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RATIONAL INNER FUNCTIONS IN THE SCHUR-AGLER CLASS OF THE POLYDISK

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Abstract

Every two-variable rational inner function on the bidisk has a special representation called a unitary transfer function realization. It is well known and related to important ideas in operator theory that this does not extend to three or more variables on the polydisk. We study the class of rational inner functions on the polydisk which *do* possess a unitary realization (the Schur-Agler class) and investigate minimality in their representations. Schur-Agler class rational inner functions in three or more variables cannot be represented in a way that is *as* minimal as two variables might suggest.

1. Prologue

Let $\mathbb{D}$, $\mathbb{T}$, $\mathbb{D}^n$, $\mathbb{T}^n$ denote the unit disk in $\mathbb{C}$, the unit circle, the n-polydisk (or just polydisk), and the n-torus (or just torus), respectively.

A rational inner function f on the polydisk $\mathbb{D}^n$ is a rational function:

$$f = q/p \quad q, p \in \mathbb{C}[z_1, \dots, z_n]$$

where p has no zeros on $\mathbb{D}^n$, and an inner function:

$$|f| = 1 \text{ a.e. on } \mathbb{T}^n.$$

In one variable, rational inner functions are just the Blaschke products:

$$f(z) = \mu \prod_{j=1}^{N} \frac{z - a_j}{1 - \bar{a}_j z} \quad a_j \in \mathbb{D},\ \mu \in \partial\mathbb{D}.$$

Rational inner functions on the polydisk, while not as powerful a tool as Blaschke products, are still important because (1) they are dense in the topology of local uniform convergence inside the set of holomorphic

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functions on $\mathbb{D}^n$ with supremum norm at most one, and (2) they are closely related to the study of stable polynomials, polynomials whose roots do not intersect the polydisk. It is our aim to study a special class of rational inner functions, called the Schur-Agler class rational inner functions, whose definition warrants motivation.

Using matrices, Blaschke products can be represented in a way that appears analogous to a linear fractional transformation. For example,

$$\frac{z^2 - 1/4}{1 - (1/4)z^2} = A + zB(I - zD)^{-1}C$$

where A, B, C, D are the block entries of a 3×3 unitary:

$$U = \begin{array}{c c} & \begin{array}{cc} \mathbb{C} & \mathbb{C}^2 \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathbb{C}^2 \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array} = \begin{bmatrix} -1/4 & 0 & \sqrt{15}/4 \\ \sqrt{15}/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}.$$

The notation is supposed to indicate $A = -1/4$, $B = [0, \sqrt{15}/4]$, $C = [\sqrt{15}/4, 0]^T$, and $D = \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix}$. This type of representation is called a unitary transfer function realization (a term from engineering). We shall say *unitary realization* for short.

Two-variable rational inner functions can be represented in a similar way. Take for example

$$f(z_1, z_2) = \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}.$$

If we let U be the unitary matrix

$$U = \begin{array}{c c} & \begin{array}{cc} \mathbb{C} & \mathbb{C}^2 \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathbb{C}^2 \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array} = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1/2 & -1/2 \\ \sqrt{2}/2 & -1/2 & 1/2 \end{bmatrix}$$

and let

$$E(z_1, z_2) = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}$$

then writing $z = (z_1, z_2)$, it turns out

$$f(z) = A + BE(z)(I - DE(z))^{-1}C. \tag{1.1}$$

Surprisingly, not all three-variable rational inner functions have a unitary transfer function realization. This is known and related to important ideas in operator theory. What is also surprising —and one of the main points of this article— is that even if a three-variable rational inner function has a unitary realization, it cannot always be represented as minimally as the two-variable theory would suggest. Something we intend to show is that the following rational inner function

$$g(z) = \frac{3z_1z_2z_3 - z_1z_2 - z_1z_3 - z_2z_3}{3 - z_1 - z_2 - z_3}$$

can be represented in the form

$$g(z) = A + BE(z)(I - DE(z))^{-1}C$$

where

$$U = \begin{array}{cc} & \begin{array}{cc} \mathbb{C} & \mathbb{C}^N \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathbb{C}^N \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array}$$

is a block unitary matrix and $E(z)$ is an $N \times N$ diagonal matrix with z_1, z_2, z_3 on the diagonal (in some combination). Naively extrapolating from the previous two-variable example, one might expect that N could be chosen to equal $N = 3$. This is not the case. Instead, we show that $6 \leq N \leq 9$.

