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A Chinese Mathematician in Plato's Cave. Virtual/Real Dimensions of Internet Epistemology

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The internet is a digital universe governed by algorithms, where numbers and the discrete operate to give the illusion of a continuum. Who can say where the internet is, even though it connects all of us to a seemingly bondless expanse of information? When considering the topic of this VIIth Congress on Ontology, I began to think about what existence means in this digitalized world. Prior to 1987, my own research focused primarily on three figures:

- Georg Cantor, who systematically developed set theory and created an accompanying theory of transfinite numbers;
- Charles S. Peirce, the founder of Pragmatism, who among many other accomplishments developed a non-rigorous treatment of infinitesimals; and
- Abraham Robinson, who introduced a rigorous theory of infinitesimals in the context of model theory and mathematical logic.

All were concerned with problems of continuity and the infinite. But beginning in 1987, my research has also turned to another very different world, at least in some respects, of mathematics—namely to the history of mathematics in China, both ancient and modern. Consequently, I thought for this VIIth International Ontology Congress it might be worthwhile to explore briefly how the virtual environment created by the internet might be viewed from a Chinese perspective, and just to maintain the symmetry of ancient and modern, I plan to consider the views on mathematics and ontology of the 3rd-century Wei Dynasty Chinese mathematician Liu Hui of the Warring States Period, with those of the best-known living mathematician in China, Wu Wen-Tsun.

Extraction of Roots and Approximations for the Square-Root of 2

If we are interested in the connections between number and continuity, the first of many related problems were discovered at their earliest by the ancient Greeks. The Pythagoreans, who struggled with these concepts immediately come to mind—their initial hypothesis that all things could be expressed through numbers, by which they meant either whole numbers, the integers, or ratios of integers, i.e. fractions, seemed reasonable enough. However, this fundamental assumption was soon challenged by the counter-intuitive discovery of incommensurable magnitudes, and the realization that the numerical length of some magnitudes could only be approximated but never determined exactly by any rational number or finite decimal fraction.

What did the ancient Chinese have to say about such lengths as the diagonal of the square, or the ratio of the circumference to the diameter of the circle? Basically, Chinese mathematicians operated in a digitalized world, a mathematics of numbers rather than magnitudes. The earliest approximation method we have from the most ancient yet-known Chinese source, the 算数書 *Suan shu shu* (Book on Numbers and Computations, ca. 186 BCE) approximates the square-root of a number as follows. The example is from a problem devoted to方田 *Fang Tian* (Square Fields) [see Zhangjiashan 2001, p. 272; Peng 2001, p. 124; Wenwu 2000, p. 82; and Dauben 2007b]:

> [Problem 54]: (Given) a field of 1 *mu*, how many [square] *bu* are there? (The answer) says[a]: 14 15/31 [square] *bu*. The method says[b]: a square 15 *bu* (on each side) is deficient by 15 [square] *bu*; a square of 16 *bu* (on each side) is in excess by 16 [square] *bu*. (The method) says: combine the excess and deficiency as the divisor; (taking) the deficiency numerator multiplied by the excess denominator and the excess numerator times the deficiency denominator, combine them as the dividend.[c] Repeat this, as in the "method of finding the width."[d]

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a. This problems proceeds on the assumption that there are 240 *bu*^{2} to one *mu*. These were standard measures at the time the *Suan Shu Shu* was written, and the *bu*, a unit of length, is not differentiated in the text by a special designation when it is used as a measure of area, square *bu*, or *bu*^{2}. The *mu*, however, is a measure of area, like the acre, and so the situation here is similar to calculating the number of square feet or meters per acre.

b. The section of the text is not really devoted to the "method" of the problem, which actually begins with the following sentence. What follows here is simply a statement of the amount of excess or deficiency with respect to one square *mu* (240 square *bu*) of squares 14 and 15 *bu* on a side, respectively. The actual "method" of the problem is given in the following sentence.

c. This method reflects the way in which this problem is worked out on the counting board. If the numbers for the "deficiency" are put down on the left, those for the "excess" on the right, the top numbers are the lengths of the two squares of sides 15 and 16 *bu* each; under these are their respective amounts of deficiency or excess. The layout:

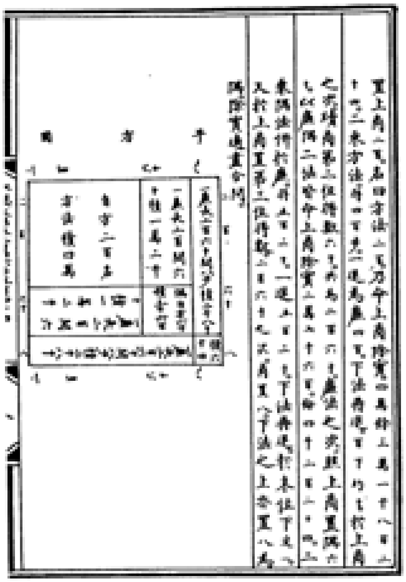
deficiency numerator excess numerator	不足子	*bu zu zi*	15 16	贏子	*ying zi*
deficiency denominator excess denominator	不足母	*bu zu mu*	15 16	贏母	*ying mu*

d. The appropriate method is actually found in the *Qi Cong* Problem 65: "Finding the Length," rather than in the *Qi Guang* Problem 64: "Finding the Width." According to the method described here, using the method of excess and deficiency to solve this problem leads to the following computation: (15×16+16×15)/(16+15) = 480/31 = 15 15/31 *bu* [for additional details, see Dauben 2007b].

Here the square root of 240 is simply approximated using the method of excess and deficiency, and while 15 15/31 is not an exact result, it is close enough to suit the needs of the *Suan shu shu*.

By the time Liu Hui wrote his commentary on the *Nine Chapters* in 263 CE, an algorithm had been developed to approximate square roots as precisely was one might wish. Liu Hui appreciated the fact that some numbers have exact square roots, some not. For those numbers for which an integer root could not be exhausted, there was a special term or expression, 不可開 *bu ke kai* (it cannot be extracted), i.e. it "does not end" or "it is not exact." For mathematicians who found 3 a good-enough approximation for *pi*, taking 3 as the square root for √10 was perhaps also good enough.

But this was much too inaccurate for Liu Hui, who describes the following procedure for extracting the root of a given number in his commentary on problem X in chapter Y of the *Nine Chapters*. Let the given number be N; let a be the largest integer such that $a^2=A<N$. To illustrate the method Liu Hui describes, the following diagram speaks for itself. This diagram is from the 永樂大典 *Yongle dadian* (Yongle Encyclopedia, 1403 CE), a massive Ming Dynasty compendium completed in the early 15th century; the illustration is meant to show how the square root of *N* is to be extracted [Lam and Ang 2004, p. 106].



The first step just described is to determine the largest integer *a* that when squared is the unshaded square in the diagram such that $a^2=A<N$. The difference, N-A, is then represented by the shaded portion in the diagram. One then finds the next largest integer b such that $A+(2ba+b^2)$ does not exceed N. Then N-$(A+(2ba+b^2))$ leaves another gnomon, and the algorithm continues analogously until one either finds an exact value for the square roof of *N*, in which case the algorithm stops, or one stops at some convenient point with a fractional root with a gnomon-remainder and the comment "bu ke kai"—it does not end, meaning no exact root has been found [Qian 1963, vol. 1, p. 150]. However, and this deserves emphasis: <u>No proof is offered that if one chose to continue this process, an exact root might yet be determined, *or not*</u>.

Did Ancient Chinese Mathematicians Know that √2 is Irrational?

Recently, some historians of Chinese mathematics have suggested that by virtue of his reference to "bu ke kai," Liu Hui was not only aware of the limitations of square root approximations, but he was also aware that some magnitudes were in fact incommensurable. This argument was first made by Alexei Volkov in the West [Volkov 1985], by the Chinese historian of mathematics Li Jimin in the East [Li 1990], and most recently and forcefully by Karine Chemla and Agathe Keller [Chemla and Keller 2002].

In commenting on the case when a given number *N* for which the square root is sought is not exhausted by the square root algorithm, Liu Hui considers fractional approximations, "one of which he finds to be always smaller $(a+(A-a^2)/2a+1)$ and the other always larger $(a+(A-a^2)/2a)$, than the root," [Chemla and Keller 2002, p. 103]. Chemla and Keller then quote Liu Hui as follows:

> One *cannot determine its value* (*shu*, i.e., the value, the quantity of the root). Therefore, *it is only when* "one names it with 'side'" that one does not make any mistake (or, that there is no error) (emphasis ours) [Qian 1963: 15], [Chemla and Keller 2002, p. 117].

