of structures for a finite vocabulary τ, where the structures in C have universe $\{1, \ldots, n\}$ for some finite $n > 0$ and where C is *invariant*, i.e., closed under isomorphism. To avoid trivialities we also assume that C contains arbitrarily large structures. Some examples of such classes are:

1. The classes SET, BOOLE, FIELD, GROUP, ABELIAN and CYCLIC of sets (structures of empty vocabulary), Boolean algebras, fields, groups, abelian groups, and cyclic groups, respectively.
2. The class GRAPH of (undirected and simple) graphs.
3. The class ORD of linear orderings.
4. The classes LOP of linear orderings with a distinguished point and LOU of linear orderings with a unary relation.

Let C and D be classes. We say that C is *strongly isomorphism reducible to* D and write $C \leq_{\text{iso}} D$, if there is a function $f\colon C \to D$ computable in polynomial time such that for all $\mathcal{A}, \mathcal{B} \in C$, $\mathcal{A} \simeq \mathcal{B}$ iff $f(\mathcal{A}) \simeq f(\mathcal{B})$. We then say that f is a *strong isomorphism reduction* from C to D and write $f : C \leq_{\text{iso}} D$. If $C \leq_{\text{iso}} D$ and $D \leq_{\text{iso}} C$, denoted by $C \equiv_{\text{iso}} D$, then C and D *have the same strong isomorphism degree.*

Examples:
(a) The map sending a field to its multiplicative group shows that FIELD $\leq_{\text{iso}}$ CYCLIC.
(b) CYCLIC $\leq_{\text{iso}}$ ABELIAN $\leq_{\text{iso}}$ GROUP; more generally, if $C \subseteq D$, then $C \leq_{\text{iso}} D$ via the identity.
(c) SET $\equiv_{\text{iso}}$ FIELD $\equiv_{\text{iso}}$ ABELIAN $\equiv_{\text{iso}}$ CYCLIC $\equiv_{\text{iso}}$ ORD $\equiv_{\text{iso}}$ LOP. (For the proof see [2].)

Proposition 19. *$C \leq_{\text{iso}}$ GRAPH for all classes C.*

The structure of $\leq_{\text{iso}}$ between LOU and GRAPH is linked with central open problems of descriptive complexity. Before turning to that I'll first consider the structure below LOU. That structure, even below LOP, is quite rich.

Theorem 20. *The partial ordering of the countable atomless Boolean algebra is embeddable into the partial ordering induced by $\leq_{\text{iso}}$ on the degrees of strong isomorphism reducibility below LOP. More precisely, let $\mathcal{B}$ be the countable atomless Boolean algebra. Then there is a one-to-one function $b \mapsto C_b$ defined on B such that for all $b, b' \in B$:*

(i) C_b is a subclass of LOP
(ii) $b \leq b'$ iff $C_b \leq_{\text{iso}} C_{b'}$.

This result is obtained by comparing the number of isomorphism types of structures with universe of bounded cardinality in different classes. For a class C we let $C(n)$ be the subclass consisting of all structures in C with universe of cardinality $\leq n$ and we let $\#C(n)$ be the number of isomorphism types of structures in $C(n)$. Examples:

$\#\text{BOOLE}(n) = [\log n]$, $\#\text{CYCLIC}(n) = n$, $\#\text{SET}(n) = \#\text{ORD}(n) = n + 1$.
$\#\text{LOP}(n) = \sum_{i=1}^{n} i = (n + 1) \cdot n/2$ and $\#\text{LOU}(n) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.
$\#\text{GROUP}(n)$ is superpolynomial but subexponential (more precisely, it is bounded by $n^{O(\log^2 n)}$). See [1].

A class C is *potentially reducible* to a class D, written $C \leq_{\text{pot}} D$, iff there is some polynomial p such that $\#C(n) \leq \#D(p(n))$ for all $n \in \mathbb{N}$. Of course, by $C \equiv_{\text{pot}} D$ we mean $C \leq_{\text{pot}} D$ and $D \leq_{\text{pot}} C$.

