DOUBLE ADJUNCTIONS

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ABSTRACT. We characterize double adjunctions in terms of presheaves and universal squares, and then apply these characterizations to free monads and Eilenberg–Moore objects in double categories. We improve upon an earlier result of Fiore–Gambino–Kock in [7] to conclude: if a double category with cofolding admits the construction of free monads in its horizontal 2-category, then it also admits the construction of free monads as a double category horizontally and vertically, and also in its vertical 2-category. We also prove that a double category admits Eilenberg–Moore objects if and only if a certain parameterized presheaf is representable. Along the way, we develop parameterized presheaves on double categories and prove a double Yoneda Lemma.

CONTENTS

1. INTRODUCTION

Although 2-categories and double categories were conceived at about the same time and the same place (Bénabou [1] in 1967 and Ehresmann [5] in 1963, respectively), the theory of 2-categories has had a huge advantage in development, whereas the theory of double categories has been slower to really take off. One reason is that 2-categories behave and feel much more like 1-categories, whereas double categories exhibit many new and strange phenomena. However, the past decade has seen a certain renaissance of double categories, and double-categorical structures are finding application more and more frequently in many different areas.

We became interested in double categories through work in conformal field theory, topological quantum field theory, operad theory, and type theory. In all these cases, the double-categorical structures come about in situations where there are two natural kinds of morphisms, typically some complicated morphisms (like spans of sets or bimodules) and some more elementary ones (like functions between sets or ring homomorphisms), and the double-categorical aspects concern the interplay between such different kinds of morphisms. While it often provides great conceptual insight to have everything encompassed in a double category, one is often confronted with the lack of machinery for dealing with double categories, and a need is being felt for a more systematic theory of double categories.

This paper can be seen as a small step in that direction: although our work is motivated by some concrete questions about monads, we try to develop rather systematically the basics of adjunctions between double categories: we introduce parametrized presheaves, prove a double Yoneda Lemma, characterize adjunctions in several ways, and go on to study double categories with further structure — foldings or cofoldings — for which we study the question of existence of free monads and Eilenberg–Moore objects. This was our original motivation, and in that sense the present paper is a sequel to our previous paper [7] about *monads in double categories*, although logically the present paper is rather a precursor: with the theory we develop here, some of the results from our previous paper can be strengthened and simplified at the same time.

In some regards, double adjunctions express universality in the ways one expects based on experience with 1-categories, as we prove in Theorem 5.2. A double adjunction may be given by double functors F and G with horizontal natural transformations η and ε satisfying the two triangle identities, or by double functors F and G with a universal horizontal natural transformation (η or ε), or by a single double functor F or G equipped with appropriate universal squares compatible with vertical composition, or by a bijection between sets of squares compatible with vertical composition.

However, the characterizations of adjointness in 1-category theory in terms of representability do not carry over to double category theory in a straightforward way, and instead require a new notion of *presheaf on a double category.* Namely, to prove that an ordinary 1-functor $F\colon \mathbf{A} \to \mathbf{X}$ admits a right adjoint, it is sufficient to show that the presheaf $\mathbf{A}(F-, A)$ is representable for each object A separately. But to establish that a *double* functor F admits a right *double* adjoint, two new requirements arise: 1) we must consider how the analogous presheaves *vertically combine*, and 2) we must consider the representability of all the analogous presheaves *simultaneously rather than separately.* The first requirement (that we must consider how the analogous presheaves vertically combine) forces presheaves on double categories to be *vertically lax* and to *take values in* *the double category* $\mathbb{S}\mathrm{pan}^t$ *of vertical spans*, as opposed to the 1-category **Set**. We prove a Yoneda Lemma for such $\mathbb{S}\mathrm{pan}^t$-valued presheaves in Proposition 3.10. The second requirement (that we must consider all the analogous presheaves simultaneously) forces us to consider *parameterized* presheaves on double categories. With these notions we establish the double-categorical analogue of the representability characterisation of adjunctions in Theorem 5.3, namely a double functor admits a right horizonal adjoint if and only if a certain parameterized $\mathbb{S}\mathrm{pan}^t$-valued presheaf is representable. Parameterized presheaves also play a role in the proof of Theorem 5.2.

Many double categories that arise in practice have convenient additional structure: vertical morphisms often map to horizontal morphisms, and squares can often be *folded* (or *cofolded*) into horizontal 2-cells. Foldings and cofoldings are recalled in Definitions 6.2 and 6.8. One utility of these extra structures is that questions about a double category with folding or cofolding can sometimes be reduced to questions about its horizontal 2-category. Pseudo double categories which admit both a folding and a cofolding are essentially the same as the *framed bicategories* of Shulman [10]. In this article we work with foldings and cofoldings separately because some examples, including our motivating examples, admit one or the other but not both.

As an example of the principle of reduction to the horizontal 2-category in the presence of a folding or cofolding, our Proposition 6.10 states that two double functors F and G compatible with foldings (or cofoldings) are horizontal double adjoints if and only if their underlying horizontal 2-functors are 2-adjoints. If vertical morphisms are further assumed to map fully faithfully to horizontal morphisms, then a double functor F compatible with the foldings (or cofoldings) admits a horizontal right double adjoint if and only if F admits a right adjoint in every other reasonable sense, see Corollary 6.13, and its cofolding counterpart Corollary 6.16.

A further instance of reduction to the horizontal 2-category concerns monads in double categories, and is one the main applications of the present paper. In our earlier paper [7] we showed how to associate to a double category $\mathbb{D}$ a double category $\mathbb{E}\mathrm{nd}(\mathbb{D})$ of endomorphisms in $\mathbb{D}$ and a double category $\mathbb{M}\mathrm{nd}(\mathbb{D})$ of monads in $\mathbb{D}$. The double categories $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ are extensions of Street's 2-categories of endomorphisms and monads in [12] in the sense that if $\mathcal{K}$ is a 2-category and $\mathbb{H}(\mathcal{K})$ is $\mathcal{K}$ viewed as a vertically trivial double category, then the horizontal 2-categories of $\mathbb{E}\mathrm{nd}(\mathbb{H}(\mathcal{K}))$ and $\mathbb{M}\mathrm{nd}(\mathbb{H}(\mathcal{K}))$ are Street's 2-categories $\mathrm{End}(\mathcal{K})$ and $\mathrm{Mnd}(\mathcal{K})$. In [7, Theorem 3.7] we established a fairly technical criterion which allows one to conclude the existence of free monads in a double-categorical sense from the existence of free monads in the underlying horizontal 2-category. The basic assumptions are that the double category is a framed bicategory and the appropriate substructures admit 1-categorical equalizers and coproducts. In the present paper we clarify and generalize this substantially, using the theory of double adjunctions and cofoldings.

A double category $\mathbb{D}$ is said to *admit the construction of free monads* if the forgetful functor $\mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ admits a vertical left double adjoint such that the underlying vertical morphism of each unit component is the identity. This is somewhat more stringent than our earlier definition in [7], where we required only a vertical left double adjoint. Our main application, Theorem 9.5, states that a double category with cofolding admits the construction of free monads if its horizontal 2-category admits the construction of free monads. This improves [7, Theorem 3.7], since it removes nearly all the hypotheses and also strengthens the conclusion. A main step is our Proposition 7.5, which states that a cofolding on a double category $\mathbb{D}$ induces cofoldings on $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$. (The corresponding statement for foldings is false.)

An example is the free–forgetful adjunction between $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ and $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$, where $\mathbb{S}\mathrm{pan}$ is the double category of horizontal spans. Endomorphisms in $\mathbb{S}\mathrm{pan}$ are directed graphs and monads in $\mathbb{S}\mathrm{pan}$ are categories. This example is worked out in detail in Section 8.

Returning to general double categories without cofolding, we now describe our second main application. Theorem 10.3 states that a double category $\mathbb{D}$ admits Eilenberg–Moore objects if and only if the parameterized presheaf is representable which assigns to a monad (X, S) and an object I in $\mathbb{D}$ the set $S\text{-}\boldsymbol{Alg}_I$ of S-algebra structures on I. The proof is quite short, since most of the work was done in the earlier sections.

An overview of the paper is as follows. In Section 3 we introduce parameterized presheaves on double categories and their representability, and prove the Double Yoneda Lemma. In Sections 4, 5, and 6 we introduce universal squares, prove the various characterizations of double adjunctions, and consider the special case of double adjunctions compatible with foldings and cofoldings. In Section 7 we prove that $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ admit cofoldings when $\mathbb{D}$ does. Section 8 works out the double adjunction between $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ and $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ explicitly. Sections 9 and 10 are applications of the results on double adjunctions to the construction of free monads in double categories with cofolding and to a characterization of the existence of Eilenberg–Moore objects in a general double category.

2. Notational Conventions

We begin by fixing some notation concerning double categories.

A *double category* is a categorical structure consisting of objects, horizontal morphisms, vertical morphisms, squares, the relevant source and target functions, compositions, and units. Succinctly, a double category is an internal category in **Cat**. The theory of double categories was pioneered by A. Ehresmann and C. Ehresmann, beginning with [5]. We indicate double categories with blackboard letters, such as $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{E}$.

If $\mathbb{D}$ is a double category, then $\mathrm{Hor}\,\mathbb{D}, \mathrm{Ver}\,\mathbb{D}$, and $\mathrm{Sq}\,\mathbb{D}$, signify the collections of horizontal morphisms, vertical morphisms, and squares in $\mathbb{D}$. To specify the set of horizontal respectively vertical morphisms from an object D_1 to an object D_2, we write Hor $\mathbb{D}(D_1, D_2)$ and Ver $\mathbb{D}(D_1, D_2)$. Similarly, the notation Hor $\mathbb{D}(f, g)$ indicates the function Hor $\mathbb{D}(D_1, D_2) \to$ Hor $\mathbb{D}(D_1', D_2')$ obtained by pre- and postcomposition with the horizontal morphisms f and g. The function Ver $\mathbb{D}(j, k)$ is defined analogously. To indicate the collection of squares with fixed left vertical boundary j and fixed right vertical boundary k, we write

$$\mathbb{D}(j,k) = \left\{ \alpha \in \mathrm{Sq}\ \mathbb{D} \;\middle|\; \alpha \text{ has the form } \begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ \cdot & \longrightarrow & \cdot \end{array} \right\}. \tag{1}$$

For example, for the vertical identities $1^v_{D_1}$ and $1^v_{D_2}$, the set $\mathbb{D}(1^v_{D_1}, 1^v_{D_2})$ consists of the 2-cells between morphisms $D_1 \to D_2$ in the horizontal 2-category of $\mathbb{D}$. In general, the squares in $\mathbb{D}(j, k)$ may not compose vertically. Also in analogy to the hom-notation, the notation $\mathbb{D}(\alpha, \beta)$ means horizontal pre- and postcomposition by squares α and β. For any double category $\mathbb{D}$, the *horizontal opposite* $\mathbb{D}^{\mathrm{horop}}$ is formed by switching horizontal source and target for both horizontal morphisms and squares in $\mathbb{D}$. More precisely, the horizontal 1-category of $\mathbb{D}^{\mathrm{horop}}$ is equal to the opposite of the horizontal 1-category of $\mathbb{D}$, the vertical 1-category of $\mathbb{D}^{\mathrm{horop}}$ is the same as that of $\mathbb{D}$, and the category (Ver $\mathbb{D}^{\mathrm{horop}}$, Sq $\mathbb{D}^{\mathrm{horop}}$) is equal to the opposite category of (Ver $\mathbb{D}$, Sq $\mathbb{D}$).

We may associate various substructures to a double category. We indicate the horizontal and vertical 2-categories of a double category $\mathbb{D}$ by $\mathbf{H}\mathbb{D}$ and $\mathbf{V}\mathbb{D}$. The double category $\mathbb{V}_1\mathbb{D}$ has vertical 1-category the vertical 1-category of $\mathbb{D}$ and everything else trivial, that is, there are no non-trivial squares and no non-trivial horizontal morphisms in $\mathbb{V}_1\mathbb{D}$.

A 2-category gives rise to various double categories. If $\mathbf{C}$ is a 2-category, the double category $\mathbb{H}\mathbf{C}$ has $\mathbf{C}$ as its horizontal 2-category and only trivial vertical morphisms. Similarly, the double category $\mathbb{V}\mathbf{C}$ has $\mathbf{C}$ as its vertical 2-category and only trivial horizontal morphisms.

The pseudo double category $\mathbb{S}$pan will play a special role in this paper. It is the pseudo double category in which objects are sets, the horizontal morphisms are spans of sets, the vertical morphisms are functions, and the squares are the morphisms of spans. The pseudo double category $\mathbb{S}\mathrm{pan}^t$ is its transpose, which means the horizontal and vertical data are exchanged (including boundaries of squares). Note that $\mathbb{S}$pan is horizontally weak while $\mathbb{S}\mathrm{pan}^t$ is vertically weak.

3. Parameterized Presheaves on Double Categories, Representability, and the Double Yoneda Lemma

In this section we develop the basic theory of parametrized presheaves on a double category and prove a Yoneda Lemma for double categories. The Double Yoneda Lemma and the characterization of left double adjoints in Theorem 5.3 require parameterized $\mathbb{S}\mathrm{pan}^t$-valued presheaves, as explained in the Introduction. The covariant Double Yoneda Lemma for presheaves on a double category $\mathbb{D}$ says that morphisms from the represented presheaf $\mathbb{D}(R,-)$ to a presheaf K on $\mathbb{D}^{\text{horop}}$ are in bijective correspondence with the set $K(R)$.

A presheaf on a double category assigns to objects sets, to horizontal morphisms functions, to vertical morphisms spans of sets, and to squares morphisms of spans. Moreover, these target spans are equipped with a kind of composition provided by the vertical laxness of the presheaf.

Definition 3.1. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A *presheaf on* $\mathbb{D}$ *parameterized by* $\mathbb{E}$ is a vertically lax double functor $\mathbb{D}^{\text{horop}} \times \mathbb{E} \to \mathbb{S}\text{pan}^t$. We synonymously use the term *presheaf on* $\mathbb{D}$ *indexed by* $\mathbb{E}$. A *presheaf on a double category* $\mathbb{D}$ is a presheaf on $\mathbb{D}$ parameterized by the terminal double category, that is, a *presheaf on* $\mathbb{D}$ is a vertically lax double functor $\mathbb{D}^{\text{horop}} \to \mathbb{S}\text{pan}^t$.

Example 3.2. The first example is delivered by the hom-sets of a double category $\mathbb{D}$. Namely, a presheaf on $\mathbb{D}$ indexed by $\mathbb{D}$ is defined on objects and horizontal morphisms by

$$\begin{aligned} \mathbb{D}(-,-)\colon\ \mathbb{D}^{\text{horop}} \times \mathbb{D} &\longrightarrow \mathbb{S}\text{pan}^t \\ (D_1, D_2) &\longmapsto \text{Hor}\,\mathbb{D}(D_1, D_2) \\ (f, g) &\longmapsto \text{Hor}\,\mathbb{D}(f, g)\ . \end{aligned}$$

On vertical morphisms (j,k), it is the vertical span

$$\begin{array}{c} \text{Hor}\,\mathbb{D}(s^v j, s^v k) \\ \Big\uparrow{\scriptstyle s^v} \\ \mathbb{D}(j,k) \\ \Big\downarrow{\scriptstyle t^v} \\ \text{Hor}\,\mathbb{D}(t^v j, t^v k), \end{array}$$

which we often denote simply by $\mathbb{D}(j,k)$. On squares (α, β), the vertically lax double functor $\mathbb{D}(-,-)$ is the morphism of vertical spans induced by $\mathbb{D}(\alpha,\beta)(\gamma) = [\alpha\ \ \gamma\ \ \beta]$ as well as $\text{Hor}\,\mathbb{D}(s^v\alpha, s^v\beta)$ and $\text{Hor}\,\mathbb{D}(t^v\alpha, t^v\beta)$.

