THE NON-ABSOLUTENESS OF MODEL EXISTENCE IN UNCOUNTABLE CARDINALS FOR $L_{\omega_1,\omega}$

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Abstract. For sentences ϕ of $L_{\omega_1,\omega}$, we investigate the question of absoluteness of ϕ having models in uncountable cardinalities. We first observe that having a model in $\aleph_1$ is an absolute property, but having a model in $\aleph_2$ is not as it may depend on the validity of the Continuum Hypothesis. We then consider the GCH context and provide sentences for any $\alpha \in \omega_1 \setminus \{0, 1, \omega\}$ for which the existence of a model in $\aleph_\alpha$ is non-absolute (relative to large cardinal hypotheses).

Throughout, we assume ϕ is an $L_{\omega_1,\omega}$ sentence which has infinite models. By Löwenheim-Skolem, ϕ must have a countable model, so the property "having a countable model" is an absolute property of such sentences in the sense that its validity does not depend on the properties of the set theoretic universe we work in, i.e. it is a consequence of ZFC. A main tool for absoluteness considerations is Shoenfields absoluteness Theorem (Theorem 25.20 in [7]). It states that any property expressed by either a $\mathbf{\Sigma}^1_2$ or a $\mathbf{\Pi}^1_2$ formula is absolute between transitive models of ZFC.

The purpose of this paper is to investigate the question of how far we can replace "countable" by higher cardinalities. As John Baldwin observed in [2], it follows from results of [6] that the property of ϕ having arbitrarily large models is absolute (it can be expressed in form of the existence of an infinite indiscernible sequence, which by Shoenfield absoluteness is absolute). Consequently, since the Hanf number of the logic $L_{\omega_1,\omega}$ equals $\beth_{\omega_1}$, the existence of models in cardinalities above that number is absolute. Therefore the context we are interested in is where ϕ (absolutely) does not have a model of size $\beth_{\omega_1}$.

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1. The case $\aleph_1$

For *complete* sentences ϕ (meaning that any model of ϕ satisfies the same $L_{\omega_1,\omega}$ sentences), having a model in $\aleph_1$ is an absolute notion. We have the following characterization (see also [2]) of ϕ having a model of size $\aleph_1$ (which is a $\mathbf{\Sigma}^1_1$ property and therefore absolute by Shoenfield's absoluteness Theorem):

> $(*)$ There exist two countable models M, N of ϕ such that M is a proper elementary (in the fragment of ϕ) substructure of N.

To see that this is a characterization, note first that if ϕ has an uncountable model, $(*)$ holds by Löwenheim-Skolem. For the converse, we use the completeness of ϕ which implies that any two countable models of ϕ are isomorphic (by Scotts isomorphism Theorem, since ϕ must imply Scott sentences of countable models). Then, as $N \cong M$, we can find a proper countable $L_{\omega_1,\omega}$-elementary extension of N as well and continue this procedure ω_1 many times (taking unions at limit stages). The union of this elementary chain will then be a model of ϕ of size $\aleph_1$.

If the sentence is not complete, there might be examples of ϕ having an uncountable model, where $(*)$ fails (Gregory claimed the existence of such an example in [5]). However having a model of size $\aleph_1$ turns out to be absolute in general[1]. We have to provide a slightly more subtle criteria to deal with possibly incomplete ϕ. To state it, we have to regard the sentence ϕ as a set theoretic object using standard coding of formulas of $L_{\omega_1,\omega}$. ϕ can thus be regarded as a hereditarily countable set.

The following property which (again by Shoenfield) is absolute, characterizes ϕ having a model of size $\aleph_1$:

> $(**)$ There is a countable model U of ZFC$^-$ (ZFC without the power set axiom) containing ϕ with $U \models$ "ω_1 exists and there is a model of ϕ with universe ω_1".

First, suppose ϕ has a model M of size $\aleph_1$, say one with universe ω_1. As both ϕ and M are elements of H_{ω_2} (the collection of sets hereditarily of size at most $\aleph_1$), we have $H_{\omega_2} \models$ ZFC$^-$ + "there is a model of ϕ with universe ω_1". Now it suffices to take a countable (first order) elementary substructure $U \prec H_{\omega_2}$ containing ϕ, and U will have the properties of $(**)$.