We now introduce the rest of the paper in a more general framework.

2. Introduction

Let us introduce three properties.

Definition 2.1. If $f\colon \mathbb{D}^n \to \mathbb{D}$ is holomorphic, then we say f is *satisfies the von Neumann inequality* or f is in the *Schur-Agler class* if

$$||f(T)|| \leq 1$$

for all commuting n-tuples of strict contractions $T = (T_1, \ldots, T_n)$.

Definition 2.2. A function $f\colon \mathbb{D}^n \to \mathbb{D}$ possesses an *Agler decomposition* if there exist positive semi-definite kernels $K_j\colon \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{C}$, $j = 1, \ldots, n$, such that

$$1 - f(z)\overline{f(\zeta)} = \sum_{j=1}^{n} (1 - z_j\bar{\zeta}_j)K_j(z, \zeta).$$

Recall that a function $K(z, \zeta)$ is positive semi-definite if for every finite set F the matrix

$$(K(z, \zeta))_{z,\zeta \in F}$$

is positive semi-definite. (We would need an ordering to form an actual matrix, but this is unimportant.) For more information on positive semi-definite kernels, refer to [**2**, Section 2.7].

Definition 2.3. A function $f\colon \mathbb{D}^n \to \mathbb{D}$ has a *unitary realization* if there is a Hilbert space decomposed into n orthogonal summands

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n$$

and a unitary operator V in $\mathcal{B}(\mathbb{C} \oplus \mathcal{H})$ which we write as

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathcal{B}(\mathbb{C}, \mathbb{C})$, $B \in \mathcal{B}(\mathcal{H}, \mathbb{C})$, $C \in \mathcal{B}(\mathbb{C}, \mathcal{H})$, $D \in \mathcal{B}(\mathcal{H}, \mathcal{H})$, such that

$$f(z) = A + BE(z)(I - DE(z))^{-1}C \tag{2.1}$$

where $E(z)$ is the diagonal matrix with block diagonal entries $z_1 I_{\mathcal{H}_1}$, $z_2 I_{\mathcal{H}_2}, \dots, z_n I_{\mathcal{H}_n}$.

When the Hilbert spaces are finite dimensional, we shall refer to the *size* of the realization as

$$\sum_{j=1}^{n} \dim \mathcal{H}_j.$$

Note $\mathcal{B}(\mathcal{H}, \mathcal{K})$ represents the set of bounded linear operators from Hilbert space $\mathcal{H}$ to Hilbert space $\mathcal{K}$. Also, $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$.

The connection between operator inequalities, positive semi-definite decompositions, and realizations was made by J. Agler.

Theorem 2.4 ([**1**])**.** *Let $f\colon \mathbb{D}^n \to \mathbb{D}$ be holomorphic. The following are equivalent:*

(1) *f satisfies a von Neumann inequality.*
(2) *f has an Agler decomposition.*
(3) *f has a unitary realization.*

In particular, all three conditions are automatically true when $n = 1$ or 2, and only then, because of the following theorems.

Theorem 2.5 ([**14**])**.** *Every $f\colon \mathbb{D} \to \mathbb{D}$ holomorphic satisfies the von Neumann inequality.*

Theorem 2.6 ([**3**])**.** *Every $f\colon \mathbb{D}^2 \to \mathbb{D}$ holomorphic satisfies the von Neumann inequality.*

Theorem 2.7 ([**13**] and [**9**])**.** *Not every $f\colon \mathbb{D}^n \to \mathbb{D}$ holomorphic satisfies the von Neumann inequality when $n > 2$.*

Accordingly, a function of n variables that satisfies any of the above properties will be called a Schur-Agler class function. We shall abbreviate this to just *Agler class*. The Agler class is natural because of its interaction with operator theory and it is possible to write down many examples of Agler class functions simply by writing down a unitary realization. On the other hand, it is difficult to determine whether a given function is in the Agler class and it is difficult to write down Agler decompositions explicitly (even in two variables). For more general information on Theorem 2.4 see [**7**] or the book [**2**]. For more detailed information about the theorem see [**5**].

