HANKEL OPERATORS ON HOLOMORPHIC HARDY-ORLICZ SPACES

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ABSTRACT. We characterize the symbols of Hankel operators that extend into bounded operators from the Hardy-Orlicz $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ in the unit ball of $\mathbb{C}^n$, in the case where the growth functions Φ_1 and Φ_2 are either concave or convex. The case where the growth functions are both concave has been studied by Bonami and Sehba. We also obtain several weak factorizations theorems for functions in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$, with concave growth function, in terms of products of Hardy-Orlicz functions with convex growth functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball of $\mathbb{C}^n (n \geq 1)$. We denote by $d\nu$ the Lebesgue measure on $\mathbb{B}^n$ and $d\sigma$ the normalized measure on $\mathbb{S}^n = \partial\mathbb{B}^n$ the boundary of $\mathbb{B}^n$. By $\mathcal{H}(\mathbb{B}^n)$, we denote the space of holomorphic functions on $\mathbb{B}^n$.

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we let

$$\langle z, w \rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$

so that $|z|^2 = \langle z, z \rangle = |z_1|^2 + \cdots + |z_n|^2$.

We say a function Φ is a growth function, if Φ is a continuous and nondecreasing function from $[0, \infty)$ onto itself. We say that Φ is of lower type if we can find $p > 0$ and $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$(1) \qquad\qquad \Phi(st) \leq Ct^p\Phi(s).$$

We say that Φ is of upper type if we can find $q > 0$ and $C > 0$ such that, for $s > 0$ and $t \geq 1$,

$$(2) \qquad\qquad \Phi(st) \leq Ct^q\Phi(s).$$

We say that Φ is of lower type p (resp. upper type q) when (1) (resp. (2)) is satisfied. Also, we say that Φ satisfies the Δ_2- condition if there exists a constant $K > 1$ such that, for any $t \geq 0$,

$$(3) \qquad\qquad \Phi(2t) \leq K\Phi(t).$$

Observe the equivalence between the properties (2) and (3).

For Φ a growth function, we denote by $\mathcal{H}^\Phi(\mathbb{B}^n)$ the Hardy-Orlicz space consisting of holomorphic function f in the unit ball $\mathbb{B}^n$ such that if the functions f_r are defined by $f_r(z) = f(rz)$ then

$$||f||_{\mathcal{H}^\Phi}^{lux} := \sup_{r<1} ||f_r||_{L^\Phi}^{lux} < \infty \,,$$

where

$$||f||_{L^\Phi}^{lux} := \inf \left\{ \lambda > 0 : \int_{\mathbb{S}^n} \Phi\left(\frac{|f(\xi)|}{\lambda} \right) d\sigma(\xi) \leq 1 \right\}$$

is the Luxemboug (quasi-)norm of f in the Orlicz space $L^\Phi(\mathbb{S}^n)$. We will also often consider the following (quasi-)norm on $\mathcal{H}^\Phi(\mathbb{B}^n)$, namely

$$||f||_{\mathcal{H}^\Phi} := \sup_{r<1} \int_{\mathbb{S}^n} \Phi(|f(r\xi)|) d\sigma(\xi)$$

which is finite for $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$. For $0 < p < \infty$, when $\Phi(t) = t^p$, the above space corresponds to the usual Hardy space $\mathcal{H}^p(\mathbb{B}^n)$, that is the space of all $f \in \mathcal{H}(\mathbb{B}^n)$ such that

$$||f||_p^p := \sup_{0<r<1} \int_{\mathbb{S}^n} |f(r\xi)|^p d\sigma(\xi) < \infty.$$

Two growth functions Φ_1 and Φ_2 are said equivalent if there exists some constant c such that

$$c\Phi_1(ct) \leq \Phi_2(t) \leq c^{-1}\Phi_1(c^{-1}t).$$

Such equivalent growth functions define the same Orlicz space. We denote by $\mathcal{H}^\infty(\mathbb{B}^n)$, the space of bounded holomorphic functions in $\mathbb{B}^n$.

The following is proved in [14]:

PROPOSITION 1.1. *For Φ_1 and Φ_2 two growth functions, the bilinear map $(f,g) \mapsto fg$ sends $L^{\Phi_1} \times L^{\Phi_2}$ onto L^Φ, with the inverse mappings of Φ_1, Φ_2 and Φ related by*

$$(4) \qquad\qquad \Phi^{-1} = \Phi_1^{-1} \times \Phi_2^{-1}.$$

Moreover, there exists some constant c such that

$$||fg||_{L^\Phi}^{lux} \leq c||f||_{L^{\Phi_1}}^{lux}||g||_{L^{\Phi_2}}^{lux}.$$

Let us define two classes of growth functions of our interest in this paper.

DEFINITION 1.2. *We call $\mathscr{L}_p$ the set of growth functions Φ of lower type p, $(0 < p \leq 1)$, such that the function $t \mapsto \frac{\Phi(t)}{t}$ is non-increasing.*

DEFINITION 1.3. *We call $\mathscr{U}^q$ the set of growth functions Φ of upper type q, $(q \geq 1)$, such that the function $t \mapsto \frac{\Phi(t)}{t}$ is non-decreasing.*

Clearly, functions in $\mathscr{L}_p$ and $\mathscr{U}^q$ satisfy the Δ_2- condition. Note that if $\Phi \in \mathscr{U}^q$ then Φ is of lower type 1. For $\Phi \in \mathscr{L}_p$ (resp. $\mathscr{U}^q$), without loss of generality, eventually replacing Φ by the equivalent growth function $\int_0^t \frac{\Phi(s)}{s}ds$, we

can assume that Φ is concave (resp. convex) and Φ is a $\mathcal{C}^1$ function with derivative $\Phi'(t) \simeq \frac{\Phi(t)}{t}$.

For any $\xi \in \mathbb{S}^n$ and $\delta > 0$, let

$$B_\delta(\xi) = \{w \in \mathbb{S}^n : |1 - \langle w, \xi \rangle| < \delta\}.$$

We call a weight ϱ, any continuous increasing function from $[0, \infty)$ onto itself, which is of upper type α on $[0, 1]$, that is,

$$(5) \qquad\qquad\qquad \varrho(st) \le s^\alpha \varrho(t)$$

for $s > 1$, with $st \le 1$. Given a weight ϱ, we define the space $BMO(\varrho)$ as the subspace of $L^2(\mathbb{S}^n)$ consisting of those $f \in L^2(\mathbb{S}^n)$ such that

$$(6) \qquad\qquad \sup_B \inf_{R \in \mathcal{P}_N(B)} \frac{1}{(\varrho(\sigma(B)))^2 \sigma(B)} \int_B |f - R|^2 d\sigma = C < \infty,$$

where, for $B = B_\delta(\xi_0)$, the space $\mathcal{P}_N(B)$ is the space of polynomials of order $\le N$ in the $(2n-1)$ last coordinates related to an orthonormal basis whose first element is ξ_0 and second element $\Im\xi_0$. Here N is taken larger than $2n\alpha - 1$. We note $\|f\|_{BMO(\varrho)} := \|f\|_2 + C$, where C is given in the definition (6) of $BMO(\varrho)$. The space $BMOA(\varrho)$ is then the space of function $f \in \mathcal{H}^2(\mathbb{B}^n)$ such that

$$\sup_{r<1} \|f_r\|_{BMO(\varrho)} < \infty.$$

Clearly, $BMOA(\varrho)$ coincides with the space of holomorphic functions in $\mathcal{H}^2(\mathbb{B}^n)$ such that their boundary values lie in $BMO(\varrho)$. When $\varrho = 1$, $BMOA(\varrho)$ is the usual space of holomorphic functions with bounded mean oscillation $BMOA$.

