FOUNDATIONAL IMPLICATIONS OF THE INNER MODEL HYPOTHESIS

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1. Introduction.

The goal of this paper is to bring the *Inner Model Hypothesis* (IMH), an axiomatic approach formulated by the second author in [Fri06], into the current debate on the implications of independence results in set theory. We argue that the IMH provides an alternative to the two main contenders in this debate: the view that the universe of sets is *inherently undetermined*, its essential features being exhausted by the axioms of ZFC, and the opposing view that the next step toward the goal of making our knowledge of the universe of sets more determinate consists in the search for a suitable extension of the system *ZFC + large cardinal axioms*. Both perspectives are objectionable in principle and the Inner Model Hypothesis confirms this in fact.

A brief overview of the current situation with regard to independence in set theory is given in section 2. Section 3 illustrates the main views in the current debate on the implications of independence phenomena. Criticism against these views is presented in section 4, while the implications of the Inner Model Hypothesis are discussed in section 5.

2. A puzzling state of affairs.

As a consequence of Gödel's construction of the *inner model L* and Cohen's introduction of *forcing* techniques in set theory, the existence of alternative universes satisfying the accepted axioms (i.e., the axioms of the system ZFC) has emerged as an inescapable fact. In addition to ZFC, the universe L of constructible sets satisfies the Generalized Continuum Hypothesis (GCH) (and therefore the Singular Cardinal Hypothesis (SCH)), the assertion that there is a definable non-measurable set of reals, and the Singular Square Principle; it fails to satisfy the

Suslin Hypothesis, the Whitehead conjecture, the Borel Conjecture and the existence of a Borel bijection between any two non-Borel analytic sets.[1] On the other hand, many of these principles behave differently in forcing extensions of L and, relative to the existence of large cardinals, they all behave differently in some model of ZFC.[2]

As a natural move in the attempt to decide statements independent from ZFC and thereby make our picture of the universe of sets more determinate, candidate axioms for extending ZFC have been proposed and investigated. In line with a suggestion of Gödel, a prominent role in this investigation has been played by *large cardinal axioms*.[3] With reference to such axioms Gödel says:

> It is not impossible that [...] some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets. ([FDK⁺90], 150-3)

What came to be known as "Gödel's program for new axioms" did not however produce the desired results as far as independence is concerned. The statement of greatest interest which is independent from ZFC, Cantor's Continuum Hypothesis, is also independent from "ZFC + large cardinal axioms". But a relevant general fact emerged: The study of large cardinal axioms took the form of a strictly mathematical venture ("the theory is assumed and theorems are proved in the ordinary mathematical manner", [FK10], ix), and its *mathematical success* was used as a source of evidence in set theory. *Success* is meant here as Gödel intended it, i.e. as consisting in axioms being "fruitful in consequences, exactly in 'verifiable' consequences, i.e. consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover [...]"[4], as well as in axioms shedding light "upon a whole discipline, and furnishing [...] powerful methods for solving given problems"([Goe47], 183).[5]

It is however worth noting that mathematical success can be reasonably ascribed to extensions of ZFC incompatible with "ZFC + large cardinal axioms". ZFC

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[1]GCH is the assertion that for any cardinal number κ, $2^\kappa = \kappa^+$, while the SCH, implied by GCH, is the same assertion for κ a singular strong limit cardinal. For the other principles mentioned see [Jec03].

[2]Specifically, there are forcing extensions of L in which GCH is false, definable sets of reals are measurable and the Suslin Hypothesis, Whitehead Conjecture and the Borel Conjecture are true. Models of the negation of SCH, the negation of the Singular Square Principle and the existence of a Borel bijection between any two non-Borel analytic sets can be obtained assuming the existence of a hypermeasurable cardinal, a supercompact cardinal and a measurable cardinal, respectively.

[3]Large cardinal axioms assert the existence of cardinals κ with various strong properties, always implying that the family of sets of hereditary cardinality $< \kappa$ is a model of ZFC.