We are interested in *rational* inner Agler class functions. Let us state what holds in two variables.

Theorem 2.8. *Let $f\colon \mathbb{D}^2 \to \mathbb{D}$ be a rational inner function and write $f = q/p$ with $q, p \in \mathbb{C}[z_1, z_2]$ of degree in z_1 at most d_1 and degree in z_2 at most d_2.*

(1) *A sums of squares decomposition holds. There exist polynomials $A_1, \dots, A_{d_1}, B_1, \dots, B_{d_2} \in \mathbb{C}[z_1, z_2]$ such that*

$$|p(z)|^2 - |q(z)|^2 = (1 - |z_1|^2) \sum_{j=1}^{d_1} |A_j(z)|^2 + (1 - |z_2|^2) \sum_{j=1}^{d_2} |B_j(z)|^2.$$

(2) *([**11**]) f has a finite dimensional unitary realization. There exists a finite dimensional Hilbert space with direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and a unitary matrix $U\colon \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$*

$$U = \begin{array}{cc} & \begin{array}{cc} \mathbb{C} & \mathcal{H} \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathcal{H} \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array}$$

such that $\dim \mathcal{H}_j \leq d_j$ *for $j = 1, 2$ and*

$$f(z) = A + BE(z)(I - DE(z))^{-1}C$$

where $E(z)$ is the block diagonal matrix:

$$E(z) = \begin{bmatrix} z_1 I_{\mathcal{H}_1} & 0 \\ 0 & z_2 I_{\mathcal{H}_2} \end{bmatrix}.$$

Unknown to most of the mathematics community, Kummert [**11**] proved the second item (which is well known to be equivalent to the first). This was pointed out to us by Ball [**4**]. Cole and Wermer [**8**] proved this result using Agler's theorem (without concern for degree bounds) and showed that the above result is essentially equivalent to Andô's inequality. For a direct proof of this result and more discussion see [**6**] or [**10**].

The fundamental question for this article is:

> To what extent does Theorem 2.8 carry over to n variables if we stipulate that our rational inner function is in the Agler class?

The two-variable arguments in [**8**] can be used to establish the following theorem. A result of this type was announced by Ball [**4**]. We need to use some aspects of the proof so we sketch the proof later on.

Theorem 2.9. *Let $f\colon \mathbb{D}^n \to \mathbb{D}$ be an Agler class rational inner function and write $f = q/p$, with $q, p \in \mathbb{C}[z_1, \dots, z_n]$. Then,*

(1) *A sums of squares decomposition holds. There exist integers $N_1, \dots, N_n$ such that*

$$|p(z)|^2 - |q(z)|^2 = \sum_{j=1}^{n}(1 - |z_j|^2) \sum_{k=1}^{N_j} |A_{j,k}(z)|^2$$

where $A_{j,k} \in \mathbb{C}[z_1, \dots, z_n]$.

(2) *f has a finite dimensional unitary realization.*

One cannot control the number of terms in the sums of squares (and the dimension of the unitary realization) as precisely as in two variables. To emphasize this point, observe that if we write down a finite dimensional unitary realization as in Definition 2.3

$$f(z_1, \dots, z_n) = A + BE(z)(I - DE(z))^{-1}C$$

where we assume $\dim \mathcal{H}_j \le d_j$, then $f = q/p$ is a rational function where q, p each have degree at most d_j in the variable z_j. This follows from Cramer's rule. (It can also be shown by direct calculation that f is indeed inner.)

Conversely, if one starts with an Agler class rational inner function $f = q/p$ where q, p each have degree at most d_j in the variable z_j, then something surprising occurs. One cannot in general achieve $\dim \mathcal{H}_j \le d_j$ in the unitary realization. The dimension of $\mathcal{H}_j$ may need to be chosen larger than d_j. Theorem 2.10 presents the bound we can prove on $\dim \mathcal{H}_j$ and Theorem 2.11 gives an example which shows the bound $\dim \mathcal{H}_j \le d_j$ is not in general possible.

