APPLICATION OF THE COMBINED INTEGRAL METHOD TO STEFAN PROBLEMS

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Abstract. In this paper we present a new, accurate form of the heat balance integral method, termed the Combined Integral Method (or CIM). The application of this method to Stefan problems is discussed. For simple test cases the results are compared with exact and asymptotic limits. In particular, it is shown that the CIM is more accurate than the second order, large Stefan number, perturbation solution for a wide range of Stefan numbers. In the initial examples it is shown that the CIM reduces the standard problem, consisting of a PDE defined over a domain specified by an ODE, to the solution of one or two algebraic equations. The latter examples, where the boundary temperature varies with time, reduce to a set of three first order ODEs.

1. Introduction

The heat balance integral method (HBIM) is a well-known method for finding approximate solutions to thermal problems [6]. It has proved particularly valuable in the solution of Stefan problems, where few analytical solutions exist. However, it has also been applied to problems such as thermal explosions, the Korteweg-de-Vries equation, microwave heating of grain and re-wetting of surfaces [2, 19, 24, 25]. Given that it is really an application of Karman-Pohlhausen integral method [20] for analysing boundary layers in fluid flow, it is also suitable for certain problems in viscous flow, see [18, 23].

The popularity of the HBIM is due primarily to its simplicity. However, the method has various well-known drawbacks [14, 16, 17]. For example, the choice of approximating function is arbitrary and this is key to the method's accuracy. To compound the ambiguity there are often different ways to formulate even the most basic problem, see [14], and this also affects the accuracy. Furthermore, there is no agreed method for measuring the accuracy and so many authors limit their studies to test problems with either exact or numerical solutions. Indeed, they often motivate the approximating function through an exact or numerical solution, which is then typically used to check the accuracy of the approximate method. One of the objections raised in [16, 17] is that the existence of an exact or numerical solution makes the approximate solution redundant.

Key words and phrases. Heat Balance Integral Method; Stefan problems; Phase change; Perturbation solution.

Recently Myers [16] has addressed the issue of the choice of approximating function as well as measuring the accuracy. The new method can be illuminated through the simple one-dimensional thermal problem specified by $u_t = u_{xx}$ and subject to $u(0,t) = 1$, $u(\infty,t) = u(x,0) = 0$. The standard HBIM proceeds by defining a polynomial approximating function

$$u = \left(1 - \frac{x}{\delta}\right)^n \ . \tag{1}$$

The heat penetration depth, $\delta(t)$, indicates the point where the temperature rise above the initial value is negligible and the standard approach of Goodman uses $n = 2$, although there are examples with $n = 3, 4, 7$ in the literature [2, 14, 13]. The feature that distinguishes the method of [16] from previous authors is that n is not specified *a priori* or chosen by comparison with exact solutions or boundary conditions. Instead it is left arbitrary and another equation is then required to close the system. The extra equation comes from minimising a least-squares error

$$E_n = \int_0^\delta (u_t - u_{xx})^2 \, \mathrm{d}x = \int_0^\delta f(x,t)^2 \, \mathrm{d}x \ . \tag{2}$$

If u is an exact solution of the heat equation then $E_n = 0$, otherwise $E_n > 0$ [11]. The value of E_n also provides a measure of the *relative* accuracy of the method, without the need for an exact solution. We emphasise the word relative since this criteria is usually time-dependent and for small times may indicate high errors even when the approximation appears accurate. However, it will demonstrate whether one approximation is more accurate than another. Minimising E_n to determine n means that n is found through a global constraint, *i.e.* the choice is based on providing an accurate solution over the whole domain, as opposed to matching the solution at a single point. Note, one could equally well deal with an error $E_n = \int_0^\delta |f(x,t)| \, dx$. This reduces the algebra associated with squaring the expression for f, but then one has to determine where f is positive and negative and split up the integration accordingly.

For the thermal problem discussed above, this method leads to $n \approx 2.2$ and the improvement from Goodmans standard form with $n = 2$ is slight (where the improvement is based on the values of E_n for $n = 2$ and 2.2). However, when the condition $u(0,t) = 1$ is replaced by a constant flux boundary condition, $u_x(0,t) = -1$, E_n is minimised when $n \approx 3.6$ and the error is reduced by a factor of 30 from that obtained with $n = 2$. The method of [16] was extended to Stefan problems in [17]. For melting with a constant temperature boundary condition the error in the growth rate compared to taking $n = 2$ was reduced by a factor of 130 (when compared with the exact solution).

In general the method introduced in [16, 17] will lead to the value of n which produces the lowest error according to the criteria of equation (2). In those papers the appropriate value of n for different scenarios was given. Hence the solution is as simple as that of Goodman's to apply. For example, in the problem discussed

above the only change is to set $n = 2.2$ in equation (1), while for constant flux and Newton cooling thermal problems setting $n = 3.6$ will significantly improve results. Similarly, for Stefan problems the appropriate values of n and approximating function are summarised in [17]. However, in these papers it was also pointed out that there exist difficulties when the optimal choice of n depends on time. These typically occur with time-dependent boundary conditions. Theoretically the method can deal with $n = n(t)$ but the algebra becomes extremely cumbersome (and so removes the basic appeal of the HBIM). To overcome this drawback Mitchell & Myers [14] recently developed an alternative approach that deals more easily with $n(t)$.

An extension to the HBIM, known as the Refined Integral Method (RIM), was presented by Sadoun & Si-Ahmed [21]. It simply involves integrating the heat equation twice. In fact, this method has appeared at least twice before these authors published their results. Whilst analysing an oxygen diffusion problem Gupta & Banik [7, 8, 9] developed a semi-analytical method, denoted the *constrained integral method*. They employed a polynomial of even degree with four unknown functions. After the application of boundary conditions these reduced to two unknowns which were then found from applying what they termed the zeroth and first moments: these are precisely the HBIM and RIM formulations respectively. Hill's book [10] terms the RIM the *integral formulation by integration*. He also extends the method to provide an iterative scheme. For a Stefan problem with a constant temperature boundary he demonstrates that the growth rate is more accurately captured by the iterative method than by HBIM or RIM solutions. The most accurate HBIM or RIM formulation depends on the value of the Stefan number (in agreement with the conclusions of [14]).

For certain situations the RIM can improve on the HBIM, but there is no set rule on when it will be the best method. The relative merits of each method are discussed in [14]. The approach adopted by Mitchell & Myers [12] was to use the HBIM with an unknown exponent, as in [16, 17], and then use RIM to give the extra equation to determine n. Again this is a global constraint rather than one based on matching at a point. In general it results in a similar level of accuracy as that of [16, 17] but the algebra to determine n is simpler. Whilst there is no check on the error, as occurs when the analysis is based on minimising E_n, this new method will always be as accurate as the better of the HBIM and RIM solutions and will usually be better than both. For the examples investigated in [12] the error was of a similar magnitude to that obtained in [16, 17]. As will be seen later it also removes some of the ambiguity in the formulation. This is a similar approach to the constrained integral method considered by [7, 8, 9]. In their case they employed the extra RIM equation to permit an extra term (and hence unknown coefficient) in the temperature expression. In our case we use it to determine the unknown exponent.

In [12] only thermal problems were investigated; in the following work we will extend this new integral method, hereafter referred to as the Combined Integral

Method (CIM), to deal with Stefan problems. We will also investigate the errors by comparison with exact, limiting and numerical solutions. In particular we will show that, for the cases examined, the CIM is more accurate than a second order perturbation solution for a wide range of Stefan numbers.

For the Stefan problems considered in the following sections there are two basic types of approximating profile. In a semi-infinite subcooled solid where s denotes the position of the moving front and $s < x$, $u \to -1$ as $x \to \infty$ the temperature is specified by

$$u = -1 + \left(\frac{\delta - x}{\delta - s}\right)^n \ . \tag{3}$$

This satisfies $u(s,t) = 0$ and $u(\delta,t) = -1$, where s, δ are unknown functions of time, and n may be a function of time depending on the problem. In a melt region a typical approximation is

$$u = a\left(1 - \frac{x}{s}\right) + (1-a)\left(1 - \frac{x}{s}\right)^m \ , \tag{4}$$

where in this case $u(0,t) = 1$ and $u(s,t) = 0$. The linear term is required otherwise $u_x(s,t) = 0$ for $m > 1$ and, at least for a material initially at the melting temperature, without this term the front would remain stationary for all time. For certain problems the values of a and m may vary with time. However, to simplify the analysis we will always first assume that they are constant, and only if this assumption turns out to be inconsistent with the solutions obtained will we look for varying values.