Conversely, assuming $(**)$ holds for some countable U, we can take an elementary extension U' of U where all (in the sense of U) countable sets are unchanged and all (in U) uncountable ones become sets of size $\aleph_1$ (using Corollary A of Theorem 36 in [9]). In particular this is true for the ω_1 of U' on which we know a model M of ϕ lives (note that $U' \models (M \models \phi)$ implies that $M \models \phi$ in the real universe; to see this, consider U' as a transitive model (via the Mostowski-collapse)

[1]This has also been observed recently by Paul Larson. His argument uses iterated generic ultrapowers. Rami Grossberg points out, he knew of this fact already in the 1980's but did not publish it, and that others like Shelah, Barwise and Keisler most likely knew of it even earlier.

and use that satisfaction can be expressed by a Δ_0-formula). So we get a model of ϕ of size $\aleph_1$.

There is another absolute criteria characterizing ϕ having an uncountable model, but it requires going beyond the logic $L_{\omega_1,\omega}$. Let us consider the extension $L_{\omega_1,\omega}(Q)$ of $L_{\omega_1,\omega}$ obtained by adding an extra quantifier Q with the semantics "there exist uncountably many". As is shown in [3], $L_{\omega_1,\omega}(Q)$ admits a completeness theorem which actually has a very natural (absolute) deduction calculus. Now the statement

$(***)$ There is a proof of $\neg Qx(x = x)$ starting from ϕ

characterizes ϕ having only countable models. Thus the negation of $(***)$ is an (absolute) property characterizing ϕ having an uncountable model. Note that this argument shows that model existence in $\aleph_1$ is absolute even for $L_{\omega_1,\omega}(Q)$ sentences.

2. Going beyond $\aleph_1$

It is not generally true that the existence of a model of size $\aleph_2$ is an absolute property.

A very simple way to see this is to take any sentence ϕ that has models exactly up to size continuum. We easily find even *complete* sentences with this property. Then clearly, ϕ has a model of size $\aleph_2$ if and only if the continuum hypothesis fails.

More generally, such a sentence has a model of size $\aleph_\alpha$ if and only if $2^{\aleph_0} \geq \aleph_\alpha$. So for any $\alpha > 1$, the existence of a model of size $\aleph_\alpha$ is non-absolute.

There are many examples of complete $L_{\omega_1,\omega}$-sentences in the literature having models exactly up to size continuum, but they are mostly more complicated than necessary for our purposes, because their authors have been interested in additional properties. Therefore we provide here a very simple such example:

Let the language L consist of countably many binary relation symbols E_n $(n < \omega)$, and let $\sigma \in L_{\omega_1,\omega}$ be the conjunction of

- All E_n are equivalence relations such that E_0 has two classes and each E_n-class is the union of exactly two E_{n+1}-classes.
- $\forall x, y((\bigwedge_{n<\omega} E_n(x, y)) \to x = y)$

It is an easy back-and-forth argument to show that any two countable models of σ are isomorphic, so σ is complete. Every model represents a set of branches through a full binary tree, so there cannot be models greater than the continuum. On the other hand, the Cantor space 2^ω together with the relations "$E_n(x, y)$ if and only if x and y coincide on the $n + 1$ first components" is a model of σ of size continuum.

3. Going beyond $\aleph_1$ under the assumption of GCH

As we have seen, playing with the cardinal exponential function provides trivial examples for the non-absoluteness of the existence of models of cardinality greater than $\aleph_1$. A next natural question is if this is the only non-absoluteness phenomenon there is. That is, under the additional assumption of GCH does the existence of models in cardinalities greater than $\aleph_1$ become an absolute notion?

We will now provide two sequences of examples which answer this question negatively for all cardinalities $\aleph_\alpha$ where $\alpha > 1$ is a countable ordinal, not equal to ω. The first sequence of examples works for α not limit or successor of a limit ordinal, the second one for all $\alpha > \omega$. Before we start either construction, we will define sentences which will be used in both cases.