As pointed out in [1, 4], from Viviani's results [13], $BMOA(\rho)$ spaces appear as duals of particular Hardy-orlicz spaces.

THEOREM 1.4. *Let $\Phi \in \mathscr{L}_p$. The topological dual space $(\mathcal{H}^\Phi(\mathbb{B}^n))^*$ of $\mathcal{H}^\Phi(\mathbb{B}^n)$ identifies with the space $BMOA(\rho)$ (with equivalent norms) under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1} \int_{\mathbb{S}^n} f(r\xi)\overline{g(r\xi)}d\sigma(\xi),$$

when Φ and ρ are related by

$$(7) \qquad\qquad\qquad \rho(t) := \rho_\Phi(t) = \frac{1}{t\Phi^{-1}(1/t)}.$$

In order to give the dual of $\mathcal{H}^\Phi(\mathbb{B}^n)$ when $\Phi \in \mathscr{U}^q$, we need to recall the notion of complementary function of a growth function. For Φ a growth function, the complementary function, $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, is defined by

$$(8) \qquad\qquad\qquad \Psi(s) = \sup_{t \in \mathbb{R}_+} \{ts - \Phi(t)\}.$$

We may verify that if $\Phi \in \mathscr{U}^q$, then Ψ is also a growth function of lower type such that $t \mapsto \frac{\Psi(t)}{t}$ is non-decreasing but which may not satisfy the Δ_2−conditon.

The fact that Ψ also satisfy the Δ_2-conditon is of importance in our results here. We thus introduce another class of growth function. We say that a growth function Φ satisfies the $\bigtriangledown_2-$condition whenever its complementary satisfies the Δ_2- conditon. There are several characterizations of growth function Φ such that its complementary function Ψ satisfies the Δ_2-conditon. One of these conditions is the Dini condition which we recall here. We say that $\Phi \in \mathscr{U}^q$ satisfies the Dini condition if there exists a constant $C > 0$ such that, for $t > 0$,

$$(9) \qquad \int_0^t \frac{\Phi(s)}{s^2} ds \leq C \frac{\Phi(t)}{t}.$$

So when Φ satisfies (9), then Φ satisfies the $\bigtriangledown_2-$condition. From the duality result in [9], we obtain the following result.

THEOREM 1.5. *Let $\Phi \in \mathscr{U}^q$ and Ψ its complementary function. Suppose that Φ satisfies the Dini condition (9). The topological dual space $(\mathcal{H}^\Phi(\mathbb{B}^n))^*$ of $\mathcal{H}^\Phi(\mathbb{B}^n)$ identifies with $\mathcal{H}^\Psi(\mathbb{B}^n)$ (with equivalent norms) under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1} \int_{\mathbb{S}^n} f(r\xi) \overline{g(r\xi)} d\sigma(\xi).$$

The orthogonal projection of $L^2(\partial \mathbb{B}^n)$ onto $\mathcal{H}^2(\mathbb{B}^n)$ is called the Szegö projection and denoted P. It is given by

$$(10) \qquad P(f)(z) = \int_{\partial \mathbb{B}^n} S(z, \xi) f(\xi) d\sigma(\xi),$$

where $S(z, \xi) = \frac{1}{(1 - \langle z, \xi \rangle)^n}$ is the Szegö kernel on $\partial \mathbb{B}^n$. We denote as well by P its extension to $L^1(\partial \mathbb{B}^n)$.

For $b \in \mathcal{H}^2(\mathbb{B}^n)$, the small Hankel operator with symbol b is defined for f a bounded holomorphic function by $h_b(f) := P(b\overline{f})$.

We are interested here to the boundedness of the small Hankel operators h_b from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ to $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$.

In the one dimensional case, that is the unit disc of the complex plane $\mathbb{C}$, boundedness of the small Hankel operator between Hardy spaces has been considered in [7] and completely solved in [12]. A. Bonami and S. Madan in [3] used the so called "balayage" of Carleson measures to characterize symbols of bounded Hankel operators between Hardy-Orlicz spaces in the unit disc of $\mathbb{C}$. It is well-known that h_b extends as a bounded operator on $\mathcal{H}^p(\mathbb{B}^n)$ for $p > 1$ if and only if b is in $BMOA$ (see [5]). Recently, some of the one dimensional results have been extended to the unit ball $\mathbb{B}^n$. First, using some simple techniques, A. Bonami, S. Grellier and the first author proved in [2] that h_b is bounded on $\mathcal{H}^1(\mathbb{B}^n)$ if and only if $b \in BMOA(\varrho)$ with $\varrho(t) = \frac{1}{\log(\frac{4}{t})}$. In [1], A. Bonami and S. Grellier using weak factorization results were able to characterize symbols of bounded Hankel operators from the space $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$, where $\Phi \in \mathscr{L}_p$. The two last works have been extended in [4] to the case of Hankel operators

between two Hardy-Orlicz spaces $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ with $\Phi_i \in \mathscr{L}_p$; $i = 1, 2$. In both [1] and [4], because of the restriction (concavity) on the Orlicz functions, there are several cases of boundedness which are not taken into account. In [11], the authors provided with a characterization of bounded Hankel operators, h_b, from $\mathcal{H}^{\Phi_p}(\mathbb{B}^n)$ to $\mathcal{H}^q(\mathbb{B}^n)$, in terms of the appartenance of the symbols in some weighted Lipschtiz spaces, where $\Phi_p = \left(\frac{t}{\log(e+t)}\right)^p$, $p \le 1$, $0 < q < \infty$. We remark that $\Phi_p \in \mathscr{L}_p$ and for $q > 1$, $\Phi_2(t) = t^q$ is in $\mathscr{U}^q$. In this paper we consider the boundedness of h_b between the Hardy-Orlicz spaces $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$, where Φ_1 and Φ_2 are either in $\mathscr{L}_p$ or $\mathscr{U}^q$ but not both in $\mathscr{L}_p$. When the functions $\Phi_2 \in \mathscr{U}^q$, we restrict to those satisfying

$$(11) \qquad \lim_{x \to \infty} \frac{\Phi_2(x)}{x} = \infty.$$

In fact, if $\Phi_2 \in \mathscr{U}^q$ does not satisfy (11) then Φ_2 is equivalent to $\Phi(x) = x$ so that $\mathcal{H}^{\Phi_2}(\mathbb{B}^n) = \mathcal{H}^1(\mathbb{B}^n)$. This case has been settled in [1] and note that (11) implies that $q > 1$.

The simple and direct approach used in [11] seems difficult to be used here for this general situation. We will be inspired instead by the techniques and methods in [1, 4]. The main tool is the use of the molecular decomposition of Hardy-Orlicz spaces given in [1] to obtain our needed weak factorization. The fact that molecules in the molecular decomposition in [1] can have arbitrary large order will be crucial. In the next section we will prove the following results.

THEOREM 1.6. *Let $\Phi_1 \in \mathscr{L}_p$ and $\Phi_2 \in \mathscr{U}^q$, $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$ and Ψ_2 the complementary of Φ_2. Then the product of two functions, one in $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and the other one in $\mathcal{H}^{\Psi_2}(\mathbb{B}^n)$, is in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$, with Φ such that*

$$(12) \qquad \rho_\Phi := \frac{\rho_1}{\rho_2},$$

or, equivalently,

$$(13) \qquad \Phi^{-1}(t) := \Phi_1^{-1}(t)\Psi_2^{-1}(t).$$

Moreover, functions in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ can be weakly factorized in terms of products of functions of $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and $\mathcal{H}^{\Psi_2}(\mathbb{B}^n)$.