"Shu" here means "number"—but all this suggests is that Liu Hui appreciated the fact that his algorithm had failed to determine an exact value for the root; this is not, however, the same as proving that the root is actually irrational. With the above passage in mind, Chemla and Keller continue with their commentary as follows:

> We deduce that, in this case, the root being sought has a "value," a "quantity" (*shu*), even if it cannot be expressed in a way that "exhausts the inner constitution" of the magnitude considered with respect to unit. Note that our interpretation of the impossibility of "exactly" "exhausting the inner constitution" above fits with Liu Hui's discarding of fractional quantities as a possible result in such cases here. Moreover, the only solution for stating the value in an exact way is to introduce a way of naming it, as "side of *N*." However, one can also express the inner constitution of the magnitude with respect to unity approximately, by a pair of integers [Chemla and Keller 2002, p. 117].

Clearly Liu Hui appreciates that the *lü* or ratio of numbers he obtains to approximate the root does not in fact serve to do so exactly. But all that he knows from this is that the ratio or *lü* he has constructed, meant to express the ratio of the side to the diagonal of the square, does not give a precise value for the ratio of the actual magnitudes in question. What Liu Hui in fact says is that:

> If by extraction, (the number) is not used up, this means that one cannot extract (its root). You must then call it (the number with side (面 mian) [Qian 1963, p. 150].

What the Pythagoreans did in establishing the existence of incommensurable magnitudes was to assume that all magnitudes could be expressed as the ratio of two numbers a/b. Then, probably by an anthyphairetic argument, they found that there were cases in which given two magnitudes, a common unit could not be found. This is analogous to the assumption that the ratio 1/3, when computed, would lead to some finite results, but instead, 1/3 in its decimal expansion is .3333…, *bu ke kai*. But this "bu ke kai" does not prove the existence of irrational quantities, for 1/3 is clearly rational, but the division it represents never terminates, it never ends.

What is missing from this account is any argument, not to say "proof"—that incommensurable magnitudes or irrational numbers exist. Liu Hui never proves (or argues) that there is no *lü* for $\sqrt{2}$. This is an ontological problem that proved, as we know, to be of profound significance for the ancient Greeks.[1]

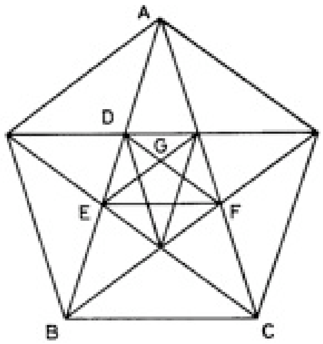
1 Perhaps Chemla and Keller, despite their arguments to the contrary, admit as much when they write: "This helps to make clearer the statement in Liu Hui's commentary that *comes closest* to an assertion of the irrational character of some ratios between magnitudes" (Chemla and Keller 2002, p. 117, emphasis added). Somewhat later in their paper they also admit that "The sharp demarcation that ancient Greek mathematical authors made between number and magnitude and that relates to this treatment of irrationality is not to be met with in ancient Chinese and Indian texts" [Chemla and Keller 2002, p. 122]. Realizing that there were magnitudes for which they could associate no corresponding rational number, the Greeks knew their arithmetic of ratios of whole numbers was incomplete. Had Chinese and Indian mathematicians dealing with square and cube roots also understood that there were geometric magnitudes for which there were no corresponding numerical equivalents in their system of numbers and ratios of numbers, they should have articulated a similar distinction. That they did not may in part be explained by the fact that the Chinese had no clear concept, mathematically, of magnitude in the sense that Euclid, for the most part, develops his principles of geometry, i.e. the *Elements*, in terms of magnitudes and not arithmetic, the limitations of which were dramatically revealed by the discovery of incommensurable magnitudes. For details on this subject, see "Does the *Nine Chapters* Include the Concept of Irrational Number?" [in Xu 2005, p. 62-89].