[4]"[...] and make it possible to condense into one proof many different proofs", [Goe47], 183.

[5]On the success of large cardinal axioms see [Hau04] and [Arr07].

$+ V = L$, for instance, is fruitful in consequences, furnishes powerful methods for solving problems and introduces the concept of *constructibility*, important throughout set theory.[6] Of course this theory is incompatible with ZFC + "there exists a measurable cardinal".[7]

How the mathematical success of large cardinal axioms is related to the program of making the picture of the set-theoretical universe more determinate – and, more generally, to the aim of producing *definitive* set-theoretical hypotheses – is discussed in the next two sections.

3. Reactions.

Faced with the situation described in section 2, set-theorists show diverse reactions. The existence of mutually incompatible, successful extensions of ZFC led some to the conclusion that the notion *set-theoretic universe* is inherently undetermined. This position is clearly expressed by Shelah in [She03]:

> I do not feel "a universe of ZFC" is like "the Sun", it is rather like "a human being" or "a human being of some fixed nationality." [...] You may think "does CH, i.e., $2^{\aleph_0} = \aleph_1$ hold?" is like "Can a typical American be Catholic?" ([She03], 211)

A different attitude is endorsed by those who, due to the success of large cardinal axioms, regard ZFC as "the twentieth century choice" for the axioms of set theory and consider "ZFC + large cardinal axioms" to be the contemporary theory of sets, "to be adopted by all, as part of a broadest point of view".[8] In fact these authors do not draw conclusions similar to Shelah's from the fact that large cardinals are preserved under forcing, and hence models of "ZFC + large cardinal axioms" exist in which mutually exclusive propositions are true. They put stress not on the failure of large cardinals to produce a determinate picture of the universe of sets but instead on the mathematical success of large cardinal axioms, and explicitly take this as providing evidence for the *correctness* (or *truth*) of these axioms, even regarding them as definitive hypotheses.[9] At the

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[6]Inner and core models for large cardinals can be regarded as generalizations of the universe L of constructible sets. See [Jen95].

[7]That if a *measurable cardinal* exists, then $V \neq L$ was proved by Scott in 1961. See [Jec03] for details.

[8]See, respectively, [Woo01] and [Ste00], where the point is made that the "broadest point of view" proviso is meant to exclude from attention the temporary adoption of restrictive assumptions as a convenient device for avoiding irrelevant structure" (e.g.. "$V = L$ is often temporarily assumed for such reasons by set-theorists who do not believe it [...]", [Ste00], 422).

[9]E.g. *projective Determinacy* (PD), implied by the existence of infinitely many Woodin cardinals, is said in [Woo01] to be "the *correct* axiom for the projective sets", yielding forcing invariant answers to questions independent of ZFC (e.g. the measurability of projective sets), which, when first formulated, were considered unsolvable. See [Woo01], 570. By *forcing invariance* is here meant that no sentence in the language of second order arithmetic, in which properties of projective sets are formulated, can be shown to be independent of the existence

same time the hope is expressed that new correct (true) axioms will emerge that decide questions independent from the system "ZFC + large cardinals". As a result, the program of making the picture of the set-theoretical universe more determinate is placed in the restricted form: find suitable axiomatic extensions of "ZFC + large cardinals".

This forms part of Woodin's conclusions in [Woo01], where an axiomatic proposal is advanced that is intended to play the same role with regard to third order number theory, in which the Continuum Hypothesis (CH) can be formulated, that is played by large cardinal axioms with regard to second order number theory.[10]

> So, is the Continuum Hypothesis solvable? Perhaps I am not completely confident that the "solution" I have sketched is the solution, but it is for me convincing evidence that there *is* a solution. [...] The universe of all sets is a large place. We have just barely begun to understand it. ([Woo01], 690)

Both Shelah's and Woodin's positions are not immune to criticism. Objections to them are advanced in the next section.