3.1. **Auxiliary sentences.** Let $L_0^\alpha = \{Q_\beta, a_n, <, F\}_{\beta\leq\alpha;\ n<\omega}$, where the Q_β are unary predicates, the a_n are constant symbols, $<$ is a binary and F a ternary relation symbol.

Let $\sigma_0^\alpha \in (L_0^\alpha)_{\omega_1,\omega}$ be the conjunction of the following sentences:

- The universe is the union of all Q_β.
- $Q_0 = \{a_n | n < \omega\}$ where all a_n designate distinct elements.
- For any $\beta < \alpha$, $Q_{\beta+1}$ is disjoint from any Q_γ for all $\gamma \leq \beta$.
- For any limit ordinal $\beta \leq \alpha$, $Q_\beta = \bigcup\limits_{\gamma<\beta} Q_\gamma$.
- $<$ linearly orders $Q_{\beta+1}$ for every $\beta < \alpha$ and $x < y$ implies that for some $\beta < \alpha$, both x and y belong to $Q_{\beta+1}$.
- $F(a, b, c)$ implies that for some $\beta < \alpha$, $a \in Q_{\beta+1}$, $b < a$ and $c \in Q_\beta$.
- For every $\beta < \alpha$ and every $a \in Q_{\beta+1}$, $F(a, \cdot, \cdot)$ defines a total injective function from $\{x|x < a\}$ into Q_β.

Note that for β a limit ordinal or zero, Q_β is not ordered by $<$ and if $\alpha = 0$, both $<$ and F are empty relations.

Definition 1. *Let κ be an infinite cardinal and $<$ be a total linear ordering on some set X. This ordering is called* κ-like, *if all proper initial segments have cardinality strictly less than κ.*

Q_0 is countable by definition and thus the ordering on Q_1 is ω_1-like by the properties of F. Also, for any $\beta < \alpha$, $|Q_{\beta+1}| \leq |Q_\beta|^+$ because F forces the ordering on $Q_{\beta+1}$ to be $|Q_\beta|^+$-like. Consequently, the cardinality of Q_β is at most $\aleph_\beta$ for all $\beta \leq \alpha$ and the maximum possible cardinality of a model of σ_0^α is $\aleph_\alpha$.

3.2. **Coding Kurepa-families.** Suppose in this section that there is some countable ordinal δ such that $\alpha = \delta + 2$. First, we recall a classical definition.

Definition 2. *Let κ be any infinite cardinal. A κ^+* Kurepa-family *is a family $\mathcal{F}$ of subsets of some set A with $|A| = \kappa^+$, such that $|\mathcal{F}| > \kappa^+$ and for any subset $B \subset A$ with $|B| = \kappa$, $|\{X \cap B | X \in \mathcal{F}\}| \leq \kappa$.*

Let KH_{κ^+} be the statement that there exists a κ^+-Kurepa-family.

Let $L_1^\alpha = L_0^\alpha \cup \{R, T\}$ where R is a binary and T a ternary relation symbol. We will now define a sentence $\sigma_1^\alpha \in (L_1^\alpha)_{\omega_1,\omega}$ with the following properties: every model of σ_1^α is a model of σ_0^α with the additional property that if in a model of σ_1^α for some β we have $|Q_{\beta+2}| > |Q_{\beta+1}| > |Q_\beta|$, then the elements of $Q_{\beta+2}$ will code (via R seen as a relation between $Q_{\beta+2}$ and $Q_{\beta+1}$) a $|Q_{\beta+1}|$ Kurepa-family on $Q_{\beta+1}$. The relation T will be used to code for any initial segment I of $(Q_{\beta+1}, <)$ $|I|$ many subsets of I so that we can axiomatize the property of any set coded by R to have an intersection with I coinciding with one of those sets.

Let $\sigma_1^\alpha \in (L_1^\alpha)_{\omega_1,\omega}$ be the conjunction of σ_0^α and the following:

- $T(a, b, c)$ implies that for some $\beta < \alpha$, $a, b, c \in Q_{\beta+1}$
- $R(a, b)$ implies that for some β with $\beta+1 < \alpha$, $a \in Q_{\beta+2}$ and $b \in Q_{\beta+1}$ and noting $A_q = \{x \in Q_{\beta+1} | R(q, x)\}$ for q in $Q_{\beta+2}$, $q \neq q'$ implies $A_q \neq A_{q'}$.
- For all β with $\beta + 1 < \alpha$ and $a \in Q_{\beta+2}$, $b \in Q_{\beta+1}$, there is some $c < b$ such that $A_a \cap \{x | x < b\} = \{x | T(b, c, x)\}$.