Theorem 2.10. *Using the assumptions and notation of Theorem 2.9, assume the degree of q, p is at most d_j in the variable z_j for $j = 1, \dots, n$. Write $d = (d_1, d_2, \dots, d_n)$. Then,*

(1) *Each $A_{j,k}$ (from Theorem 2.9) satisfies*

$$\deg_{z_i} A_{j,k} \le \begin{cases} d_i & i \neq j \\ d_i - 1 & i = j. \end{cases}$$

As a result, the integers $N_1, \dots, N_n$ in Theorem 2.9 can be bounded as follows

$$N_j \le d_j \prod_{k \neq j} (d_k + 1).$$

(2) *The unitary realization of f can be chosen so that the dimensions of the blocks satisfy*

$$\dim \mathcal{H}_j \le d_j \prod_{k \neq j} (d_k + 1).$$

In particular, f has a unitary realization of size

$$\sum_{j=1}^{n} d_j \prod_{k \neq j} (d_k + 1).$$

Theorem 2.11. *The rational inner function*

$$f(z) = \frac{3z_1z_2z_3 - z_1z_2 - z_2z_3 - z_1z_3}{3 - z_1 - z_2 - z_3}$$

is in the Agler class. It has a unitary realization of size 9 *but it cannot be realized with size less than* 6.

3. Proof of Theorems 2.9 and 2.10

Claim 1. If we have a sums of squares decomposition, then we automatically have a finite dimensional unitary realization.

Proof: This is the well-known lurking isometry argument. So, suppose $f = q/p$ is rational, inner, and Agler class, and

$$|p(z)|^2 - |q(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2) ||\vec{F}_j(z)||^2$$

where $\vec{F}_j \in \mathbb{C}^{N_j}[z]$ is a vector polynomial (the notation is simpler if we use vector polynomials in place of sums of squares).

Rearranging we get

$$|p(z)|^2 + \sum_{j=1}^{n} |z_j|^2 ||\vec{F}_j(z)||^2 = |q(z)|^2 + \sum_{j=1}^{n} ||\vec{F}_j(z)||^2.$$

By the polarization theorem for holomorphic functions

$$p(z)\overline{p(\zeta)} + \sum_{j=1}^{n} \langle z_j \vec{F}_j(z), \zeta_j \vec{F}_j(\zeta)\rangle = q(z)\overline{q(\zeta)} + \sum_{j=1}^{n} \langle \vec{F}_j(z), \vec{F}_j(\zeta)\rangle.$$

This formula can be used to show that the map which sends

$$\begin{bmatrix} p(z) \\ z_1\vec{F}_1(z) \\ \vdots \\ z_n\vec{F}_n(z) \end{bmatrix} \mapsto \begin{bmatrix} q(z) \\ \vec{F}_1(z) \\ \vdots \\ \vec{F}_n(z) \end{bmatrix}$$

is a well-defined linear and isometric map (initially defined on the span of the elements of the form given on the left into the span of the elements of the given form on the right). It may be extended (if necessary) to a unitary matrix U of dimensions $1 + \sum_{j=1}^{n} N_j$ which we write in block form

$$U = \begin{array}{cc} & \begin{array}{cc} \mathbb{C} & \mathbb{C}^N \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathbb{C}^N \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array}$$

where $N = \sum_j N_j$. Let us write

$$\vec{F}(z) = \begin{bmatrix} \vec{F}_1(z) \\ \vdots \\ \vec{F}_n(z) \end{bmatrix}$$

and let $E(z)$ be the block $N \times N$ diagonal matrix with block diagonal entries $z_1 I_{N_1}, z_2 I_{N_2}, \dots, z_n I_{N_n}$. Then, by construction of U

$$Ap(z) + BE(z)\vec{F}(z) = q(z)$$
$$Cp(z) + DE(z)\vec{F}(z) = \vec{F}(z).$$

If one first solves for $\vec{F}(z)$ using the second equation, and then inserts this into the first equation, we arrive at

$$q/p(z) = A + BE(z)(I - DE(z))^{-1}C$$

as desired. □

Next, we rehash the arguments of Cole and Wermer (which were originally applied to two variables) in the n-variable context to prove Theorem 2.9. This repetition is necessary because we need some of the details of the proof in order to keep track of degrees in Theorem 2.10.