The Pythagoreans' Discovery of Incommensurability

How was incommensurability discovered? Kurt von Fritz, in examining both the Greek construction of the pentagon or pentagram, argued several decades ago that in either case, an infinite descent argument proceeds to establish the existence of incommensurable magnitudes [von Fritz 1945]. The process is based on the Euclidean algorithm, *anthyphairesis*, namely given two homogenous magnitudes A > B, subtracting the smaller from the larger gives a remainder C. If C > B, subtracting B from C leaves a remainder D. If C < B, subtracting C from B leaves a remainder D'. The process continues in this manner, and in the case of numbers, the algorithm terminates in a finite number of steps yielding the greatest common divisor of the two numbers. Euclid proved these results in Book VII, propositions 1 and 2 in the *Elements*. In Book X, proposition 3, the same is established for commensurable magnitudes, resulting in the greatest common measure between the two magnitudes (for details, see [Knorr 1975, p. 29]). In the case of incommensurable magnitudes, however, this process proceeds *ad infinitum*, and no common multiple will ever be reached, since the algorithm never terminates, but proceeds always proceeds to yield the same ratio of incommensurable magnitudes.

According to von Fritz, this was how the ancient Pythagoreans originally must have discovered the existence of incommensurable magnitudes, specifically from their construction of the regular pentagon or pentagram, and their realization that if two magnitudes are in mean and extreme ratio, applying the Euclidean algorithm, subtracting the smaller from the larger to find a greatest common measure, led to a succession of magnitudes always in the same mean and extreme ratio. Consequently, the *anthyphairesis* in this case continues with no end. Thus any two magnitudes in mean and extreme ratio were necessarily incommensurable, and there was no least common multiple of the two magnitudes because there was no end to the anthyphairetic algorithm.

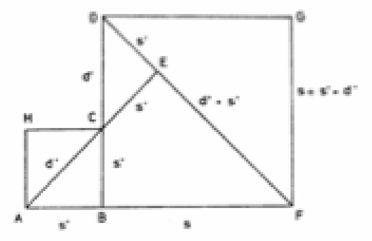
In considering the diagram of inscribed pentagons and pentagrams on the right, the diagonal AB and side BC are in mean and extreme ratio, meaning that AB is the mean proportional between BC and their sum AB+BC, or equivalently, BC:AB=AB:(AB+BC), or AB^2=BC(AB+BC). Realizing that the point E divides the diagonal AB into mean and extreme ratio, and that the ratio of the diagonal EF and the side DE are also in mean and extreme ratio, continuing this process always leads to another and another, and indeed, to an infinite sequence of inscribed figures whose sides and diagonals are always in mean and extreme ratio, *ad infinitum*.



If we now apply the Euclidean algorithm in hopes of finding a unit that will serve as a common multiple of both AB and BC, or EF and DE, we will generate an endless series of pentagons/pentagrams where EF:DE (in the diagram above) will always be in the same ratio as that of AB:BC. If AB and BC, or EF and DE were in numerical ratio, or were commensurable, the Euclidean algorithm generating successively inscribed pentagons or pentagrams would terminate after a finite number of steps, yielding their common measure. The fact that this does not happen means that magnitudes in mean and extreme ratio are incommensurable. Thus the Euclidean algorithm will never come to an end, and there will not be a common unit magnitude that may serve as a multiple of both AB and BC or EF and DE.

Slightly more than a decade after von Fritz advanced the idea that the ancient Pythagoreans discovered incommensurability via the anthyphairetic algorithm applied to two magnitudes in mean and extreme ration, Siegfried Heller suggested that it was more likely made as a result of applying the same anthyphairetic process to the side and diagonal of the square [Heller 1958].

The outlines of the argument may be seen with reference to the diagram below [from Knorr 1975, p. 32]. Applying the Euclidean algorithm to find a common magnitude to serve as a multiple of both the side and diagonal, first subtract the length of the side BF from the diagonal DF, and the remainder s□ is the side of another square with diagonal d□ . Subtracting this side s□ from the diagonal d□ leaves another remainder s□ , which again constitutes a square of side s□ and diagonal d□ .