2. One phase semi-infinite material at melting temperature

Consider a semi-infinite material which is everywhere at the melting temperature. At the boundary $x = 0$ a heat source is applied which results in melting. The position of the melt front is denoted $x = s(t)$. The basic one-dimensional problem is specified by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ , \qquad 0 < x < s(t) \tag{5}$$

$$u(0,t) = 1 \ , \qquad u(s,t) = 0 \ , \qquad \beta \frac{\mathrm{d}s}{\mathrm{d}t} = -\left.\frac{\partial u}{\partial x}\right|_{x=s} \ , \qquad s(0) = 0 \ . \tag{6}$$

This is scaled in such a way that the only remaining parameter is the Stefan number $\beta = L_m/(c_p\ \Delta u)$, where L_m is the latent heat of melting, c_p the specific heat capacity and Δu the temperature variation.

The system (5, 6) has the exact solution

$$u(x,t) = 1 - \frac{\mathrm{erf}\big[x/(2\sqrt{t})\big]}{\mathrm{erf}(\alpha)} \ , \qquad s(t) = 2\alpha\sqrt{t} \ , \tag{7}$$

where u is the temperature in the melt and α satisfies the transcendental equation

$$\sqrt{\pi}\beta\alpha\,\mathrm{erf}(\alpha)e^{\alpha^2} = 1\ . \tag{8}$$

To apply the integral methods we approximate the temperature with the form (4). For the HBIM the heat equation is integrated over $x \in [0, s]$ to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^s u\,\mathrm{d}x = \left.\frac{\partial u}{\partial x}\right|_{x=s} - \left.\frac{\partial u}{\partial x}\right|_{x=0}\ . \tag{9}$$

Further details may be found in [14, 17]. As discussed in the introduction it is assumed that a, m are constant and so substituting for u from (4) gives

$$\frac{a(m+1)+2(1-a)}{2(m+1)}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{m(1-a)}{s}\ . \tag{10}$$

The Stefan condition (6b) becomes

$$\beta\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{a}{s}\ , \tag{11}$$

hence $s = \sqrt{2at/\beta}$. Obviously, since $t, \beta \geq 0$, and to ensure that the front velocity is positive, we require $a > 0$.

With the RIM there is a choice between taking the first integral over $x \in [0, \xi]$ or $[\xi, s]$ and then taking the second for $\xi \in [0, s]$. The first choice leads to

$$s\frac{\mathrm{d}}{\mathrm{d}t}\int_0^s u(x,t)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^s xu(x,t)\,\mathrm{d}x = -1 - s\left.\frac{\partial u}{\partial x}\right|_{x=0}\ , \tag{12}$$

while the second gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^s xu(x,t)\,\mathrm{d}x = 1 + s\left.\frac{\partial u}{\partial x}\right|_{x=s}\ . \tag{13}$$

This demonstrates a typical ambiguity with the standard method. There are now two possible formulations which will exhibit different levels of accuracy, as examined in [14]. However, with the current method the HBIM and RIM formulations are combined and equation (13) can be obtained from (12) by substituting for $u_x(0,t)$ via equation (9). Hence solving equations (9) and (12) or (9) and (13) is equivalent and this particular ambiguity is removed. From now on we will deal with equation (13) which involves less integration. It is also directly what is termed the first moment [7, 8, 9], as discussed in the introduction: multiplying both sides of the heat equation (5) by x and integrating over $x \in (0, s)$ gives precisely (13).

Substituting for u from equation (4) the RIM formulation leads to

$$\frac{a(m+1)(m+2)+6(1-a)}{3(m+1)(m+2)}s\frac{\mathrm{d}s}{\mathrm{d}t} = 1 - a\ . \tag{14}$$

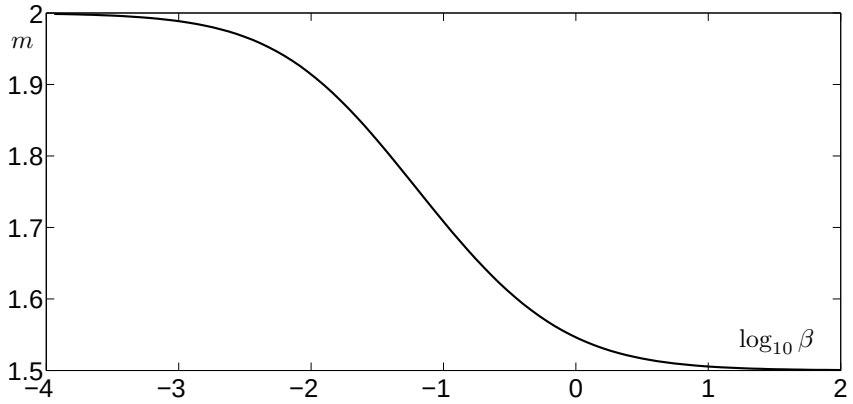

FIGURE 1. *Variation of m with β.*

The key to the CIM is equating the RIM and HBIM formulations, therefore we equate (10) and (14) to remove ss_t and determine a relation between a and m

$$a = \frac{6(2-m)}{(m-1)(2m^2+5m-6)} \ . \tag{15}$$

The Stefan condition (11) may be used to eliminate ss_t in either of (10) or (14) and then combining with (15) results in a nonlinear equation for m which may be written in the form

$$2m(m+1)\left[\beta(2m-3)(2+m)(1+m)(2m^2+5m-6)+12(m-2)\right] = 0 \ . \tag{16}$$

Hence the problem of solving the PDE (5) over a moving domain specified by (6b) is reduced to solving an algebraic equation for m, with the Stefan number as the only parameter. Once m is known the value of a is given by equation (15) and then the position of the front through $s = \sqrt{2at/\beta}$.

To simplify the calculations we may neglect the physically unrealistic solutions of equation (16) where $m = -1, 0$. The approximate value for m is then found from the numerical solution of the quintic equation in square brackets in (16). As previously stated, we expect $m > 1$, in fact from equation (15) we can see that the requirement $a > 0$ restricts $m \in (1, 2)$. Finally, noting that $\beta > 0$ the quintic term of (16) then fixes $m \in [1.5, 2)$. The solution of the quintic equation is plotted in Figure 1. It is worth pointing out that when $\beta > 1$ the best value of m predicted by this method is close to 1.5. Most standard analyses using the HBIM assume $m = 2$, see [14] for example. From Figure 1 it is clear that only when $\beta \ll 1$ is this a good choice. Note, for water and paraffin wax $\beta \in [1, 10]$, for metals $\beta \in [0.1, 1]$ and it is even smaller for silicates [1, 10, 17].

