OPTIMAL ASYMPTOTIC BOUNDS FOR SPHERICAL DESIGNS

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ABSTRACT. In this paper we prove the conjecture of Korevaar and Meyers: for each $N \geq c_d t^d$ there exists a spherical t-design in the sphere S^d consisting of N points, where c_d is a constant depending only on d.

1. INTRODUCTION

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with the Lebesgue measure μ_d normalized by $\mu_d(S^d) = 1$.

A set of points $x_1, \ldots, x_N \in S^d$ is called a *spherical t-design* if

$$\int_{S^d} P(x)\, d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d+1$ variables, of total degree at most t. The concept of a spherical design was introduced by Delsarte, Goethals, and Seidel [12]. For each $t, d \in \mathbb{N}$ denote by $N(d,t)$ the minimal number of points in a spherical t-design in S^d. The following lower bound

$$N(d,t) \geq \begin{cases} \binom{d+k}{d} + \binom{d+k-1}{d} & \text{if } t = 2k, \\ 2\binom{d+k}{d} & \text{if } t = 2k+1, \end{cases} \tag{1}$$

is proved in [12].

Spherical t-designs attaining this bound are called tight. The vertices of a regular $t+1$-gon form a tight spherical t-design in the circle, so $N(1,t) = t+1$. Exactly eight tight spherical designs are known for $d \geq 2$ and $t \geq 4$. All such configurations of points are highly symmetrical, and optimal from many different points of view (see Cohn, Kumar [8] and Conway, Sloane [11]). Unfortunately, tight designs rarely exist. In particular, Bannai and Damerell [2, 3] have shown that tight spherical designs with $d \geq 2$ and $t \geq 4$ may exist only for $t = 4$, 5, 7 or 11. Moreover, the only tight 11-design is formed by minimal vectors of the Leech lattice in dimension 24. The bound (1) has been improved by Delsarte's linear programming method for most pairs $(d,\, t)$; see [22].

On the other hand, Seymour and Zaslavsky [20] have proved that spherical t-designs exist for all d, $t \in \mathbb{N}$. However, this proof is nonconstructive and gives no idea of how big $N(d,t)$ is. So, a natural question is to ask how $N(d,t)$ differs from the tight bound (1). Generally, to find the exact value of $N(d,t)$ even for small d and t is a surprisingly hard problem. For example, everybody believes that 24 minimal vectors of the D_4 root lattice form a 5-design with minimal number of points in S^3, although it is only proved that $22 \le N(3,5) \le 24$; see [6]. Further, Cohn, Conway, Elkies, and Kumar [7] conjectured that every spherical 5-design consisting of 24 points in S^3 is in a certain 3-parametric family. Recently, Musin [17] has solved a long standing problem related to this conjecture. Namely, he proved that the kissing number in dimension 4 is 24.

In this paper we focus on asymptotic upper bounds on $N(d,t)$ for fixed $d \ge 2$ and $t \to \infty$. Let us give a brief history of this question. First, Wagner [21] and Bajnok [1] proved that $N(d,t) \le C_d t^{Cd^4}$ and $N(d,t) \le C_d t^{Cd^3}$, respectively. Then, Korevaar and Meyers [14] have improved these inequalities by showing that $N(d,t) \le C_d t^{(d^2+d)/2}$. They have also conjectured that

$$N(d,t) \le C_d t^d.$$

Note that (1) implies $N(d,t) \ge c_d t^d$. Here and in what follows we denote by C_d and c_d sufficiently large and sufficiently small positive constants depending only on d, respectively.

The conjecture of Korevaar and Meyers attracted the interest of many mathematicians. For instance, Kuijlaars and Saff [19] emphasized the importance of this conjecture for $d = 2$, and revealed its relation to minimal energy problems. Mhaskar, Narcowich, and Ward [16] have constructed positive quadrature formulas in S^d with $C_d t^d$ points having *almost* equal weights. Very recently, Chen, Frommer, Lang, Sloan, and Womersley [9, 10] gave a computer-assisted proof that spherical t-designs with $(t+1)^2$ points exist in S^2 for $t \le 100$.

For $d = 2$, there is an even stronger conjecture by Hardin and Sloane [13] saying that $N(2,t) \le \frac{1}{2}t^2 + o(t^2)$ as $t \to \infty$. Numerical evidence supporting the conjecture was also given.

In [4], we have suggested a nonconstructive approach for obtaining asymptotic bounds for $N(d,t)$ based on the application of the Brouwer fixed point theorem. This led to the following result:

For each $N \ge C_d t^{\frac{2d(d+1)}{d+2}}$ there exists a spherical t-design in S^d consisting of N points.

Instead of the Brouwer fixed point theorem we use in this paper the following result from the Brouwer degree theory [18, Th. 1.2.6, Th. 1.2.9].