Claim 2. Suppose $f = q/p$ is rational inner Agler class and let r be the maximum of the total degrees of p and q. Then f has a sums of squares decomposition:

$$|p(z)|^2 - |q(z)|^2 = \sum_{j=1}^{n}(1 - |z_j|^2)||\vec{F}_j(z)||^2$$

where each $\vec{F}_j$ is a vector polynomial of total degree less than or equal to $r - 1$. Every such decomposition must satisfy this degree bound.

Proof: By Agler's theorem, f has an Agler decomposition:

$$1 - f(z)\overline{f(\zeta)} = \sum_{j=1}^{n}(1 - z_j\bar{\zeta}_j)K_j(z,\zeta) \tag{3.1}$$

where each K_j is a positive semi-definite kernel.

Observe that

$$\begin{aligned}\frac{1}{\prod_{j=1}^{n}(1 - z_j\bar{\zeta}_j)} &\geq \frac{1 - f(z)\overline{f(\zeta)}}{\prod_{j=1}^{n}(1 - z_j\bar{\zeta}_j)}\\ &\geq \frac{K_j(z,\zeta)}{\prod_{i\neq j}(1 - z_i\bar{\zeta}_i)}\\ &\geq K_j(z,\zeta)\end{aligned}$$

in the sense of positive semi-definite kernels (i.e. $K \geq L$ means $K - L$ is positive semi-definite in this situation). It follows from standard facts about reproducing kernels that each K_j is the reproducing kernel of a space of analytic functions contractively contained in $H^2(\mathbb{D}^n)$ and that for each j there is a Hilbert space $\mathcal{H}_j$ and an $\mathcal{H}_j$-valued analytic function $\vec{F}_j \colon \mathbb{D}^n \to \mathcal{H}_j$ such that

$$K_j(z,\zeta) = \langle \vec{F}(z), \vec{F}(\zeta)\rangle.$$

(See [**8**] for more on the details of this argument.)

Let us multiply equation (3.1) by $p(z)\overline{p(\zeta)}$ and absorb this factor into the definition of $\vec{F}_j(z)$ so that we really have

$$p(z)\overline{p(\zeta)} - q(z)\overline{q(\zeta)} = \sum_{j=1}^{n}(1 - z_j\bar{\zeta}_j)\langle\vec{F}_j(z), \vec{F}_j(\zeta)\rangle.$$

Now we let $z = \zeta = t\mu$ where $t \in \mathbb{D}$ and $\mu \in \mathbb{T}^n$:

$$\frac{|p(t\mu)|^2 - |q(t\mu)|^2}{1 - |t|^2} = \sum_{j=1}^{n} ||\vec{F}_j(t\mu)||^2. \tag{3.2}$$

The left hand side is a polynomial in t, $\bar{t}$ and a trigonometric polynomial in μ. To see this, recall from [**12**, Theorem 5.2.5] that rational inner functions in the polydisk, such as our f, have the form

$$\frac{M(z)\overline{p(1/\bar{z}_1, \dots, 1/\bar{z}_n)}}{p(z)}$$

where $M(z) = z^d$ is a monomial of sufficiently high degree. Here we are writing $d = (d_1, \dots, d_n)$. So, $q(z) = M(z)\overline{p(1/\bar{z})}$, and it is then possible to see by a direct calculation that $(1 - |t|^2)$ divides the coefficients of μ in $|p(t\mu)|^2 - |t^{|d|}\mu^d\overline{p((1/\bar{t})\mu)}|^2$.

Write

$$p(z) = \sum_{\alpha} p_\alpha z^\alpha$$

$$q(z) = \sum_{\alpha} q_\alpha z^\alpha$$

$$\vec{F}_j(z) = \sum_{\alpha} \vec{F}_{j,\alpha} z^\alpha.$$

(We are using multi index notation to write polynomials and power series.)