THEOREM 1.7. *Let $\Phi_1 \in \mathscr{L}_p$ and $\Phi_2 \in \mathscr{U}^q$, $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$ and assume that Φ_2 satisfies the Dini condition (9). Then the Hankel operator h_b extends into a bounded operator from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ if and only if its symbol b belongs to $BMOA(\rho_\Phi) = (\mathcal{H}^{\Phi}(\mathbb{B}^n))^*$, where*

$$\rho_\Phi := \frac{\rho_1}{\rho_2}.$$

In section 3, we study the boundedness of the Hankel operator h_b from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$, when $\Phi_i \in \mathscr{U}^q$, $i = 1, 2$. To deal with this situation, because of the

convexity of both growth functions, we will have to rewrite, in a slightly general form, the molecular decomposition in [1]. This will allow us to obtain other weak factorizations results for functions in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$, with $\Phi \in \mathscr{L}_p$, in terms of products of functions of $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$, with $\Phi_i \in \mathscr{U}^q$, $i = 1, 2$. This generalizes the classical result in [5, 6]. We obtain in this situation the following result.

THEOREM 1.8. *Let Φ_1 and Φ_2 in $\mathscr{U}^q$, and $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$.*
We suppose that:

    (i) Φ_2 *satisfies the Dini condition (9)*
    (ii) $\frac{\Phi_1^{-1}(t)\Psi_2^{-1}(t)}{t}$ *is non-decreasing or $\Phi_1 = \Phi_2$.*

Then the Hankel operator h_b extends into a bounded operator from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ if and only if its symbol b belongs to $BMOA(\rho_\Phi)$, where

$$\rho = \rho_\Phi := \frac{\rho_1}{\rho_2}.$$

Finally, all over the text, C will be a constant not necessary the same at each occurrence. We will also use the notation $C(k)$ to express the fact that the constant depends on the underlined parameter. Given two positive quantities A and B, the notation $A \lesssim B$ means that $A \leq CB$ for some positive constant C. When $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$.

2. BOUNDEDNESS OF h_b: $\mathcal{H}^{\Phi_1}(\mathbb{B}^n) \to \mathcal{H}^{\Phi_2}(\mathbb{B}^n)$; $(\Phi_1, \Phi_2) \in \mathscr{L}_p \times \mathscr{U}^q$

The section is devoted to the proof of Theorem 1.6 and Theorem 1.7.

2.1. **Some properties of growth functions.** We collect in this subsection few properties of growth functions we shall used later.

We start with this useful proposition which gives relations between functions in the classes $\mathscr{L}_p$ and $\mathscr{U}^q$.

PROPOSITION 2.1. *The following assertion holds:*

$$\Phi \in \mathscr{L}_p \text{ if and only if } \Phi^{-1} \in \mathscr{U}^{1/p}.$$

Proof. It is clear that Φ is a growth function if and only if Φ^{-1} is a growth function. Also, it is clear that $\frac{\Phi(t)}{t}$ is non-increasing if and only if $\frac{\Phi^{-1}(t)}{t}$ is non-decreasing. So what is left to show is that Φ is of lower type p if and only if Φ^{-1} is of upper type $1/p$. Suppose there exists $C > 1$ so that for every $s \leq 1$ and all $t > 0$, we have

(14) $\Phi(st) \leq Cs^p \Phi(t).$

Let $x \geq 1$ and $y > 0$. We have, applying inequality (14) to $s = \frac{1}{(Cx)^{1/p}}$ and $t = \Phi^{-1}(y)(Cx)^{1/p}$,

$$y \leq \frac{1}{x} \Phi \left(\Phi^{-1}(y)(Cx)^{1/p} \right),$$

thus, $\Phi^{-1}(xy) \leq Ax^{1/p}\Phi^{-1}(y)$. Hence Φ^{-1} is of upper type $1/p$. The arguments could be reversed. This ends the proof of the proposition. $\square$

LEMMA 2.2. *Let $\Phi_1 \in \mathscr{L}_p$ and $\Phi_2 \in \mathscr{U}^q$, and Ψ_2 the complementary function of Φ_2. Let Φ be such that*

$$\Phi^{-1}(t) := \Phi_1^{-1}(t)\Psi_2^{-1}(t).$$

Then $\Phi \in \mathscr{L}_r$ for some $r \leq p$.

Proof. Let us write $\frac{\Phi^{-1}(t)}{t} = \left(\frac{\Phi_1^{-1}(t)}{t} \right) \Psi_2^{-1}(t)$ and remark that $\frac{\Phi_1^{-1}(t)}{t}$ and $\Psi_2^{-1}(t)$ are non-decreasing. We deduce easily that $\frac{\Phi^{-1}(t)}{t}$ is non-decreasing.

By Proposition 2.1, it just remains to show that Φ^{-1} is of upper type $1/r$. Let $s \geq 1$ and $t > 0$, applying Proposition 2.1 to Φ_1 and using the fact that $\frac{\Psi_2^{-1}(t)}{t}$ is non-increasing, we have

$$\Phi^{-1}(st) = \Phi_1^{-1}(st)\Psi_2^{-1}(st)$$
$$\lesssim s^{1/p}\Phi_1^{-1}(t)st\frac{\Psi_2^{-1}(st)}{st}$$
$$\lesssim s^{\frac{p+1}{p}}\Phi^{-1}(t). \qquad \square$$

LEMMA 2.3. *Let Φ_1 be a growth function and $\Phi_2 \in \mathscr{U}^q$, $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$ and Ψ_2 the complementary of Φ_2. Then, if*

$$\rho_\Phi := \frac{\rho_1}{\rho_2},$$

we also have

$$\Phi^{-1}(t) \simeq \Phi_1^{-1}(t)\Psi_2^{-1}(t)$$

and vice-versa.

Proof. It is enough to prove that

(15) $$\rho_2(t) \simeq \Psi_2^{-1}(1/t).$$

This follows easily from the fact that

(16) $$t \leq \Phi_2^{-1}(t)\Psi_2^{-1}(t) \leq 2t. \qquad \square$$

LEMMA 2.4. *Let $\Phi \in \mathscr{U}^q$. Then for $0 < m < \infty$, $\Phi_m(t) := \Phi(t^{1/m})$ is of lower type $1/m$ and if moreover m is large enough, then $\frac{\Phi(t)}{t^m}$ is non-increasing.*

Proof. Clearly, Φ_m is of lower type $1/m$ since Φ is of lower type 1. Let us recall that there are constants $c_1 > 0$ and $c_2 > 0$ such that for any $t > 0$,

$$c_1 \frac{\Phi(t)}{t} \leq \Phi'(t) \leq c_2 \frac{\Phi(t)}{t}.$$

Now, put $g(t) = \frac{\Phi(t)}{t^m}$ with $m \geq c_2$. We obtain easily that

$$\begin{aligned} g'(t) &= \frac{t\Phi'(t) - m\Phi(t)}{t^{m+1}} \\ &\leq \frac{(c_2 - m)\Phi(t)}{t^{m+1}} \\ &\leq 0. \end{aligned} \qquad \square$$

LEMMA 2.5. *Let Φ_1 and Φ_2 be in $\mathscr{U}^q$, and Ψ_2 the complementary function of Φ_2. Let Φ be such that $\Phi^{-1}(t) = \Phi_1^{-1}(t)\Psi_2^{-1}(t)$. We suppose that Φ_2 satisfies the Dini condition (9) and that*

$$\frac{\Phi_2^{-1} \circ \Phi_1(t)}{t} \quad \text{is non-increasing.}$$

Then $\Phi \in \mathscr{L}_p$ for some $p > 0$.