A popular route to determine the approximate solution to Stefan problems, when $\beta \gg 1$, is to use a perturbation method. We now examine this and compare the accuracy of the CIM to that of the perturbation solution. In the limit $\beta \to \infty$

it is simple to show that the temperature profile becomes linear, $u \to 1 - x/s$ and then the Stefan condition gives $s \to \sqrt{2t/\beta}$. This is the leading order of the perturbation solution. Higher order corrections may be found by expanding in terms of the small parameter $1/\beta$, see [1, 10] for example. To determine the perturbation solution we must first re-scale time $t = \beta\tau$ so that

$$\frac{1}{\beta}\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \,, \qquad \frac{\mathrm{d}s}{\mathrm{d}\tau} = -\left.\frac{\partial u}{\partial x}\right|_{x=s} \,. \tag{17}$$

To determine a correction to the leading order linear form a straightforward expansion on (17), namely $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 \cdots$, where $\epsilon = 1/\beta$, leads to a second order term involving $s_{\tau\tau}$. Since there is only one initial condition the problem of finding a second initial condition may be avoided by first setting $y = x - s(\tau)$. Then, since s is monotonic in τ, it is possible to work in terms of s as the time variable, i.e. $\tau = \tau(s)$. Denoting $u(x, \tau) = U(y, s)$ the heat equation becomes

$$\frac{\partial^2 U}{\partial y^2} = \epsilon \frac{\mathrm{d}s}{\mathrm{d}\tau}\left(\frac{\partial U}{\partial s} - \frac{\partial U}{\partial y}\right) = -\epsilon \left.\frac{\partial U}{\partial y}\right|_{y=0}\left(\frac{\partial U}{\partial s} - \frac{\partial U}{\partial y}\right) \,. \tag{18}$$

Now we take an expansion

$$U = U_0(y, s) + \epsilon U_1(y, s) + \epsilon^2 U_2(y, s) + \cdots \,, \tag{19}$$

subject to $U_0(-s, s) = 1$, $U_0(0, s) = 0$ and $U_1 = U_2 = 0$ at $y = -s, 0$. This leads to

$$U_0 = -\frac{y}{s} \,, \qquad U_1 = \left(\frac{y^3}{6s^3} + \frac{y^2}{2s^2} + \frac{y}{3s}\right) \tag{20}$$

$$U_2 = -\frac{y^5}{40s^5} - \frac{y^4}{8s^4} - \frac{5y^3}{18s^3} - \frac{y^2}{3s^2} - \frac{7y}{45s} \,. \tag{21}$$

The Stefan condition becomes

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = -\left.\frac{\partial U}{\partial y}\right|_{y=0} = \frac{1}{s} - \frac{\epsilon}{3s} + \frac{7\epsilon^2}{45s} \,. \tag{22}$$

Hence

$$s = \sqrt{2\tau\left(1 - \frac{\epsilon}{3} + \frac{7\epsilon^2}{45} + \cdots\right)} = \sqrt{\frac{2t}{\beta}}\left(1 - \frac{1}{6\beta} + \frac{23}{360\beta^2} + \cdots\right) \,. \tag{23}$$

Of course the perturbation solution must coincide with the expansion of the exact solution. We may see this by first noting that in the limit $\beta \to \infty$ the two sides of the transcendental equation, (8), balance provided $\alpha \to 0$ [5]. The small α expansion of (8) is

$$\beta\left[2\,\alpha^2 + \frac{4}{3}\,\alpha^4 + \frac{8}{15}\alpha^6 + O\left(\alpha^8\right)\right] = 1 \,. \tag{24}$$

The leading order solution is $\alpha_0 = 1/\sqrt{2\beta}$ which motivates the expansion

$$\alpha = \alpha^* = \frac{1}{\sqrt{2\beta}}\left(1 + \frac{\alpha_1}{\beta} + \frac{\alpha_2}{\beta^2} + \cdots\right) , \tag{25}$$

where α^* denotes an approximation to α. Substituting this back into (24) leads to $\alpha_1 = -1/6$, $\alpha_2 = 23/360$ and then using the expression for $s(t)$, equation (7b), reproduces (23).

We may compare this with the large β expansion of the CIM. Allowing $\beta \gg 1$ in equation (16) leads to $m = 3/2 + 6/(105\beta) - 1898/(128625\beta^2)$. This may be used to determine a from equation (15) and then the CIM gives

$$s \approx \sqrt{\frac{2t}{\beta}}\left(1 - \frac{1}{6\beta} + \frac{149}{2520\beta^2}\cdots\right) . \tag{26}$$

The CIM therefore agrees with the expansion of the exact solution up to first order, with a 7% error at second order.

A small β expansion of the exact solution may also be obtained (although beyond leading order the terms are more difficult to obtain)

$$2\alpha^2 = 2(\alpha^*)^2 = 2\ln\left[\sqrt{\frac{2}{\pi}}\left(1 + \frac{1}{\beta}\right)\right] - \ln\left\{2\ln\left[\sqrt{\frac{2}{\pi}}\left(1 + \frac{1}{\beta}\right)\right]\right\} , \tag{27}$$

further details may be found in [5, 10].

Since the exact and the combined integral methods both lead to solutions of the form $s \sim \sqrt{t}$ the error in s is independent of time and depends only on a constant factor. If we write the solution to the Stefan condition, equation (11) in the form of equation (7a), $s = 2\mu\sqrt{t}$ where $\mu = \sqrt{a/(2\beta)}$, then the difference between the exact and CIM solutions is reflected in the difference between α and μ: the former depends solely on β and the latter on β and m. A similar expression for the large β perturbation solution may be obtained from equation (23) and from (27) for small β. The percentage difference from the exact solution for the CIM and large and small β expansions ($100(\alpha - \mu)/\alpha$ or $100(\alpha - \alpha^*)/\alpha$) is displayed in Figure 2. For $\beta > 7$ the second order perturbation is the most accurate, however, the errors for either method are negligible (below 0.009%). For lower β values the perturbation solution quickly loses accuracy. For example, when $\beta = 1$ the perturbation error is 2.3% while the CIM error is 0.25%. Of the three solutions shown, the CIM remains the most accurate down to $\beta = 0.07$ where the small β solution applies. In practical terms, it is clear that the CIM is sufficiently accurate for most realistic applications.

In comparison to the most accurate HBIM formulation with $m = 2$ (as determined in the review of [14]) in general the CIM is also significantly more accurate. For example, when $\beta = 5$, the percentage error for the CIM is 0.0163% whilst for the HBIM it is 0.75%, *i.e. an improvement of a factor 46.* For $\beta = 1$ the accuracy is improved by a factor 10, whilst for $\beta = 0.1$ where we find $m \sim 2$ the errors are similar (although the CIM is still the most accurate).

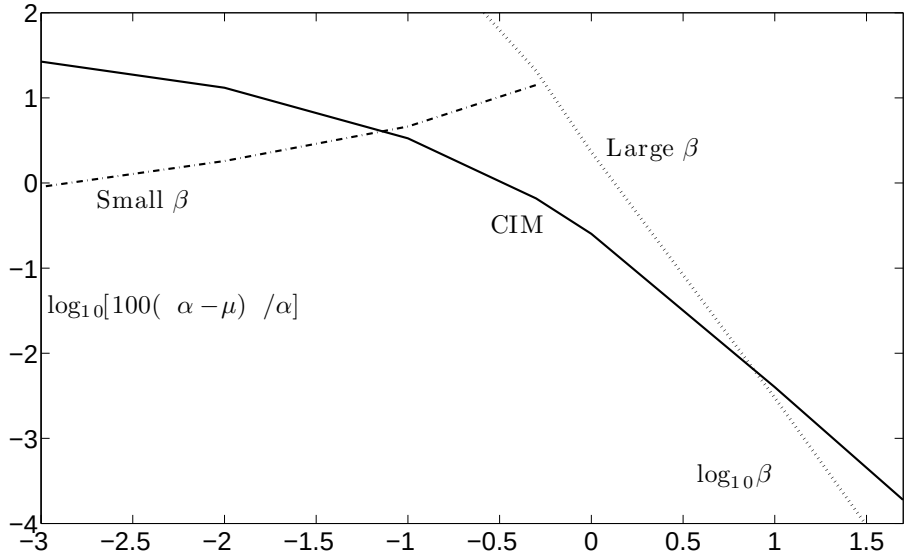

FIGURE 2. *Percentage difference between exact and CIM and large and small β expansions for varying β.*

3. Two phase semi-infinite, subcooled material

The basic two-phase problem considered in the following section is defined as

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\ , \qquad 0 < x < s \qquad\qquad \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}\ , \qquad x > s \tag{28}$$

$$u = v = 0 \qquad\qquad \beta\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\ , \qquad \text{at } x = s \tag{29}$$

$$u(0,t) = 1 \qquad v(x,0) = -1\ , \qquad v = -1 \text{ as } x \to \infty\ , \tag{30}$$

where we assume that the thermal properties remain constant through the phase change. This assumption is easily modified, see [14]. This set of equations describes the melting of a subcooled material, where the melting is driven by the temperature specified at $x = 0$. Since $u(0,t) > u(s,t) > v(\infty,t)$ the melt front moves with a positive velocity, $s_t > 0$. This system has the exact solution

$$u = 1 - \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)}{\operatorname{erf}\alpha}\ , \quad 0 < x < s\ , \qquad v = -1 + \frac{\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)}{\operatorname{erfc}\ \alpha}\ , \quad s < x < \infty\ , \tag{31}$$

where the moving front satisfies $s(t) = 2\alpha\sqrt{t}$ and α is determined using the Stefan condition (29b)

$$\beta\alpha\sqrt{\pi} = \frac{e^{-\alpha^2}}{\operatorname{erf}\alpha} - \frac{e^{-\alpha^2}}{\operatorname{erfc}\ \alpha}\ , \tag{32}$$

see [4].