Lemma 3. *Suppose for some $\beta < \alpha$, $|Q_{\beta+1}| > |Q_\beta|$ (i.e. $|Q_{\beta+1}| = |Q_\beta|^+$). Then the ordering on $Q_{\beta+1}$ must have cofinality $|Q_{\beta+1}|$.*

Proof. Each inital segment $I_b = \{x | x < b\}$ of the order on $Q_{\beta+1}$ has size at most $|Q_\beta|$ by the properties of F. If $(b_i)_{i<\lambda}$ is cofinal in that order, $Q_{\beta+1} = \bigcup_{\gamma<\lambda} I_{b_\gamma}$ implies $|Q_{\beta+1}| = \lambda \cdot |Q_\beta|$ which must equal λ (because we assume $|Q_{\beta+1}| > |Q_\beta|$). □

Lemma 4. *Suppose β is such that $\beta + 1 < \alpha$ and let $\kappa = |Q_\beta|$. Suppose the cardinalities increase in the next two steps, i.e. that $|Q_{\beta+1}| = \kappa^+$ and $|Q_{\beta+2}| = \kappa^{++}$. Then $\mathcal{F} = \{A_q | q \in Q_{\beta+2}\}$ is a κ^+-Kurepa-family (on $Q_{\beta+1}$).*

Proof. Let $X \subset Q_{\beta+1}$ be of cardinality at most κ. Since κ^+ is regular and the cofinality of the order on $Q_{\beta+1}$ equals κ^+ (by Lemma 3, using $|Q_\beta| = \kappa$ and $|Q_{\beta+1}| = \kappa^+$), X is included in some initial segment $I_a = \{x | x < a\}$ of $(Q_{\beta+1}, <)$. Using the properties of R and T, $|\{Y \cap X | Y \in \mathcal{F}\}| \leq |\{Y \cap I_a | Y \in \mathcal{F}\}| \leq |I_a| \leq |Q_\beta| = \kappa$. Thus, since $|\mathcal{F}| = |Q_{\beta+2}| = \kappa^{++} > \kappa^+$, $\mathcal{F}$ satisfies the definition of a κ^+-Kurepa-family. □

Proposition 5. *Suppose for some $\alpha < \omega_1$, KH_κ is true for every successor cardinal $\kappa < \aleph_\alpha$. Then σ_1^α has a model of cardinality $\aleph_\alpha$.*

Proof. We can explicitly construct such a model for every α using the given Kurepa-families.

Let the Q_β be any sets (with the disjointness and union properties formulated in σ_0^α) of cardinality $\aleph_\beta$ and order them in the order type of $\aleph_\beta$ (except for β zero or limit). We obviously have no trouble defining the F relations at this point.

Next, define the R relations such that the sets A_q form Kurepa-families isomorphic to the given ones in the corresponding cardinalities.

Finally, the T-relations have to be defined such that the sets coded on the initial segments by them capture all possibilities for intersections of sets A_q with that initial segment. $\square$

Proposition 6. *Suppose for some $\delta < \omega_1$, $\mathrm{KH}_{\aleph_{\delta+1}}$ fails. Then $\sigma_1^{\delta+2}$ has no model of cardinality $\aleph_{\delta+2}$.*

Proof. Assume the contrary for some $\delta < \omega_1$ and let $M \models \sigma_1^{\delta+2}$ be of cardinality $\aleph_{\delta+2}$. Clearly, $Q_{\delta+k}$ must be exactly of cardinality $\aleph_{\alpha+k}$ for all $k \in \{0,1,2\}$ (since these are the maximum possible cardinalities for those sets). Now we get a $\aleph_{\delta+1}$-Kurepa-family by Lemma 4, contradicting our assumption. $\square$

In conclusion, for any $\alpha < \omega_1$ which is not a limit or successor of a limit ordinal, we get an example of an (incomplete) $L_{\omega_1,\omega}$-sentence such that the existence of a model in $\aleph_\alpha$ is non-absolute, even under the assumption of GCH, as the existence of the model depends on the existence of Kurepa-families.