Proof. Using (16) and Proposition 2.1, it is enough to show that

$$\phi^{-1}(t) = \frac{t\Phi_1^{-1}(t)}{\Phi_2^{-1}(t)} \quad \text{is in} \quad \mathscr{U}^{1/p}.$$

We have $\phi^{-1}(\Phi_1(t)) = \frac{t\Phi_1(t)}{\Phi_2^{-1}\circ\Phi_1(t)}$, so that ϕ^{-1} is a growth function such that $\frac{\phi^{-1}(t)}{t}$ is non-decreasing. It is left to show that ϕ^{-1} is of upper type. Let $s \geq 1$ and $t > 0$, using the fact $\frac{\Psi_2^{-1}(t)}{t}$ and $\frac{\Phi_1^{-1}(t)}{t}$ are non-increasing, we have

$$\begin{aligned} \phi^{-1}(st) &\simeq \Phi_1^{-1}(st)\Psi_2^{-1}(st) \\ &\lesssim \qquad\qquad\qquad s\Phi_1^{-1}(t)s\Psi_2^{-1}(s) \\ &\lesssim s^2\phi^{-1}(t). \end{aligned}$$

Hence $\phi^{-1} \in \mathscr{U}^2$. This ends the proof. $\qquad \square$

2.2. Proof of Theorem 1.6 and Theorem 1.7. We recall the following definition of molecule (see [1] and the references therein).

DEFINITION 2.6. *A holomorphic function $A \in \mathcal{H}^2(\mathbb{B}^n)$ is called a molecule of order L, associated to the ball $B := B(z_0, r_0) \subset \mathbb{S}^n$, if it satisfies*

$$(17) \quad ||A||_{mol(B,L)} := \left(\sup_{r<1} \int_{\mathbb{S}^n} \left(1 + \frac{d(z_0, r\xi)^{L+n}}{r_0^{L+n}} \right) |A(r\xi)|^2 \frac{d\sigma(\xi)}{\sigma(B)} \right)^{1/2} < \infty.$$

We have used the notation $d(z, w) := |1 - z\overline{w}|$ for $z, w \in \overline{\mathbb{B}^n}$.

It is proved in [1] that for $\Phi \in \mathscr{L}_p$, every molecule A of order L so that $L > L_p := 2n(1/p - 1)$ belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$ with

(18) $$||A||_{\mathcal{H}^\Phi} \lesssim \Phi(||A||_{mol(B,L)})\sigma(B).$$

The following molecular decomposition for functions in some Hardy-Orlicz spaces is proved in [1].

THEOREM 2.7. *Let* $\Phi \in \mathscr{L}_p$. *For any* $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$, *there exist molecules* A_j *of order* $L > L_p$, *associated to the balls* B_j, *so that* f *may be written as*

$$f = \sum_j A_j$$

with $||f||_{\mathcal{H}^\Phi} \simeq \sum_j \Phi(||A_j||_{mol(B_j,L)})\sigma(B_j)$.

We have the following generalization of [8, Lemma 3.9].

LEMMA 2.8. *Let* $(\Omega, \mathbb{P})$ *be a probability space,* Φ *a convex function such that* $\Phi(0) = 0$, *and* $0 < p < \infty$. *Then for every* $g \in L^\infty(\Omega)$,

$$||g||_{\mathcal{H}^{\Phi^p}}^{lux} \leq \frac{||g||_\infty}{\Phi^{-1}(||g||_\infty/||g||_{L^p})}.$$

Proof. The proof follows exactly as in [8]. We give it here for completeness. We may assume that $||g||_\infty = 1$. Since Φ is convex with $\Phi(0) = 0$ and $|g| \leq 1$, we obtain for every $C > 0$,

$$\int_\Omega \Phi^p(\frac{|g|}{C})d\mathbb{P} \leq \int_\Omega |g|^p\Phi^p(\frac{1}{C})d\mathbb{P} = ||g||_{L^p}^p\Phi^p(\frac{1}{C}).$$

Now one sees that $||g||_{L^p}^p\Phi^p(\frac{1}{C}) \leq 1$ if and only if $C \geq 1/\Phi^{-1}(1/||g||_{L^p})$, and from this follows the proof of the lemma. $\square$

The following is a direct consequence of the above lemma.

LEMMA 2.9. *Suppose that* $\Phi \in \mathscr{U}^q$ *and let* Ψ *be its complementary. For each* $a \in \mathbb{B}^n$ *and* $0 < p < \infty$, *let*

$$g(z) = \Psi^{-1}\left(\frac{1}{(1 - |a|)^{n/p}}\right)\left(\frac{1 - |a|}{1 - \langle z, a\rangle}\right)^{\frac{2n}{p}}.$$

We have $||g||_{\mathcal{H}^{\Psi^p}}^{lux} \lesssim 1$.

We now describe a factorization of each molecule. This is the main ingredient in the proof of our results in the case we are concerned here.

THEOREM 2.10. *Let* $\Phi_1 \in \mathscr{L}_p$ *and* $\Phi_2 \in \mathscr{U}^q$, $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$ *and* Ψ_2 *the complementary of* Φ_2. *Let* Φ *be such that* $\rho(t) = \rho_\Phi(t) := \frac{\rho_1}{\rho_2}$. *We assume moreover that* Φ_2 *satisfies the Dini condition (9). Then a molecule* A *associated to the ball*

B may be written as fg, where f is a molecule and $g \in \mathcal{H}^{\Psi_2}(\mathbb{B}^n)$. Moreover, for L, L' given with $L' + 4n \leq L$, f and g may be chosen such that

$$(19) \qquad ||g||_{\mathcal{H}^{\Psi_2}} \lesssim 1, \qquad ||f||_{mol(B,L')} \lesssim \frac{||A||_{mol(B,L)}}{\rho_2(\sigma(B))},$$

or such that

$$(20) \qquad ||g||_{\mathcal{H}^{\Psi_2}} \lesssim 1, \qquad ||f||^{lux}_{\mathcal{H}^{\Phi_1}} \lesssim ||A||_{mol(B,L)}\sigma(B)\rho(\sigma(B)).$$

Proof. Suppose $B := B(z_0, r) \subset \mathbb{S}^n$, with $z_0 \in \mathbb{S}^n$ and $r < 1$. Let $a = (1-r)z_0$ and take

$$g(z) = \Psi_2^{-1}\left(\frac{1}{(1-|a|)^n}\right)\left(\frac{1-|a|}{1-\langle z, a\rangle}\right)^{2n}.$$

So, by Lemma 2.9, we have $||g||_{\mathcal{H}^{\Psi_2}} \lesssim 1$ and using (15) we also have that

$$(21) \qquad |g(z)| \simeq \rho_2(\sigma(B))\frac{r^{2n}}{|1-z\overline{a}|^{2n}}.$$

We know that $|1-z\overline{a}|^{2n} \lesssim (d(z,z_0)+r)^{2n}$. Hence

$$|g(z)| \gtrsim \rho_2(\sigma(B))\frac{r^{2n}}{(d(z,z_0)+r)^{2n}}.$$

Having this and under the condition on L and L', the rest of the proof is just imitating the proof of the analogue result in [4, Theorem 4.3], we omit these details. $\qquad\square$

Having Theorem 2.10 and the techniques in [4], the proof of Theorem 1.6 and Theorem 1.7 is now routine. Indeed, the sufficient part of Theorem 1.6 is an application of Proposition 1.1 and the sufficiency of Theorem 1.7 follows from Theorem 1.5, Theorem 1.4 and Proposition 1.1. The necessity part follows as in the proof of Theorem 1.8 and 1.9 in [4] where we use Theorem 2.10 instead of Theorem 4.3 in [4], we omit the details. This ends the proof of Theorem 1.6 and Theorem 1.7.