In the melt region the problem is similar to that of §2, with the exception of the v_x term in the Stefan condition. Hence, for the HBIM the appropriate governing equation in the melt is equation (9). With the RIM, depending on the initial interval of integration, the governing equation is either (12) or (13); the two formulations are equivalent once coupled with (9). In the solid appropriate HBIM and RIM formulations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_s^\delta v\,\mathrm{d}x + \frac{\mathrm{d}\delta}{\mathrm{d}t} = -\left.\frac{\partial v}{\partial x}\right|_{x=s} \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_s^\delta xv(x,t)dx - s\frac{\mathrm{d}}{\mathrm{d}t}\int_s^\delta v dx + (\delta - s)\frac{\mathrm{d}\delta}{\mathrm{d}t} = 1\ . \tag{34}$$

The polynomial approximations in the two phases are given by equation (4) in the melt and (3) in the solid. With these profiles the Stefan condition may be written

$$\beta\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{a}{s} - \frac{n}{\delta - s}\ . \tag{35}$$

Since this melting problem has $s_t > 0$ and $\delta > s$ we require $a > 0$. Equation (35) obviously admits solutions of the form $s = 2\mu\sqrt{t}$, $\delta = 2\lambda\sqrt{t}$ and this leads to

$$(2\beta\mu^2 - a)(\lambda - \mu) = -n\mu\ . \tag{36}$$

Substituting for u from equation (4) the HBIM and RIM equations (9, 12) reduce to

$$\mu^2 = \frac{m(m+1)(1-a)}{a(m-1)+2}\ , \qquad \mu^2 = \frac{3(1-a)(2+m)(1+m)}{2\left[a(m+4)(m-1)+6\right]}\ . \tag{37}$$

Using the temperature profile (3) in the solid the HBIM and RIM equations (33, 34) are

$$n\mu + \lambda = \frac{n(n+1)}{2(\lambda-\mu)}\ , \qquad n\mu + 2\lambda = \frac{(n+1)(n+2)}{2(\lambda-\mu)}\ . \tag{38}$$

Of course, the analysis could stop here. The system of PDEs specified by equations (28) subject to the Stefan condition (29b) has now been reduced to solving a set of five algebraic equations, (36–38). However, root finding techniques are more reliable with fewer equations, consequently we rearrange equations (37) to find

$$a = \frac{6(2-m)}{(m-1)(2m^2+5m-6)}\ , \qquad \mu = \sqrt{\frac{(m+1)(m+2)(2m-3)}{4(m-1)}}\ , \tag{39}$$

where the positive square root is taken since $\mu > 0$ (to allow the melt front to move in the positive x direction). While equations (38) may be rearranged to

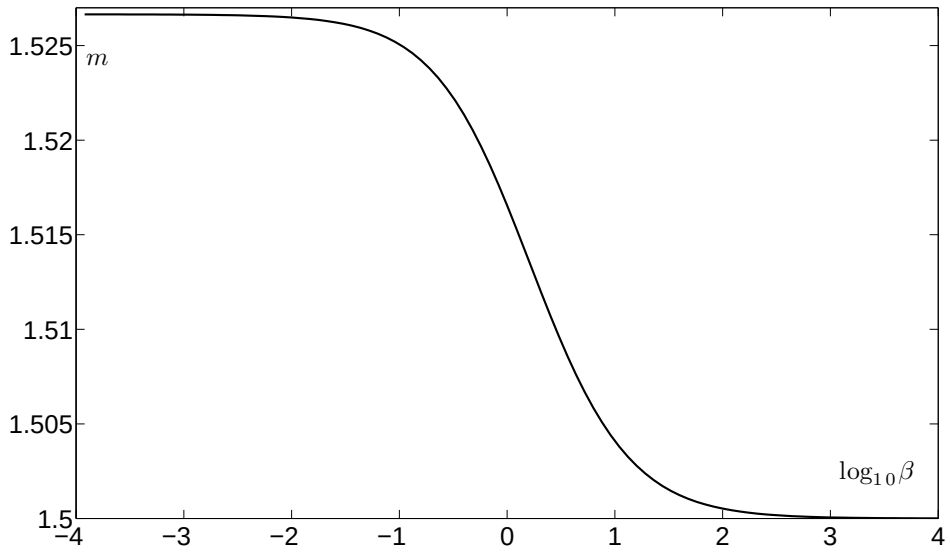

FIGURE 3. *Variation of m with β.*

give

$$\mu = (n-2)\sqrt{\frac{(n+1)}{2n(n+2)}}\ , \qquad \lambda = \sqrt{\frac{2n(n+1)}{(n+2)}}\ . \tag{40}$$

Equating the expressions for μ, (39b, 40a), gives a relation solely between m, n

$$2(n-2)^2(n+1)(m-1) = n(n+2)(m+1)(m+2)(2m-3)\ . \tag{41}$$

Substituting for μ, λ, a in equation (36) provides a second relation between m, n (and also shows the dependency on β). Hence the problem is now reduced to solving two simultaneous algebraic equations

$$2(n-2)^2(n+1)(m-1) = n(n+2)(m+1)(m+2)(2m-3) \tag{42}$$

$$\left[\frac{\beta(m+1)(m+2)(2m-3)}{2} - \frac{6(2-m)}{2m^2+5m-6}\right](n+2) = n(2-n)(m-1)\ . \tag{43}$$

As initial guesses for m, n we note that since $m, n > 1$ then from equation (39a) the requirement $a > 0$ determines $m < 2$, from (39b) $\mu > 0$ requires $m > 3/2$ while (40a) requires $n > 2$. Consequently the governing equations are solved with initial guesses $(m, n) = (1.75, 3)$.

The variation of m, n with β is shown in Figures 3, 4. From Figure 3 it is clear that m varies only slightly with β and in general $m \in [1.5, 1.527]$ (note, for the

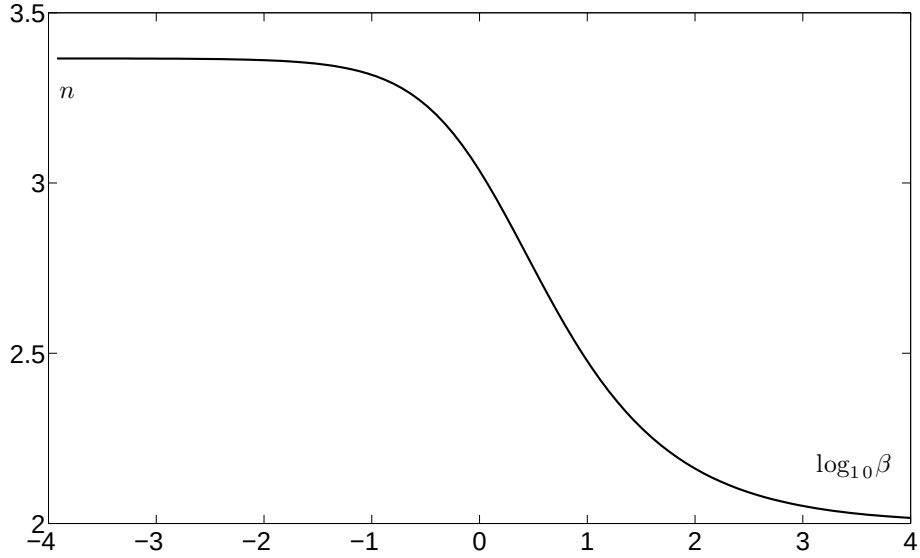

FIGURE 4. *Variation of n with β.*

one-phase problem $m \in [1.5, 2)$). Figure 4 shows that n has a greater variation $n \in [2, 3.37]$, and it also has a slower decay towards the large β limit.