For the vertically lax double functor $\mathbb{D}(-,-)$, the composition coherence square in $\mathbb{S}\text{pan}^t$

$$\begin{bmatrix} \mathbb{D}(j,k) \\ \mathbb{D}(\ell,m) \end{bmatrix} \longrightarrow \mathbb{D}\left(\begin{bmatrix} j \\ \ell \end{bmatrix}, \begin{bmatrix} k \\ m \end{bmatrix}\right)$$

is simply composition in $\mathbb{D}$. More precisely, on elements we have

$$\begin{array}{ccc} \begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle j}\downarrow & \xi_1 & \downarrow{\scriptstyle k} \\ \cdot & \longrightarrow & \cdot \\ {\scriptstyle \ell}\downarrow & \xi_2 & \downarrow{\scriptstyle m} \\ \cdot & \longrightarrow & \cdot \end{array} & \longmapsto & \begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle \left[\begin{smallmatrix} j \\ \ell \end{smallmatrix}\right]}\Bigg\downarrow & \left[\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right] & \Bigg\downarrow{\scriptstyle \left[\begin{smallmatrix} k \\ m \end{smallmatrix}\right]} \\ \cdot & \longrightarrow & \cdot \end{array} \end{array} .$$

The unit coherence square in $\mathbb{S}\mathrm{pan}^t$ of the vertically lax double functor $\mathbb{D}(-,-)$ is simply the vertical identity square embedding

$$1^v_{\mathbb{D}(D_1,D_2)} \xrightarrow{i^v} \mathbb{D}(1^v_{D_1}, 1^v_{D_2})$$

$$f \longmapsto \begin{array}{ccc} D_1 & \xrightarrow{f} & D_2 \\ \| & i^v_f & \| \\ D_1 & \xrightarrow[f]{} & D_2 \end{array} .$$

The presheaf $\mathbb{D}(-,-)$ may also be considered as a presheaf on $\mathbb{D}^{\mathrm{horop}}$ indexed by $\mathbb{D}^{\mathrm{horop}}$. This completes the example $\mathbb{D}(-,-)$.

Example 3.3. As a special case of Example 3.2, we may fix the first variable to be an object R in $\mathbb{D}$ and we obtain a presheaf on $\mathbb{D}^{\mathrm{horop}}$, namely

$$\mathbb{D}(R,-)\colon\ \mathbb{D} \longrightarrow \mathbb{S}\mathrm{pan}^t .$$

This presheaf is *represented* by the object R. We shall discuss a notion of representability for parameterized presheaves in Definition 3.7, as they will be a key ingredient in our characterizations of double adjunctions in Theorem 5.2 (vi) and Theorem 5.3.

We write out the features of Example 3.2 for this special case, since we will need these represented presheaves in the Double Yoneda Lemma. Like any double functor, this presheaf consists of an object functor and a morphism functor

$$\mathbb{D}(R,-)^{\mathrm{Obj}}\colon\ (\mathrm{Obj}\ \mathbb{D}_0, \mathrm{Obj}\ \mathbb{D}_1) \longrightarrow (\mathrm{Sets}, \mathrm{Functions})$$

$$\mathbb{D}(R,-)^{\mathrm{Mor}}\colon\ (\mathrm{Mor}\ \mathbb{D}_0, \mathrm{Mor}\ \mathbb{D}_1) \longrightarrow (\mathrm{Spans}, \mathrm{Morphisms\ of\ Spans}) .$$

The object functor is the usual represented presheaf on the horizontal 1-category, namely

$$\mathbb{D}(R,D)^{\mathrm{Obj}} := \{f\colon\ R \to D \mid f \text{ horizontal morphism in } \mathbb{D}\} = \mathrm{Hor}\ \mathbb{D}(R,D)$$

$$\mathbb{D}(R,g)^{\mathrm{Obj}}(f) := [f\ \ g] .$$

The morphism functor, on the other hand, takes a vertical morphism $j\colon\ D \to D'$ in $\mathbb{D}$ to the (vertical) span $\mathbb{D}(R,j)^{\mathrm{Mor}}$ defined as

$$\begin{array}{c} \mathbb{D}(R,D)^{\mathrm{Obj}} \\ \uparrow s^v \\ \mathbb{D}(1^v_R, j) \\ \downarrow t^v \\ \mathbb{D}(R,D')^{\mathrm{Obj}}, \end{array}$$

and on a square β we have the morphism of spans $\mathbb{D}(R,\beta)^{\mathrm{Mor}}$ induced by $\mathbb{D}(R,\beta)^{\mathrm{Mor}}(\alpha) = [\alpha\ \ \beta]$.

The composition coherence square in $\mathbb{S}\mathrm{pan}^t$

$$\begin{bmatrix}\mathbb{D}(R,j)^{\mathrm{Mor}} \\ \mathbb{D}(R,k)^{\mathrm{Mor}}\end{bmatrix} \longrightarrow \mathbb{D}(R, \left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right])$$

of the vertically lax double functor $\mathbb{D}(R,-)$ is simply composition in $\mathbb{D}$. More precisely, on elements we have

$$\begin{array}{ccc} R & \longrightarrow & \\ \Vert & \xi_1 & \downarrow j \\ R & \longrightarrow & \\ \Vert & \xi_2 & \downarrow k \\ R & \longrightarrow & \end{array} \quad \longmapsto \quad \begin{array}{ccc} R & \longrightarrow & \\ \Vert & \left[\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right] & \downarrow \left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right] \\ R & \longrightarrow & \end{array} .$$

The unit coherence square in $\mathbb{S}\mathrm{pan}^t$ of the vertically lax double functor $\mathbb{D}(R,-)$ is simply the identity embedding

$$1^v_{\mathbb{D}(R,D)^{\mathrm{Obj}}} \xrightarrow{i^v} \mathbb{D}(R, 1^v_D)^{\mathrm{Mor}}$$

$$f \longmapsto \begin{array}{ccc} R & \xrightarrow{f} & D \\ \Vert & i^v_f & \Vert \\ R & \xrightarrow[f]{} & D \end{array} .$$

Example 3.4. If $\mathbf{C}$ is a 1-category, then a classical presheaf on $\mathbf{C}$ may be considered a presheaf on $\mathbb{H}\mathbf{C}$ in the following way. A classical presheaf on $\mathbf{C}$ is the same thing as a strictly unital double functor $F\colon \mathbb{H}\mathbf{C}^{\mathrm{horop}} \to \mathbb{S}\mathrm{pan}^t$ which has composition coherence morphism for $F(1^v_C) \circ F(1^v_C) \to F(1^v_C)$ given by the projection of the diagonal of $FC \times FC$ to FC. Any presheaf on $\mathbb{H}\mathbf{C}$ restricts to a classical presheaf on $\mathbf{C}$ by forgetting $F(1^v_C)$ for each C and the composition and identity coherences.

Example 3.5. A presheaf on the (opposite of the) terminal double category 1 is the same as a category, since a vertically lax double functor from 1 into $\mathbb{S}\mathrm{pan}^t$ is the same as a (horizontal) monad in $\mathbb{S}\mathrm{pan}$, which is the same as a category.

Example 3.6. Let $\mathbf{C}$ be a 1-category. Then $\mathbf{C}(-,-)$ is a presheaf on $\mathbf{C}$ indexed by Obj $\mathbf{C}$. This is a way to consider all the presheaves $\mathbf{C}(-,C)$ simultaneously. Similarly, by parametrizing via the vertical 1-category of $\mathbb{D}$, the indexed presheaf $\mathbb{D}(-,-)\colon \mathbb{D}^{\mathrm{horop}} \times \mathbb{V}_1\mathbb{D} \to \mathbb{S}\mathrm{pan}^t$ is a way of considering all presheaves $\mathbb{D}(-,R)$ *simultaneously and how they combine vertically* (recall the notation $\mathbb{V}_1\mathbb{D}$ from Section 2). This point of view will become important for our characterization of double adjunctions in Theorems 5.2 and 5.3.

Definition 3.7. A parameterized presheaf $F\colon \mathbb{D}^{\text{horop}} \times \mathbb{E} \to \mathbb{S}\text{pan}^t$ in the sense of Definition 3.1 is *representable* if there exists a double functor $G\colon \mathbb{E} \to \mathbb{D}$ such that F is horizontally naturally isomorphic to the parameterized presheaf $\mathbb{D}(-, G-)\colon \mathbb{D}^{\text{horop}} \times \mathbb{E} \to \mathbb{S}\text{pan}^t$. "Horizontal naturally isomorphic" means horizontal naturally isomorphic as vertically lax double functors.

Example 3.8. The presheaf $\mathbb{D}(-, R)\colon \mathbb{D}^{\text{horop}} \to \mathbb{S}\text{pan}^t$ is represented by the double functor $* \to \mathbb{D}$ that is constant R. The indexed presheaf $\mathbb{D}(-,-)\colon \mathbb{D}^{\text{horop}} \times \mathbb{V}_1\mathbb{D} \to \mathbb{S}\text{pan}^t$ is represented by the inclusion of the vertical 1-category of $\mathbb{D}$ into $\mathbb{D}$.

Definition 3.9. A *morphism of presheaves* is a horizontal natural transformation of vertically lax double functors $\mathbb{D}^{\text{horop}} \to \mathbb{S}\text{pan}^t$.

We next prove the Double Yoneda Lemma. For simplicity, we do the covariant version rather than the contravariant version.

Proposition 3.10 (Double Yoneda Lemma)**.** *Let $\mathbb{D}$ be a small double category, R an object of $\mathbb{D}$, and $K\colon \mathbb{D} \to \mathbb{S}\text{pan}^t$ a vertically lax double functor. Then the map*

$$\theta_{R,K}\colon \text{HorNat}(\mathbb{D}(R,-),K) \longrightarrow KR$$
$$\alpha \longmapsto \alpha_R(1_R^h)$$

is a bijection. Further, this bijection is a horizontal natural isomorphism of double functors HN and E

$$HN, E\colon \mathbb{D} \times \mathbb{D}\text{blCat}_{vert.lax}(\mathbb{D}, \mathbb{S}\text{pan}^t) \longrightarrow \mathbb{S}\text{pan}^t$$

$$HN(R,K) := \text{HorNat}(\mathbb{D}(R,-),K)$$

$$E(R,K) := K(R).$$

Proof: This is an extension of the proof of [2, Theorem 1.3.3]. We define $\theta_{R,K}(\alpha) = \alpha(1_R^h) \in K(R)$ and for $a \in K(R)$ we define a horizontal natural transformation $\tau(a)\colon \mathbb{D}(R,-) \Rightarrow K$. To each object $D \in \mathbb{D}$ we have the horizontal morphism in $\mathbb{S}\text{pan}^t$

$$\tau(a)_D\colon \mathbb{D}(R,D) \longrightarrow KD$$
$$f \longmapsto K(f)(a).$$

and to each vertical morphism j in $\mathbb{D}$ we have the square $\tau(a)_j$ in $\mathbb{S}\mathrm{pan}^t$

$$\begin{array}{ccccccc} \mathbb{D}(R,D)^{\mathrm{Obj}} & \xrightarrow{\tau(a)_D} & K(D) & & \mathbb{D}(R,D)^{\mathrm{Obj}} & \xrightarrow{K(-)(a)} & K(D) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbb{D}(1^v_R, j) & \xrightarrow{\tau(a)_j} & K(j) & = & \mathbb{D}(1^v_R, j) & \xrightarrow{K(-)(\delta^K_R(a))} & K(j) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}(R,D')^{\mathrm{Obj}} & \xrightarrow[\tau(a)_{D'}]{} & K(D') & & \mathbb{D}(R,D')^{\mathrm{Obj}} & \xrightarrow[K(-)(a)]{} & K(D'). \end{array} \tag{2}$$

These squares commute, because for $\begin{array}{ccc} R & \longrightarrow & D \\ \| & \xi & \downarrow j \\ R & \longrightarrow & D' \end{array} \in \mathbb{D}(1^v_R, j)$ the squares

$$\begin{array}{ccccc} K(R) & = & K(R) & \xrightarrow{K(f)} & K(D) \\ \| & & \uparrow & & \uparrow \\ K(R) & \xrightarrow{\delta^k_R} & K(1^v_R) & \xrightarrow{K(\xi)} & K(j) \\ \| & & \downarrow & & \downarrow \\ K(R) & = & K(R) & \xrightarrow[K(f')]{} & K(D') \end{array} \tag{3}$$

commute. For example, the top square in (2) evaluated on ξ is the same as the top half of (3) evaluated on a.

The naturality of $\tau(a)$, τ, and θ is proved as in [2, Theorem 1.3.3]. Alternatively, the proposition follows from the *internal Yoneda Lemma*. □

Corollary 3.11. *For objects $R, S \in \mathbb{D}$, each horizontal natural transformation $\mathbb{D}(R,-) \Rightarrow \mathbb{D}(S,-)$ has the form $\mathbb{D}(h,-)$ for a unique horizontal arrow $h\colon S \to R$.*

Remark 3.12. If k is a vertical morphism in $\mathbb{D}$, then

$$\mathbb{D}(k,-)\colon (\mathrm{Ver}\,\mathbb{D}, \mathrm{Sq}\,\mathbb{D}) \longrightarrow (\mathrm{Sets}, \mathrm{functions})$$

$$\ell \longmapsto \mathbb{D}(k,\ell)$$

is an ordinary presheaf on $(\mathrm{Ver}\,\mathbb{D}, \mathrm{Sq}\,\mathbb{D})^{\mathrm{op}}$.

4. Universal Squares in a Double Category

The components of the unit or counit of any 1-adjunction are universal arrows. Conversely, a 1-adjunction can be described in terms of such universal arrows. In this section we introduce universal squares in a double category, with a view towards the analogous characterizations of horizontal double adjunctions in Theorem 5.2.

Definition 4.1. If $S\colon \mathbb{D} \to \mathbb{C}$ is a double functor, then a *(horizontally) universal square from the vertical morphism* j *to* S is a square μ in $\mathbb{C}$ of the form

$$\begin{array}{ccc} C_1 & \xrightarrow{u_1} & SR_1 \\ {\scriptstyle j}\downarrow & \mu & \downarrow{\scriptstyle Sk} \\ C_2 & \xrightarrow[u_2]{} & SR_2 \end{array}$$

such that the map

$$\begin{array}{ccc} \mathbb{D}(k,\ell) & \longrightarrow & \mathbb{C}(j,S\ell) \\ \beta' & \longmapsto & [\mu \;\; S\beta'] \end{array} \tag{4}$$

is a bijection for all vertical morphisms ℓ. There is of course a dual notion of *(horizontally) universal square from a double functor* S *to a vertical morphism* j.