3. BOUNDEDNESS OF h_b: $\mathcal{H}^{\Phi_1}(\mathbb{B}^n) \to \mathcal{H}^{\Phi_2}(\mathbb{B}^n)$; $\Phi_i \in \mathscr{U}^q$, $i = 1, 2$.

This section is devoted to the study of the boundedness of the Hankel operator h_b, between two Hardy-Orlicz spaces $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ with $\Phi_i \in \mathscr{U}^q$; $i = 1, 2$. The main tools we need are in [1] where atomic and molecular decomposition for functions in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ with $\Phi \in \mathscr{L}_p$ are described. But since we are dealing here with convex functions, we will need to consider some simple generalizations of those results in order to get rid of the present difficulty. These extensions are explained in the next subsection and in most case the proofs just follow the lines of the ones in [1] where we will refer for further details.

3.1. **Generalization of atomic and molecular decomposition.** For $\Phi \in \mathscr{L}_p$, the atomic decomposition and molecular decomposition for $\mathcal{H}^\Phi(\mathbb{B}^n)$ are described using square integrable function [1]. One classical result used often in various arguments there is the fact that the Szegö projection is bounded in $L^2(\mathbb{S}^n)$. It is well known that, for all $1 < m < \infty$, the Szegö projection is bounded from $L^m(\mathbb{S}^n)$ to $\mathcal{H}^m(\mathbb{B}^n)$. This fact allows us to obtain atomic and molecular decomposition for $\mathcal{H}^\Phi(\mathbb{B}^n)$ using $m-$ integrable function. We will now give precise description of what we are talking. In the sequel, $m > 1$ will be a fixed real.

DEFINITION 3.1. *A function a in $L^m(\mathbb{S}^n)$ is called an $m-$atom of order $N \in \mathbb{N}$ associated to the ball $B := B(z_0, r_0)$, for some $z_0 \in \mathbb{S}^n$, if supp $a \subset B$ and when $r_0 < \delta$,*

$$\int_{\mathbb{S}^n} a(\xi)P(\xi)d\sigma(\xi) = 0 \text{ for every } P \in \mathcal{P}_N(z_0).$$

We obtain the following atomic decomposition.

THEOREM 3.2. *Let $N \in \mathbb{N}$ be larger than $N_{p,m} := mn(1/p - 1) - 1$. Given any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$ there exist $m-$atoms a_j of order N such that (in the distribution sense)*

$$f = P\left(\sum_{j=0}^\infty a_j\right) = \sum_{j=0}^\infty P(a_j).$$

Moreover,

$$\sum_{j=0}^\infty \sigma(B_j)\Phi(||a_j||_m \sigma(B_j)^{-1/m}) \simeq ||f||_{\mathcal{H}^\Phi(\mathbb{B}^n)}.$$

DEFINITION 3.3. *A holomorphic function $A \in \mathcal{H}^m(\mathbb{B}^n)$ is called an $m-$molecule of order L, associated to the ball $B := B(z_0, r_0) \subset \mathbb{S}^n$, if it satisfies*

$$(22) \quad ||A||_{mol(B,L,m)} := \left(\sup_{r<1} \int_{\mathbb{S}^n} \left(1 + \frac{d(z_0, r\xi)^{L+n}}{r_0^{L+n}}\right) |A(r\xi)|^m \frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/m} < \infty.$$

The following proposition replaces Proposition 1.9 in [1]. The proof is similar.

PROPOSITION 3.4. *For an $m-$atom a of order N associated to a ball $B \subset \mathbb{S}^n$, its Szegö projection $P(a)$ is an $m-$molecule associated to $\tilde{B}$ of double radius, of any order $L < (m-2)n + \frac{N+1}{2}m$. It satisfies*

$$(23) \qquad\qquad ||A||_{mol(\tilde{B},L,m)} \lesssim ||a||_m \sigma(B)^{-1/m}.$$

This yields the following molecular decomposition.

THEOREM 3.5. *Let $\Phi \in \mathscr{L}_p$. For any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$, there exist $m-$molecules A_j of order $L > L_{p,m} := mn(1/p - 1)$, associated to the balls B_j, so that f may be written as*

$$f = \sum_j A_j$$

with $||f||_{\mathcal{H}^\Phi} \simeq \sum_j \Phi(||A_j||_{mol(B_j,L,m)})\sigma(B_j)$.

Let ρ be a weight, we define a family of weighted BMOA spaces $BMOA(\rho, m)$.

$$(24) \qquad BMOA(\rho, m) := \left\{ f \in \mathcal{H}^m(\mathbb{B}^n) : \quad \sup_{r<1} ||f_r||_{BMO(\rho,m)} < \infty \right\},$$

where

$$||f||_{BMO(\rho,m)} = \left(\sup_B \inf_{R \in \mathcal{P}_N(B)} \frac{1}{(\varrho(\sigma(B)))^m \sigma(B)} \int_B |f - R|^m d\sigma \right)^{1/m}.$$

The work of Viviani [13] allows to obtain that $BMOA(\rho, m)$ are all the same as $m \geq 1$ so that the dual $(\mathcal{H}^\Phi(\mathbb{B}^n))^*$ of $\mathcal{H}^\Phi(\mathbb{B}^n)$ identifies with $BMOA(\rho, m)$, for each fixed m.

The following proposition will be used to replace the inequality (18) in the case of Orlicz function $\Phi \in \mathscr{U}^q$.

PROPOSITION 3.6. *Let $\Phi \in \mathscr{U}^q$ and $0 < p \leq 1$. Then, for m large enough, any $m-$molecule A of order L associated to a ball B, such that $L > \frac{nm}{p} - 2n$, belongs to $\mathcal{H}^{\Phi^p}(\mathbb{B}^n)$ with*

$$(25) \qquad\qquad\qquad ||A||_{\mathcal{H}^{\Phi^p}} \lesssim \Phi^p(||A||_{mol(B,L,m)})\sigma(B).$$

Proof. We first remark that by Lemma 2.4, $\Phi_m = \Phi(t^{1/m}) \in \mathscr{L}_{1/m}$, so that we may suppose that Φ_m is concave and so does Φ_m^p. The proof then follows as in the proof of Proposition 1.10 in [1], where we just need to use often the Jensen Inequality in the following way:

$$(26) \qquad \int_X \Phi^p(f)d\mu = \int_X \Phi_m^p(f^m)d\mu \leq \Phi_m^p\left(\int_X f^m d\mu\right) = \Phi^p\left(||f||_{L^m(X,d\mu)}\right).$$

Here, $d\mu$ is a probability measure and f is a positive function on the measure space $(X, d\mu)$. □

3.2. Boundedness of h_b: $\mathcal{H}^{\Phi_1}(\mathbb{B}^n) \to H^{\Phi_2}(\mathbb{B}^n)$. We are now ready to give our result for the boundedness of the Hankel operator from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ in the case where Φ_1 and Φ_2 are both convex. As in the previous section, we will follow the techniques in [4], the generalization of molecular decomposition described in the previous subsection will be used in the present situation.

We first describe a factorization of each $m-$molecule as a product of functions in Hardy-Orlicz spaces with power of convex growth functions.