Lemma 21. *If $C \leq_{\text{iso}} D$, then $C \leq_{\text{pot}} D$.*

Proof. Let $f : C \leq_{\text{iso}} D$. As f is computable in polynomial time, there is a polynomial p such that for all $\mathcal{A} \in C$ we have $|f(A)| \leq p(|A|)$, where $f(A)$ denotes the universe of $f(\mathcal{A})$. As f strongly preserves isomorphisms, it therefore induces a one-to-one map from $\{\mathcal{A} \in C \mid |A| \leq n\}/_{\simeq}$ to $\{\mathcal{B} \in D \mid |B| \leq p(n)\}/_{\simeq}$. □

We state some consequences of this simple observation:

Proposition 22. *1. CYCLIC* $\not\leq_{\text{iso}}$ *BOOLE and LOU* $\not\leq_{\text{iso}}$ *LOP.*
2. $C \leq_{\text{pot}}$ *LOU for all classes* C *and LOU* $\equiv_{\text{pot}}$ *GRAPH.*
3. The strong isomorphism degree of GROUP is strictly between that of LOP and GRAPH.
4. The potential reducibility degree of $GROUP$ *is strictly between that of LOP and* LOU.

The following concepts are used in the proof of Theorem 20. We call a function $f : \mathbb{N} \to \mathbb{N}$ *value-polynomial* iff it is increasing and $f(n)$ can be computed in time $f(n)^{O(1)}$. Let VP be the class of all value-polynomial functions. For $f \in \text{VP}$ the set $C_f = \{\mathcal{A} \in \text{LOP} \mid |A| \in \text{im}(f)\}$ is in polynomial time and is closed under isomorphism. As there are exactly $f(k)$ pairwise non-isomorphic structures of cardinality $f(k)$ in LOP, we get

$$\#\mathcal{C}_f(n) = \sum_{k \in \mathbb{N} \text{ with } f(k) \leq n} f(k).$$

The following proposition contains the essential idea underlying the proof of Theorem 20. Loosely speaking, it says that if the gaps between consecutive values of $f \in \text{VP}$ "kill" every polynomial, then there are classes C and D with $C \not\leq_{\text{pot}} D$.

Proposition 23. *Let* $f \in VP$ *and assume that for every polynomial* $p \in \mathbb{N}[X]$ *there is an* $n \in \mathbb{N}$ *such that*

$$\sum_{k \in \mathbb{N} \textit{ with } f(2k) \leq n} f(2k) > \sum_{k \in \mathbb{N} \textit{ with } f(2k+1) \leq p(n)} f(2k+1).$$

Then $\mathcal{C}_{g_0}$ *is not potentially reducible to* $\mathcal{C}_{g_1}$*, where* $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ *are defined by* $g_0(n) := f(2n)$ *and* $g_1(n) := f(2n+1)$.

Proof. For contradiction assume that there is some polynomial p such that $\#\mathcal{C}_{g_0}(n) \leq \#\mathcal{C}_{g_1}(p(n))$ for all $n \in \mathbb{N}$. Choose n to satisfy the hypothesis. Then

$$\#\mathcal{C}_{g_0}(n) = \sum_{f(2k) \leq n} f(2k) > \sum_{f(2k+1) \leq p(n)} f(2k+1) = \#\mathcal{C}_{g_1}(p(n)),$$

a contradiction. □

The other needed ingredient for the proof of Theorem 20 is:

Lemma 24. *The images of the functions in VP together with the finite subsets of* $\mathbb{N}$ *are the elements of a countable Boolean algebra* $\mathcal{V}$ *(under the usual set-theoretic operations). The factor algebra* $\mathcal{V}/_{\equiv_{\text{pot}}}$*, where for* $b, b' \in V$

$$b \equiv b' \iff (b \setminus b') \cup (b' \setminus b) \text{ is finite},$$

is a countable atomless Boolean algebra.

This lemma shows that the set of images of functions in VP has a rich structure. To complete the proof of Theorem 20, the functions in VP are composed with a "stretching" function h, which guarantees that the gaps between consecutive values "kill" every polynomial. Then we can apply the idea of the proof of Proposition 23 to show that the set of the $\leq_{\text{pot}}$-degrees has a rich structure too. For the details see [2].

So far, in all concrete examples of classes C and D for which we know the status of $C \leq_{\text{iso}} D$ and of $C \leq_{\text{pot}} D$, we have $C \leq_{\text{iso}} D$ iff $C \leq_{\text{pot}} D$. So the question arises whether the relations of strong isomorphism reducibility and potential reducibility coincide. We believe that they are distinct but have only the following partial result:

Theorem 25. *If UEEXP* $\cap$ *coUEEXP* $\neq$ *EEXP, then the relations of strong isomorphism reducibility and that of potential reducibility are distinct.*

Recall that $\text{EEXP} = \text{DTIME}\left(2^{2^{n^{O(1)}}}\right)$ and $\text{NEEXP} := \text{NTIME}\left(2^{2^{n^{O(1)}}}\right)$.