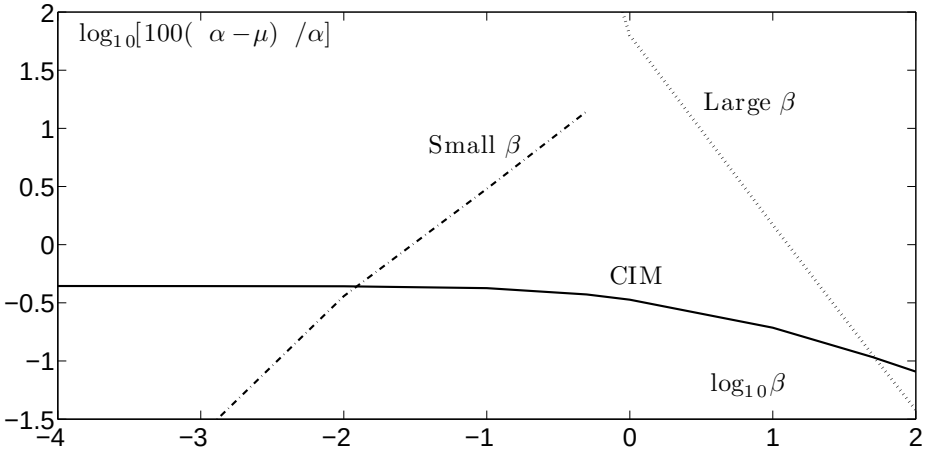

FIGURE 5. *Percentage difference between exact and CIM and large and small β expansion for varying β.*

As in §2 we now consider the behaviour as $\beta \to \infty$. To save time we do not show the perturbation method but merely write out the large β (small α)

expansion of (32) to second order,

$$\alpha = \sqrt{\frac{1}{2\beta}}\left(1 - \frac{1}{\sqrt{2\pi\beta}} - \frac{(9+2\pi)}{12\pi\beta} + \cdots\right) . \tag{44}$$

As $\beta \to \infty$ obviously $\alpha \to 0$, that is the boundary velocity tends to zero. We therefore expect the equivalent CIM term, μ, to also tend to zero. Equations (39b,40a) therefore motivate expansions with leading order terms $m = 3/2$ and $n = 2$. At higher order we find

$$m = \frac{3}{2} + \frac{2}{35\beta} + \mathcal{O}(\beta^{-3/2}) , \qquad n = 2 + 2\sqrt{\frac{2}{3\beta}} - \frac{1}{9\beta} + \mathcal{O}(\beta^{-3/2}) . \tag{45}$$

Note that, unlike the n expansion, the expression for m has no term involving $\beta^{-1/2}$. This is in keeping with the numerical results shown in Figures 3, 4 where m decays much more rapidly as $\beta \to \infty$ than n. Substituting the expressions for m, n into the expressions for μ, λ then gives

$$\mu = \frac{1}{\sqrt{2\beta}}\left(1 - \frac{1}{\sqrt{6\beta}} + \mathcal{O}(\beta^{-1})\right) , \qquad \lambda = \sqrt{3} + 7\frac{\sqrt{2}}{12\beta} + \mathcal{O}(\beta^{-3/2}) . \tag{46}$$

The expression for μ differs from that of α at first order ($\sqrt{6}$ as compared to $\sqrt{2\pi}$ or a 2.3% difference), so we would expect reasonably accurate results (but not as accurate as the one-phase problem where the difference occurred at second order). The leading order expression for λ is independent of β, reflecting the fact that even when the melting occurs very slowly the heat is still transmitted through the solid. Again we can also examine a small β limit. In this case, if we let $\alpha \to \infty$ in the transcendental equation (32) we find the dominant balance is incorrect (the erfc term tends to minus infinity and cannot balance with the positive left hand side) and we are forced to balance the two error function terms. This must be solved numerically to give $\alpha \approx 0.4769$. In Figure 5 we compare the accuracy of the CIM and large and small β solutions against the exact solution for α. As in the previous example the CIM is the most accurate over a wide range of β. In this case, only when $\beta > 51.5$ does the large β expansion improve on the CIM accuracy and by then the error is below 0.1%. In contrast to the example of §2 the CIM retains excellent accuracy even for small β. The solution for $\beta \to 0$ is most accurate for $\beta < 0.012$. Below this value the CIM error remains approximately constant at around 0.44%.

To demonstrate more clearly the accuracy of the CIM in Figure 6 we present temperature profiles corresponding to $\beta = 1$ respectively, at times $t = 0.1, 1, 5$. Obviously the agreement between the exact solution (solid line) and CIM solution (dashed line) is excellent at all times.

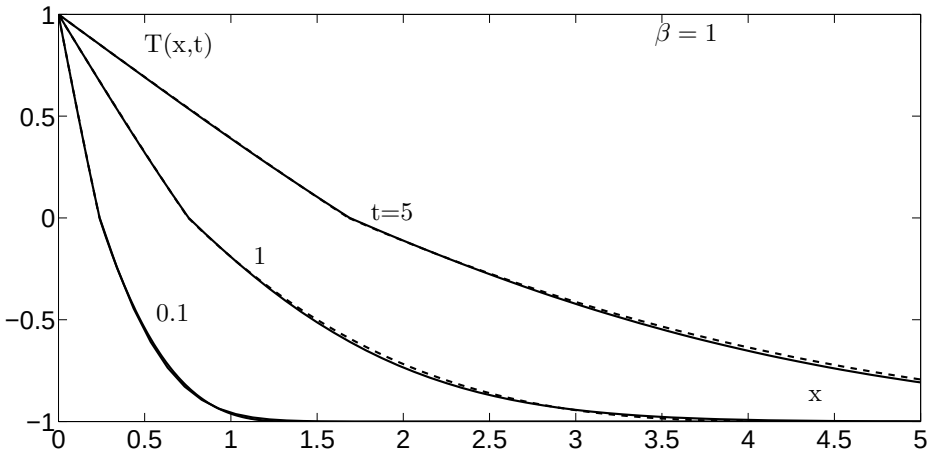

FIGURE 6. *Temperature at* $t = 0.1, 1, 5$ *when* $\beta = 0.1$.

4. One phase semi-infinite material with a constant flux boundary condition

In §2 we studied the one-phase problem with a constant temperature boundary condition. In the following analysis we deal with a similar problem, the difference being that the boundary condition is modified to one of constant flux, that is, we replace $u(0,t) = 1$ with

$$\frac{\partial u}{\partial x} = -1 \ , \qquad \text{at } x = 0 \ . \tag{47}$$

The cases examined in the previous two sections led to approximating polynomials with constant exponent m. However, in [16, 17] a number of examples were shown where $m = m(t)$. This made the analysis too complicated for practical use, so the author worked with the constant value $m(0)$. In the following we will demonstrate that the CIM deals more easily with such cases. As well as being an example with varying m, this case also has no exact analytical solution and so is well suited to approximate techniques such as the CIM.

The approximating profile for u is similar to (4) and given by

$$u = a\left(1 - \frac{x}{s}\right) - \frac{a-s}{m}\left(1 - \frac{x}{s}\right)^m \ , \tag{48}$$

which satisfies $u(s,t) = 0$ and (47). After substituting into the HBIM and RIM formulations we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{as}{2} - \frac{(a-s)s}{m(m+1)}\right] = \frac{s-a}{s} \tag{49}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{as^2}{6} - \frac{(a-s)s^2}{m(m+1)(m+2)}\right] = \frac{s-a}{m} \ , \tag{50}$$

and these must be coupled to the Stefan condition (11).