Proposition 4.2. *Suppose* $S\colon \mathbf{D} \to \mathbf{C}$ *is a 2-functor and* $u\colon C \to SR$ *is a morphism in* $\mathbf{C}$. *Then* $\mu := i^v_u$ *is universal from* 1^v_C *to* $\mathbb{H}S$ *if and only if the functor*

$$\begin{array}{ccc} \mathbf{D}(R,D) & \xrightarrow{S(-)\circ u} & \mathbf{C}(C,SD) \\ f' & \longmapsto & [u \;\; Sf'] \end{array}$$

is an isomorphism of categories. In other words, the square i^v_u *in* $\mathbb{H}\mathbf{C}$ *is universal if and only if the morphism* u *of* $\mathbf{C}$ *is 2-universal.*

Proof: In this situation the assignment $\beta' \mapsto [\mu \;\; \mathbb{H}S\beta']$ is a functor, namely whiskering with u. Then the claim follows from the observation that the morphism part of a functor is bijective if and only if the functor is an isomorphism of categories. $\square$

Proposition 4.3. *The bijection in* (4) *is a natural transformation of functors*

$$\mathbb{D}(k,-) \Longrightarrow \mathbb{C}(j,S-)\;. \tag{5}$$

Conversely, given k *and* j, *any natural bijection of functors as in* (5) *arises in this way from a unique square* $\mu \in \mathbb{C}(j,Sk)$ *which is universal from* j *to* S.

Proof: The proof is very similar to that of [9, Proposition 1, page 59]. The bijection is natural because

$$[\mu \;\; S\,[\beta' \;\; \gamma']] = [\mu \;\; [S\beta' \;\; S\gamma']]\,.$$

For the converse, let $\phi\colon \mathbb{D}(k,-) \Rightarrow \mathbb{C}(j,S-)$ be a natural bijection, and define $\mu := \phi_k(i^h_k)$. The naturality diagram for ϕ and β' yields $[\mu \;\; S\beta'] = \phi_\ell(\beta')$, which in turn implies that (4) is a bijection, since ϕ_ℓ is a bijection. $\square$

For later use, we record the dual to Proposition 4.3 using the inverse bijection.

Proposition 4.4. *Universal squares in* $\mathbb{C}(Sk, j)$ *from* $S\colon \mathbb{D} \to \mathbb{C}$ *to* j *are in bijective correspondence with natural bijections*

$$\mathbb{C}(S-, j) \Longrightarrow \mathbb{D}(-, k)\ .$$

5. Double Adjunctions

In any 2-category $\mathcal{K}$, there is a notion of adjunction. Namely, two 1-morphisms $f\colon A \to B$ and $g\colon B \to A$ in $\mathcal{K}$ are adjoint if there exist 2-cells $\eta\colon 1_A \Rightarrow gf$ and $\varepsilon\colon fg \Rightarrow 1_B$ satisfying the triangle identities. In particular, if we consider adjunctions in the 2-category $\mathbf{DblCat_h}$ of small double categories, double functors, and horizontal natural transformations, we arrive at the notion of horizontal double adjunction in Definition 5.1. The 2-category $\mathbf{DblCat_h}$ is the same as the 2-category $\mathsf{Cat}(\mathbf{Cat})$ of internal categories in $\mathbf{Cat}$, internal functors, and internal natural transformations, so we may expect various characterizations of horizontal double adjunctions in terms of universal arrows and bijections of hom-sets, along the lines of Theorem 2 on page 83 of Mac Lane's book [9]. The characterizations of horizontal double adjunctions in Theorems 5.2 and 5.3 are the main results of this section. In Section 8 we present a completely worked example of a vertical double adjunction: the free and forgetful double functors between endomorphisms and monads in $\mathbb{S}\mathrm{pan}$. This is an extension of the classical adjunction between small directed graphs and small categories.

Definition 5.1. Let $\mathbb{A}$ and $\mathbb{X}$ be double categories. A *horizontal double adjunction from* $\mathbb{X}$ *to* $\mathbb{A}$ consists of double functors

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{A} \tag{6}$$

and horizontal natural transformations

$$\eta\colon 1_{\mathbb{X}} \Longrightarrow GF$$

$$\varepsilon\colon FG \Longrightarrow 1_{\mathbb{A}}$$

such that the composites

$$G \xrightarrow{\eta * i_G} GFG \xrightarrow{i_G * \varepsilon} G$$

$$F \xrightarrow{i_F * \eta} FGF \xrightarrow{\varepsilon * i_F} F$$

are the respective identity horizontal natural transformations. In this case we say F and G are *horizontal double adjoints* and we write $F \dashv G$.

A horizontal double adjunction from $\mathbb{X}$ to $\mathbb{A}$ is precisely an *adjunction in the 2-category* $\mathbf{DblCat_h} = \mathsf{Cat}(\mathbf{Cat})$ from $\mathbb{X}$ to $\mathbb{A}$. A *vertical double adjunction* is an adjunction in the 2-category $\mathbf{DblCat_v}$, which has objects small double categories, morphisms double functors, and 2-cells vertical natural transformations. Our convention is that a double adjunction is assumed to be horizontal when neither "horizontal" nor "vertical" is written.

Theorem 5.2. *A double adjunction $F \dashv G$ is completely determined by the items in any one of the following lists.*

(i) *Double functors F, G as in* (6) *and a horizontal natural transformation $\eta\colon 1_{\mathbb{X}} \Rightarrow GF$ such that for each vertical morphism j in $\mathbb{X}$, the square η_j is universal from j to G.*

(ii) *A double functor G as in* (6) *and functors*

$$F_0\colon (\mathrm{Obj}\,\mathbb{X}, \mathrm{Ver}\,\mathbb{X}) \longrightarrow (\mathrm{Obj}\,\mathbb{A}, \mathrm{Ver}\,\mathbb{A})$$

$$\eta\colon (\mathrm{Obj}\,\mathbb{X}, \mathrm{Ver}\,\mathbb{X}) \longrightarrow (\mathrm{Hor}\,\mathbb{X}, \mathrm{Sq}\,\mathbb{X})$$

such that for each vertical morphism j in $\mathbb{X}$ the square η_j is of the form

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF_0X \\ {\scriptstyle j}\downarrow & \eta_j & \downarrow{\scriptstyle GF_0j} \\ Y & \xrightarrow[\eta_Y]{} & GF_0Y \end{array}$$

and is universal from j to G. Then the double functor F is defined on vertical arrows by F_0 and on squares χ by universality via the equation $[\eta_{s\chi}\ \ GF\chi] = [\chi\ \ \eta_{t\chi}]$.

(iii) *Double functors F, G as in* (6) *and a horizontal natural transformation $\varepsilon\colon FG \Rightarrow 1_{\mathbb{A}}$ such that for each vertical morphism k in $\mathbb{A}$, the square ε_k is universal from F to k.*

(iv) *A double functor F as in* (6) *and functors*

$$G_0\colon (\mathrm{Obj}\,\mathbb{A}, \mathrm{Ver}\,\mathbb{A}) \longrightarrow (\mathrm{Obj}\,\mathbb{X}, \mathrm{Ver}\,\mathbb{X})$$

$$\varepsilon\colon (\mathrm{Obj}\,\mathbb{A}, \mathrm{Ver}\,\mathbb{A}) \longrightarrow (\mathrm{Hor}\,\mathbb{A}, \mathrm{Sq}\,\mathbb{A})$$

such that for each vertical morphism k in $\mathbb{A}$ the square ε_k is of the form

$$\begin{array}{ccc} FG_0A & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle FG_0k}\downarrow & \varepsilon_k & \downarrow{\scriptstyle k} \\ FG_0B & \xrightarrow[\varepsilon_B]{} & B \end{array}$$

and is universal from F to k. Then the double functor G is defined on vertical morphisms by G_0 and on squares α by universality via the equation $[FG\alpha\ \ \varepsilon_{t\alpha}] = [\varepsilon_{s\alpha}\ \ \alpha]$.

(v) *Double functors F, G as in* (6) *and a bijection*

$$\varphi_{j,k}\colon\ \mathbb{A}(Fj,k) \longrightarrow \mathbb{X}(j,Gk)$$

natural in the vertical morphisms j and k and compatible with vertical composition. Naturality here means natural isomorphism of functors

$$(\mathrm{Ver}\ \mathbb{X}, \mathrm{Sq}\ \mathbb{X})^{\mathrm{op}} \times (\mathrm{Ver}\ \mathbb{A}, \mathrm{Sq}\ \mathbb{A}) \longrightarrow \mathbf{Set}\ ,$$

and compatibility with vertical composition means

$$\varphi\left(\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right) = \begin{bmatrix}\varphi(\alpha)\\ \varphi(\beta)\end{bmatrix}.$$

(vi) *Double functors F, G as in* (6) *and a horizontal natural isomorphism between the vertically lax double functors (parameterized presheaves)*

$$\mathbb{A}(F-,-)\colon\ \mathbb{X}^{\mathrm{horop}} \times \mathbb{A} \longrightarrow \mathbb{S}\mathrm{pan}^t$$

$$\mathbb{X}(-,G-)\colon\ \mathbb{X}^{\mathrm{horop}} \times \mathbb{A} \longrightarrow \mathbb{S}\mathrm{pan}^t.$$

Proof: We first prove Definition 5.1 is equivalent to (v), then we use this equivalence to prove the other equivalences. In each equivalence, we omit the proof that the two procedures are inverse to one another.

Definition 5.1 $\Rightarrow$ (v). Suppose $\langle F, G, \eta, \varepsilon\rangle$ is a double adjunction. Then for any square γ of the form

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle j}\big\downarrow & \gamma & \big\downarrow{\scriptstyle \ell} \\ \cdot & \longrightarrow & \cdot \end{array}$$

we have $[\eta_j\ \ GF\gamma] = [\gamma\ \ \eta_\ell]$ by the horizontal naturality of η. We define $\varphi_{j,k}$ and $\varphi_{j,k}^{-1}$ by

$$\varphi_{j,k}(\alpha) := [\eta_j\ \ G\alpha]$$
$$\varphi_{j,k}^{-1}(\beta) := [F\beta\ \ \varepsilon_k].$$

Then we have

$$\begin{aligned} \varphi\varphi^{-1}\beta &= \varphi\,[F\beta\ \ \varepsilon_k] \\ &= [\eta_j\ \ GF\beta\ \ G\varepsilon_k] \\ &= [\beta\ \ \eta_{Gk}\ \ G\varepsilon_k]\ \text{(by horizontal naturality)} \\ &= \beta\ \text{(by triangle identity)} \end{aligned}$$

and similarly $\varphi^{-1}\varphi(\alpha) = \alpha$. Clearly $\varphi_{j,k}$ is natural in j and k and compatible with vertical composition, and we now have $\langle F, G, \varphi\rangle$ as in (v).

(v) $\Rightarrow$ Definition 5.1. From $\langle F, G, \varphi\rangle$ as in (v), we define horizontal natural transformations by

$$\eta_j := \varphi(i^h_{Fj})$$
$$\varepsilon_k := \varphi^{-1}(i^h_{Gk}).$$

They are natural and satisfy the triangle identities by the same arguments as in Mac Lane's book [9, pages 81-82], and we obtain $\langle F, G, \eta, \varepsilon \rangle$ as in Definition 5.1.

(i) $\Rightarrow$ (v). Suppose we have $\langle F, G, \eta \rangle$ as in (i). The universality of η_j says that

$$\begin{aligned} \mathbb{A}(Fj,k) &\longrightarrow \mathbb{X}(j,Gk) \\ \alpha &\longmapsto [\eta_j \;\; G\alpha] \end{aligned} \tag{7}$$

is a bijection. Clearly this bijection is natural in j and k, and compatible with vertical composition, so we obtain $\langle F, G, \varphi \rangle$ as in description (v).

(v) $\Rightarrow$ (i). From the first part, we know that Definition 5.1 is equivalent to (v) and that $\varphi_{j,k}(\alpha) = [\eta_j \;\; G\alpha]$. This gives us F, G, and η. The universality of η_j then follows, because the map in (7) is equal to $\varphi_{j,k}$ and is therefore bijective.

(i) $\Rightarrow$ (ii). The data in (ii) are just a restriction of the data in (i).

(ii) $\Rightarrow$ (i). The universality of η_j guarantees that for each square χ in $\mathbb{X}$ there is a unique square $F\chi$ such that $[\eta_{s\chi} \;\; GF\chi] = [\chi \;\; \eta_{t\chi}]$. This defines F on squares χ in $\mathbb{X}$, and we take F to be F_0 on the vertical morphisms of $\mathbb{X}$. Then F is a double functor by the universality and the hypothesis that F_0 and η are functors. Finally, η is natural because of the defining equation $[\eta_{s\chi} \;\; GF\chi] = [\chi \;\; \eta_{t\chi}]$.

5.1 $\Leftrightarrow$ (iii). The proof of the equivalence Definition 5.1 $\Leftrightarrow$ (iii) is dual to the proof the equivalence Definition 5.1 $\Leftrightarrow$ (i).

(iii) $\Leftrightarrow$ (iv). The proof of the equivalence (iii) $\Leftrightarrow$ (iv) is dual to the proof of the equivalence (i) $\Leftrightarrow$ (ii).

(v) $\Leftrightarrow$ (vi). This is just the definitions.

This completes the proof of the equivalence of Definition 5.1 with each of (i), (ii), (iii), (iv), (v), and (vi). □

See Section 8 for a completely worked example of a double adjunction in terms of Theorem 5.2 (v). For a characterization of adjunctions between pseudo double categories in terms of certain kinds of graphs, see Garner [8, Appendix A]. For a discussion of framed adjunctions between framed bicategories, see Shulman [10, Section 8].

In ordinary 1-category theory, a functor $F\colon \mathbf{A} \to \mathbf{X}$ admits a right adjoint if and only if each presheaf $\mathbf{A}(F-, A)$ is representable for each A. But for double categories and double functors $F\colon \mathbb{A} \to \mathbb{X}$, we must consider the representability of the parameterized $\mathbb{S}\mathrm{pan}^t$-valued presheaf $\mathbb{A}(F-,-)$.

Theorem 5.3 (Characterization of horizontal left double adjoints in terms of parameterized representability)**.** *Let $F\colon \mathbb{X} \to \mathbb{A}$ be a double functor. Then F admits a horizontal right double adjoint if and only if the parameterized presheaf* *on* $\mathbb{X}$

$$\mathbb{A}(F-,-)\colon\ \mathbb{X}^{\mathrm{horop}}\times\mathbb{V}_1\mathbb{A}\longrightarrow\mathbb{S}\mathrm{pan}^t$$

is represented by a double functor $G_0\colon\ \mathbb{V}_1\mathbb{A}\to\mathbb{X}$ *(see Definitions 3.1 and 3.7 as well as Section 2). That is, the double functor* F *admits a horizontal right double adjoint if and only if for every vertical morphism* k *in* $\mathbb{A}$*, the classical presheaf*

$$\mathbb{A}(F-,k)\colon\ (\mathrm{Ver}\,\mathbb{X},\mathrm{Sq}\,\mathbb{X})^{\mathrm{op}}\longrightarrow\mathbf{Set}$$

is representable in a way compatible with vertical composition.

Proof: Suppose that a horizontal right double adjoint G exists. Then by Theorem 5.2 (vi) the parameterized presheaves $\mathbb{A}(F-,-)$ and $\mathbb{X}(-,G-)$ are horizontally naturally isomorphic as vertically lax functors on $\mathbb{X}^{\mathrm{horop}}\times\mathbb{A}$, so their restrictions to $\mathbb{X}^{\mathrm{horop}}\times\mathbb{V}_1\mathbb{A}$ are also horizontally naturally isomorphic. The double functor G_0 is simply the restriction of G. We have represented $\mathbb{A}(F-,-)$ by G_0.