The complexity class $UEEXP$ consists of those $Q \in \text{NEEXP}$ for which there is a non-deterministic Turing machine of type NEEXP that for every $x \in Q$ has exactly one accepting run. Finally, $\text{coUEEXP} := \{\sim Q \mid Q \in \text{UEEXP}\}$.

Here is the idea of the proof: Assume $Q \in \text{UEEXP} \cap \text{coUEEXP}$. We construct classes C and D which contain structures in the same cardinalities and which contain exactly two non-isomorphic structures in these cardinalities. Therefore they are potentially reducible to each other. While it is trivial to exhibit two non-isomorphic structures in C of the same cardinality, from any two non-isomorphic structures in D we obtain information on membership in Q for all strings of a certain length. If $C \leq_{\text{iso}} D$ held, then we would get non-isomorphic structures in D (in time allowed by EEXP) by applying the strong isomorphism reduction to two non-isomorphic structures in C and therefore obtain $Q \in \text{EEXP}$.

In the other direction we have:

Theorem 26. *If strong isomorphism reducibility and potential reducibility are distinct, then* $P \neq \#P$.

Recall that $P = \#P$ means that for every polynomial time non-deterministic Turing machine $\mathbb{M}$ the function $f_{\mathbb{M}}$ such that $f_{\mathbb{M}}(x)$ is the number of accepting runs of $\mathbb{M}$ on $x \in \Sigma^*$ is computable in polynomial time. The class $\#P$ consists of all the functions $f_{\mathbb{M}}$.

Until now we have focused exclusively on isomorphism relations on invariant polynomial time classes of finite structures. But this theory can be put into the broader context of NP equivalence relations in general. If E and E' are NP equivalence relations, then we say that E is *strongly equivalence reducible to* E', and write $E \leq_{\mathrm{eq}} E'$, iff there is a function f computable in polynomial time such that for all strings x, y: xEy iff $f(x)E'f(y)$. We then say that f is a *strong equivalence reduction* from E to E' and write $f : E \leq_{\mathrm{eq}} E'$. The following natural question then arises: Is there a *maximal* NP equivalence relation under the reducibility $\leq_{\mathrm{eq}}$? The final section of [2] relates this question to enumerations of clocked Turing machines, to p-optimal proof systems as well as to other central questions in complexity theory.

Another natural question is whether, in analogy to the computability theory context, every NP equivalence relation is reducible to an isomorphism relation on a polynomial time invariant class of finite structures, or equivalenty, whether graph isomorphism is $\leq_{\mathrm{eq}}$ complete among NP equivalence relations. For this we have the following partial result:

Proposition 27. *(*[2]*) Assume that the polynomial time hierarchy does not collapse. Then* not *every NP equivalence relation reduces to graph isomorphism.*

Indeed there are many worthy open questions in this area waiting to be explored.

In conclusion

After decades of work focusing on the "unary" case, definability theory has been dramatically deepened by the study of binary relations, most importantly equivalence relations. An important step in this process was taken in Harvey's fundamental paper with Lee Stanley [8]. The extent to which the different areas of logic have been enriched through the study of analogues of Harvey's idea is only now being understood, and I look forward to seeing much exciting work in this direction during the coming years.

References

[1] H.U. Besche, B. Eick and E.A. O'Brien, *The groups of order at most 2000*, Electronic Research announcements of the American Mathematical Society, **7** (2001), 1–4.

[2] S. Buss, Y. Chen, J. Flum, S. Friedman and M. Müller, *Strong isomorphism reductions in complexity theory*, Journal of Symbolic Logic, December 2011.

[3] E. Fokina, S. Friedman, *Equivalence relations on classes of computable structures*, Proceedings of "Computability in Europe 2009", Heidelberg, Germany, Lecture Notes in Computer Science, **5635** (2009), 198–207.

[4] E. Fokina, S. Friedman, *On* Σ^1_1 *equivalence relations over the natural numbers*, to appear, Mathematical Logic Quarterly.

[5] E. Fokina, S. Friedman, V. Harizanov, J. Knight, C. McCoy and A. Montalban, *Isomorphism relations on computable structures*, Journal of Symbolic Logic, March 2012.

[6] E. Fokina, S. Friedman, A. Törnquist, *The effective theory of Borel equivalence relations*, Annals of Pure and Applied Logic, **161** (2010), 837–850.