The problem has now been reduced to solving three ODEs for a, s and m. In the previous examples we assumed m to be constant and then $a = a(m)$ was also constant. Eliminating s from the equations led to an algebraic relation involving only β, *i.e.* $m = m(\beta)$. In the present case if we set a, m to be constant a similar elimination process results in $m = m(s_t)$ and it is therefore only independent of time if $s \propto t$. (As will be seen later, the small time solution has $s \propto t$ and then m is constant at leading order). If we attempt a solution of the form $s = Ct$ then either of (49) or (50) leads to $m = m(t)$ and so we are forced to treat a and m as time-dependent. This requires further initial conditions. Since $s(0) = 0$ it follows from (48) that $a(0) = 0$ to ensure $u(x, 0) = 0$.

We may determine the initial condition for m in a number of ways. A formal expansion of the governing equations for small t may be found in Appendix 1. However, we may also exploit our knowledge of the solution behaviour. For small time, both a and $s \to 0$, while m tends to a non-zero constant. So, for small t we let $m = m_0$, $s = s_0 t^p$. Equation (11) then shows $a = \beta s_0^2 p t^{2p-1}$ and equation (49) becomes

$$\frac{d}{dt}\left[\frac{\beta s_0^3 p t^{3p-1}}{2} + \frac{s_0^2 t^{2p} - \beta s_0^3 p t^{3p-1}}{m_0(m_0+1)}\right] = 1 - \beta p s_0 t^{p-1} \ , \tag{51}$$

or

$$\frac{\beta s_0^3 p(3p-1)t^{3p-2}}{2} + \frac{s_0^2 2pt^{2p-1} - \beta s_0^3 p(3p-1)t^{3p-2}}{m_0(m_0+1)} = 1 - \beta p s_0 t^{p-1} \ . \tag{52}$$

The only physically sensible balance is between the two terms on the right hand side, giving $p = 1$ and $s(0) \to 0$. There are other balances between terms, namely, $p = -1, 1/2, 1/3$: the first gives $s(0) \to \infty$ the other two lead to the final term on the right hand side, t^{p-1}, dominating and so must be discounted. Setting $p = 1$ and balancing the two terms on the right hand side requires $s_0 = 1/\beta$, and hence $a = t/\beta$. To find higher order terms the leading order solutions motivate the expansion

$$s = \frac{t}{\beta} + s_1 t^2 + s_2 t^3 \ , \qquad a = \frac{t}{\beta} + 3s_1 t^2 + (2\beta s_1^2 + 4s_2)t^3 \ , \tag{53}$$

$$m = m_0 + m_1 t + m_2 t^2 \ . \tag{54}$$

Substituting into (49) and (50) leads to

$$s = \frac{t}{\beta}\left(1 - \frac{t}{2\beta^2} + \frac{5t^2}{6\beta^4} + \cdots\right) \tag{55}$$

$$a = \frac{t}{\beta}\left(1 - \frac{3t}{2\beta^2} + \frac{23t^2}{6\beta^4} + \cdots\right) \tag{56}$$

$$m = 2 + \frac{t^2}{3\beta^4} + \cdots \ . \tag{57}$$

Hence, as $t \to 0$ our initial conditions are determined by $s(0) = a(0) \to t/\beta, m \to 2$. These values are in agreement with the more formal analysis of Appendix 1.

Since there is no exact solution to this problem we now consider the large β perturbation, as in §2. We follow the analysis of §2 and first re-scale time $t = \beta\tau$ and denote $u(x,\tau) = U(y,s)$ to obtain (18). With the expansion for U given by (19) subject to the boundary conditions

$$\frac{\partial U_0}{\partial y} = -1 \ , \qquad \frac{\partial U_1}{\partial y} = \frac{\partial U_2}{\partial y} = 0 \ , \tag{58}$$

at $y = -s$ and $U_0 = U_1 = U_2 = 0$ at $y = 0$ we find

$$U_0 = -y \ , \qquad U_1 = \frac{1}{2}y^2 + sy \ , \qquad U_2 = -sy^2 - 2s^2 y \ . \tag{59}$$

Substituting these into the Stefan condition gives

$$s = \tau\left(1 - \frac{1}{2}\epsilon\tau + \frac{5}{6}\epsilon^2\tau^2 + \cdots\right) = \frac{t}{\beta}\left(1 - \frac{t}{2\beta^2} + \frac{5t^2}{6\beta^4} + \cdots\right) \ . \tag{60}$$

Note, since t appears solely in the form t/β the large β expansion is equivalent to the small t expansion and so the large β expansion of the CIM solution is identical to equation (55).

Figure 7 shows plots of m and a against t for various values of β. We see that $m \to 2$ as $t \to 0$ or $\beta \to \infty$, while $a \to 0$ as $t \to 0$ and $a \propto t$ as $\beta \to \infty$ which confirms the asymptotic results in (55) and (56). In the left plot in Figure 8 we show s against t for various β. The solid line is the CIM solution, whilst the perturbation solution of (60) is the dotted line. For the two highest values of β the agreement with the CIM solution is very close and the curves are virtually indistinguishable. However, at $\beta = 1$ the perturbation solution clearly breaks down after a very short time. The right plot in Figure 8 gives the error in s against β at $t = 1$ between the asymptotic and CIM solutions when compared to a numerical solution, similar to that described in Mitchell & Vynnycky [15]. In this case, only when $\beta > 24$ does the large β expansion improve on the CIM accuracy and by then the error is below 0.001%. This again demonstrates that CIM is more accurate over a wide range of β. Note we cut off the solutions for $\beta > 7$ due to numerical difficulties.

5. Time-dependent boundary condition

Finally we highlight the application of the CIM to a problem with an explicit time-dependence at the boundary $x = 0$. Consider the one-phase problem of §2 but with a time dependent boundary temperature $u(0,t) = f(t)$. The approximating profile for u is now

$$u = a\left(1 - \frac{x}{s}\right) + (f - a)\left(1 - \frac{x}{s}\right)^m \ . \tag{61}$$