4. CRITICISM.

Let us start with positions like those expressed by Woodin regarding extensions of "ZFC + large cardinal axioms". According to them, successful, hence correct (true), set-theoretic axioms (large cardinal axioms) have been discovered that settle some notable questions independent from ZFC. This implies that the program for making the picture of the universe more determined cannot but consist in extending "ZFC + large cardinal axioms". We argue that the implication "success → correctness (or truth)" presupposed by this view is objectionable, and makes it ultimately untenable.

Observe first that by assuming the implication: "success → correctness (or truth)", one cannot do justice to the existence of mutually incompatible successful systems of set theory (like "ZFC + large cardinal axioms" and "ZFC + $V = L$"). For correctness (truth) is commonly intended as a matter of all or nothing, ruling out the possibility of equally correct (true) but mutually exclusive axiomatic systems. This would be the case, though, if evidence due to success were to imply correctness (truth) in set theory. On the other hand,

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of large cardinals implying PD by the method of set-forcing. In fact, by a theorem of Woodin, if you suppose that every set belongs to an iterable inner model satisfying "there are ω Woodin cardinals", then, if M and N are set-generic extensions of V, you have $L(\mathbb{R})^M \equiv L(\mathbb{R})^N$. See [Woo01].

[10]Second and third order number theory are presented in [Woo01] as the theories of the structures $< H(\omega_1), \in >$ and $< H(\omega_2), \in >$. See [Woo01] for details.

assuming the implication "success → correctness (or truth)" and denying correct-
ness or truth to e.g. "ZFC + $V = L$", one would *ipso facto* deny its mathematical
success, which is undeniable.

The success of the axiom of constructibility ($V = L$) is often regarded as a
counterexample to the view that success is all there is to correctness and truth
in set theory.

> A favorite example against the pragmatic view that we accept an
> axiom because of its elegance (simplicity) and power (usefulness)
> is the constructibility hypothesis. It should be accepted accord-
> ing to the pragmatic view but is not generally accepted as true.
> ([Wan74], 196)

Wang suggests what would be necessary and sufficient conditions for an axiomatic
system to be accepted (as correct or true). Beyond being successful, the system
should be explicitly suggested by the meaning of set.

> $[V = L]$ is likely to be false according to the iterative concept
> of set. Basically it is felt that the pragmatic view leaves out the
> criterion of intuitive plausibility. ([Wan74], 196)

Wang's argument, however, does not apply to most large cardinal axioms and,
especially, to the ones discussed by Woodin. "Correct" ("true") principles like
Projective Determinacy, and the large cardinal axioms implying it, lack any clear
direct link to the iterative concept, which Wang calls upon as the meaning of
set. In fact referring to these axioms, and explicitly describing them as "true",
Woodin comments:

> There are natural questions about $H(\omega_1)$ which are not solvable
> from ZFC. However, there are axioms for $H(\omega_1)$ which resolve
> these questions [...] and which are clearly *true*. But the truth
> of these axioms became evident only *after* a great deal of work.
> ([Woo01], 569)

Moreover, also the implication "success *and* intuitive plausibility (adherence to
the iterative concept) → correctness (truth)" is objectionable. For it can be
plausibly suggested that the iterative concept is a concept that arose alongside
successful set-theoretic developments, and as such is a metaphorical reformula-
tion of the insights delivered by the latter.[11] The same holds for methodolog-
ical maxims that are often presented as inspired by the iterative concept, like
e.g. "maximize", the view that the universe of sets should be high and wide, so
"the more sets one proves to exist, the better". A mathematical concept could
only be attached to the sentence "the universe is maximal" only after Scott's
result that if a measurable cardinal exists then $V \neq L$ was obtained. Viewing
the iterative concept and methodological principles like "maximization" in this

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[11]See [Arr07] and [Arr].

way leads one to reject Wang's suggestion that "intuitive plausibility" (i.e. adherence to the iterative concept or "maximization") is sufficient, in conjunction with success, to produce truth or correctness in set theory. For, along with every system of set theory that turns out to be successful (according to Gödel's characterization of success), a distinguished concept of set and a system of preferred methodological maxims is likely to emerge.[12] Since competing successful systems of axioms exist in set theory, taking the conjunction "success *and* intuitive plausibility" to imply correctness (or truth), would still leave one with mutually exclusive, correct (true) systems of axioms. This contrasts with how the term *correct* (*true*) is meant to be used.