Theorem A. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and Ω an open bounded subset, with boundary $\partial\Omega$, such that $0 \in \Omega \subset \mathbb{R}^n$. If $(x, f(x)) > 0$ for all $x \in \partial\Omega$, then there exists $x \in \Omega$ satisfying $f(x) = 0$.*

We employ this theorem to prove the conjecture of Korevaar and Meyers.

Theorem 1. *For each $N \geq C_d t^d$ there exists a spherical t-design in S^d consisting of N points.*

Note that Theorem 1 is slightly stronger than the original conjecture because it guarantees the existence of spherical t-designs for *each* N greater than $C_d t^d$.

This paper is organized as follows. In Section 2 we explain the main idea of the proof. Then in Section 3 we present some auxiliary results. Finally, we prove Theorem 1 in Section 4.

2. Preliminaries and the main idea

Let $\mathcal{P}_t$ be the Hilbert space of polynomials P on S^d of degree at most t such that

$$\int_{S^d} P(x) d\mu_d(x) = 0,$$

equipped with the usual inner product

$$(P, Q) = \int_{S^d} P(x)Q(x) d\mu_d(x).$$

By the Riesz representation theorem, for each point $x \in S^d$ there exists a unique polynomial $G_x \in \mathcal{P}_t$ such that

$$(G_x, Q) = Q(x) \text{ for all } Q \in \mathcal{P}_t.$$

Then a set of points $x_1, \ldots, x_N \in S^d$ forms a spherical t-design if and only if

$$G_{x_1} + \cdots + G_{x_N} = 0. \tag{2}$$

For a differentiable function $f \colon \mathbb{R}^{d+1} \to \mathbb{R}$ denote by

$$\frac{\partial f}{\partial x}(x_0) := \left(\frac{\partial f}{\partial \xi_1}(x_0), \ldots, \frac{\partial f}{\partial \xi_{d+1}}(x_0) \right)$$

the gradient of f at the point $x_0 \in \mathbb{R}^{d+1}$.

For a polynomial $Q \in \mathcal{P}_t$ we define the spherical gradient as follows:

$$\nabla Q(x) := \frac{\partial}{\partial x} Q\left(\frac{x}{|x|} \right), \tag{3}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{d+1}$.

We apply Theorem A to the open subset Ω of a vector space $\mathcal{P}_t$,

$$\Omega := \left\{ P \in \mathcal{P}_t \;\middle|\; \int_{S^d} |\nabla P(x)| d\mu_d(x) < 1 \right\}. \tag{4}$$

Now we observe that the existence of a continuous mapping $F \colon \mathcal{P}_t \to (S^d)^N$, such that for all $P \in \partial\Omega$

$$\sum_{i=1}^{N} P(x_i(P)) > 0, \text{ where } F(P) = (x_1(P), ..., x_N(P)), \tag{5}$$

readily implies the existence of a spherical t-design in S^d consisting of N points. Consider a mapping $L\colon (S^d)^N \to \mathcal{P}_t$ defined by

$$(x_1, \dots, x_N) \xrightarrow{L} G_{x_1} + \dots + G_{x_N},$$

and the following composition mapping $f = L \circ F\colon \mathcal{P}_t \to \mathcal{P}_t$. Clearly

$$(P, f(P)) = \sum_{i=1}^{N} P(x_i(P))$$

for each $P \in \mathcal{P}_t$. Thus, applying Theorem A to the mapping f, the vector space $\mathcal{P}_t$, and the subset Ω defined by (4), we obtain that $f(Q) = 0$ for some $Q \in \mathcal{P}_t$. Hence, by (2), the components of $F(Q) = (x_1(Q), ..., x_N(Q))$ form a spherical t-design in S^d consisting of N points.

The most naive approach to construct such F is to start with a certain well-distributed collection of points x_i $(i = 1, \dots, N)$, put $F(0) := (x_1, \dots, x_N)$, and then move each point along the spherical gradient vector field of P. Note that this is the most greedy way to increase each $P(x_i(P))$ and make $\sum_{i=1}^{N} P(x_i(P))$ positive for each $P \in \partial\Omega$. Following this approach we will give an explicit construction of F in Section 4, which will immediately imply the proof of Theorem 1.

3. Auxiliary results

To construct the corresponding mapping F for each $N \geq C_d t^d$ we extensively use the following notion of an area-regular partition.