THEOREM 3.7. *Let Φ_1 and $\Phi_2 \in \mathscr{U}^q$, $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$, $0 < p \leq 1$, and Ψ_2 the complementary function of Φ_2. Let Φ be such that $\rho(t) = \rho_\Phi(t) := \frac{\rho_1}{\rho_2}$. Then for m large enough, an $m-$molecule A associated to the ball B may be written as*

fg, where f is an $m-$molecule and $g \in \mathcal{H}^{\Psi_2^p}(\mathbb{B}^n)$. Moreover, for L, L' given with $L' + \frac{2mn}{p} \leq L$, f and g may be chosen such that

$$(27) \qquad ||g||_{\mathcal{H}^{\Psi_2^p}} \lesssim 1, \qquad ||f||_{mol(B,L',m)} \lesssim \frac{||A||_{mol(B,L,m)}}{\rho_2(\sigma(B)^{1/p})},$$

or such that

$$(28) \qquad ||g||_{\mathcal{H}^{\Psi_2^p}} \lesssim 1, \qquad ||f||^{lux}_{\mathcal{H}^{\Phi_1^p}} \lesssim ||A||_{mol(B,L,m)} \sigma(B)^{1/p} \rho(\sigma(B)^{1/p}).$$

Proof. Suppose $B := B(z_0, r) \subset \mathbb{S}^n$, with $z_0 \in \mathbb{S}^n$ and $r < 1$. Let $a = (1-r)z_0$ and take

$$g(z) = \Psi_2^{-1}\left(\frac{1}{(1-|a|)^{n/p}}\right)\left(\frac{1-|a|}{1-\langle z, a\rangle}\right)^{2n/p}.$$

So, by Lemma 2.9, we have $||g||_{\mathcal{H}^{\Psi_2^p}} \lesssim 1$ and using (15) we also have that

$$(29) \qquad |g(z)| \gtrsim \rho_2(\sigma(B)^{1/p}) \frac{r^{\frac{2n}{p}}}{(d(z,z_0)+r)^{\frac{2n}{p}}}, \quad z \in \overline{\mathbb{B}^n}.$$

We pose $f = A/g$ and $\tilde{B} = B(z_0, 2r)$ and proceed to prove the second part of (27).

$$||f||^m_{mol(B,L',m)} = \int_{\mathbb{S}^n}\left(1 + \frac{d(z_0,\xi)^{L'+n}}{r^{L'+n}}\right)|f(\xi)|^m \frac{d\sigma(\xi)}{\sigma(B)}$$
$$\simeq \int_{\tilde{B}}|f(\xi)|^m\frac{d\sigma(\xi)}{\sigma(B)} + \int_{\mathbb{S}^n\setminus\tilde{B}}\left(\frac{d(z_0,\xi)}{r}\right)^{L'+n}|f(\xi)|^m\frac{d\sigma(\xi)}{\sigma(B)}$$
$$= I + II,$$

where

$$I = \int_{\tilde{B}}|f(\xi)|^m\frac{d\sigma(\xi)}{\sigma(B)} \quad \text{and} \quad II = \int_{\mathbb{S}^n\setminus\tilde{B}}\left(\frac{d(z_0,\xi)}{r}\right)^{L'+n}|f(\xi)|^m\frac{d\sigma(\xi)}{\sigma(B)}$$

In $\tilde{B}$, we have $|g(\xi)| \gtrsim \rho_2(\sigma(B)^{1/p})$ so that

$$I \lesssim \frac{1}{(\rho_2(\sigma(B)^{1/p}))^m}\int_{\tilde{B}}|A(\xi)|^m\frac{d\sigma(\xi)}{\sigma(B)} \lesssim \frac{||A||^m_{mol(B,L,m)}}{(\rho_2(\sigma(B)^{1/p}))^m}.$$

In $\mathbb{S}^n\setminus\tilde{B}$, we have $|g(\xi)| \gtrsim \rho_2(\sigma(B)^{1/p})\frac{r^{2n/p}}{d(z,z_0)^{2n/p}}$ so that we have

$$II = \int_{\mathbb{S}^n\setminus\tilde{B}}\left(\frac{d(z_0,\xi)}{r}\right)^{L'+n}|f(\xi)|^m\frac{d\sigma(\xi)}{\sigma(B)}$$
$$\lesssim \frac{1}{(\rho_2(\sigma(B)^{1/p}))^m}\int_{\mathbb{S}^n\setminus\tilde{B}}\left(\frac{d(z_0,\xi)}{r}\right)^{L'+n+\frac{2nm}{p}}|A(\xi)|^m\frac{d\sigma(\xi)}{\sigma(B)}$$
$$\lesssim \frac{||A||^m_{mol(B,L,m)}}{(\rho_2(\sigma(B)^{1/p}))^m}.$$

This proves (27). It remains to prove (28). By homogeneity, it is sufficient to prove that, for $||A||_{mol(B,L,m)}\sigma(B)^{1/p}\rho(\sigma(B)^{1/p}) = 1$, the function that has been chosen is such that $\int_{\mathbb{S}^n} \Phi_1^p(|f|)d\sigma \lesssim 1$. By (25), it is enough to prove that $||f||_{mol(B,L',m))} \lesssim \Phi_1^{-1}\left(\frac{1}{\sigma(B)^{1/p}}\right) = \frac{1}{\sigma(B)^{1/p}\rho_1(\sigma(B)^{1/p})}$. This holds by (27) and the definition of ρ. □

Taking $\Phi_1 = \Phi_2$, we obtain in particular the following.

COROLLARY 3.8. Let $\Phi \in \mathscr{U}^q$, $0 < p \le 1$, $\rho(t) = \frac{1}{t\Phi^{-1}(1/t)}$, and Ψ the complementary function of Φ. Then for m large enough, an $m-$molecule A associated to the ball B may be written as fg, where f is an $m-$molecule and $g \in \mathcal{H}^{\Psi p}(\mathbb{B}^n)$. Moreover, for L, L' given with $L' + \frac{2mn}{p} \le L$, f and g may be chosen such that

$$(30) \qquad ||g||_{\mathcal{H}^{\Psi p}} \lesssim 1, \qquad ||f||_{mol(B,L',m)} \lesssim \frac{||A||_{mol(B,L,m)}}{\rho(\sigma(B)^{1/p})},$$

or such that

$$(31) \qquad ||g||_{\mathcal{H}^{\Psi p}} \lesssim 1, \qquad ||f||^{lux}_{\mathcal{H}^{\Phi p}} \lesssim ||A||_{mol(B,L,m)}\sigma(B)^{1/p}.$$

This leads to the following generalization of the weak factorization of the Hardy spaces $\mathcal{H}^p(\mathbb{B}^n)$, $0 < p \le 1$, obtained in [5, 6]

THEOREM 3.9. Let $0 < p \le 1$, $\Phi \in \mathscr{U}^q$, $\rho(t) = \frac{1}{t\Phi^{-1}(1/t)}$, and Ψ the complementary function of Φ. Then given any $f \in \mathcal{H}^p(\mathbb{B}^n)$ there exist $f_j \in \mathcal{H}^{\Phi p}(\mathbb{B}^n)$, $g_j \in \mathcal{H}^{\Psi p}(\mathbb{B}^n)$, $j \in \mathbb{N}$ such that

$$f = \sum_{j=0}^{\infty} f_j g_j$$

and

$$(32) \qquad \sum_j ||g_j||^{lux}_{\mathcal{H}^{\Psi p}}||f_j||^{lux}_{\mathcal{H}^{\Phi p}} \lesssim ||f||_p.$$

Proof. Applying Theorem 3.5 to $\Phi(t) = t^p$, we know that there exist $m-$molecules A_j of order $L > L_{p,m} := mn(1/p - 1)$, associated to the balls B_j, so that f may be written as

$$f = \sum_j A_j$$

with $||f||_{\mathcal{H}^p} \simeq \sum_j ||A_j||^p_{mol(B_j,L,m)}\sigma(B_j)$.