In the other direction, suppose that the parameterized presheaf on $\mathbb{X}$

$$\mathbb{A}(F-,-)\colon\ \mathbb{X}^{\mathrm{horop}}\times\mathbb{V}_1\mathbb{A}\longrightarrow\mathbb{S}\mathrm{pan}^t$$

is representable by a double functor $G_0\colon\ \mathbb{V}_1\mathbb{A}\to\mathbb{X}$, and let

$$\varphi\colon\ \mathbb{A}(F-,-)\Longrightarrow\mathbb{X}(-,G_0-)$$

be a horizontally natural isomorphism between vertically lax functors. For vertical morphisms (j,k), we then have an isomorphism of spans in **Set**.

$$\begin{array}{ccc}
\mathbb{A}(Fs^vj,s^vj) & \xrightarrow{\varphi(s^vj,s^vj)} & \mathbb{X}(s^vj,G_0s^vj)\\
{\scriptstyle s^v}\uparrow & & \uparrow{\scriptstyle s^v}\\
\mathbb{A}(Fj,j) & \xrightarrow{\varphi(j,k)} & \mathbb{X}(j,G_0j)\\
{\scriptstyle t^v}\downarrow & & \downarrow{\scriptstyle t^v}\\
\mathbb{A}(Ft^vj,t^vj) & \xrightarrow[\varphi(t^vj,t^vj)]{} & \mathbb{X}(t^vj,G_0t^vj)\,.
\end{array}$$

Since $\mathbb{V}_1\mathbb{A}$ has no nontrivial horizontal morphisms or squares, the condition of horizontal naturality in k is satisfied vacuously. So, essentially we have horizontally natural bijections $\varphi(-,k)\colon\ \mathbb{A}(F-,k)\Rightarrow\mathbb{X}(-,G_0k)$, and these correspond to universal squares from F to k of the form

$$\begin{array}{ccc}
FG_0A & \xrightarrow{\varepsilon(A)} & A\\
{\scriptstyle FG_0k}\downarrow & \varepsilon(k) & \downarrow{\scriptstyle k}\\
FG_0B & \xrightarrow[\varepsilon(B)]{} & B
\end{array}$$

by Proposition 4.4. The assignments of $\varepsilon(A)$ and $\varepsilon(k)$ to A and k form a functor

$$\varepsilon\colon\ (\mathrm{Obj}\ \mathbb{A}, \mathrm{Ver}\ \mathbb{A}) \longrightarrow (\mathrm{Hor}\ \mathbb{X}, \mathrm{Sq}\ \mathbb{X})$$

because of the compatibility of φ with the vertical laxness of the parameterized presheaves. Finally, the characterization in Theorem 5.2 (iv) tells us that G_0 extends to a right adjoint G, defined on squares α using universality and the equation $\big[FG\alpha\ \ \varepsilon(t^h\alpha)\big] = \big[\varepsilon(s^h\alpha)\ \ \alpha\big]$. □

6. Compatibility with Foldings or Cofoldings

Many double categories that arise in practice are equipped with additional structures which make them easier to work with. In some instances, this extra structure allows one to reduce questions about the double category to questions about the horizontal 2-category. In this section we investigate to what extent adjunctions in the 2-category of double categories with folding/cofolding can be reduced to the horizontal 2-adjunction.

Extra structures on double categories have various equivalent formulations, each with its own motivating point of view. Perhaps the earliest extra structure on a double category is a *connection pair* in the sense of Brown–Spencer [4], which was shown to be equivalent to *thin structures* by Brown–Mosa in [3]. The *foldings* of Fiore in [6] are another structure on double categories, and these are equivalent to connection pairs without assuming the horizontal and vertical 1-categories are the same. Similarly, *cofoldings* are equivalent to *coconnection pairs*. Foldings and cofoldings translate squares of a double category into 2-cells in the horizontal 2-category, in the same way that the quintet squares of Examples 6.1 and 6.7 are 2-cells in the horizontal 2-category.

Foldings and cofoldings are recalled in Definitions 6.2 and 6.8. They can also be adapted to pseudo double categories, as was sketched in [6] for foldings. Pseudo double categories with both folding and cofolding are essentially the same as the *framed bicategories* of [10]. However, in this article we work with foldings and cofoldings individually, rather than assuming a framing, because several important examples admit either a folding or a cofolding, but not both. This is the case for the double categories of endomorphisms and monads, $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$, in Section 7: if $\mathbb{D}$ admits a cofolding, then so do $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ in Proposition 7.5, but the analogous statement for foldings is not true.

Our results concerning double adjunctions between double categories with extra structure are summarized as follows. If F and G are double functors between double categories with foldings, and F and G preserve the foldings, then F and G are horizontally double adjoint if and only if the horizontal 2-functors $\mathbf{H}F$ and $\mathbf{H}G$ are 2-adjoint, as we prove in Proposition 6.10. If we make an additional assumption on the foldings, namely that their holonomies are fully faithful, then a double functor F preserving foldings admits a horizontal right double adjoint if and only if F admits a right adjoint in any other sense. That is, F admits a horizontal right double adjoint if and only if the horizontal 2-functor $\mathbf{H}F$ admits a right 2-adjoint if and only if the vertical 2-functor $\mathbf{V}F$ admits a right 2-adjoint if and only if the double functor F admits a vertical right double adjoint, as we prove in Corollary 6.13. We have analogous statements for double categories with cofoldings: Proposition 6.10 also holds for double categories with cofolding, and Corollary 6.13 has its analogue in Corollary 6.16 for fully faithful coholonomies.

The *holonomy* is the part of a folding which translates vertical morphisms to horizontal morphisms in a functorial way preserving source and target. Fully faithfulness of the holonomy is a very strong assumption, as it implies that the double category is isomorphic to the quintets $\mathbb{Q}\mathbf{C}$ for some 2-category $\mathbf{C}$. Most double categories are obviously not of the form $\mathbb{Q}\mathbf{C}$. However, the idea for Corollary 6.16 surprisingly also works for the forgetful double functor $U\colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$, even though the coholonomy is not fully faithful. We prove in Theorem 9.5 that a double category $\mathbb{D}$ with cofolding admits the construction of free monads horizontally if and only if its horizontal 2-category $\mathbf{H}\mathbb{D}$ admits the construction of free monads if and only if its vertical 2-category $\mathbf{V}\mathbb{D}$ admits the construction of free monads if and only if the double category $\mathbb{D}$ admits the construction of free monads vertically.

We begin the detailed discussion of foldings with the example of quintets.

Example 6.1 (Quintets)**.** A special double category associated to a 2-category $\mathbf{C}$ is Ehresmann's double category $\mathbb{Q}\mathbf{C}$ of *quintets of* $\mathbf{C}$. Its objects are the objects of $\mathbf{C}$, horizontal and vertical morphisms are the morphisms of $\mathbf{C}$, and the squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

are the 2-cells $\alpha\colon\ k \circ f \Rightarrow g \circ j$ in $\mathbf{C}$. The horizontal 2-category of $\mathbb{Q}\mathbf{C}$ is $\mathbf{C}$. The vertical 2-category of $\mathbb{Q}\mathbf{C}$ is $\mathbf{C}$ with the 2-cells reversed.

The double category $\mathbb{Q}\mathbf{C}$ is entirely determined by its horizontal 2-category. Similarly, any double category with folding is determined by its vertical 1-category and horizontal 2-category. Brown–Mosa's notion of folding in [3] was extended in [6] to non edge-symmmetric double categories.

Definition 6.2 (Definition 3.16 of [6])**.** A *folding on a double category* $\mathbb{D}$ consists of the following.

(i) A 2-functor $\overline{(-)}\colon\ (\mathbf{V}\mathbb{D})_0 \to \mathbf{H}\mathbb{D}$ which is the identity on objects. In other words, to each vertical morphism $j\colon\ A \to C$, there is associated a horizontal morphism $\overline{j}\colon\ A \to C$ with the same domain and range in a functorial way. We call this 2-functor $j \mapsto \overline{j}$ the *holonomy*, following the terminology of Brown-Spencer in [4], who first distinguished the notion.

(ii) Bijections $\Lambda^{f,k}_{j,g}$ from squares in $\mathbb{D}$ with boundary

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow[\;g\;]{} & D \end{array} \tag{8}$$

to squares in $\mathbb{D}$ with boundary

$$\begin{array}{ccc} A & \xrightarrow{[f\ \bar{k}]} & D \\ \Vert & & \Vert \\ A & \xrightarrow[{[\bar{j}\ g]}]{} & D. \end{array} \tag{9}$$

These bijections are required to satisfy the following axioms.

(i) Λ is the identity if j and k are vertical identity morphisms.

(ii) Λ preserves horizontal composition of squares, that is,

$$\Lambda\left(\begin{array}{ccccc} A & \xrightarrow{\;f_1\;} & B & \xrightarrow{\;f_2\;} & C \\ {\scriptstyle j}\big\downarrow & \alpha & \big\downarrow{\scriptstyle k} & \beta & \big\downarrow{\scriptstyle \ell} \\ D & \xrightarrow[\;g_1\;]{} & E & \xrightarrow[\;g_2\;]{} & F \end{array}\right) = \begin{array}{ccc} A & \xrightarrow{[f_1\ f_2\ \bar{\ell}]} & F \\ \Vert & [i^v_{f_1}\ \ \Lambda(\beta)] & \Vert \\ A & \xrightarrow{[f_1\ \bar{k}\ g_2]} & F \\ \Vert & [\Lambda(\alpha)\ \ i^v_{g_2}] & \Vert \\ A & \xrightarrow[{[\bar{j}\ g_1\ g_2]}]{} & F. \end{array}$$

(iii) Λ preserves vertical composition of squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle j_1}\big\downarrow & \alpha & \big\downarrow{\scriptstyle k_1} \\ C & \xrightarrow{\;g\;} & D \\ {\scriptstyle j_2}\big\downarrow & \beta & \big\downarrow{\scriptstyle k_2} \\ E & \xrightarrow[\;h\;]{} & F, \end{array}\right) = \begin{array}{ccc} A & \xrightarrow{[f\ \bar{k}_1\ \bar{k}_2]} & F \\ \Vert & [\Lambda(\alpha)\ \ i^v_{\bar{k}_2}] & \Vert \\ A & \xrightarrow{[\bar{j}_1\ g\ \bar{k}_2]} & F \\ \Vert & [i^v_{\bar{j}_1}\ \ \Lambda(\beta)] & \Vert \\ A & \xrightarrow[{[\bar{j}_1\ \bar{j}_2\ h]}]{} & F. \end{array}$$

(iv) Λ preserves identity squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & = & A \\ {\scriptstyle j}\downarrow & i^h_j & \downarrow{\scriptstyle j} \\ B & = & B \end{array}\right) = \begin{array}{ccc} A & \xrightarrow{\bar{j}} & B \\ \| & i^v_{\bar{j}} & \| \\ A & \xrightarrow[\bar{j}]{} & B. \end{array}$$

The notion of folding on a double category can be packaged succinctly as a double functor $\Lambda\colon \mathbb{D} \to \mathbb{QHD}$ which is the identity on the horizontal 2-category $\mathbf{H}\mathbb{D}$ of $\mathbb{D}$ and is fully faithful on squares.

Example 6.3. The double category $\mathbb{S}\mathrm{pan}$ admits a folding. The holonomy is

$$\left(A \xrightarrow{j} C\right) \longmapsto \left(A \xleftarrow{1^h_A} A \xrightarrow{j} C\right)$$

and the folding is

$$\left(\begin{array}{ccccc} A & \xleftarrow{f_0} & Y & \xrightarrow{f_1} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle k} \\ C & \xleftarrow[g_0]{} & Z & \xrightarrow[g_1]{} & D \end{array}\right) \longmapsto \left(\begin{array}{ccccc} A & \xleftarrow{f_0} & Y & \xrightarrow{kf_1} & D \\ \| & & \downarrow{\scriptstyle (f_1,\alpha)} & & \| \\ A & \xleftarrow[\mathrm{pr}_1]{} & A \times_D Z & \xrightarrow[g_1 \circ \mathrm{pr}_2]{} & D \end{array}\right).$$

Remark 6.4. If a double category $\mathbb{D}$ is equipped with a folding, then 2-cell composition in the vertical 2-category $\mathbf{V}\mathbb{D}$ corresponds to 2-cell composition in the horizontal 2-category $\mathbf{H}\mathbb{D}$. More precisely, if f_1, f_2, g_1, g_2 are identities in Definition 6.2 (ii), then $[\,\alpha \;\; \beta\,]$ is the vertical composition $\beta \odot \alpha$ in the 2-category $\mathbf{V}\mathbb{D}$, and compatibility with horizontal composition says $\Lambda(\beta \odot \alpha) = \Lambda(\alpha) \odot \Lambda(\beta)$. Concerning vertical composition in the 2-category $\mathbf{V}\mathbb{D}$, if f, g, h in Definition 6.2 (iii), then $\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]$ is the horizontal composition $\beta * \alpha$ in the 2-category $\mathbf{V}\mathbb{D}$, and $\Lambda(\beta * \alpha) = \Lambda(\beta) * \Lambda(\alpha)$.

Definition 6.5. Let $\mathbb{C}$ and $\mathbb{D}$ be double categories with folding. A double functor $F\colon \mathbb{C} \to \mathbb{D}$ is *compatible with the foldings* if

$$F(\bar{j}) = \overline{F(j)} \qquad \text{and} \qquad F(\Lambda^{\mathbb{C}}(\alpha)) = \Lambda^{\mathbb{D}}(F(\alpha))$$

for all vertical morphisms j and squares α in $\mathbb{C}$.

Definition 6.6. Let $F, G\colon \mathbb{C} \to \mathbb{D}$ be morphisms of double categories with folding. A horizontal natural transformation $\theta\colon F \Rightarrow G$ is *compatible with the foldings* if for all vertical morphisms j in $\mathbb{C}$ the following equation holds.

$$
(10)\qquad \Lambda\left(\begin{array}{ccc} FA & \xrightarrow{\theta A} & GA \\ {\scriptstyle Fj}\downarrow & \theta j & \downarrow{\scriptstyle Gj} \\ FC & \xrightarrow[\theta C]{} & GC \end{array}\right) = \begin{array}{ccc} FA & \xrightarrow{[\theta A\ G\overline{j}]} & GC \\ \Vert & i^v_{[\theta A\ G\overline{j}]} & \Vert \\ FA & \xrightarrow[{[G\overline{j}\ \theta C]}]{} & GC \end{array}
$$

A vertical natural transformation $\sigma\colon F \Rightarrow G$ is *compatible with the foldings* if for all vertical morphisms j the following equation holds.

$$
(11)\qquad \Lambda\left(\begin{array}{ccc} FA & \xrightarrow{F\overline{j}} & FC \\ {\scriptstyle \sigma A}\downarrow & \sigma\overline{j} & \downarrow{\scriptstyle \sigma C} \\ GA & \xrightarrow[G\overline{j}]{} & GC \end{array}\right) = \begin{array}{ccc} FA & \xrightarrow{[F\overline{j}\ \overline{\sigma C}]} & GC \\ \Vert & i^v_{[F\overline{j}\ \sigma C]} & \Vert \\ FA & \xrightarrow[{[\overline{\sigma A}\ G\overline{j}]}]{} & GC \end{array}
$$

Some double categories admit a cofolding rather than a folding, as the following variant of Example 6.1 illustrates. In double adjunctions between double categories of monads and double categories of endomorphisms, cofoldings are more relevant than foldings, because cofoldings produce a map from vertical monad maps to horizontal monad maps.