Let $\mathcal{R} = \{R_1, \dots, R_N\}$ be a finite collection of closed sets $R_i \subset S^d$ such that $\cup_{i=1}^{N} R_i = S^d$ and $\mu_d(R_i \cap R_j) = 0$ for all $1 \leq i < j \leq N$. The partition $\mathcal{R}$ is called area-regular if $\mu_d(R_i) = 1/N$, $i = 1, \dots, N$. The partition norm for $\mathcal{R}$ is defined by

$$\|\mathcal{R}\| := \max_{R \in \mathcal{R}} \operatorname{diam} R,$$

where $\operatorname{diam} R$ stands for the maximum geodesic distance between two points in R. We need the following fact on area-regular partitions (see Bourgain, Lindenstrauss [5] and Kuijlaars, Saff [15]):

Theorem B. *For each $N \in \mathbb{N}$ there exists an area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ with $\|\mathcal{R}\| \leq B_d N^{-1/d}$ for some constant B_d large enough.*

We will also use the following spherical Marcinkiewicz–Zygmund type inequality:

Theorem C. *There exists a constant r_d such that for each area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m}$, each collection of points $x_i \in R_i$ $(i = 1, \dots, N)$, and each algebraic polynomial P of total degree m, the inequality*

$$\frac{1}{2} \int_{S^d} |P(x)| d\mu_d(x) \leq \frac{1}{N} \sum_{i=1}^{N} |P(x_i)| \leq \frac{3}{2} \int_{S^d} |P(x)| d\mu_d(x) \tag{6}$$

holds.

Theorem C follows naturally from the proof of Theorem 3.1 in [16].

Corollary 1. *For each area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m+1}$, each collection of points $x_i \in R_i$ $(i = 1, \dots, N)$, and each algebraic polynomial P of total degree m,*

$$\frac{1}{3\sqrt{d}} \int_{S^d} |\nabla P(x)| d\mu_d(x) \le \frac{1}{N} \sum_{i=1}^{N} |\nabla P(x_i)| \le 3\sqrt{d} \int_{S^d} |\nabla P(x)| d\mu_d(x). \tag{7}$$

Proof. Since $|\nabla P| = \sqrt{P_1^2 + \dots + P_{d+1}^2}$ in S^d, where P_j are polynomials of total degree $m+1$, Corollary 1 is an immediate consequence of (6) applied to P_j, $j = 1, \dots, d+1$. □

4. Proof of Theorem 1

In this section we construct the map F introduced in Section 2 and thereby finish the proof of Theorem 1.

For $d, t \in \mathbb{N}$, take $C_d > (54dB_d/r_d)^d$, where B_d is as in Theorem B and r_d is as in Theorem C, and fix $N \ge C_d t^d$. Now we are in a position to give an exact construction of the mapping $F \colon \mathcal{P}_t \to (S^d)^N$ which satisfies condition (5). Take an area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ with

$$\|\mathcal{R}\| \le B_d N^{-1/d} < \frac{r_d}{54dt} \tag{8}$$

as provided by Theorem B, and choose an arbitrary $x_i \in R_i$ for each $i = 1, \dots, N$. Put $\varepsilon = \frac{1}{6\sqrt{d}}$ and consider the function

$$h_\varepsilon(u) := \begin{cases} u & \text{if } u > \varepsilon, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Take a mapping $U \colon \mathcal{P}_t \times S^d \to \mathbb{R}^{d+1}$ such that

$$U(P, y) = \frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)}.$$

For each $i = 1, \dots, N$ let $y_i \colon \mathcal{P}_t \times [0, \infty) \to S^d$ be the map satisfying the differential equation

$$\frac{d}{ds} y_i(P, s) = U(P, y_i(P, s)) \tag{9}$$

with the initial condition

$$y_i(P, 0) = x_i,$$

for each $P \in \mathcal{P}_t$. Note that each mapping y_i has its values in S^d by definition of spherical gradient (3). Since the mapping $U(P, y)$ is Lipschitz continuous in both P and y, each y_i is well defined and continuous in both P and s, where the metric on $\mathcal{P}_t$ is given by the inner product. Finally put

$$F(P) = (x_1(P), \dots, x_N(P)) := \Big(y_1\Big(P, \frac{r_d}{3t}\Big), \dots, y_N\Big(P, \frac{r_d}{3t}\Big)\Big). \tag{10}$$

By definition the mapping F is continuous on $\mathcal{P}_t$. So, as explained in Section 2, to finish the proof of Theorem 1 it suffices to prove

Lemma 1. *Let $F\colon \mathcal{P}_t \to (S^d)^N$ be the mapping defined by* (10)*. Then for each $P \in \partial\Omega$,*

$$\frac{1}{N}\sum_{i=1}^{N} P(x_i(P)) > 0,$$

where Ω is given by (4)*.*

Proof. Fix $P \in \partial\Omega$. For the sake of simplicity we write $y_i(s)$ in place of $y_i(P,s)$. By the Newton-Leibniz formula we have

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} P(x_i(P)) &= \frac{1}{N}\sum_{i=1}^{N} P(y_i(r_d/3t))\\ &= \frac{1}{N}\sum_{i=1}^{N} P(x_i) + \int_0^{r_d/3t} \frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N} P(y_i(s))\right] ds. \qquad (11)\end{aligned}$$