It is also worth noting that methodological maxims are very far from suggesting unique proposals for axiomatic extensions of ZFC. E.g. "maximization" may suggest the principle "there exists a $j \colon V \to V$", which is incompatible with the *Axiom of Choice*, also in line with maximality considerations.[13] The *Inner Model Hypothesis*, incompatible with large cardinal axioms, offers yet another example of the ambiguity of the concept of "maximization" (see the next section).

One might still object to our criticism by asserting that success comes in degrees in set theory, making it possible to draw a distinction between incompatible successful systems according to their degree of success, and suggest that it is only the most successful set-theoretic system that deserves to be regarded as correct or true. That mathematical success comes in degrees seems to be the case. According to Gödel's characterization of success, in fact, the term "successful" is to be applied to mathematical developments through which a link is established between formerly unrelated mathematical facts. A link may consist in one theory's enabling the interpretation of another in its own terms. Under these circumstances, the former would reveal itself to be "more successful" than the latter. In fact, as an implication of Scott's theorem, the universe L could be seen as a proper sub-universe of V and studied "from within" V under large cardinal axioms, thereby convincing some of the superior success of "ZFC + large cardinals" over "ZFC + $V = L$". Supposing "maximality" to be essentially a matter of maximizing interpretative power, Steel says the following with regard to ZFC + $V = L$:

> In this light we can see why most set-theorists reject $V = L$ as restrictive: adopting it restricts the interpretative power of the language of set theory. The language of set theory as used by the believer of $V = L$ can certainly be translated into the language of set theory as used by the believer in measurable cardinals, via the translation $\phi \mapsto \phi^L$. There is no translation in the other direction.

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[12]This view is presented and motivated in [Arr].

[13]This point is made in [Hau05]. The principle "there exists a $j \colon V \to V$" (there is a nontrivial elementary embedding of the universe into itself) was proved to be contradictory with Choice by Kunen. See [Jec03] for details.

> While it is true that adopting $V = L$ enables one to settle new
> formal sentences, this is in fact a completely sterile move, because
> one settles ϕ by giving it the same interpretation as ϕ^L which can
> be settled in anyone's theory. ([Ste00], 423)

Yet it remains that while one may accept that success comes in degrees, this
is usually not the case as far as correctness and truth are concerned. Accordingly, correctness (truth) might well be supposed to be an attribute of the "most
successful system of set theory", but this could not be done by arguing that correctness (truth) is an implication of success. The only possible way for one to
coherently say that a successful axiomatic system for sets is correct (true) seems
to be that of explicitly presenting one's position as a deliberate act, an act based
on the decision to attach correctness (truth) to success "at the highest degree",
as well as on a shared agreement as to what the most successful axiomatic system
for sets currently is. However, at the moment, there is no agreement among set-theorists as to what the most successful theory of sets is.[14] Skeptical positions
on the status of large cardinal axioms have been expressed (see e.g. [She03]).
Arguments like Steel's to the effect that an interpretation of "ZFC + $V = L$"
in terms of "ZFC + large cardinal axioms" is possible but not vice-versa, have
been contested as well. Jensen, for instance, maintains that the relation between
"ZFC + large cardinal axioms" and "ZFC + $V = L$" is one of mutual inter-pretability. For L itself can see the existence of "natural" models for large cardinal axioms if there are such cardinals in V. As a consequence of *Shoenfield's
Absoluteness Lemma*, in fact, L and V have transitive countable models for the
same large cardinal hypotheses.[15] "Hence we could just assume ourselves to be
in a countable segment of L when we assume H."[16]

To sum up: the view that success furnishes evidence for correctness (truth),
though not *per se* contradictory, does not help in defending the view that the
program for making our picture of the universe more determinate must consist in
finding suitable extensions of "ZFC + large cardinal axioms". At most it suggests
that one should be cautious in taking as correct (true) what one regards as the
most successful axiomatic system for sets, as there exist views about success that
run contrary to one's own.