Example 6.7. If $\mathbf{C}$ is a 2-category, let $\overline{\mathbb{Q}}\mathbf{C}$ be the double category in which the objects are the objects of $\mathbf{C}$, the horizontal 1-category is the underlying 1-category of $\mathbf{C}$, the vertical 1-category is the *opposite* of the underlying 1-category of $\mathbf{C}$, and the squares $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j^{\mathrm{op}}}\downarrow & \alpha & \downarrow{\scriptstyle k^{\mathrm{op}}} \\ C & \xrightarrow[g]{} & D \end{array}$ are 2-cells of the form $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\uparrow & \searrow\!\!\!\searrow^{\alpha} & \uparrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$ in $\mathbf{C}$.
The double category $\overline{\mathbb{Q}}\mathbf{C}$ admits a *cofolding* in the following sense.

Definition 6.8. A *cofolding on a double category* $\mathbb{D}$ consists of the following.

(i) A 2-functor $(-)^*\colon (\mathbf{V}\mathbb{D})_0^{\mathrm{op}} \to \mathbf{H}\mathbb{D}$ which is the identity on objects. In other words, to each vertical morphism $j\colon A \to C$, there is associated a horizontal morphism $j^*\colon C \to A$ in a functorial way. We call the 2-functor $j \mapsto j^*$ the *coholonomy*.

(ii) Bijections $\Lambda^{f,k}_{j,g}$ from squares in $\mathbb{D}$ with boundary

(12)
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

to squares in $\mathbb{D}$ with boundary

(13)
$$\begin{array}{ccc} C & \xrightarrow{[j^* \ f]} & B \\ \| & & \| \\ C & \xrightarrow[{[g \ k^*]}]{} & B. \end{array}$$

These bijections are required to satisfy the following axioms.

(i) Λ is the identity if j and k are vertical identity morphisms.

(ii) Λ preserves horizontal composition of squares, that is,

$$\Lambda\left(\begin{array}{ccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} & \beta & \downarrow{\scriptstyle \ell} \\ D & \xrightarrow[g_1]{} & E & \xrightarrow[g_2]{} & F \end{array}\right) = \begin{array}{ccc} D & \xrightarrow{[j^* \ f_1 \ f_2]} & C \\ \| & [\Lambda(\alpha) \ i^v_{f_2}] & \| \\ D & \xrightarrow{[g_1 \ k^* \ f_2]} & C \\ \| & [i^v_{g_1} \ \Lambda(\beta)] & \| \\ D & \xrightarrow[{[g_1 \ g_2 \ \ell^*]}]{} & C. \end{array}$$

(iii) Λ preserves vertical composition of squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j_1}\downarrow & \alpha & \downarrow{\scriptstyle k_1} \\ C & \xrightarrow{g} & D \\ {\scriptstyle j_2}\downarrow & \beta & \downarrow{\scriptstyle k_2} \\ E & \xrightarrow[h]{} & F, \end{array}\right) = \begin{array}{ccc} E & \xrightarrow{[j_2^* \ j_1^* f]} & B \\ \| & [i^v_{j_2^*} \ \Lambda(\alpha)] & \| \\ E & \xrightarrow{[j_2^* \ g \ k_1^*]} & B \\ \| & [\Lambda(\beta) \ i^v_{k_1^*}] & \| \\ E & \xrightarrow[{[h \ k_2^* \ k_1^*]}]{} & B. \end{array}$$

(iv) Λ preserves identity squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & = & A \\ {\scriptstyle j}\downarrow & i^h_j & \downarrow{\scriptstyle j} \\ B & = & B \end{array}\right) = \begin{array}{ccc} B & \xrightarrow{j^*} & A \\ \| & i^v_{j^*} & \| \\ B & \xrightarrow[j^*]{} & A. \end{array}$$

Example 6.9. The double category $\mathbb{S}\mathrm{pan}$ admits a cofolding. The coholonomy is

$$\left(A \xrightarrow{j} C\right) \longmapsto \left(C \xleftarrow{j} A \xrightarrow{1^h_A} A\right).$$

The cofolding is similar to the folding described in Example 6.3.

If two given double functors F and G are compatible with the foldings (respectively cofoldings), then one can reduce the question of horizontal double adjointness to the question of 2-adjointness for the underlying 2-functors $\mathbf{H}F$ and $\mathbf{H}G$.

Proposition 6.10. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories with folding (respectively cofolding) and consider double functors F and G compatible with the foldings (respectively cofoldings).*

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{A} \tag{14}$$

Then F and G are horizontal double adjoints if and only if their horizontal 2-functors $\mathbf{H}F$ and $\mathbf{H}G$ are 2-adjoints.

Proof: If F and G are double adjoints, then $\mathbf{H}F$ and $\mathbf{H}G$ are 2-adjoints, since the 2-functor $\mathbf{H}\colon \mathbf{DblCat_h} \to \mathbf{2\text{-}Cat}$ preserves adjoints, as does any 2-functor.

For the converse, suppose that F and G are compatible with the foldings and $\varphi_{X,A}\colon \mathbf{H}\mathbb{A}(FX, A) \to \mathbf{H}\mathbb{X}(X, GA)$ is a natural isomorphism of categories. For vertical morphisms j and k in $\mathbb{X}$ and $\mathbb{A}$ respectively, we define a bijection

$$\varphi_{j,k}\colon\ \mathbb{A}(Fj, k) \longrightarrow \mathbb{X}(j, Gk)$$

$$\varphi_{j,k}(\alpha) := \left(\Lambda^{f^\dagger, Gk}_{j, g^\dagger}\right)^{-1} \varphi_{sj,tk}\left(\Lambda^{f,k}_{Fj,g}(\alpha)\right).$$

Here $f^\dagger$ and $g^\dagger$ are the transposes of the horizontal morphisms f and g with respect to the underlying 1-adjunction. The naturality of $\varphi_{X,A}$ guarantees that the boundaries are correct.

The bijection $\varphi_{j,k}$ is compatible with vertical composition for the following reasons:

(i) $\varphi_{X,A}$ is compatible with the vertical composition of 2-cells in $\mathbf{H}\mathbb{X}$ and $\mathbf{H}\mathbb{A}$
(ii) the isomorphism $\varphi_{X,A}$ is natural in X and A, and
(iii) the foldings are compatible with vertical composition as in Definition 6.2 (iii).

The naturality of $\varphi_{j,k}$ in j and k similarly follows from (i) and (ii) above, and the compatibility of the foldings with horizontal composition in Definition 6.2 (ii).

These natural bijections $\varphi_{j,k}$ compatible with vertical composition are equivalent to a unit η and counit ε in a double adjunction by Theorem 5.2 (v), so we are finished.

The analogous proof works for the cofolding claim. □

Remark 6.11. In Proposition 6.10, note that the horizontal natural transformations η and ε which make F and G into horizontal double adjoints are not required to be compatible with the foldings, though if η and ε exist, they can be replaced by horizontal natural transformations compatible with the foldings. Note also that the holonomy (respectively coholonomy) is not required to be fully faithful.

Proposition 6.10 allows us to draw conclusions about double adjointness when both double functors F and G are already given, and are compatible with the foldings. It would be more desirable to have a statement reducing the existence of a horizontal right double adjoint for a given double functor F (compatible with foldings) to the existence of a right 2-adjoint for $\mathbf{H}F$, without referencing G at the outset. For such a statement, we need the strengthened hypothesis that the holonomy is fully faithful. The result is Corollary 6.13, which says that F admits a horizontal right double adjoint if and only if $\mathbf{H}F$ admits a right 2-adjoint, and this occurs if and only if the analogous vertical statements hold. To prove this, we use the 2-fully faithfulness of $\mathbf{H}$ and $\mathbf{V}$ in Proposition 6.12.

Let $\mathbf{DblCatFoldHol_h}$ denote the 2-category of small double categories with folding and fully faithful holonomy, double functors compatible with foldings, and horizontal natural transformations compatible with folding (see Definitions 6.2, 6.5, and 6.6). Let $\mathbf{DblCatFoldHol_v}$ denote the 2-category of small double categories with folding and fully faithful holonomy, double functors compatible with foldings, and vertical natural transformations compatible with folding (see Definitions 6.2, 6.5, and 6.6).

Proposition 6.12. *The forgetful 2-functors*

$$\mathbf{H}\colon \mathbf{DblCatFoldHol_h} \longrightarrow \mathbf{2Cat} \tag{15}$$

$$\mathbf{V}\colon \mathbf{DblCatFoldHol_v} \longrightarrow \mathbf{2Cat} \tag{16}$$

are 2-fully faithful.

Proof: Suppose $F, G\colon \mathbb{C} \to \mathbb{D}$ are double functors compatible with foldings, and in particular compatible with the fully faithful holonomy, and suppose $\mathbf{H}F = \mathbf{H}G$. Then the double functors F and G agree on the horizontal 2-categories. If j is a vertical morphism in $\mathbb{C}$, then $\overline{F(j)} = F(\overline{j}) = G(\overline{j}) = \overline{G(j)}$, and $F(j) = G(j)$. The double functors F and G similarly agree on squares because of the folding bijections. Conversely, if a 2-functor is defined on horizontal 2-categories, then it can be extended to the double categories using the bijective holonomy and then the foldings. Thus $\mathbf{H}$ in (15) and $\mathbf{V}$ in (16) are bijective on the objects of hom-categories.

Similar arguments hold for injectivity on horizontal respectively vertical natural transformations.

For surjectivity of (15) on 2-natural transformations, suppose $\theta\colon \mathbf{H}F \Rightarrow \mathbf{H}G$ is a 2-natural transformation. We extend θ to a horizontal natural transformation: for a vertical morphism j in $\mathbb{C}$, define θj by equation (10). Then $\Lambda(\theta k) = i^v_{[\,\theta B \;\; G\overline{k}\,]}$ and $\Lambda(\theta j) = i^v_{[\,\theta A \;\; G\overline{j}\,]}$ and the equation

$$\begin{bmatrix} i^v_{Ff} & \Lambda(\theta k) \\ \Lambda(F\alpha) & i^v_{\theta C} \end{bmatrix} = \begin{bmatrix} i^v_{\theta A} & \Lambda(G\alpha) \\ \Lambda(\theta j) & i^v_{Gg} \end{bmatrix} \tag{17}$$

holds by 2-naturality. The double naturality $[\, F\alpha \;\; \theta k \,] = [\, \theta j \;\; G\alpha \,]$ requirement for θ then follows from an application of Λ^{-1} to (17).

For surjectivity of (16) on 2-natural transformations, suppose $\sigma\colon \mathbf{V}F \Rightarrow \mathbf{V}G$ is a 2-natural transformation. We extend σ to a vertical natural transformation: for any horizontal morphism $\overline{j}$ in $\mathbb{C}$, define $\sigma\overline{j}$ by equation (11). Recall that the holonomy is fully faithful, so any horizontal morphism is of the form $\overline{j}$ for a unique vertical morphism j. Another consequence of the fully faithful holonomy, and compatibility with horizontal composition of 2-cells in the vertical 2-category in Remark 6.4, is that the holonomy and folding transform 2-natural transformations $\mathbf{V}F \Rightarrow \mathbf{V}G$ into 2-natural transformations $\mathbf{H}F \Rightarrow \mathbf{H}G$. With this fact, the proof for surjectivity of (16) on 2-natural transformations proceeds like that of (15). □

Corollary 6.13. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories with folding and fully faithful holonomies. Let $F\colon \mathbb{X} \to \mathbb{A}$ be a double functor compatible with the foldings. Then the following are equivalent.*

(i) *The double functor F admits a horizontal right double adjoint (not necessarily compatible with the foldings).*
(ii) *The 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{X} \to \mathbf{H}\mathbb{A}$ admits a right 2-adjoint.*
(iii) *The double functor F admits a vertical right double adjoint (not necessarily compatible with the foldings).*
(iv) *The 2-functor $\mathbf{V}F\colon \mathbf{V}\mathbb{X} \to \mathbf{V}\mathbb{A}$ admits a right 2-adjoint.*

Proof: By Proposition 6.12, the 2-functor $\mathbf{H}$ in (15) is 2-fully faithful, so F admits a horizontal right double adjoint compatible with the foldings if and only if $\mathbf{H}F$ admits a right 2-adjoint. But if F admits a horizontal right double adjoint G not necessarily compatible with the foldings, then $\mathbf{H}G$ is still a right 2-adjoint to $\mathbf{H}F$, and Proposition 6.12 applies to extend the 2-adjunction $\mathbf{H}F \dashv \mathbf{H}G$ to a horizontal double adjunction with left horizontal double adjoint F. Thus (i)⇔(ii) and similarly (iii)⇔(iv).

To complete the proof, we observe (ii) ⇔ (iv), because the fully faithful holonomy and folding provide a 1-1 correspondence between 2-natural transformations $\mathbf{V}F_1 \Rightarrow \mathbf{V}F_2$ and 2-natural transformations $\mathbf{H}F_1 \Rightarrow \mathbf{H}F_2$, see Remark 6.4. □

Example 6.14. The double category of quintets $\mathbb{Q}\mathbf{C}$ in Example 6.1 admits a fully faithful holonomy. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a 2-functor. Then the double functor $\mathbb{Q}F\colon \mathbb{Q}\mathbf{C} \to \mathbb{Q}\mathbf{D}$ admits a horizontal right double adjoint if only if F admits a right 2-adjoint if and only if $\mathbb{Q}F$ admits a vertical right double adjoint if and only if $\mathbf{V}\mathbb{Q}F = F^{\mathrm{co}}$ admits a right 2-adjoint.

We also state the analogues of Proposition 6.12 and Corollary 6.13 for double categories with cofoldings and fully faithful coholonomies. The coholonomy is contravariant, so the horizontal and vertical double adjunctions go in the opposite directions in Corollary 6.16.

Let $\mathbf{DblCatCoFoldCoHol_h}$ denote the 2-category of small double categories with cofolding and fully faithful coholonomy, double functors compatible with cofoldings, and horizontal natural transformations compatible with cofolding (see Definition 6.8 and consider the cofolding analogues of Definitions 6.5 and 6.6). Let $\mathbf{DblCatCoFoldCoHol_v}$ denote the 2-category of small double categories with cofolding and fully faithful coholonomy, double functors compatible with cofoldings, and vertical natural transformations compatible with cofolding (see Definition 6.8 and consider the cofolding analogues of Definitions 6.5 and 6.6).

Proposition 6.15. *The forgetful 2-functors*

$$\mathbf{H}\colon \mathbf{DblCatCoFoldCoHol_h} \longrightarrow \mathbf{2Cat} \tag{18}$$

$$\mathbf{V}\colon \mathbf{DblCatCoFoldCoHol_v}^{\mathrm{co}} \longrightarrow \mathbf{2Cat} \tag{19}$$

are 2-fully faithful.