The weak factorization then follows from the factorization of each molecule as obtained in Corollary 3.8 with

$$||g_j||_{\mathcal{H}^{\Psi p}} \lesssim 1, \qquad ||f_j||^{lux}_{\mathcal{H}^{\Phi p}} \lesssim ||A_j||_{mol(B_j,L,m)}\sigma(B_j)^{1/p}.$$

The inequality (32) follows easily from this.

$$\sum_j ||g_j||_{\mathcal{H}^{\Psi p}}^{lux} ||f_j||_{\mathcal{H}^{\Phi p}}^{lux} \lesssim \sum_j ||f_j||_{\mathcal{H}^{\Phi p}}^{lux}$$

$$\lesssim \sum_j ||A_j||_{mol(B_j,L,m)} \sigma(B_j)^{1/p}$$

$$\lesssim \left(\sum_j ||A_j||_{mol(B_j,L,m)}^p \sigma(B_j) \right)^{1/p}$$

$$\lesssim ||f||_p.$$

The proof is complete. $\square$

Using Theorem 3.7, we are ready to prove the following result about boundedness of Hankel operator h_b in the case where the growth functions are both convex.

THEOREM 3.10. *Let Φ_1 and Φ_2 in $\mathscr{U}^q$, and $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$.*
We suppose that:

   (i) *Φ_2 satisfies the Dini condition (9)*
   (ii) *$\frac{\Phi_1^{-1}(t)\Psi_2^{-1}(t)}{t}$ is non-decreasing or $\Phi_1 = \Phi_2$.*

Then the Hankel operator h_b extends into a bounded operator from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ if and only if its symbol b belongs to $BMOA(\rho)$, where

$$\rho = \rho_\Phi := \frac{\rho_1}{\rho_2}.$$

Proof. We first remark that using Proposition 2.1, we can see that (ii) implies that $\Phi \in \mathscr{L}_p$ for some p so that the dual $(\mathcal{H}^\Phi(\mathbb{B}^n))^* = BMOA(\rho_\Phi)$ and we have the factorization of $m-$molecules. The sufficient part follows from Theorem 1.5, Theorem 1.4 and Proposition 1.1. Indeed,

$$||h_b(f)||_{\mathcal{H}^{\Phi_2}(\mathbb{B}^n)}^{lux} = \sup_{||g||_{\mathcal{H}^{\Psi_2}(\mathbb{B}^n)}^{lux}=1} |\langle h_b(f), g \rangle| = \sup_{||g||_{\mathcal{H}^{\Psi_2}(\mathbb{B}^n)}^{lux}=1} |\langle b, fg \rangle|$$

$$\lesssim \sup_{||g||_{\mathcal{H}^{\Psi_2}(\mathbb{B}^n)}^{lux}=1} \left(||b||_{BMOA(\rho_\Phi)} ||fg||_{\mathcal{H}^\Phi(\mathbb{B}^n)}^{lux} \right)$$

$$\lesssim \sup_{||g||_{\mathcal{H}^{\Psi_2}(\mathbb{B}^n)}^{lux}=1} \left(||b||_{BMOA(\rho_\Phi)} ||f||_{\mathcal{H}^{\Phi_1}(\mathbb{B}^n)}^{lux} ||g||_{\mathcal{H}^{\Psi_2}(\mathbb{B}^n)}^{lux} \right)$$

$$\lesssim ||b||_{BMOA(\rho_\Phi)} ||f||_{\mathcal{H}^{\Phi_1}(\mathbb{B}^n)}^{lux}.$$

Now, we assume that h_b is bounded from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ and prove that b belongs to $BMOA(\rho)$. Since $BMOA(\rho) = BMOA(\rho, m)$ for any $m \geq 1$, it is

sufficient to prove that there exist constants C and m' such that, for each ball B we can find a polynomial $R \in \mathcal{P}_N(B)$ such that

$$(33) \qquad \left(\int_B |b - R|^{m'} d\sigma \right)^{1/m'} \leq C \left(\sigma(B) \right)^{1/m'} \rho(\sigma(B)).$$

Let $B = B(z_0, r)$ be a ball in $\mathbb{S}^n$, we take for R the orthogonal projection of b onto $\mathcal{P}_N(B)$, and let $a := \chi_B(b - R)$ so that a is an $m'-$ atom.

For $l \in L^m(\mathbb{S}^n, d\sigma)$ we will denote by $R(l)$ the orthogonal projection of l onto $\mathcal{P}_N(B)$, so that the function $a_l = \chi_B(l - R(l))$ is an $m-$atom (see Definition 3.1) associated to B. We claim that there exists an absolute constant $C = C(N, m, n)$ such that

$$(34) \qquad ||\chi_B(l - R(l))||_m \leq C||\chi_B l||_m.$$

Assume that (34) holds. From Proposition 3.4, one knows that $A_l := P(a_l)$ is an $m-$molecule associated to $\tilde{B}$, with $||A_l||_{mol(B,L,m)} \lesssim ||a_l||_m \sigma(B)^{-1/m}$. From Theorem 3.7 we know that A_l may be written as $f_l g_l$, with

$$||g_l||_{\mathcal{H}^{\Psi_2}} \lesssim 1, \qquad\qquad ||f_l||_{\mathcal{H}^{\Phi_1}}^{lux} \lesssim ||A_l||_{mol(B,L,m)} \sigma(B) \rho(\sigma(B)).$$

From this, we have , using (34)

$$
\begin{aligned}
||a||_{m'} &= \sup_{||\chi_B l||_m = 1} |\langle a, l \rangle| = \sup_{||\chi_B l||_m = 1} |\langle a, l - R(l) \rangle| \\
&= \sup_{||\chi_B l||_m = 1} |\langle b, \chi_B(l - R(l)) \rangle| = \sup_{||\chi_B l||_m = 1} |\langle b, a_l \rangle| \\
&= \sup_{||\chi_B l||_m = 1} |\langle b, P(a_l) \rangle| = \sup_{||\chi_B l||_m = 1} |\langle h_b(f_l), g_l \rangle| \\
&\lesssim \sup_{||\chi_B l||_m = 1} \left(||h_b|| ||f_l||_{\mathcal{H}^{\Phi_1}}^{lux} ||g_l||_{\mathcal{H}^{\Psi_2}}^{lux} \right) \\
&\lesssim \sup_{||\chi_B l||_m = 1} \left(||h_b|| ||a_l||_m \sigma(B)^{1/m'} \rho(\sigma(B)) \right) \\
&\lesssim ||h_b|| \left(\sigma(B) \right)^{1/m'} \rho(\sigma(B)).
\end{aligned}
$$