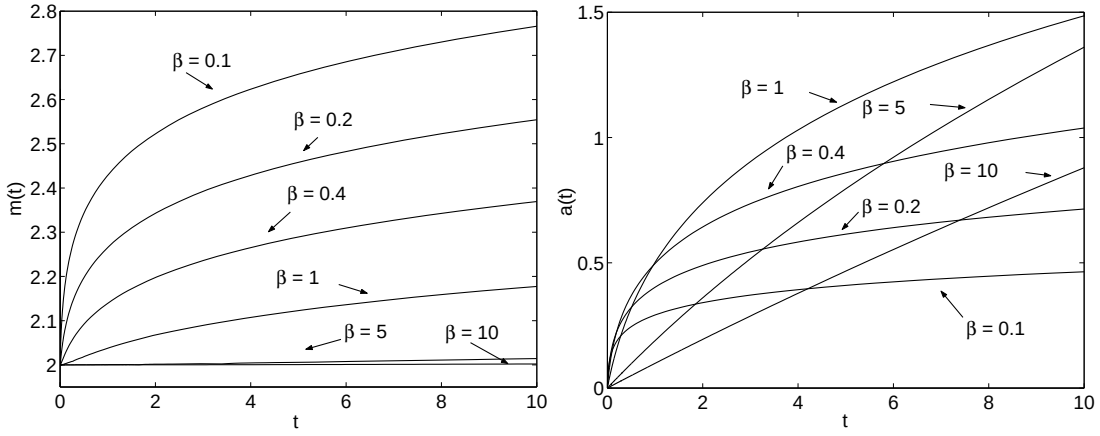

FIGURE 7. *$m(t)$ and $a(t)$ for the constant flux boundary condition.*

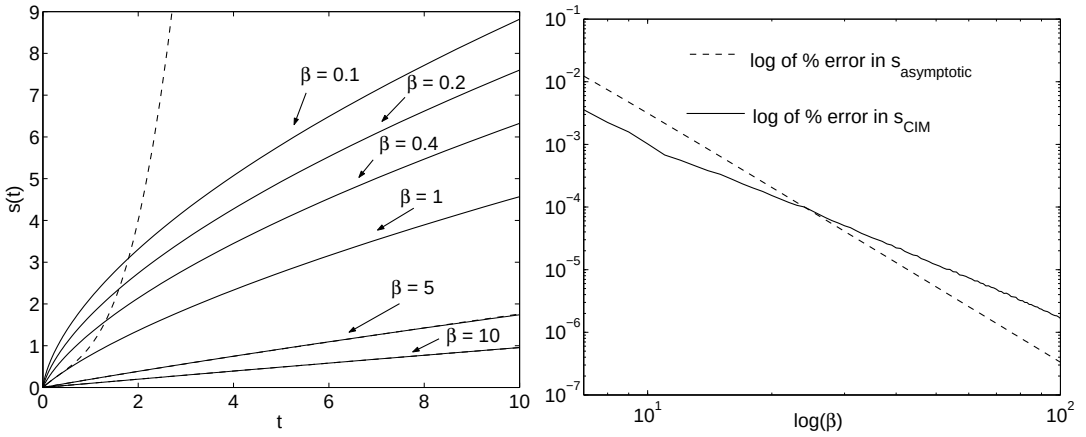

FIGURE 8. *$s(t)$ and the error in $s(1)$ against β for the constant flux boundary condition.*

The HBIM and RIM formulations are given by equation (9) and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^s xu\,\mathrm{d}x = f(t) + s\frac{\partial u}{\partial x}\bigg|_{x=s} . \tag{62}$$

Substituting the profile u from (61) into these leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{as}{2} + \frac{(f-a)s}{(m+1)}\right] = \frac{m(f-a)}{s} \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{as^2}{6} + \frac{(f-a)s^2}{(m+1)(m+2)}\right] = f - a\ , \tag{64}$$

or

$$\big[(m-1)a+2f\big]\frac{\mathrm{d}s}{\mathrm{d}t}+(m-1)s\frac{\mathrm{d}a}{\mathrm{d}t}+\frac{2s(a-f)}{m+1}\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$=\frac{2m(m+1)(f-a)}{s}-2s\frac{\mathrm{d}f}{\mathrm{d}t} \tag{65}$$

$$\big[2(m^2+3m-4)a+12f\big]\frac{\mathrm{d}s}{\mathrm{d}t}+(m^2+3m-4)s\frac{\mathrm{d}a}{\mathrm{d}t}+\frac{6(2m+3)s(a-f)}{(m+1)(m+2)}\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$=\frac{6(m+1)(m+2)(f-a)}{s}-6s\frac{\mathrm{d}f}{\mathrm{d}t}\ . \tag{66}$$

For this problem the PDE formulation is reduced to the solution of three first order ODEs (65, 66, 11), involving the unknowns a, s, m. To determine the initial conditions for a and m we note that as $t \to 0$ the boundary temperature also tends to a constant $f \to f_0$. With the correct scaling we may set $f_0 = 1$ and so $a(0), m(0)$ are determined from the constant boundary temperature problem given by (15, 16).

The left plot in Figure 9 shows s against t for the case $f(t) = 1+t$ and various β. The dashed line denotes the CIM solution and the solid line denotes the numerical solution [15]. As we found for the examples considered in the previous sections, the CIM shows excellent agreement for large β but breaks down as $\beta \to 0$: for the worst case shown, the error in s at $t = 5$ when $\beta = 0.1$ is around 3%. In Figure 9b we compare temperature profiles in the melt for three values of β. Only in the case $\beta = 0.1$ can the CIM and numerical curves be distinguished. For the sake of brevity we do not show plots of a and m, but note that both increase monotonically. For large β m varies between 1.5 and a value slightly above 2 while a varies approximately between 1 and 5. For $\beta = 0.1$ the variation in m is greater, approximately between 1.75 and 3.25 while a remains close to 0.5.

This example shows that the CIM can be applied to certain problems with time dependent boundary conditions. Here we used $f(t) = 1 + t$ which is a monotonically increasing function of t. In Sadoun *et al* [22] the RIM (with $m = 2$) is applied to boundary conditions of the form $f(t) = 1 - 0.2t$ and $f(t) = 1 - \epsilon \sin(\omega t)$. We deliberately avoid these boundary conditions since heat balance integral methods are not well suited to problems where the temperature tends to zero or has oscillations. The plots of $s(t)$ given in [22] show good agreement with numerical solutions, although with the oscillating function s increases monotonically, following the peaks of the numerical solution. However, for a sufficiently long time or sufficiently large ϵ, the temperature profile will generally diverge from the true profile, and this is not discussed at all in [22].

For thermal problems this issue is discussed in detail in [12]. Their argument may be summarised as follows. When f is a decreasing function of time the temperature at $x = 0$ decreases with time. Since initially the temperature within the medium is everywhere below the boundary value, for small times it decreases

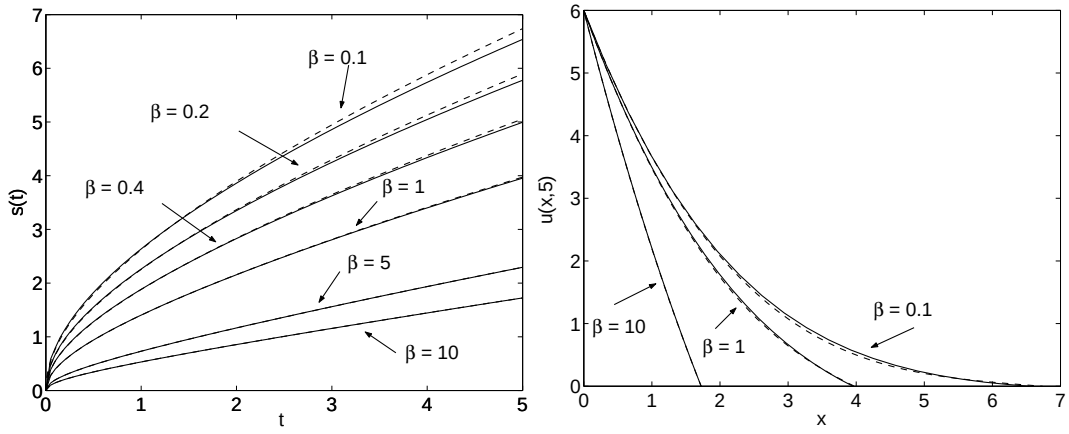

FIGURE 9. *$s(t)$ and $u(x,5)$ for the time dependent boundary condition $u(0,t) = 1 + t$.*

monotonically to the far field temperature. However, as the initial energy propagates into the material there comes a time when the internal temperature is greater than that at the boundary. This first occurs when $u_x(0,t) = 0$ and subsequently there is a maximum at some point $x = p(t)$ where $0 < p < \delta$. The standard polynomial profiles do not permit a turning point, $u_x \neq 0$ (except when $x = \delta$), and consequently, the HBIM or RIM will fail when describing any profile with a turning point. If, in some manner, the boundary temperature could reach zero without the solution breaking down then with $u(0,t) = u(\delta,t) = 0$ the HBIM and RIM will both predict $u(x,t) = 0$ everywhere. So, in the case $f = 1 - 0.2t$ the integral methods will predict $u(x,t) = 0$ at $t = 5$ even though heat has propagated into the medium up until this time. Obviously an oscillatory boundary temperature will lead to unreliable temperature predictions for sufficiently large ϵ. In our case we find the CIM breaks down even quicker than the other integral methods because our method has both a and m adjusting to provide the best approximation to the temperature. The solutions for s of [22] do not capture the oscillations correctly, whereas our method does for small time and will consequently fail earlier. The problem is easily spotted in the exponent m which decreases and, once it is below one, the temperature profile becomes unrealistic. Methods with a fixed m do not suffer in this way. This is not to say that integral methods should not be used on such boundary conditions, however both the temperature and position of the moving front should be monitored closely.