Corollary 6.16. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories with cofolding and fully faithful coholonomies. Let $F\colon \mathbb{X} \to \mathbb{A}$ be a double functor compatible with the cofoldings. Then the following are equivalent.*

(i) *The double functor F admits a horizontal right double adjoint (not necessarily compatible with the cofoldings).*
(ii) *The 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{X} \to \mathbf{H}\mathbb{A}$ admits a right 2-adjoint.*
(iii) *The double functor F admits a vertical left double adjoint (not necessarily compatible with the cofoldings).*
(iv) *The 2-functor $\mathbf{V}F\colon \mathbf{V}\mathbb{X} \to \mathbf{V}\mathbb{A}$ admits a left 2-adjoint.*

Example 6.17. The quintet variant $\overline{\mathbb{Q}}\mathbf{C}$ in Example 6.7 admits a fully faithful coholonomy. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a 2-functor. Then the double functor $\overline{\mathbb{Q}}F\colon \overline{\mathbb{Q}}\mathbf{C} \to \overline{\mathbb{Q}}\mathbf{D}$ admits a horizontal right double adjoint if and only if F admits a right 2-adjoint if and only if $\overline{\mathbb{Q}}F$ admits a vertical left double adjoint if and only if $\mathbf{V}\overline{\mathbb{Q}}F = F^{\mathrm{op}}$ admits a left 2-adjoint.

7. Endomorphisms and Monads in a Double Category

One of the goals of this paper is to simultaneously remove several hypotheses from the main theorem of [7] and strengthen its conclusion to obtain Theorem 9.5 of this paper, which says that if a double category $\mathbb{D}$ with cofolding admits the construction of free monads in its horizontal 2-category, then $\mathbb{D}$ admits the construction of free monads in every other sense. Towards that goal, we prove in this section a cofolding on $\mathbb{D}$ induces a cofolding on the double categories $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ of endomorphisms and monads in $\mathbb{D}$, see [7, Definitions 2.3 and 2.4]. We also give examples of $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ in the case that $\mathbb{D}$ is the double category of quintets. Another goal of this paper is Theorem 10.3, the characterization of the existence of Eilenberg–Moore objects in a double category in terms of representability of certain parameterized presheaves. For that we also need an understanding of the double category $\mathbb{M}\mathrm{nd}(\mathbb{D})$.

Example 7.1 (Endomorphisms in Quintets)**.** To a 2-category $\mathbf{C}$, we may associate two possible 2-categories of endomorphisms, depending on the choice of 2-cell direction in the notion of endomorphism map. The double category $\mathbb{E}\mathrm{nd}(\mathbb{Q}\mathbf{C})$ contains both of these 2-categories in the following way (recall the double category of quintets $\mathbb{Q}\mathbf{C}$ from Example 6.1). The objects of $\mathbb{E}\mathrm{nd}(\mathbb{Q}\mathbf{C})$ are the endomorphisms in $\mathbf{C}$, while a horizontal morphism $(F,\phi)\colon (X,P) \to (Y,Q)$ is a traditional endomorphism map in $\mathbf{C}$, that is, a morphism $F\colon X \to Y$ in $\mathbf{C}$ equipped with a 2-cell $\phi\colon QF \Rightarrow FP$. A vertical endomorphism map, on the other hand, is a morphism in $\mathbf{C}$ equipped with a 2-cell in the opposite direction as ϕ. Squares with vertically trivial boundaries or horizontally trivial boundaries are the 2-cells in the two possible 2-categories of endomorphisms in $\mathbf{C}$.

Example 7.2. On the other hand, for the quintet variant $\overline{\mathbb{Q}}\mathbf{C}$ in Example 6.7, the objects of $\mathbb{E}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C})$ are the endomorphisms in $\mathbf{C}$, and the horizontal morphisms in $\mathbb{E}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C})$ are the traditional endomorphism maps in $\mathbf{C}$. The vertical 1-category of $\mathbb{E}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C})$ is the opposite of the category of endomorphisms in $\mathbf{C}$ and traditional endomorphism maps.

Example 7.3 (Monads in Quintets)**.** To a 2-category $\mathbf{C}$, we may associate two possible 2-categories of monads, depending on the choice of 2-cell direction in the notion of monad map. The double category $\mathbb{M}\mathrm{nd}(\mathbb{Q}\mathbf{C})$ contains both of these 2-categories in the following way (recall the double category of quintets $\mathbb{Q}\mathbf{C}$ from Example 6.1). The objects of $\mathbb{M}\mathrm{nd}(\mathbb{Q}\mathbf{C})$ are the monads in $\mathbf{C}$, while a horizontal morphism $(F,\phi)\colon (X,P) \to (Y,Q)$ is a traditional monad map in $\mathbf{C}$, that is, a morphism $F\colon X \to Y$ equipped with a 2-cell $\phi\colon QF \Rightarrow FP$ compatible with the monad structure maps. A vertical endomorphism map, on the other hand, is a morphism equipped with a 2-cell in the opposite direction as ϕ and also compatible with the monad structure maps. Squares with vertically trivial boundaries or horizontally trivial boundaries are the 2-cells in the two possible 2-categories of monads in $\mathbf{C}$.

Example 7.4. On the other hand, for the quintet variant $\overline{\mathbb{Q}}\mathbf{C}$ in Example 6.7, the objects of $\mathbb{M}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C})$ are the monads in $\mathbf{C}$, the horizontal morphisms in $\mathbb{M}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C})$ are the traditional monad maps in $\mathbf{C}$. The vertical 1-category of $\mathbb{M}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C})$ is the opposite of the category of the monads and traditional monad maps in $\mathbf{C}$.

We now turn to the main point of this section: a cofolding on $\mathbb{D}$ induces a cofolding on $\mathbb{M}\mathrm{nd}(\mathbb{D})$ and $\mathbb{E}\mathrm{nd}(\mathbb{D})$.

Proposition 7.5. *If $(\mathbb{D}, \Lambda^{\mathbb{D}})$ is a double category with cofolding (Definition 6.8), then the double categories $\mathbb{M}\mathrm{nd}(\mathbb{D})$ and $\mathbb{E}\mathrm{nd}(\mathbb{D})$ inherit cofoldings from $\mathbb{D}$ and the forgetful double functor $U\colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ preserves them. The coholonomies and cofoldings are defined as in (i).*

(i) (a) *If $(u, \bar{u})\colon (X, P) \to (X', P')$ is a vertical endomorphism map, then the corresponding horizontal endomorphism map $(u, \bar{u})^*$ under the coholonomy is*

$$(u^*, \Lambda^{\mathbb{D}}(\bar{u}))\colon (X', P') \to (X, P).$$

(b) *If α is an endomorphism square, then the corresponding endomorphism 2-cell is the $\mathbb{D}$-cofolding of α, namely $\Lambda^{\mathbb{D}}(\alpha)$.*

(ii) *The assignments in (i) are compatible with vertical monad maps and monad squares:*

(a) *If (X, P) and (X', P') are monads, and $(u, \bar{u})$ is vertical monad map, then $(u, \bar{u})^* = (u^*, \Lambda^{\mathbb{D}}(\bar{u}))$ is a horizontal monad map.*

(b) *If α is a monad square, then $\Lambda^{\mathbb{D}}(\alpha)$ is a monad 2-cell.*

(iii) *Moreover, if $u\colon X \to X'$ is a fixed vertical morphism in $\mathbb{D}$, then*

$$(u, \bar{u}) \mapsto (u^*, \Lambda(\bar{u}))$$

is a bijection between vertical endomorphism maps $(X, P) \to (X', P')$ with underlying vertical morphism u and horizontal endomorphism maps $(X', P') \to (X, P)$ with underlying horizontal morphism u^*. If (X, P) and (X', P') are monads, we have a similar one-to-one correspondence between vertical monad maps with underlying morphism u and horizontal monad maps with underlying morphism u^*.

(iv) *If the coholonomy on $\mathbb{D}$ is fully faithful, then the coholonomies on $\mathbb{M}\mathrm{nd}(\mathbb{D})$ and $\mathbb{E}\mathrm{nd}(\mathbb{D})$ are also fully faithful.*

Compare with [7, Lemma 3.4].

Proof: Statement (i) defines a coholonomy and cofolding by the functoriality of the coholonomy on $\mathbb{D}$ and the compatibility of $\Lambda^{\mathbb{D}}$ with horizontal and vertical composition of squares. Compatibility with monad structure maps in (ii)(a) also follows from the compatibility of $\Lambda^{\mathbb{D}}$ with horizontal and vertical composition of squares. The bijection in (iii) for a fixed vertical morphism u (and hence also fixed u^*) follows from the bijectivity of the cofolding. Statement (iv) clearly follows from the definitions in (i) using the assumed fully faithfulness of the coholonomy on $\mathbb{D}$ and the bijectivity of the cofolding $\Lambda^{\mathbb{D}}$. □

It is worth pointing out that Statement (iii) does *not* say the coholonomies in $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ are fully faithful. Rather, Statement (iii) says if u is a *fixed* vertical morphism in $\mathbb{D}$, then we have the indicated bijections. Though Statement (iii) is weaker than fully faithfulness of the coholonomies, it does allow us to make strong conclusions. For example, Statement (iii) will be used many times in the proof of Theorem 9.5.

8. Example: Endomorphisms and Monads in $\mathbb{S}\mathrm{pan}$

Our next topic is a detailed example which illustrates the local description of double adjunctions in Theorem 5.2 (v), makes concrete the double categories of endomorphisms and monads in Section 7, and motivates Theorem 9.5 on free monads. We consider the double category $\mathbb{S}\mathrm{pan}$ in which objects are small sets, horizontal morphisms are spans, vertical morphisms are functions, and squares are span morphisms, and we consider the forgetful-free *vertical* double adjunction

$$\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan}) \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan}). \qquad \perp \tag{20}$$

Though the double categories $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ and $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are horizontally weak (horizontally they are bicategories), the double functors F and G strictly preserve all compositions and units. The 1-adjunction

$$\mathbf{DirGraph} \underset{Forget}{\overset{Free}{\rightleftarrows}} \mathbf{Cat} \qquad \perp$$

is the *vertical* 1-category part of (20).

Remark 8.1. Since the double adjunction (20) is *vertical* rather than horizontal, we use the transpose of the characterizations in Theorem 5.2. We cannot simply transpose the double categories and double functors in (20) in order to apply the non-transposed Theorem 5.2, because our notions of monads in a double category and their various morphisms prefer the horizontal direction as distinguished.

We begin by spelling out the double categories $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ and $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ recalled in Section 7. Objects and vertical morphisms of $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are directed graphs $G_0 \leftarrow G_1 \rightarrow G_0$ and morphisms of directed graphs. A horizontal morphism $(U, \phi)\colon G_* \rightarrow G'_*$ in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ is a span $U\colon G_0 \leftarrow U_1 \rightarrow G'_0$ equipped with a chosen (not necessarily bijective) function $\phi\colon U_1 \times_{G'_0} G'_1 \rightarrow G_1 \times_{G_0} U_1$, namely a square in $\mathbb{S}\mathrm{pan}$ as below.

(21)
$$\begin{array}{ccccccccc}
G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & G'_1 & \longrightarrow & G'_0 \\
\| & & & & \phi & & & & \| \\
G_0 & \longleftarrow & G_1 & \longrightarrow & G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 \,.
\end{array}$$

Horizontal composition of horizontal morphisms is by pullback. The associated ϕ-part of the composite is the vertical composition of the following squares.

(22)
$$\begin{array}{ccccccccccccc}
G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & V_1 & \longrightarrow & G''_0 & \longleftarrow & G''_1 & \longrightarrow & G''_0 \\
\| & & 1_U & & \| & & & & \psi & & & & \| \\
G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & G'_1 & \longrightarrow & G'_0 & \longleftarrow & V_1 & \longrightarrow & G''_0 \\
\| & & & & \phi & & & & \| & & 1_V & & \| \\
G_0 & \longleftarrow & G_1 & \longrightarrow & G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & V_1 & \longrightarrow & G''_0 \,.
\end{array}$$

A square in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$

(23)
$$\begin{array}{ccc}
G_* & \xrightarrow{U} & G'_* \\
{\scriptstyle J_*}\big\downarrow & \alpha & \big\downarrow{\scriptstyle J'_*} \\
H_* & \xrightarrow[V]{} & H'_*
\end{array}$$

is a square in $\mathbb{S}\mathrm{pan}$

$$\begin{array}{ccccc}
G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 \\
\big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & \big\downarrow \\
H_0 & \longleftarrow & V_1 & \longrightarrow & H'_0
\end{array}$$

such that the cube with ϕ on top and ϕ' on bottom commutes. Horizontal and vertical composition of squares in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are the horizontal and vertical compositions of the underlying squares in $\mathbb{S}\mathrm{pan}$, for example, horizontal composition is defined via pullback.

The other double category in the adjunction (20), namely $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$, is the double category of monads in the double category of spans. Objects and vertical morphisms are categories and functors. The horizontal morphisms of $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are the same as Street's morphisms of monads in a 2-category [12]. Namely, a horizontal monad morphism $U\colon C_* \to D_*$ is a span $C_0 \leftarrow U_1 \to D_0$ and a square in $\mathbb{S}\mathrm{pan}$

$$\begin{array}{ccccccccc} C_0 & \longleftarrow & U_1 & \longrightarrow & D_0 & \longleftarrow & D_1 & \longrightarrow & D_0 \\ \| & & & & \phi & & & & \| \\ C_0 & \longleftarrow & C_1 & \longrightarrow & C_0 & \longleftarrow & U_1 & \longrightarrow & D_0 \end{array}$$

such that

$$\begin{bmatrix} [\ 1^v_U \ \ \eta^D\ ] \\ \phi \end{bmatrix} = [\ \eta^C \ \ 1^v_U\ ]$$

and

$$\begin{bmatrix} & \phi & 1^v_D \\ 1^v_C & & \phi \\ & \mu^C & 1^v_U \end{bmatrix} = \begin{bmatrix} [\ 1^v_U \ \ \mu^D\ ] \\ \phi \end{bmatrix}.$$

In other words, we have a function $\phi\colon U_1 \times_{D_0} D_1 \to C_1 \times_{C_0} U_1$ such that

$$\phi(u, 1_{tu}) = (1_{su}, u) \tag{24}$$

for all $u \in U_1$ and

$$\phi^C(\phi^U(u,d),d') \circ \phi^C(u,d) = \phi^C(u, d' \circ d) \tag{25}$$

$$\phi^U(\phi^U(u,d),d') = \phi^U(u, d' \circ d). \tag{26}$$

Note that if D and K have just one object, then equation (26) and the unit equation (24) essentially say ϕ^U defines a left monoid action of D_1 on U_1. Horizontal composition of horizontal morphisms in $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ is by pullback, and the ϕ-parts compose as in equation (22).

Remark 8.2. One way to think of ϕ is as an assignment that converts a path

$$\xrightarrow{\in U_1} \xrightarrow{\in D_1}$$

to a path

$$\xrightarrow{\in C_1} \xrightarrow{\in U_1}$$

in a way compatible with unit and composition.