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[14]Nor is there, one may guess, on the "conventional" view of correctness and truth introduced
here.

[15]In fact, if the hypothesis H holds in V, then by reflection H should have a model that
is a level V_κ of V (note this informal step in the argument) for some cardinal κ. By the
Löwenheim-Skolem theorem, there is a countable elementary sub-model of V_κ, call it N, in
which H holds. By *Mostowski's Collapsing Theorem* there is a transitive $\overline{N}$ that is a countable
model of H. Let be $a \in \mathbf{R}$ be a code for $\overline{N}$. The formula asserting the existence of such an
a is Σ_2^1. By *Shoenfield's Absoluteness Lemma*, it is true in L. I.e. L sees the existence of a
transitive countable set model for H.

[16]Quoted with permission from the handout of a talk given by Jensen in Krakow in 1999.

Let us add that, in fact, neither a simple identification of correctness (truth) with success, nor the view that correctness (truth) is conventionally attached to success "at the highest degree", seems to underlie positions like Woodin's. A Platonistic attitude appears to be at work. This is explicitly admitted by Foreman in [FK10]; with regard to consistency results involving large cardinal axioms, he observes:

> This type of unifying deep structure is taken as strong evidence that the axioms proposed reflect some underlying reality and so is often cited as a primary reason for accepting the existence of large cardinals. ([FK10], x)

Under these circumstances, correctness (truth) rests no longer on success. Success may well be regarded as a clue to it – if it is supposed that it is correctness (truth), meant as a matter of "reflecting some underlying reality", that ultimately implies success (or, better, success "at the highest degree"). Moreover, by regarding correctness (truth) as sufficent, as opposed to necessary, to success, an explanation would be given, too, for the existence of mutually exclusive successful systems of set theory. For, under these circumstances, the existence of successful set theories that cannot be said to be correct (true) is no longer contradictory. However, one should still justify Platonism in order for this position to be sound. This is no easy task. Neither pursuing this justification nor criticizing it belongs to the aims of the present paper.

Having focused on the positions of Woodin, Steel and Foreman, let us now return to Shelah's views. Here one abdicates the search for new axioms that may yield solutions to questions independent of ZFC, solutions to which correctness or truth can be attached as the end-stage of a process through which a shared consensus is reached that certain mathematical developments, and the axioms that make them possible, are the most successful ones. This abdication may have positive consequences. It may work as a heuristic for exploiting the available resources (ZFC), to the effect that light is shed on still undiscovered implications of them, perhaps relevant with regard to independence phenomena. Shelah's *pcf theory*, developed entirely within ZFC, has a bearing on questions of cardinal arithmetic like the Generalized Continuum Hypothesis.[17] However, it might be felt that whereas positions like Shelah's are supported by the existence of incompatible successful set-theoretical developments, they also prescribe a halt to such developments by regarding ZFC as all there is to be said about sets. Shelah's conclusions also sound arbitrary. Why should the view that a universe of ZFC be not like "the Sun" but like "a human being of some fixed nationality" be a definitive one? Why not regard it as a description of a state of affairs that need not be permanent, merely reflecting the *actual* situation in set theory, where no development stands out as the most successful (and hence, one may add, the

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[17]See [She03], 220: "Cardinal arithmetic is loaded with consistency results because we ask the wrong questions. [...]. We should replace cardinality by cofinality, as explained below (pcf theory)".

correct or true) one? As it seems premature to say that convincing evidence is available that the correct answer to the question "Is CH true/false?" is given by a suitable extension of "ZFC + large cardinal axioms", so seems it premature to rule this out and be content with the view that the notion *universe of all sets* is inherently undetermined.

As a case study supporting the above criticisms, we discuss the second author's *Inner Model Hypothesis* (IMH) in the following section. The IMH also provides a striking example of a phenomenon alluded to above, the ambiguity of the concept of "maximization".