Since

$$|p(t\mu)|^2 = \sum_{\alpha,\beta} p_\alpha \bar{p}_\beta \mu^{\alpha-\beta} t^{|\alpha|} \bar{t}^{|\beta|}$$

(and by performing similar computations for $|q(t\mu)|^2$ and $||\vec{F}_j(t\mu)||^2$), we are able to compute the zero-th Fourier coefficient of (3.2) when viewed as a Fourier series in μ:

$$\frac{\sum_\alpha |t|^{2|\alpha|}(|p_\alpha|^2 - |q_\alpha|^2)}{1 - |t|^2} = \sum_{j=1}^{n} \sum_{\alpha} ||\vec{F}_{j,\alpha}||^2 |t|^{2|\alpha|}. \tag{3.3}$$

Recall r denotes the maximum of the total degrees of p and q. Now, $|t|^2$ does not occur to any power larger than $r - 1$ in (3.3) and therefore

$$||\vec{F}_{j,\alpha}||^2 = 0$$

whenever $|\alpha| \geq r$.

This implies each $\vec{F}_j(z)$ is a Hilbert space valued *polynomial.* It then follows that $||\vec{F}_j(z)||^2$ can be replaced with the square of a finite dimensional vector-valued polynomial since span$\{\vec{F}_{j,\alpha} : |\alpha| < r\}$ is finite dimensional. □

These two claims prove Theorem 2.9. To prove the bounds in Theorem 2.10, we assume p, q have multidegree at most $d = (d_1, \dots, d_n)$. Let $|d| = \sum_j d_j$, which is an upper bound on the total degree of p and q.

Consider again:

$$p(z)\overline{p(\zeta)} - q(z)\overline{q(\zeta)} = \sum_{j=1}^{n}(1 - z_j\bar{\zeta}_j)\langle\vec{F}_j(z), \vec{F}_j(\zeta)\rangle,$$

where we now know each $\vec{F}_j(z)$ must be a vector polynomial of total degree at most $|d|-1$. Let us focus on degree bounds for z_1; our argument applies by symmetry to the other variables.

Let M be a positive integer (which we use to amplify the degree of z_1). Replacing z and ζ in the last equation with $(z_1^M, z_2, \dots, z_n) = (z_1^M, z')$, we have

$$\begin{aligned}(3.4)\quad & |p(z_1^M, z')|^2 - |q(z_1^M, z')|^2 \\ &= (1-|z_1|^2)\left(\sum_{j=0}^{M-1}|z_1|^{2j}\right)||\vec{F}_1(z_1^M, z')||^2 + \sum_{j=2}^{n}(1-|z_j|^2)||\vec{F}_j(z_1^M, z')||^2.\end{aligned}$$

We apply Claim 2 to $f(z_1^M, z')$. Since the left hand side has total degree at most $d_1M + d_2 + \cdots + d_n = d_1(M-1) + |d|$ in (z_1, z'), the sums of squares polynomials on the right hand side have total degree at most $d_1(M-1) + |d| - 1$.

Suppose z^α has a nonzero coefficient in the Taylor expansion of $\vec{F}_1$ and write $\alpha = (\alpha_1, \dots, \alpha_n)$. Since $|z_1^{M-1}\vec{F}_1(z_1^M, z')|^2$ appears as a sums of squares term in (3.4), our degree bound from Claim 2 says

$$M - 1 + M\alpha_1 + \sum_{j\geq 2}\alpha_j \leq d_1(M-1) + |d| - 1$$

and letting M go to infinity we get $\alpha_1 \leq d_1 - 1$.

Similarly, suppose z^α has a nonzero coefficient in the Taylor expansion of $\vec{F}_j$, $j \neq 1$. Then, looking at $\vec{F}_j$ in (3.4), our degree bound gives

$$M\alpha_1 + \sum_{j\geq 2}\alpha_j \leq d_1(M-1) + |d| - 1.$$

Letting M go to infinity we get $\alpha_1 \leq d_1$.

The same argument applies to other variables. This shows $\vec{F}_j$ has multidegree at most $d - e_j$, with e_j the multi-index with 1 in the j-th position and zeros elsewhere.

Therefore, $||\vec{F}_j(z)||^2$ is a reproducing kernel for a space of polynomials of dimension at most

$$N_j = d_j \prod_{k \neq j} (d_k + 1)$$

and can therefore be written as the square of a vector polynomial with at most N_j components. (See the appendix of [**8**] for some background.)

This proves Theorem 2.10.