Returning to the description of $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$, a square

$$\begin{array}{ccc} A_* & \xrightarrow{(U,\phi)} & B_* \\ {\scriptstyle (J_1,J_0)}\downarrow & \alpha & \downarrow{\scriptstyle (K_1,K_0)} \\ C_* & \xrightarrow[(V,\psi)]{} & D_* \end{array} \tag{27}$$

in $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ is a square α in $\mathbb{S}\mathrm{pan}$ such that

$$\begin{bmatrix} \phi \\ [\ J_1 \ \ \alpha\ ] \end{bmatrix} = \begin{bmatrix} [\ \alpha \ \ K_1\ ] \\ \psi \end{bmatrix},$$

in other words

$$(J_1(\phi^A(u,b)),\ \alpha(\phi^U(u,b))) = (\psi^C(\alpha(u), K(b)),\ \psi^V(\alpha(u), K(b)).$$

Now that we understand the double categories in (20), we next turn to the double functors F and G. On objects and vertical morphisms (that is, on directed graphs and their morphisms), F is the free category functor. On a horizontal morphism $(U,\phi)\colon G_* \to G'_*$ in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ as in (21), we have $F(U)_1 := U_1$. The function ϕ extends to $F(\phi)$ by Remark 8.2 and the fact that morphisms in the free category on a (non-reflexive) graph are paths of edges. On $F(U)_1 \times_{G'_0} G'_1$, the function $F(\phi)$ is simply ϕ. On $F(U)_1 \times_{G'_0} F(G'_*)_1$, the function $F(\phi)$ is defined by moving the element of U_1 across the path, one edge at a time using ϕ. For example,

$$\begin{array}{ccc} \xrightarrow{\ u\ } & \xrightarrow{\ g\ } & \xrightarrow{\ h\ } \\ \xrightarrow{\phi^G(u,g)} & \xrightarrow{\phi^U(u,g)} & \xrightarrow{\ h\ } \\ \xrightarrow{\phi^G(u,g)} & \xrightarrow{\phi^G(\phi^U(u,g),h)} & \xrightarrow{\phi^U(\phi^U(u,g),h)} \end{array} \tag{28}$$

which is the same as below.

$$\begin{array}{cc} \xrightarrow{\ u\ } & \xrightarrow{h\circ g} \\ \xrightarrow{\phi^G(u,h\circ g)} & \xrightarrow{\phi^U(u,h\circ g)} \end{array} \tag{29}$$

The equality of (28) and (29) shows that $F(\phi)$ satisfies the composition rules in (25) and (26) by definition. Similarly, (24) holds by definition and the fact that our directed graphs are non-reflexive. Concerning the definition of F on squares, the double functor F takes a square α in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ as in (23) to the square $F\alpha$ in $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ as in (27) which has the same middle function $U_1 \to V_1$ as α, but the left and right vertical morphisms are the unique functors on the free categories that extend the directed graph morphisms on the left and right of α. For this reason, F clearly preserves vertical composition of vertical morphisms and squares. It also preserves horizontal composition because the horizontal composition in both double categories is defined via pullback. Also the ϕ part of $F(V \circ U)$ is the appropriate composite of the ϕ-parts of U and V by an inductive verification using the "switching" point of view on ϕ as just discussed. Thus F is a *strict* double functor.

The double functor G is easy to describe: it is simply the forgetful double functor, and is therefore clearly a *strict* double functor.

We use the transposition of the local description of double adjunctions in Theorem 5.2 (v) to prove that $F \dashv G$ as a vertical double adjunction. To simplify our work with the transposed characterization, we introduce the notations

$$\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})\begin{pmatrix} FU \\ V \end{pmatrix} \quad \text{and} \quad \mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})\begin{pmatrix} U \\ GV \end{pmatrix}$$

to mean the set of squares in $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ with vertical source FU and vertical target V, and the set of squares in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ with vertical source U and vertical target GV. This notation is the transpose of the notation in equation (1). We define a bijection

$$\varphi^U_V \colon \mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})\begin{pmatrix} FU \\ V \end{pmatrix} \longrightarrow \mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})\begin{pmatrix} U \\ GV \end{pmatrix} \tag{30}$$

$$\begin{array}{ccc} FA_* & \xrightarrow{F(U,\phi)} & FB_* \\ {\scriptstyle J}\big\downarrow & \alpha & \big\downarrow{\scriptstyle K} \\ C_* & \xrightarrow[(V,\psi)]{} & D_* \end{array} \quad \longmapsto \quad \begin{array}{ccc} A_* & \xrightarrow{(U,\phi)} & B_* \\ {\scriptstyle J_{\mathrm{res}}}\big\downarrow & \alpha_{\mathrm{res}} & \big\downarrow{\scriptstyle K_{\mathrm{res}}} \\ GC_* & \xrightarrow[G(V,\psi)]{} & GD_* \end{array}$$

that is compatible with horizontal composition. The index res means restriction: the maps J_{res} and K_{res} are the restrictions of the functors J and K to the directed graphs A_* and B_*, while α_{res} has the same exact middle function $U_1 \to V_1$ as α does. The square α_{res} is restricted only in the sense that its horizontal source and target are restricted. Since the middle function of α is the same as that of α_{res}, the function φ^U_V is manifestly injective. If α' is a square in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})\begin{pmatrix} U \\ GV \end{pmatrix}$, then we use the bijection $J \leftrightarrow J_{\mathrm{res}}$ to find the horizontal source and target of $(\varphi^U_V)^{-1}(\alpha')$, and define the middle function of $(\varphi^U_V)^{-1}(\alpha')$ to be that of α'. This proves the surjectivity of φ^U_V.

To see that $\varphi([\,\alpha \ \ \beta\,]) = [\,\varphi(\alpha) \ \ \varphi(\beta)\,]$, one only needs to observe that $(\alpha \times_{K_0} \beta)_{\mathrm{res}}$ is the same as $\alpha_{\mathrm{res}} \times_{(K_{\mathrm{res}})_0} \beta_{\mathrm{res}}$ because the diagrams, from which we are forming the pullbacks, are exactly the same. Namely,

$$\begin{array}{ccccccccc} (FA_*)_0 & \longleftarrow & F(U)_1 & \longrightarrow & (FB_*)_0 & \longleftarrow & F(W)_1 & \longrightarrow & (FH_*)_1 \\ {\scriptstyle J_0}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle K_0} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle L_0} \\ C_0 & \longleftarrow & V_1 & \longrightarrow & D_0 & \longleftarrow & X_1 & \longrightarrow & D_0 \end{array}$$

is exactly the same as

$$\begin{array}{ccccccccc} A_0 & \longleftarrow & U_1 & \longrightarrow & B_0 & \longleftarrow & W_1 & \longrightarrow & H_1 \\ {\scriptstyle (J_{\mathrm{res}})_0}\big\downarrow & & {\scriptstyle \alpha_{\mathrm{res}}}\big\downarrow & & \big\downarrow{\scriptstyle (K_{\mathrm{res}})_0} & & \big\downarrow{\scriptstyle \beta_{\mathrm{res}}} & & \big\downarrow{\scriptstyle (L_{\mathrm{res}})_0} \\ (GC_*)_0 & \longleftarrow & (GV)_1 & \longrightarrow & (GD_*)_0 & \longleftarrow & (GX)_1 & \longrightarrow & (GD_*)_0. \end{array}$$

It only remains to check the naturality of φ^U_V in U and V, but that is similar to the naturality of the ordinary free category functor-forgetful functor adjunction, the only difference is that here we use *vertical* pre- and post-composition of *squares*.

In summary, the bijection ϕ_V^U in (30) is compatible with horizontal composition and natural in the horizontal morphisms U and V, so $F \dashv G$ in (20) is a vertical double adjunction by Theorem 5.2 (v).

Another important example is the horizontal double adjunction between the "underlying" and the inclusion double functor

$$\mathbb{M}\mathrm{nd}(\mathbb{D}) \underset{\mathrm{Inc}_{\mathbb{D}}}{\overset{\mathrm{Und}}{\underset{\perp}{\rightleftarrows}}} \mathbb{D}.$$

The inclusion double functor $\mathrm{Inc}_{\mathbb{D}}$ always admits Und as a horizontal left double adjoint, but $\mathrm{Inc}_{\mathbb{D}}$ may or may not admit a right double adjoint. When $\mathrm{Inc}_{\mathbb{D}}$ admits a horizontal right double adjoint, we say that $\mathbb{D}$ *admits Eilenberg–Moore objects*. We will return to this topic in Section 10. Next we consider double adjoints to the forgetful double functor from monads in $\mathbb{D}$ to endomorphisms in $\mathbb{D}$.

9. Free Monads in Double Categories with Cofolding

In this section we remove several hypotheses from the main theorem of [7] and strengthen its conclusion to obtain Theorem 9.5, which says that if a double category $\mathbb{D}$ with cofolding admits the construction of free monads in its horizontal 2-category, then $\mathbb{D}$ admits the construction of free monads in every other sense. We first recall free monads on endomorphisms in a 2-category.

Definition 9.1. Let $\mathcal{K}$ be a 2-category. We say $\mathcal{K}$ *admits the construction of free monads* if either of the two following equivalent conditions hold.

(i) For every endomorphism (Y, Q) there exists a monad (Y, Q^{free}) and a 2-cell $\iota\colon Q \to Q^{\mathrm{free}}$ in $\mathcal{K}$ such that the endomorphism map $(1_Y, \iota_Q)\colon (Y, Q^{\mathrm{free}}) \to (Y, Q)$ is universal in the sense that for every monad (X, P), post-composition with $(1_Y, \iota_Q)$ induces an isomorphism of categories

$$\mathrm{Mnd}_{\mathcal{K}}((X,P),(Y,Q^{\mathrm{free}})) \xrightarrow{(1_Y,\iota_Q)\circ U(-)} \mathrm{End}_{\mathcal{K}}(U(X,P),(Y,Q)),$$

where $U\colon \mathrm{Mnd}(\mathcal{K}) \to \mathrm{End}(\mathcal{K})$ is the forgetful 2-functor.

(ii) The forgetful functor $U\colon \mathrm{Mnd}(\mathcal{K}) \to \mathrm{End}(\mathcal{K})$ admits a *right* 2-adjoint $R\colon \mathrm{End}(\mathcal{K}) \to \mathrm{Mnd}(\mathcal{K})$ with a counit ε such that the underlying morphism in $\mathcal{K}$ of each counit component $\varepsilon_{(Y,Q)}\colon UR(Y,Q) \to (Y,Q)$ is 1_Y.

Remark 9.2. Definition 9.1 is due to Staton [11, Theorem 6.1.5] in the case $\mathcal{K} = \mathbf{Cat}$, and is treated in general in our first paper [7, Theorem 1.1]. The reason Definition 9.1 requires a *right* adjoint to the forgetful functor (as opposed to an expected *left* adjoint) is the choice of the direction of 2-cell in the definition of endomorphism map and monad map, as we now explain. This right adjoint restricts to a left adjoint when we consider monads and endomorphisms on a fixed object Y.

Consider a fixed object Y of the 2-category $\mathcal{K}$. The *category of endomorphisms on* Y, denoted $\mathrm{End}(Y)$, has objects endomorphisms on Y. The morphisms in $\mathrm{End}(Y)$ are endomorphism maps with underlying morphism the identity on Y, that is, endomorphism maps of the form $(1_Y, \phi)\colon (Y, Q_1) \to (Y, Q_2)$. We follow the convention of Street [12] for the 2-cell ϕ, namely $\phi\colon Q_2 1_Y \to 1_Y Q_1$. There are no compatibility requirements on ϕ. The *category of monads on* Y, denoted $\mathrm{Mnd}(Y)$, has objects monads on Y. The morphisms in $\mathrm{Mnd}(Y)$ are monad maps with underlying morphism the identity on Y, that is, morphisms are monad maps of the form $(1_Y, \psi)\colon (Y, M_1) \to (Y, M_2)$. Again, we follow Street's convention in [12] for the 2-cell ψ, namely $\psi : M_2 1_Y \to 1_Y M_1$. The 2-cell ψ is required to be compatible with the unit and multiplication of the monads M_1 and M_2.

The variance in Definition 9.1 restricts to the expected one for monads on the fixed object Y, that is, the 2-category $\mathcal{K}$ *is said to admit the construction of free monads on* Y if the forgetful functor $U_Y\colon \mathrm{Mnd}(Y) \to \mathrm{End}(Y)$ admits a *left* adjoint. If $\mathcal{K}$ admits the construction of free monads in the sense of Definition 9.1, then $\mathcal{K}$ admits the construction of free monads on each object Y.

Remark 9.3. In Definition 9.1 (i), the isomorphism of categories commutes with the evident forgetful functors

$$\begin{array}{ccc} \mathrm{Mnd}(\mathcal{K})((X,P),(Y,Q^{\mathrm{free}})) & \xrightarrow{\cong} & \mathrm{End}(\mathcal{K})(U(X,P),(Y,Q)) \\ & \searrow \quad \swarrow & \\ & \mathcal{K}(X,Y), & \end{array}$$

since the underlying morphisms and 2-cells in $\mathcal{K}$ are composed with (whiskered with) 1_Y.

We refine [7, Definition 2.8] to the more stringent Definition 9.4. In Theorem 9.5 we then prove that if a double category $\mathbb{D}$ with cofolding admits the construction of free monads in its horizontal 2-category, then the horizontal 2-adjunction extends to a horizontal double adjunction as well as a vertical double adjunction, and $\mathbb{D}$ admits the construction of free monads as a double category.

Definition 9.4. A double category $\mathbb{D}$ is said to *admit the construction of free monads* if the forgetful double functor $U\colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ admits a vertical left double adjoint R with a unit η such that the underlying vertical morphism in $\mathbb{D}$ of each unit component $\eta_{(Y,Q)}\colon (Y,Q) \to UR(Y,Q)$ is 1^v_Y.

Theorem 9.5 (Reduction of construction of free monads to horizontal 2-category)**.** *Let* $\mathbb{D}$ *be a double category with cofolding* Λ*. If the horizontal 2-category of* $\mathbb{D}$ *admits the construction of free monads in the sense of Definition 9.1, then the double category* $\mathbb{D}$ *admits the construction of free monads in the sense of Definition 9.4.*

More precisely, suppose that the horizontal 2-category of $\mathbb{D}$ *admits the construction of free monads in the sense of Definition 9.1, that is, the forgetful*

2-functor $\mathbf{H}U$ *admits a right 2-adjoint* R *such that the counit components of the 2-adjunction are of the form* $\varepsilon_{(Y,Q)} = (1^h_Y, \iota_Q)$. *Then the following hold.*

(i) *The 2-adjunction*

$$\mathbf{H}\mathbb{M}\mathrm{nd}(\mathbb{D}) \underset{R}{\overset{\mathbf{H}U}{\rightleftarrows}} \mathbf{H}\mathbb{E}\mathrm{nd}(\mathbb{D}) \quad (\perp)$$

in Definition 9.1 extends to a horizontal double adjunction. In particular, R *extends to a double functor, also called* R. *The double functor* R *is a horizontal right double adjoint to* U.

(ii) *The double functor* R *is a vertical left double adjoint to* U, *and the components of the unit are* $\eta_{(Y,Q)} := (1^v_Y, \iota_Q)$.

(iii) *We have a 2-adjunction*

$$\mathbf{V}\mathbb{E}\mathrm{nd}(\mathbb{D}) \underset{\mathbf{V}U}{\overset{\mathbf{V}R}{\rightleftarrows}} \mathbf{V}\mathbb{M}\mathrm{nd}(\mathbb{D}). \quad (\perp)$$

Proof: The coholonomies and cofoldings on $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ inherited from $\mathbb{D}$ in Proposition 7.5 will be used throughout. Suppose that the horizontal 2-category of $\mathbb{D}$ admits the construction of free monads in the sense of Definition 9.1.