As for the consistency of the assumptions of the Propositions 5 and 6, we remark that it is folklore that the existence of Kurepa-families in different $\aleph_\alpha$ ($\alpha < \omega_1$) is independent from one another. We will now describe the formal arguments for the cases we need (essentially the same arguments would work more generally for "switching on and off" independently the existence of Kurepa-families in different $\aleph_\alpha$). In the constructible universe, KH_{κ^+} is true for all cardinals κ (this follows from the fact that $\diamondsuit^+$ holds at successor cardinals in L, see [8]). On the other hand we have:

Theorem 7. *The consistency of "ZFC+there are uncountably many inaccessible cardinals" implies the consistency of "ZFC+GCH+$\forall\alpha < \omega_1 \neg\mathrm{KH}_{\aleph_{\alpha+1}}$"*

Proof. This is a slight generalisation of Silver's argument that if κ is inaccessible then after forcing with $\mathrm{Coll}(\omega_1, <\kappa)$, the forcing to convert κ into $\aleph_2$ with countable conditions, $\mathrm{KH}_{\aleph_1}$ fails (see [7]).

Assume GCH, let κ_0 be $\aleph_1$ and let $(\kappa_\beta)_{0<\beta<\omega_1}$ enumerate the first ω_1-many inaccessible cardinals in increasing order. Let P be the fully supported product of the forcings $\mathrm{Coll}(\kappa_\beta, <\kappa_{\beta+1})$ for $\beta < \omega_1$. Then in the extension, κ_β equals $\aleph_{\beta+1}$. We claim that $\mathrm{KH}_{\kappa_\beta}$ fails for each $\beta < \omega_1$.

Indeed, the forcing P can be factored as $P(<\beta) \times P(\geq \beta)$ where $P(<\beta)$ refers only to the collapses $\mathrm{Coll}(\kappa_\gamma, <\kappa_{\gamma+1})$ for $\gamma < \beta$ and $P(\geq \beta)$ refers only to the the collapses $\mathrm{Coll}(\kappa_\gamma, <\kappa_{\gamma+1})$ for $\gamma \geq \beta$. Similarly, $V[G]$ factors as $V[G(<\beta)][G(\geq\beta)]$. In the model $V[G(<\beta)]$, $\kappa_{\beta+1}$ is still inaccessible, so we can apply Silver's argument to conclude that $\mathrm{KH}_{\kappa_{\beta+1}}$ fails in $V[G(<\beta)][G(\geq\beta)] = V[G]$, using the closure of the forcing $P(\geq\beta)$ under sequences of length less than κ_β. $\square$

Why does our sequence of examples not work for limit cardinals or limit successors?

In the absence of any Kurepa-families, we can still have $|Q_{\beta+1}| = |Q_\beta|^+$ for some β, just not two times in a row in order to not contradict Lemma 4. For example, in the case $\kappa = |Q_\beta| = |Q_{\beta+1}|$, the ordering on $Q_{\beta+1}$ may be such that every initial segment has cardinality κ and the cofinality is κ as well. Then, assuming T codes sufficiently different sets on every initial segment of $Q_{\beta+1}$, we have $\kappa^\kappa = \kappa^+$ many possibilities for sets A_q for $q \in Q_{\beta+2}$, so $Q_{\beta+2}$ may have cardinality κ^+.

That is, still assuming no kind of Kurepa-family exists, the biggest models we can get are those where for any cardinality $\kappa > \aleph_0$, there are not more than two sets Q_β of size κ. For finite n, we see that in this context the maximum cardinality of a model of σ_1^n is $\aleph_k$ where k is the smallest integer greater or equal to $\frac{n}{2}$. In particular, we see that for any limit ordinal or successor of a limit ordinal α, σ_1^α *absolutely* has a model of cardinality $\aleph_\alpha$.