It remains to show (34). It is clear that what we have to prove is that

$$(35) \qquad ||R(l)||_m \leq C||\chi_B l||_m.$$

Without loss of generality we can assume that $z_0 = (1, 0, \cdots, 0)$, so that the coordinates related to z_0 may be taken as the ordinary ones. Otherwise we use the action of the unitary group. In the local coordinates, the ball B becomes $Q(r) = \{z = (t, x) \in \mathbb{R}^{2n-1} = \mathbb{R} \times \mathbb{R}^{2n-2} : |t| + |x|^2 < r\}$, and the measure σ the Lebesgue measure in $\mathbb{R}^{2n-1}$. In these coordinates, $\mathcal{P}_N(B)$ is the space of polynomials of degree at most N with support in $Q(r)$. This is a closed subspace of the Hilbert space $L^2(Q(r), dz = dt dx)$ with finite dimension M. So if $\{P_j\}_{\{1 \leq j \leq M\}}$ is

an orthonormal basis of $\mathcal{P}_N(B)$, we have

$$R(l) = \sum_{j=1}^{M} \langle l, P_j \rangle P_j.$$

It follows that to prove (35), it is enough to prove that for some absolute constant C (independent of P and r)

$$(36) \qquad ||P||_m ||P||_{m'} \leq C ||P||_2^2,$$

for any polynomial $P \in \mathcal{P}_N(B)$. The inequality (36) follows from the fact that there exist constants A and B depending only on N and n such that, for any polynomial $P = \sum_{|\alpha| \leq N} c_\alpha z^\alpha$

$$(37) \qquad A \int_{Q(1)} |P(z)| dz \leq \sum_{|\alpha| \leq N} |c_\alpha| \leq B \int_{Q(1)} |P(z)| dz.$$

Indeed, (37) clearly shows that for any $m \geq 1$,

$$\left(\int_{Q(1)} |P(z)|^m dz \right)^{1/m} \simeq \int_{Q(1)} |P(z)| dz$$

and the desired result then follows from the fact $\mathcal{P}_N(B)$ is stable under dilation and translation. This ends the proof of the theorem. $\qquad \square$

We can observe that in the proof of this theorem, the condition (ii) is used to ensure that the resulting growth function Φ is in some $\mathcal{L}_p$. Hence using Lemma 2.5, we have the following proposition.

PROPOSITION 3.11. *Let* Φ_1 *and* Φ_2 *in* $\mathscr{U}^q$, *and* $\rho_i(t) = \frac{1}{t \Phi_i^{-1}(1/t)}$.
We suppose that

    (i) Φ_2 *satisfies the Dini condition (9)*
    (ii) $\frac{\Phi_2^{-1} \circ \Phi_1(t)}{t}$ *is non-increasing.*

Then the Hankel operator h_b *extends into a bounded operator from* $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ *into* $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ *if and only if its symbol* b *belongs to* $BMOA(\rho_\Phi) = (\mathcal{H}^{\Phi}(\mathbb{B}^n))^*$, *where*

$$\rho_\Phi := \frac{\rho_1}{\rho_2}.$$

4. BOUNDEDNESS OF h_b: $\mathcal{H}^{\Phi_1}(\mathbb{B}^n) \to \mathcal{H}^{\Phi_2}(\mathbb{B}^n)$; $(\Phi_1, \Phi_2) \in \mathscr{U}^q \times \mathscr{L}_p$

Let us begin this section by recalling the definition of the admissible maximal function $\mathcal{M}(f)$ of a holomorphic function f. For $\xi \in \mathbb{S}^n$

$$\mathcal{M}(f)(\xi) = \sup\{|f(z)| : z \in \mathbb{B}^n, |1 - \langle \xi, z \rangle| < 1 - |z|^2\}.$$

We recall that $\mathcal{H}^1_{weak}(\mathbb{B}^n)$ consists of functions $f \in \mathcal{H}(\mathbb{B}^n)$ such that,

$$\lambda \sigma \left(\{\xi \in \mathbb{S}^n : \mathcal{M}(f)(\xi) > \lambda\} \right) \leq C \text{ for any } \lambda > 0.$$

The following result is well known.

PROPOSITION 4.1. *Let $\Phi \in \mathscr{L}_p$. Suppose that Φ satisfies the Dini's condition*

(38) $$\int_1^\infty \frac{\Phi(t)}{t^2}dt \leq C < \infty.$$

Then $\mathcal{H}^1_{weak}(\mathbb{B}^n)$ embeds continuously in $\mathcal{H}^\Phi(\mathbb{B}^n)$

Proof. It is enough to prove that for any $f \in \mathcal{H}^1_{weak}(\mathbb{B}^n)$,

$$\int_{\mathbb{S}^n} \Phi(\mathcal{M}(f)(\xi))d\sigma(\xi) \leq C.$$

$$\int_{\mathbb{S}^n} \Phi(\mathcal{M}(f)(\xi))d\sigma(\xi) = \int_0^\infty \sigma\left(\{\xi \in \mathbb{S}^n : \mathcal{M}(f)(\xi) > \lambda\}\right)\Phi'(\lambda)d\lambda$$
$$= I + J.$$

Where

$$I = \int_0^1 \sigma\left(\{\xi \in \mathbb{S}^n : \mathcal{M}(f)(\xi) > \lambda\}\right)\Phi'(\lambda)d\lambda$$

and

$$J = \int_1^\infty \sigma\left(\{\xi \in \mathbb{S}^n : \mathcal{M}(f)(\xi) > \lambda\}\right)\Phi'(\lambda)d\lambda.$$

Clearly,

$$I \leq \sigma(\mathbb{S}^n)\int_0^1 \Phi'(\lambda)d\lambda = C.$$

To estimate the integral J, we use the definition of $\mathcal{H}^1_{weak}(\mathbb{B}^n)$, the fact that $\Phi'(t) \simeq \frac{\Phi(t)}{t}$ and that Φ satisfies the Dini's condition (38) to obtain

$$J = \int_1^\infty |\{\xi \in \mathbb{S}^n : \mathcal{M}(f)(\xi) > \lambda\}|\Phi'(\lambda)d\lambda$$
$$\leq C\int_1^\infty \frac{\Phi'(\lambda)}{\lambda}d\lambda$$
$$\simeq C\int_1^\infty \frac{\Phi(\lambda)}{\lambda^2}d\lambda \leq C < \infty.$$

The proof is complete. □

We next prove a result which generalizes the case $h_b : \mathcal{H}^p(\mathbb{B}^n) \to \mathcal{H}^q(\mathbb{B}^n)$ with $1 \leq p < \infty$ and $0 < q < 1$.

THEOREM 4.2. *Let $\Phi_1 \in \mathscr{U}^q$ and $\Phi_2 \in \mathscr{L}_p$. Let Ψ_1 be the complementary function of Φ_1 and, suppose that Φ_1 satisfies the Dini's condition (9) while Φ_2 satisfies (38). Then h_b extends as a bounded operator from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ to $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ if and only if $b \in \mathcal{H}^{\Psi_1}(\mathbb{B}^n)$.*

Proof. Let us begin by proving the necessity. Suppose that h_b is bounded from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ to $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$. Then for any $f \in \mathcal{H}^{\Phi_1}(\mathbb{B}^n)$, we have

$$\left| \int_{\mathbb{S}^n} b(\xi) \overline{f(\xi)} d\sigma(\xi) \right| = |h_b f(0)| \leq C \|h_b(f)\|_{\mathcal{H}^{\Phi_2}} \leq C \|h_b\| \|f\|_{\mathcal{H}^{\Phi_1}}.$$

We have used the fact that $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ is continuously contained in $\mathcal{H}^p(\mathbb{B}^n)$ (for some $p > 0$), and the evaluation at 0 is bounded on this space. It follows that b belongs to the dual space of $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ that is $b \in \mathcal{H}^{\Psi_1}(\mathbb{B}^n)$.

Conversely, if $b \in \mathcal{H}^{\Psi_1}(\mathbb{B}^n)$, then for any $f \in \mathcal{H}^{\Phi_1}(\mathbb{B}^n)$, the product $b\overline{f}$ is in $L^1(\mathbb{S}^n)$ by Proposition 1.1. Thus, $h_b(f) := P(b\overline{f})$ is in $\mathcal{H}^1_{weak}(\mathbb{B}^n)$ and consequently in $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ by Proposition 4.1. The proof is complete. $\qquad\square$

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