4. Theorem 2.11: Three-variable example

The three-variable rational inner function on the tridisk $\mathbb{D}^3$

$$f(z_1, z_2, z_3) = \frac{3z_1z_2z_3 - z_1z_2 - z_2z_3 - z_1z_3}{3 - z_1 - z_2 - z_3}$$

is in the Agler class because we can explicitly write an Agler decomposition.

Namely, let

$$S(z, w) = |P_1(z, w)|^2 + |P_2(z, w)|^2 + |P_3(z, w)|^2$$

where

$$\begin{aligned} P_1(z, w) &= \sqrt{3}(zw - z/2 - w/2) \\ P_2(z, w) &= \sqrt{3}(1 - z/2 - w/2) \\ P_3(z, w) &= (1/\sqrt{2})(z - w). \end{aligned}$$

Then, a decomposition for f is given by

$$\begin{aligned} &|3 - z_1 - z_2 - z_3|^2 - |3z_1z_2z_3 - z_1z_2 - z_2z_3 - z_1z_3|^2 \\ &\quad = (1 - |z_1|^2)S(z_2, z_3) + (1 - |z_2|^2)S(z_1, z_3) + (1 - |z_3|^2)S(z_1, z_2). \end{aligned}$$

It remains to show that none of the sums of squares terms can be chosen to be a single square. So, suppose we have a decomposition

$$\begin{aligned} &|3 - z_1 - z_2 - z_3|^2 - |3z_1z_2z_3 - z_1z_2 - z_2z_3 - z_1z_3|^2 \\ &\qquad = (1 - |z_1|^2)SOS_1(z_2, z_3) + (1 - |z_2|^2)SOS_2(z_1, z_3) \\ &\qquad\qquad + (1 - |z_3|^2)SOS_3(z_1, z_2). \end{aligned}$$

Each SOS_j is a sum of squared moduli of polynomials. Note that by Theorem 2.10, the squared polynomials in SOS_1 must have multi-degree bounded by $(0,1,1)$ (with similar bounds for the other sums of squares terms).

Setting $|z_2| = |z_3| = 1$ yields

$$|3-z_1-z_2-z_3|^2-|3z_1z_2z_3-z_1z_2-z_2z_3-z_1z_3|^2 = (1-|z_1|^2)SOS_1(z_2,z_3)$$

and $SOS_1(z_2,z_3)$ can be solved for explicitly when $z_2,z_3 \in \mathbb{T}$. Indeed, this term has to agree with $S(z_2,z_3)$ when $z_2,z_3 \in \mathbb{T}$:

$$SOS_1(z,w) = 10 - 6\,\mathrm{Re}(z+w) + 2\,\mathrm{Re}(z\bar{w}).$$

We must show this is not a single square of a polynomial of degree $(1,1)$ (on $\mathbb{T}^2$). Supposing otherwise, we equate such an expression

$$\begin{aligned}|a+bz+cw+dzw|^2 = {} & |a|^2+|b|^2+|c|^2+|d|^2 \\ & + 2\,\mathrm{Re}\,\bar{a}(bz+cw+dzw) \\ & + 2\,\mathrm{Re}(\bar{b}c\bar{z}w+\bar{b}dw+\bar{c}dz)\end{aligned}$$

with $SOS_1(z,w)$ and get the following by matching Fourier coefficients

$$10 = |a|^2+|b|^2+|c|^2+|d|^2 \tag{4.1}$$

$$-3 = \bar{a}b+\bar{c}d \tag{4.2}$$

$$-3 = \bar{a}c+\bar{b}d \tag{4.3}$$

$$1 = b\bar{c} \tag{4.4}$$

$$0 = \bar{a}d. \tag{4.5}$$

These equations cannot all hold. One of a or d equals zero by (4.5) (but not both by (4.2)). If $d = 0$, then $b = c \in \mathbb{T}$ (by (4.2), (4.3), and (4.4)), and so $\sqrt{8} = |a|$ (by (4.1)) contradicting equation (4.2):

$$-3 = \bar{a}b.$$

The case $a = 0$ works the same.

Since the sums of squares terms must equal at least two squares, a unitary realization of f has size at least $3*2=6$. Our explicit Agler decomposition shows f has a realization of size $3*3=9$.

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