To find an example with non-absolute existence of a model in $\aleph_\omega$ for example, one strategy could be to construct sentences ϕ_n ($n < \omega$) and find two set theoretic properties A, B which are both compatible with GCH, such that for any $n < \omega$, the maximum cardinality of a model of ϕ_n is $\aleph_n$ if A holds and $\aleph_1$ if B holds. Then we could define a new sentence whose models are the union of models of ϕ_n for all n and get an example where under A there is a model in $\aleph_\omega$, and under B there is none. Using the Kurepa techniques, we cannot seem to make the "gap" bigger than from $\aleph_{\frac{n}{2}}$ to $\aleph_n$.

We also remark that there are slight variations of the sentences σ_1^α which still give non-absoluteness of model-existence in $\aleph_\alpha$, while needing weaker assumptions in the Propositions 5 and 6. For example, if we code Kurepa-families only on the top level (subsets of $Q_{\alpha-1}$) by restricting R to $Q_\alpha \times Q_{\alpha-1}$, the sentence will have a model of size $\aleph_\alpha$ if and only if an $\aleph_{\alpha-1}$-Kurepa-family exists. For model existence in $\aleph_n$ for *finite* n, we could also choose to code Kurepa-families only on the lowest level (subsets of Q_1). Then the sentence has a model in $\aleph_n$ if and only if an $\aleph_1$-Kurepa-family exists.

3.3. **Coding special Aronszajn trees.** To deal with limits (greater than ω) and limit successors, we use the concept of "special Aronszajn trees":

Definition 8. *A* tree *is a partially ordered set* $(T, <)$ *such that for any element* $t \in T$*, the set* $\{x|x < t\}$ *is well ordered by* $<$*. The* rank $\mathrm{rk}(t)$ *of* t *is the order type of* $\{x|x < t\}$*. For any ordinal* α*, let* $T_\alpha = \{t \in T|\mathrm{rk}(t) = \alpha\}$*.*

For any cardinal κ*, a* κ^+*-tree is a tree* T *such that* $T_{\kappa^+} = \emptyset$ *and for all* $\alpha < \kappa^+$*,* $0 < |T_\alpha| < \kappa^+$*.* T *is* normal*, if*

- $|T_0| = 1$
- *every element has at least two immediate successors*
- *for any* $t \in T$ *and* α *with* $\mathrm{rk}(t) < \alpha < \kappa^+$*, there is some* $t' > t$ *with* $\mathrm{rk}(t') = \alpha$*.*

A normal κ^+*-tree* T *is a* special κ^+-Aronszajn tree*, if there is some set* A *of size* κ *and a function* $f\colon T \to A$ *such that for all* $t, t' \in T$*,* $t < t'$ *implies* $f(t) \neq f(t')$*.*

Let $L_2^\alpha = L_0^\alpha \cup \{\prec, f, g, \mathrm{rk}\}$, where $\prec, f, g, \mathrm{rk}$ are binary relation symbols and let $\sigma_2^\alpha \in (L_2^\alpha)_{\omega_1,\omega}$ be the conjunction of σ_0^α and the following statements:

- The relation $\prec$ partially orders $Q_{\beta+1}$ for every $\beta < \alpha$ and $a \prec b$ implies that for some $\beta < \alpha$, $a, b \in Q_{\beta+1}$.
- For every $\beta < \alpha$ and $a \in Q_{\beta+1}$, the set $\{x | x \prec a\}$ is linearly ordered by $\prec$.
- For every $\beta < \alpha$, every element $x \in Q_{\beta+1}$ has at least two immediate $\prec$-successors.
- $\mathrm{rk} \subset \bigcup\limits_{\beta<\alpha} (Q_{\beta+1} \times Q_{\beta+1})$ defines on each $Q_{\beta+1}$ an idempotent total function (we will write $\mathrm{rk}(x) = y$ for $\mathrm{rk}(x, y)$), i.e. we have $\mathrm{rk}(\mathrm{rk}(a)) = \mathrm{rk}(a)$ for all $a \in Q_{\beta+1}$ (the idea is that the fibers $\mathrm{rk}^{-1}[\{a\}]$ are the levels of the "tree" defined by $\prec$).
- $a \prec b$ implies $\mathrm{rk}(a) < \mathrm{rk}(b)$.
- $\mathrm{rk}(a) = \mathrm{rk}(b)$ implies that for all $c \prec a$, there is some $d \prec b$ with $\mathrm{rk}(d) = \mathrm{rk}(c)$.
- For all $\beta < \alpha$ and $a, b \in Q_{\beta+1}$, if for all $c \prec a$ there exists some $d \prec b$ with $\mathrm{rk}(d) = \mathrm{rk}(c)$ and conversely for all $d \prec b$ there exists some $c \prec a$ with $\mathrm{rk}(c) = \mathrm{rk}(d)$, then we must also have $\mathrm{rk}(a) = \mathrm{rk}(b)$.
- For any a, b with $\mathrm{rk}(b) > \mathrm{rk}(a)$, there is some c with $\mathrm{rk}(c) = \mathrm{rk}(b)$ and $a \prec c$ (i.e. every element has a $\prec$-successor at every higher level).
- $f, g \subset \bigcup\limits_{\beta<\alpha} (Q_{\beta+1} \times Q_\beta)$ both define total functions from every $Q_{\beta+1}$ to Q_β.
- f restricted to any set of the form $\mathrm{rk}^{-1}[\{a\}]$ ($a \in Q_\alpha$) is injective.
- $a \prec b$ implies $g(a) \neq g(b)$.

Proposition 9. *If special* $\aleph_{\beta+1}$*-Aronszajn trees exist for every* $\beta < \alpha$*, then* σ_2^α *has a model of cardinality* $\aleph_\alpha$*.*

Proof. Start with a model M of σ_0^α of size $\aleph_\alpha$ such that $<$ has order type $\aleph_{\beta+1}$ on every $Q_{\beta+1}$ for all $\beta < \alpha$. For any $\beta < \alpha$ let $(Q_{\beta+1}, \prec^*)$ be a special $\aleph_{\beta+1}$-Aronszajn tree, witnessed by a function $h : Q_{\beta+1} \to Q_\beta$. Pick from each level γ of that tree exactly one element $t_\gamma^{\beta+1}$. We may assume that $<$ has the property that $t_\gamma^{\beta+1} < t_\delta^{\beta+1}$ if and only if $\gamma < \delta$ for all $\beta < \alpha$ (otherwise permute the ordering $\prec^*$ appropriately while keeping the ordering $<$ on Q_β fixed).

Now we will expand M to a L_2^α-structure. First, set $\prec = \prec^*$ and let $\mathrm{rk}(a) = t_\gamma^{\beta+1}$ if and only if a is an element in level γ of the tree $(Q_{\beta+1}, \prec^*)$. We then can define the function f with its property stated in σ_2^α. Finally, set $g = h$ on all $Q_{\beta+1}$ ($\beta < \alpha$).

It is straightforward to verify that $\prec, \mathrm{rk}, f, g$ satisfy all the properties of σ_2^α. □

Proposition 10. *If σ_2^α has a model of size $\aleph_\alpha$, then a special $\aleph_{\beta+1}$-Aronszajn tree exists for all $\beta < \alpha$.*

Proof. Let M be a model of σ_2^α of size $\aleph_\alpha$ and fix some $\beta < \alpha$. We find a $\gamma < \alpha$ such that $|Q_\gamma| = \aleph_\beta$ and $|Q_{\gamma+1}| = \aleph_{\beta+1}$.

$(Q_{\gamma+1}, \prec)$ need not be a special $\aleph_\alpha$-Aronszajn tree (among other things, it might be ill-founded), but we can find a *sub-tree* which is an $\aleph_\alpha$-Aronszajn tree.