[7] E. Fokina, J. Knight, C. Maher, A. Melnikov, S. Quinn, *Classes of Ulm type, and relations between the class of rank-homogeneous trees and other classes*, submitted.

[8] Friedman, H., and L. Stanley, *A Borel reducibility theory for classes of countable structures*, J. Symb. Logic, **54** (1989), 894–914.

[9] S. Friedman, *Descriptive set theory for finite structures*, Lecture at the Kurt Gödel Research Center, 2009. Available at `http://www.logic.univie.ac.at/~sdf/papers/wien-spb.pdf`.

[10] S. Friedman, T. Hyttinen and V. Kulikov, *Generalized descriptive set theory and classification theory*, submitted, see `http://www.logic.univie.ac.at/~sdf/papers/joint.tapani.vadim.pdf`.

[11] S. D. Friedman, L. Motto Ros, *Analytic equivalence relations and bi-embeddability*, Journal of Symbolic Logic, **76(1)** (2011), 1581–1587.

[12] S. Gao, *Invariant descriptive set theory*, Pure and Applied mathematics, CRC Press/Chapman & Hall, 2009.

[13] S. Gao and P.M. Gerdes, *Computably enumerable equivalence relations*, Studia Logica, **67** (2001), 27–59.

[14] L. Harrington, *McLaughlin's conjecture*, Handwritten notes, 1976.

[15] L. Harrington, *Arithmetically incomparable arithmetical singletons*, Handwritten notes, 1975.

[16] L. Harrington, A. Kechris and A. Louveau, *Glimm-Efros dichotomy for Borel equivalence relations*, Journal of the American Mathematical Society, **3(4)** (1990), 903–928.

[17] Hjorth, G., *The isomorphism relation on countable torsion-free Abelian groups*, Fund. Math., **175** (2002), 241–257.

[18] G. Hjorth, *Classification and orbit equivalence relations*, Mathematical surveys and monographs 75, American Mathematical Society, 2000.

[19] G. Hjorth and A. Kechris, *Recent developments in the theory of Borel reducibility*, Fundamenta Mathematicae, **170(1–2)** (2001), 21–52.

[20] T. Hyttinen and S. Shelah, Constructing strongly equivalent nonisomorphic models for unsuperstable theories, Part C, Journal of Symbolic Logic, **64(2)** (1999).

[21] T. Hyttinen and H. Tuuri, *Constructing strongly equivalent nonisomorphic models*, Annals of Pure and Applied Logic, **52(3)** (1991).

[22] S. Jackson, A. Kechris and A. Louveau, *Countable Borel equivalence relations*, Journal of Mathematical Logic, **2(1)** (2002), 1–80.

[23] V. Kanovei, *Borel equivalence relations. Structure and classification*, Unviersity Lecture Series, 44, American Mathematical Society, 2008.

[24] A. Kechris, *Measure and category in effective descriptive set theory*, Annals of Pure and Applied Logic, **5** (1973), 337–384.

[25] A. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Springer-Verlag, 1995.

[26] A. Kechris, *New directions in descriptive set theory*, Bull. Symbolic Logic, **5(2)** (1999), 161–174.

[27] A. Kechris, A. Louveau *The classification of hypersmooth Borel equivalence relations*, J. of the American Math. Society, **10(1)** (1997), 215–242.
[28] M. Koerwien, *A complicated ω-stable depth* 2 *theory*, to appear, Journal of Symbolic Logic.
[29] C. Laskowski, *An old friend revisited: Countable models of omega-stable theories*, Proceedings of the Vaught's Conjecture conference, Notre Dame Journal of Formal Logic, **48** (2007), 133–141.
[30] A. Louveau, C. Rosendal, *Complete analytic equivalence relations*, Trans. Amer. Math. Soc., **357(12)** (2005), 4839–4866.
[31] D. Marker, *The Borel complexity of isomorphism for theories with many types*, preprint.
[32] A. Montalbán, *On the equimorphism types of linear orderings*, Bulletin of Symbolic Logic, **13** (2007), 71–99.
[33] P. G. Odifreddi, *Classical recursion theory*, volume II, North-Holland Publishing Co., 1999.
[34] H. Rogers, *Theory of recursive functions and effective computability*, McGraw-Hill, 1967.
[35] G. Sacks, *Higher recursion theory*, Springer-Verlag, 1989.
[36] S. Shelah, *Classification theory*, revised edition, North Holland, 1990.
[37] J. H. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Annals of Mathematical Logic, **18** (1980), 1–18.

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