5. The Inner Model Hypothesis.

We begin with a restatement of our thesis. Objections can be raised against the view that the notion *universe of all sets* can only be made determinate by finding axiomatic extensions of "ZFC + large cardinal axioms" which successfully decide questions independent of the latter. In advancing this view it is assumed that mathematical success provides evidence for the correctness or truth of large cardinal axioms, which renders these axioms definitive set-theoretic principles that one can only "extend" but not contradict. In assuming that success implies correctness (truth), however, one is either tacitly committed to Platonism or faces the embarrassing situation that mutually exclusive and successful axiomatic systems for sets coexist. On the other hand, no a priori ground seems to exist for ruling out the possibility of making the notion *universe of all sets* more determinate than it is now through the introduction of new axiomatic proposals.

By advancing the *Inner Model Hypothesis*, one *de facto* remains open to the possibility of making the universe of sets more determinate. At the same time, one does not impose the restriction of consistency with "ZFC + large cardinal axioms". The approach of the Inner Model Hypothesis is not to "determine" the universe by directly postulating what sets exist in it (which is done when e.g. large cardinals are assumed to exist in V), but to state from a metatheoretical perspective what properties the universe of sets is supposed to possess.

Let us discuss the hypothesis in more detail. How can metatheoretical properties be identified which one may wish the universe V of sets to have? The suggestion made in [Fri06] is that one start from ZFC (or from a theory for sets and classes like Gödel-Bernays) and provisionally regard V as a model for it endowed with *countably many* sets (and classes). For a countable universe V many techniques are available for creating not only inner universes of V but also outer universes of V, i.e. universes V^* such that $V \subseteq V^*$, to which V can be compared. These techniques not only include (set and class) forcing, but also methods that arise from further generalizations of the forcing method (such as hyperclass forcing) or from infinitary model theory. Being able to compare V to a multitude of other universes enables one to better formulate properties that one wishes the intended

universe V to obey. The *Inner Model Hypothesis* takes advantage of this method of comparison:

If a statement ϕ without parameters holds in an inner universe of some outer universe of V (i.e. in some universe compatible with V), then it already holds in some inner universe of V.

Equivalently: statements that are *internally consistent* with respect to an outer universe of V are already *internally consistent* in V, where a statement is *internally consistent* if it holds in some inner universe. It follows that by enlarging V one gains nothing as far as internal consistency is concerned. So according to the Inner Model Hypothesis, V is *maximal* with respect to internal consistency.[18]

Although the IMH is formulated by supposing V to be countable, the Inner Model Hypothesis can also be formulated as a (weaker) hypothesis for an uncountable V. This is done by restricting the notion of outer universe to the set- and class-generic extensions of the given universe that preserve the Gödel-Bernays axioms, thereby reducing the hypothesis to a principle of ordinary class theory. Alternatively, one may regard the IMH as saying that although V itself is not countable, it should satisfy sentences that are true in countable universes which are maximal with respect to internal consistency. It is also worth noting that having the universe maximize internal consistency via the IMH *generalizes* a phenomenon known to hold for formulas (without parameters) proved to be consistent by set-forcing.[19]

One knows a lot about the consistency strength of the *Inner Model Hypothesis*. It is established by the following results.[20] 1) Assume that there is a Woodin cardinal and a larger inaccessible cardinal. Then there are universes which maximize internal consistency, so the Inner Model Hypothesis is consistent. 2) The Inner Model Hypothesis implies that there are inner models with measurable cardinals of arbitrarily large Mitchell order.