First of all, we note that the order $<$ on $Q_{\gamma+1}$ is $\aleph_\beta$-like and thus must have cofinality $\aleph_{\beta+1}$. Pick a $<$-increasing sequence $(t_\delta)_{\delta<\aleph_{\beta+1}}$ of elements in the image of rk. Define $T_0 = \{t_0\}$ and for any non-zero $\delta < \aleph_{\beta+1}$, set $T_\delta = \{a | \mathrm{rk}(a) = t_\delta,\ t_0 \prec a\}$. It is a straightforward verification that $T = \bigcup_{\delta<\aleph_{\beta+1}} T_\delta$ ordered by $\prec$ is a special $\aleph_{\beta+1}$-Aronszajn tree. □

Consequently, the existence of a model of σ_2^α of size $\aleph_\alpha$ is equivalent to the existence for all $\beta < \alpha$ of special $\aleph_{\beta+1}$-Aronszajn trees.

It is a consequence of GCH that special κ-Aronszajn trees exist for all successor cardinals κ that are not successors of limit cardinals. Moreover, in the constructible universe, special Aronszajn trees exist even in successors of limit cardinals (this is a consequence of $\Box_\kappa$, see [8]).

On the other hand, the consistency of "ZFC+$\exists\kappa$(κ supercompact)" implies the consistency of "ZFC+GCH+there are no special $\aleph_\alpha$-Aronszajn trees for all countable limit successors α", as follows from results found in [4]:

We start with a model of GCH with a supercompact cardinal κ and force with $\mathrm{Coll}(\omega_1, < \kappa)$. This forcing preserves a stationary reflection property sufficient to ensure that Weak Square fails at $\aleph_\lambda$ for λ a limit ordinal of countable cofinality. By a result of Jensen found in [8], Weak Square at a cardinal κ is equivalent to the existence of a special Aronszajn tree on κ^+.

In conclusion, assuming the consistency of supercompact cardinals, model existence in $\aleph_\alpha$ of σ_2^α is non-absolute for every countable $\alpha > \omega$.

Similarly to the last remark in section 3.2, we can also consider slight variations of the sentences σ_2^α. For example, to get non-absolutness of model-existence in $\aleph_\alpha$ where α is a countable successor ordinal greater than ω, it would suffice to code special Aronszajn trees on the top level (i.e. we may restrict $\prec$ to Q_α). However, the set-theoretic assumptions would not be weakened, as we still need the existence of a supercompact cardinal to eliminate special $\aleph_\alpha$-Aronzajn trees. Also, to achieve non-absoluteness of model existence in limit cardinals, we need to code special Aronszajn trees on every successor level.

4. Final observations

The case of model-existence in $\aleph_\omega$ remains open. Also, it is not clear if we can find *complete* sentences for which model-existence in different cardinalities is non-absolute. Take for example the sentence σ_1^2 and try to find a completion of it. First, we may replace the $Q_0 \cup Q_1$ part by Julia Knight's construction of a complete sentence characterizing $\aleph_1$ (see [10]). We can use her techniques to incorporate the T relation in a complete way, but it already seems difficult to control the properties of this relation (like e.g. making sure T codes many different sets). But even if T behaves nicely, the hard part will be to include the Q_2-part in a complete way such that it has a chance (assuming a $\aleph_1$-Kurepa-family exists) to have cardinality $\aleph_2$ in some model. Since we want a complete sentence, we would code very specific Kurepa-families (depending on how T exactly looks like) and it might be set theoretically non-trivial to find out if those are a consistent concept.

In the light of Lemma 7.1.6 from [1], a model of size $\aleph_2$ would have to be *small*, i.e. it must realize only countably many $L_{\omega_1,\omega}$ types. If we naively start with Julia Knight's example (without the T-relation) and define a Kurepa-family (that we find in the constructible universe) on it, it looks quite desperate to achieve smallness, especially when we try to define T.

It might turn out that under GCH, the existence of a model in, say, $\aleph_2$ is an absolute notion for complete $L_{\omega_1,\omega}$ sentences, but we do not have any particular evidence (besides our difficulties to complete the examples presented earlier) to support this thesis.

As a last remark, our use of the concept of Kurepa-families has the slight flaw that in order to find set theoretic universes which do not contain such families, we have to assume the existence of inaccessible cardinals. The special Aronszajn technique is even worse as we have to assume the consistency of supercompact cardinals. It would be nice to find $L_{\omega_1,\omega}$ sentences for which under GCH the existence of models of certain cardinalities is not absolute, regardless of the existence of large cardinals.

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