6. Summary and discussion

The classical HBIM analysis involved specifying an approximating function, usually a quadratic and then, after applying appropriate boundary conditions,

integrating the heat equation and solving the resultant ordinary differential equation for the heat penetration depth δ. This approach has been extended by many authors through different approximating functions, cubic, quartic, exponential *etc.* Recently a new approach was investigated in [16, 17], where the exponent of the highest order term in the polynomial was left unknown. An extra unknown requires an extra equation and this was provided by the function E_n, equation (2), which defines the least squares error of the approximate solution to the heat equation. The process followed in [16, 17] then provided values for the exponent that would minimise the error. For example, for a simple thermal problem with a constant temperature applied at the boundary setting $m = 2.233$ will provide the most accurate HBIM solution. For a fixed flux condition $m = 3.584$ is best. However, for certain problems it turns out that $m = m(t)$ and then the method of [16, 17] becomes unwieldy. This motivated the current study, to find an accurate and simple way to determine values for the unknown exponent.

In the present study we followed a similar route, in that there is an undetermined exponent. The extra governing equation is provided through the RIM approach. This method proved to be highly accurate. For the one-phase Stefan problem with a fixed boundary temperature the governing equations reduced to solving a single algebraic equation to determine $m = m(\beta)$. For the two phase case the problem reduced to solving two algebraic equations for n, m. In both cases we were able to show that the solutions were more accurate than the second order perturbation for a wide range of Stefan numbers. Only for $\beta > 7$ for the one phase and $\beta > 51$ for the two phase case does the perturbation solution become more accurate and by this stage the error is below 0.1%.

Of course, when using the error defined by (2) the exponents obtained by the method of [16, 17] will lead to even more accurate results than in the present study. However, the present method generally exhibits a similar level of accuracy and really comes to the fore when analysing more complex boundary conditions. For example in [17] the problem of ablation, with a constant flux at the boundary, is considered. This leads to a time-dependent exponent and the algebra involved in minimising E_n becomes too complex to deal with. Consequently n is set to $n(0)$ (the error is greatest at $t = 0$). However, this does mean that for all $t > 0$ the error $E_n(t)$ is no longer minimised. For the current approach the analysis remains simple, and for a constant flux problem the governing equations reduce to solving three first order ordinary differential equations. Comparison with a full numerical solution demonstrated the accuracy of the method and also showed that for $\beta < 24$ the CIM is more accurate than the second order perturbation solution. Finally, we considered a boundary temperature that increased linearly with time and again excellent agreement with a numerical solution was shown for a wide range of β.

There are difficulties associated with the present method. For example, for time-dependent parameter values the choice of initial condition can be difficult. We found values through small time expansions. However, a pragmatic approach

would be to simply specify a value, for example $m = 3$, and hope that the numerical solution quickly converges to the correct value. As discussed in the final section, when the boundary temperature approaches zero, or oscillates too much integral methods will tend to break down. The CIM may break down quicker than standard approaches which do not provide such an accurate approximation for small times and hence avoid the difficulties for longer.

Acknowledgments

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Appendix A. Asymptotic behaviour for small time for the constant flux condition

Since the melt region is initially of zero thickness, it is advantageous to work in transformed coordinates. Hence we set

$$\xi = \frac{x}{s(t)} , \qquad u = s(t)U(\xi, t) , \tag{A.1}$$

so that the governing equations and boundary conditions become

$$\frac{\partial^2 U}{\partial \xi^2} = s\left[s_t U + s\frac{\partial U}{\partial t} - \xi s_t \frac{\partial U}{\partial \xi}\right] , \qquad 0 < \xi < 1 \tag{A.2}$$

$$U = 0 , \qquad \beta s_t = -\frac{\partial U}{\partial \xi} , \qquad \text{at } \xi = 1 \tag{A.3}$$

$$\frac{\partial U}{\partial \xi} = -1 , \qquad \text{at } \xi = 0 \tag{A.4}$$

$$U = 0 , \qquad s = 0 , \qquad \text{at } t = 0 . \tag{A.5}$$

Following [3, 5], we seek a solution of (A.2)-(A.5) by expanding U and s_t in a power series in s

$$U(\xi, s) = U_0(\xi) + sU_1(\xi) + s^2 U_2(\xi) + \dots , \; s_t(s) = F_0 + sF_1 + s^2 F_2 + \dots . \tag{A.6}$$

Substituting these expansions into (A.2-A.5) gives the leading order equations:

$$\frac{\partial^2 U_0}{\partial \xi^2} = 0 , \qquad U_0(1) = 0 , \qquad \beta F_0 = -\left.\frac{\partial U_0}{\partial \xi}\right|_{\xi=1} , \qquad \left.\frac{\partial U_0}{\partial \xi}\right|_{\xi=0} = -1 , \tag{A.7}$$

which have solution

$$U_0 = 1 - \xi \ , \qquad F_0 = \frac{1}{\beta} \ . \tag{A.8}$$

The $\mathcal{O}(s)$ equations are

$$\frac{\partial^2 U_1}{\partial \xi^2} = F_0 U_0 - \xi F_0 \frac{\partial U_0}{\partial \xi} \ , \qquad U_1(1) = 0 \ , \tag{A.9}$$

$$\beta F_1 = -\frac{\partial U_1}{\partial \xi}\bigg|_{\xi=1} \ , \qquad \frac{\partial U_1}{\partial \xi}\bigg|_{\xi=0} = 0 \ , \tag{A.10}$$

and these can be solved to give

$$U_1 = \frac{1}{2\beta}(\xi^2 - 1) \ , \qquad F_1 = -\frac{1}{\beta^2} \ . \tag{A.11}$$

If we look at the next order terms, from examining (A.2) it is clear that we need to evaluate $\frac{\partial U}{\partial t}$. Now, since the expansion in (A.6) is for $U \equiv U(\xi, s)$ we have

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial s} s_t = s_t \frac{\partial}{\partial s}\big(U_0 + sU_1 + \dots\big) = s_t U_1 \ .$$

Thus the $\mathcal{O}(s^2)$ equations become

$$\frac{\partial^2 U_2}{\partial \xi^2} = 2F_0 U_1 + F_1 U_0 - \xi \left(F_0 \frac{\partial U_1}{\partial \xi} + F_1 \frac{\partial U_0}{\partial \xi} \right) \ , \quad U_2(1) = 0 \ , \tag{A.12}$$

$$\beta F_2 = -\frac{\partial U_2}{\partial \xi}\bigg|_{\xi=1} \ , \quad \frac{\partial U_2}{\partial \xi}\bigg|_{\xi=0} = 0 \ , \tag{A.13}$$

and these can be solved to give

$$U_2 = -\frac{1}{\beta^2}(\xi^2 - 1) \ , \qquad F_2 = \frac{2}{\beta^3} \ . \tag{A.14}$$

In fact, the next order terms give

$$U_3 = \frac{3\xi^2}{\beta^3} - \frac{\xi^4}{12\beta^3} - \frac{35}{12} \ , \qquad F_2 = -\frac{17}{\beta^4} \ . \tag{A.15}$$

Hence

$$s_t = \frac{1}{\beta} - \frac{1}{\beta^2} s + \frac{2}{\beta^3} s^2 - \frac{17}{3\beta^4} s^3 \ .$$

If we substitute the expansion

$$s = s_0 t + s_1 t^2 + s_2 t^3 + s_3 t^4 \ , \tag{A.16}$$

and match terms then we find that

$$s(t) = \frac{t}{\beta} - \frac{1}{2\beta^3} t^2 + \frac{5}{6\beta^5} t^3 - \frac{17}{8\beta^7} t^4 \ . \tag{A.17}$$

We can also substitute (A.16) into the HBIM, RIM equations (49, 50) and Stefan condition (11). Assuming a and m are of the form

$$a = a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^3 \ , \qquad m = m_0 + m_1 t + m_2 t^2 \ , \tag{A.18}$$

we find that s satisfies (A.17) and

$$a = \frac{t}{\beta} - \frac{3}{2\beta^3} t^2 + \frac{23}{6\beta^5} t^3 - \frac{305}{24\beta^7} t^4 \ , \qquad m = 2 + \frac{43}{2\beta^4} t^2 \ . \tag{A.19}$$

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