Note that by adopting the Inner Model Hypothesis, while not extending "ZFC + large cardinal axioms", one does appeal to large cardinals in two respects. First, large cardinal axioms are invoked for establishing its consistency strength. This acknowledges the major feature of the mathematical success of large cardinal axioms, their ability to prove consistency. The relevance of these axioms is seen here as metamathematical rather than as mathematical. Second, one asks whether the Inner Model Hypothesis has relevant implications with regard to large cardinals. This is in fact the case. Among the consequences of the Inner Model Hypothesis is that no *inaccessibles*, hence no large cardinals, exist in V and that the real numbers are not closed under the $\sharp$ operation. That is to say: not only is the Inner Model Hypothesis not an extension of the system

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[18]To put it in other terms, if $\mathfrak{L}$ = language of set/class theory and, for a universe W, $\Phi(W)$ = all sentences of $\mathfrak{L}$ which are true in some inner universe of W, then, under the Inner Model Hypothesis, if $V \subseteq W$ then $\Phi(V) = \Phi(W)$.

[19]See [Fri06] for the details of this claim.

[20]See [FWW08].

"ZFC + large cardinals"; it is also incompatible with it! The *consistency* of large cardinal axioms is however preserved under the IMH (V sees inner models for them); it is only their *existence* that is contradicted.

This latter point also has important consequences for the methodological notion of "maximization". The IMH clearly asserts a *maximal* property of the universe of sets, namely that internal consistency has been maximized. But it is at the same time in conflict with the existence of large cardinals. This is despite the fact that large cardinal axioms have also been traditionally assumed to assert a form of maximality for the universe of sets. Let us return to Gödel:

> From an axiom in some sense opposite to $[V = L]$, the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which ... would state some maximum property of the system of all sets, whereas $[V = L]$ states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set. ([Goe64], 262-3)

Note that there is no implication in this quote that "maximization" must be based on large cardinal axioms. And indeed, the IMH provides an alternative way of maximizing the universe of sets, thereby revealing the profound ambiguity of this concept.

What about questions which are independent from ZFC? Some of them are decided under the Inner Model Hypothesis, e.g. the Singular Cardinal Hypothesis and the existence of a projective non-measurable set of reals, which turn out to be true, and the existence of a Borel bijection between any two non-Borel analytic sets, which, instead, turns out to be false.[21] The Continuum Hypothesis remains undecided, though. For, suppose that V satisfies the Inner Model Hypothesis. One can create, by set forcing, a larger universe $V[G]$, in which CH is true (using a "Lévy collapse"). Since V is contained in $V[G]$, The Inner Model Hypothesis is also true in $V[G]$. So the hypothesis is consistent with CH. It cannot imply its negation. Similarly, one can create a larger universe $V[H]$ in which CH is false (by adding $\aleph_2$ Cohen reals), the Inner Model Hypothesis being true in $V[H]$. So the Inner Model Hypothesis cannot imply CH either. One needs a stronger version of the Inner Model Hypothesis to settle CH, i.e. the hypothesis for formulas

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[21]Theorem 15 in [Fri06] proves that that IMH implies the existence of a real R such that ZFC fails in $L_\alpha[R]$ for all ordinals α. This property implies that (a) for some real R, $\aleph_1 = \aleph_1^{L[R]}$, which in turn implies that (b) for some real R, $R^\sharp$ does not exist, which is equivalent to (c): for some real R, Jensen's covering property holds relative to $L[R]$ (i.e., every uncountable set of ordinals is a subset of a set in $L[R]$ of the same size). The truth of the *Singular Cardinal Hypothesis* and the *Singular Square principle* and the falsity of the existence of a Borel-isomorphism of non-Borel analytic sets (via the results presented in [Har78]) follow from (c), while the existence of a projective non-measurable set of reals (via the results in [She84]) follows from (a).

with globally absolute parameters.[22] A consistency proof for the resulting *Strong Inner Model Hypothesis* (SIMH) is however still lacking.

Let us conclude with a bold question. Will the *Inner Model Hypothesis*, and its implications, be accepted as a definitive feature of the universe, making it more determinate than it is now? According to the views presented throughout this paper, the considerable mathematical success of the IMH is to play a decisive role in this respect, whether or not one deliberately decides to attach correctness (truth) to the most successful set-theoretic hypotheses. But the philosophical implications of the IMH are clear, as it presents an important challenge to two widely-shared views in contemporary set theory.

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[22]See [Fri06].

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