(i) To extend R to a double functor, we use the cofoldings on $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ and the crucial fact that the underlying morphism of the counit is the identity morphism. The 2-functor R is defined on (horizontal) endomorphism maps $(F,\phi)\colon (X,P) \to (Y,Q)$ and endomorphism 2-cells $\alpha\colon (F_1,\phi_1) \Rightarrow (F_2,\phi_2)$ by the equations

$$[\ UR(F,\phi)\ \ (1_Y,\iota_Q)] = [\ (1_X,\iota_P)\ \ (F,\phi)\ ] \tag{31}$$

$$\Big[\ UR\alpha\ \ i^v_{(1_Y,\iota_Q)}\Big] = \big[\ i^v_{(1_X,\iota_P)}\ \ \alpha\ \big]. \tag{32}$$

If $(u,\overline{u})$ is a vertical endomorphism map, then $R(u^*,\Lambda(\overline{u})) =: (Ru^*, R\Lambda(\overline{u}))$ is defined by (31). We see from (31) that the underlying horizontal morphism of Ru^* is u^*, so by Proposition 7.5 (iii) we may apply Λ^{-1} to $R\Lambda(\overline{u})$ to obtain $R(u,\overline{u}) := (u, \Lambda^{-1}R\Lambda(\overline{u}))$ with underlying vertical morphism u. A similar argument using equation (32) defines R on squares of $\mathbb{E}\mathrm{nd}(\mathbb{D})$.

By construction, the double functors R and U are compatible with the cofoldings, so the 2-adjunction $\mathbf{H}U \dashv \mathbf{H}R$ extends to a horizontal double adjunction by Proposition 6.10.

(ii) We prove that R is a vertical left double adjoint to $U\colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ using the vertical version of Theorem 5.2 (ii), which requires functors

$$R_0\colon (\mathrm{Obj}\ \mathbb{E}\mathrm{nd}(\mathbb{D}), \mathrm{Hor}\ \mathbb{E}\mathrm{nd}(\mathbb{D})) \longrightarrow (\mathrm{Obj}\ \mathbb{M}\mathrm{nd}(\mathbb{D}), \mathrm{Hor}\ \mathbb{M}\mathrm{nd}(\mathbb{D}))$$

$$\eta\colon\ (\mathrm{Obj}\ \mathbb{E}\mathrm{nd}(\mathbb{D}), \mathrm{Hor}\ \mathbb{E}\mathrm{nd}(\mathbb{D})) \longrightarrow (\mathrm{Ver}\ \mathbb{E}\mathrm{nd}(\mathbb{D}), \mathrm{Sq}\ \mathbb{E}\mathrm{nd}(\mathbb{D}))$$

such that for each horizontal morphism (F,ϕ) in $\mathbb{E}\mathrm{nd}(\mathbb{D})$ the square $\eta_{(F,\phi)}$ is of the form

$$\begin{array}{ccc} (X,P) & \xrightarrow{(F,\phi)} & (Y,Q) \\ {\scriptstyle \eta_{(X,P)}}\big\downarrow & \eta_{(F,\phi)} & \big\downarrow{\scriptstyle \eta_{(Y,Q)}} \\ UR_0(X,P) & \xrightarrow[UR_0(F,\phi)]{} & UR_0(Y,Q) \end{array}$$

and is universal from (F,ϕ) to U.

We define R_0 as the horizontal 1-adjoint already present, namely $R_0(X,P) := (X, P^{\mathrm{free}})$ and $R_0(F,\phi)\colon\ (X,P^{\mathrm{free}}) \to (Y,Q^{\mathrm{free}})$ is the unique (horizontal) monad morphism such that $(1_Y, \iota_Q)\circ UR_0(F,\phi) = (F,\phi)\circ(1_X,\iota_P)$.

The functor η on objects is $\eta_{(X,P)} := (1^v_X, \left(\Lambda^{\mathbb{D}}\right)^{-1}(\iota_P)) = (1^v_X, \iota_P)$. Here $\Lambda^{\mathbb{D}}$ is the cofolding on $\mathbb{D}$, and we are using Proposition 7.5 (i) and (iii), and the fact that $(1^v_X)^* = 1^h_X$. For a horizontal endomorphism map (F,ϕ), we define $\eta_{(F,\phi)}$ to be $\left(\Lambda^{\mathbb{E}\mathrm{nd}(\mathbb{D})}\right)^{-1}$ of the vertical identity square

$$\begin{array}{ccccc} UL_0(X,P) & \xrightarrow{(1^h_X,\iota_P)} & (X,P) & \xrightarrow{(F,\phi)} & (Y,Q) \\ \Big\| & & i_v & & \Big\| \\ UL_0(X,P) & \xrightarrow[UL_0(F,\phi)]{} & UL_0(Y,Q) & \xrightarrow[(1^h_Y,\iota_Q)]{} & (Y,Q) \end{array}$$

in $\mathbb{E}\mathrm{nd}(\mathbb{D})$.

For the universality of $\eta_{(Y,Q)}$ concerning vertical morphisms, we must prove for each endomorphism (Y,Q) and each monad (X,P) that

$$\mathrm{Ver}_{\mathbb{M}\mathrm{nd}(\mathbb{D})}(Y,Q^{\mathrm{free}}),(X,P)) \xrightarrow{U(-)\circ(1^v_Y,\iota_Q)} \mathrm{Ver}_{\mathbb{E}\mathrm{nd}(\mathbb{D})}((Y,Q),U(X,P))$$

is a bijection. For injectivity, if $U(u,\overline{u})\circ(1^v_Y,\iota_Q) = U(v,\overline{v})\circ(1^v_Y,\iota_Q)$, then $u=v$, and the coholonomy on $\mathbb{E}\mathrm{nd}(\mathbb{D})$ gives us

$$(1^h_Y,\iota_Q)\circ U(u^*,\Lambda(\overline{u})) = (1^h_Y,\iota_Q)\circ U(v^*,\Lambda(\overline{v})),$$

so $\Lambda(\overline{u}) = \Lambda(\overline{v})$ by horizontal universality of $(1^h_Y,\iota_Q)$. Finally, $\overline{u}=\overline{v}$ by Proposition 7.5 (iii). For surjectivity, if $(w,\overline{w})\colon\ (Y,Q)\to U(X,P)$ is a vertical endomorphism map, the horizontal universality of $(1^h_Y,\iota_Q)$ guarantees a horizontal monad map $(F,\phi)\colon\ (X,P)\to(Y,Q^{\mathrm{free}})$ such that $(1^h_Y,\iota_Q)\circ U(F,\phi) = (w^*,\Lambda(\overline{w}))$. Then $F=w^*$, and we may take $(u,\overline{u}) = (w,\Lambda^{-1}([\ \phi\ \ \iota_Q\ ])$ so that $U(u,\overline{u})\circ(1^v_Y,\iota_Q) = (w,\overline{w})$, again by Proposition 7.5 (iii).

We next prove that the square $\eta_{(F,\phi)}$ is vertically universal, that is, the map

$$\mathbb{M}\mathrm{nd}(\mathbb{D})\begin{pmatrix} R_0(F,\phi) \\ (F',\phi') \end{pmatrix} \longrightarrow \mathbb{E}\mathrm{nd}(\mathbb{D})\begin{pmatrix} (F,\phi) \\ U(F',\phi') \end{pmatrix} \tag{33}$$

$$\beta \longmapsto \begin{bmatrix} \eta_{(F,\phi)} \\ U\beta \end{bmatrix}$$

is a bijection (recall Definition 4.1.). The notation $\mathbb{M}\mathrm{nd}(\mathbb{D})\begin{pmatrix} R_0(F,\phi) \\ (F',\phi') \end{pmatrix}$ indicates the set of monad squares with top horizontal arrow $R_0(F,\phi)$ and bottom horizontal arrow (F',ϕ'). The notation $\mathbb{E}\mathrm{nd}(\mathbb{D})\begin{pmatrix} (F,\phi) \\ U(F',\phi') \end{pmatrix}$ indicates the set of endomorphism squares with top horizontal arrow (F,ϕ) and bottom horizontal arrow $U(F',\phi')$.

Since we have already checked the universality of $\eta_{(Y,Q)}$ with respect to vertical morphisms, and since squares with distinct vertical arrows are distinct, it suffices to prove a bijection for monad squares which additionally have the left and right vertical arrows fixed, so we consider monad squares of the form

$$\begin{array}{ccc} (X,P) & \xrightarrow{R_0(F,\phi)} & (Y,Q^{\mathrm{free}}) \\ {\scriptstyle (u,\bar{u})}\big\downarrow & \beta & \big\downarrow{\scriptstyle (v,\bar{v})} \\ (X',P') & \xrightarrow[(F',\phi')]{} & (Y',Q'). \end{array}$$

We factor the map in (33) (for fixed $(u,\bar{u})$ and $(v,\bar{v})$), into a sequence of bijections.

$$\begin{aligned} \beta &\leftrightarrow \Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}(\beta) \\ &\leftrightarrow \begin{bmatrix} U\Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}(\beta) & i^v_{(1_Y,\iota_Q)} \end{bmatrix} \\ &\leftrightarrow \begin{bmatrix} i^v_{(u,\bar{u})^*} & i^v \\ U\Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}(\beta) & i^v_{(1_Y,\iota_Q)} \end{bmatrix} \\ &\leftrightarrow \begin{bmatrix} \eta_{(F,\phi)} \\ U\beta \end{bmatrix}. \end{aligned}$$

The last bijection is $(\Lambda^{\mathbb{E}\mathrm{nd}(\mathbb{D})})^{-1}$ and relies on the fact that U is compatible with the cofoldings $\Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}$ and $\Lambda^{\mathbb{E}\mathrm{nd}(\mathbb{D})}$.

(iii) The 2-adjunction $\mathbf{V}R \dashv \mathbf{V}U$ follows from the vertical double adjunction in (ii). □

10. Existence of Eilenberg–Moore Objects

In Street's article [12], a 2-category $\mathbf{C}$ is said to *admit the construction of algebras* if the inclusion 2-functor $\mathrm{Inc}_{\mathbf{C}}\colon \mathbf{C} \to \mathrm{Mnd}(\mathbf{C})$ admits a right 2-adjoint $\mathrm{Alg}_{\mathbf{C}}\colon \mathrm{Mnd}(\mathbf{C}) \to \mathbf{C}$. Synonymously, we say $\mathbf{C}$ *admits Eilenberg–Moore objects.* For a monad (X,S) in $\mathbf{C}$, the object $\mathrm{Alg}_{\mathbf{C}}(X,S)$ is denoted X^S. A right 2-adjoint $\mathrm{Alg}_{\mathbf{C}}$ exists if and only if for each monad (X,S), the presheaf $\mathrm{Mnd}_{\mathbf{C}}\left(\mathrm{Inc}_{\mathbf{C}}-, (X,S)\right)$ is representable. The representing object is then X^S.

The situation for monads in a double category $\mathbb{D}$ is more subtle, as representability of the individual presheaves $\mathrm{Mnd}_{\mathbb{D}}\,(\mathrm{Inc}_{\mathbb{D}}(-),(X,S))$ does not suffice, and we must consider parameterized presheaves.

Definition 10.1. Let $\mathbb{D}$ be a double category and let $\mathrm{Inc}_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{M}\mathrm{nd}(\mathbb{D})$, $I \mapsto (I, \mathrm{id}_I)$ be the inclusion double functor. We say that the double category $\mathbb{D}$ *admits Eilenberg–Moore objects* if $\mathrm{Inc}_{\mathbb{D}}$ admits a horizontal right double adjoint.

Remark 10.2. To an object I and a monad (X,S) in $\mathbb{D}$, we may associate the set $S\text{-}\boldsymbol{Alg}_I$ of S-algebra structures on I, which is the set of horizontal monad morphisms from (I,id_I) to (X,S). This assignment extends to a parameterized presheaf on $\mathbb{D}$ in the sense of Definition 3.1, namely

$$\mathbb{M}\mathrm{nd}(\mathbb{D})(\mathrm{Inc}_{\mathbb{D}}-,-)\colon\ \mathbb{D}^{\mathrm{horop}} \times \mathbb{V}_1\mathbb{M}\mathrm{nd}(\mathbb{D}) \longrightarrow \mathbb{S}\mathrm{pan}^t\ . \tag{34}$$

Recall that $\mathbb{V}_1\mathbb{M}\mathrm{nd}(\mathbb{D})$ is the double category which has the same vertical 1-category as $\mathbb{M}\mathrm{nd}(\mathbb{D})$, but everything else is trivial, as in Section 2.

Next we can turn to our main application of double adjunctions to the characterization of existence of Eilenberg–Moore objects.

Theorem 10.3 (Characterization of existence of Eilenberg–Moore objects)**.** *The inclusion double functor*

$$\mathrm{Inc}_{\mathbb{D}}\colon\ \mathbb{D} \longrightarrow \mathbb{M}\mathrm{nd}(\mathbb{D})$$

$$I \longmapsto (I, \mathrm{id})$$

admits a horizontal right double adjoint if and only if the parameterized presheaf

$$\text{-}\boldsymbol{Alg}\_\colon\ \mathbb{D}^{\mathrm{horop}} \times \mathbb{V}_1\mathbb{M}\mathrm{nd}(\mathbb{D}) \longrightarrow \mathbb{S}\mathrm{pan}^t$$

is (horizontally) representable in the sense of Definition 3.7. See Section 2 for the definition of $\mathbb{V}_1$.

Proof: By Theorem 5.3, the double functor $\mathrm{Inc}_{\mathbb{D}}$ admits a horizontal right double adjoint if and only if the parameterized presheaf (34) is representable, but **-*Alg*_** is (34) by definition. □

Example 10.4. Suppose $\mathbf{C}$ is a 2-category which admits Eilenberg–Moore objects in the sense of 2-category theory, that is, the 2-functor $\mathrm{Inc}_{\mathbf{C}}\colon \mathbf{C} \to \mathrm{Mnd}(\mathbf{C})$ admits a right 2-adjoint. Then the double category $\overline{\mathbb{Q}}\mathbf{C}$ admits Eilenberg–Moore objects since $\overline{\mathbb{Q}}\mathbf{C}$ and $\mathbb{M}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C})$ both have cofoldings with fully faithful coholonomies, $\mathrm{Inc}_{\overline{\mathbb{Q}}\mathbf{C}}$ preserves them, and $\mathbf{H}\mathrm{Inc}_{\overline{\mathbb{Q}}\mathbf{C}} = \mathrm{Inc}_{\mathbf{C}}$ admits a right 2-adjoint. See Example 6.7, Proposition 7.5, and Corollary 6.16. The representing functor $G\colon \mathbb{V}_1\mathbb{M}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{C}) \to \mathbb{S}\mathrm{pan}^t$ for **-*Alg*_** is the transposed opposite of the right adjoint to $\mathrm{Inc}_{\mathbf{C}}$.

In a future paper we will present Eilenberg–Moore objects in a double category as weighted double limits and treat the free Eilenberg–Moore completion of a double category.

Acknowledgements

We gratefully acknowledge the support and hospitality of the Centre de Recerca Matemàtica during the academic year 2007/08, in occasion of the thematic programme on Homotopy Theory and Higher Categories. Thomas M. Fiore was supported at the University of Chicago by NSF Grant DMS-0501208. At the Universitat Autònoma de Barcelona he was supported by grant SB2006-0085 of the Spanish Ministerio de Educación y Ciencia under the Programa Nacional de ayudas para la movilidad de profesores de universidad e investigadores españoles y extranjeros. Thomas M. Fiore was also supported by the Max Planck Institut für Mathematik, and he thanks MPIM for its kind hospitality during Summer 2010. Joachim Kock was partially supported by grants MTM2006-11391 and MTM2007-63277 of Spain and SGR2005-00606 of Catalonia.

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