Now to prove Lemma 1, we first estimate the value

$$\left|\frac{1}{N}\sum_{i=1}^{N} P(x_i)\right|$$

from above, and then estimate the value

$$\frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N} P(y_i(s))\right]$$

from below, for each $s \in [0, r_d/3t]$. We have

$$\begin{aligned}\left|\frac{1}{N}\sum_{i=1}^{N} P(x_i)\right| &= \left|\sum_{i=1}^{N}\int_{R_i} P(x_i) - P(x)\, d\mu_d(x)\right| \le \sum_{i=1}^{N}\int_{R_i} |P(x_i) - P(x)| d\mu_d(x)\\ &\le \frac{\|\mathcal{R}\|}{N}\sum_{i=1}^{N} \max_{z\in S^d\colon \mathrm{dist}(z,x_i)\le\|\mathcal{R}\|} |\nabla P(z)|\end{aligned}$$

where $\mathrm{dist}(z, x_i)$ denotes the geodesic distance between z and x_i. Hence, for $z_i \in S^d$ such that $\mathrm{dist}(z_i, x_i) \le \|\mathcal{R}\|$ and

$$|\nabla P(z_i)| = \max_{z\in S^d\colon \mathrm{dist}(z,x_i)\le\|\mathcal{R}\|} |\nabla P(z)|,$$

we obtain

$$\left|\frac{1}{N}\sum_{i=1}^{N} P(x_i)\right| \le \frac{\|\mathcal{R}\|}{N}\sum_{i=1}^{N} |\nabla P(z_i)|.$$

Consider another area-regular partition $\mathcal{R}' = \{R_1', \dots, R_N'\}$ defined by $R_i' = R_i \cup \{z_i\}$. Clearly $\|\mathcal{R}'\| \le 2\|\mathcal{R}\|$ and so, by (8), we get $\|\mathcal{R}'\| < r_d/(27\,d\,t)$. Applying inequality (7) to the partition $\mathcal{R}'$ and the collection of points z_i we obtain that

$$\left|\frac{1}{N}\sum_{i=1}^{N} P(x_i)\right| \le 3\sqrt{d}\,\|\mathcal{R}\| \int_{S^d} |\nabla P(x)| d\mu_d(x) < \frac{r_d}{18\sqrt{d}\,t} \tag{12}$$

for any $P \in \partial\Omega$. On the other hand, the differential equation (9) implies

$$\begin{aligned} \frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N} P(y_i(s))\right] =& \frac{1}{N}\sum_{i=1}^{N} \frac{|\nabla P(y_i(s))|^2}{h_\varepsilon(|\nabla P(y_i(s))|)} \\ \ge& \frac{1}{N}\sum_{i:\,|\nabla P(y_i(s))|\ge\varepsilon} |\nabla P(y_i(s))| \\ \ge& \frac{1}{N}\sum_{i=1}^{N} |\nabla P(y_i(s))| - \varepsilon. \end{aligned} \tag{13}$$

Since

$$\left|\frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)}\right| \le 1$$

for each $y \in S^d$, it follows again from (9) that $\left|\frac{dy_i(s)}{ds}\right| \le 1$. Hence we arrive at

$$\operatorname{dist}(x_i, y_i(s)) \le s.$$

Now for each $s \in [0, r_d/3t]$ consider the area-regular partition $\mathcal{R}'' = \{R_1'', \dots, R_N''\}$ given by $R_i'' = R_i \cup \{y_i(s)\}$. By (8) we have

$$\|\mathcal{R}''\| < \frac{r_d}{54dt} + \frac{r_d}{3t};$$

so we can apply (7) to the partition $\mathcal{R}''$ and the collection of points $y_i(s)$. This and inequality (13) yield

$$\begin{aligned} \frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N} P(y_i(s))\right] &\ge \frac{1}{N}\sum_{i=1}^{N} |\nabla P(y_i(s))| - \frac{1}{6\sqrt{d}} \\ &\ge \frac{1}{3\sqrt{d}} \int_{S^d} |\nabla P(x)| d\mu_d(x) - \frac{1}{6\sqrt{d}} = \frac{1}{6\sqrt{d}}, \end{aligned} \tag{14}$$

for each $P \in \partial\Omega$ and $s \in [0, r_d/3t]$. Finally, equation (11) and inequalities (12) and (14) imply

$$\frac{1}{N}\sum_{i=1}^{N} P(x_i(P)) > \frac{1}{6\sqrt{d}}\frac{r_d}{3t} - \frac{r_d}{18\sqrt{d}\,t} = 0. \tag{15}$$

Lemma 1 is proved. □

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