Since the side and diagonal of the successive squares determined by this application of the Euclidean algorithm are always in the same ratio, the algorithm will never reach a common magnitude that could be taken as the measure of both the side and diagonal of the square. Consequently, BF and DF must be incommensurable. QED.

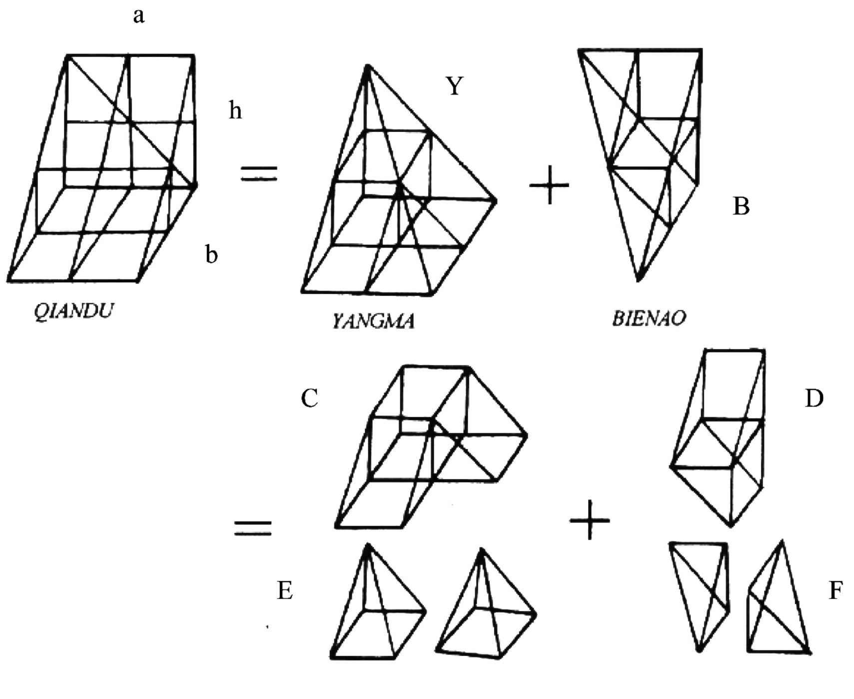
Wilbur Knorr, in his detailed study of the problem of discovery of incommensurability, however, understands the problem of *anthyphairesis* in early Greek geometry somewhat differently:

> The Pythagoreans employed the algorithm not as a theoretical device for proving the irrationality of the diameter, but as a practical device for approximating it. But it is conceivable that the discovery of such an algorithm, yielding an infinity of values always approaching but never equaling the limiting value, might initially have been misconstrued as a proof [of] incommensurability [Knorr 1975, pp. 33-34].

Knorr prefers to understand the Greek's discovery of incommensurability in terms of the side and diagonal of the square in terms of the properties of even and odd numbers, in the classic formulation given by Aristotle in *Prior Analytics* I.23, 41a29: "if the side and diameter are assumed commensurable with each other, one may deduce that odd numbers equal even numbers; this contradiction then affirms the incommensurability of the given magnitudes," [Knorr 1975, p. 23]. Knorr also offers several ingenious interpretations of how a Pythagorean poof might have proceeded, based upon the geometry of the square and the ratio of the side to the diagonal [Knorr 1975, p. 26-28]. In each case, the argument proceeds by contradiction, assuming the commensurability in question, just as Aristotle uses the example of the ratio of the side and diagonal of the square to demonstrate the technique of reasoning *per impossible* in the *Prior Analytics*.

Liu Hui's Proof for the Volume of the Right Square-Based Pyramid V=(1/3)abh

Like the Greek treatments of algorithms to determine the ratio of the side and diagonal of the square, or those of the pentagon or pentagram, involving applications of the Euclidean algorithm that continue anthyphairetically *ad infinitum*, there is an interesting example to be found among Liu Hui's comments in the *Nine Chapters*, an algorithm that does not terminate, "bu ke kai," but one that leads to a very different conclusion from the anthyphairetic case in Greek mathematics. This concerns the Chinese determination of the volume of the right pyramid with rectangular base, called a *yangma*. Liu Hui knew that the formula $V=(1/3)abh$ was exact (where *a* is the length and *b* the width of the rectangular base of the pyramid, and *h* the height as in the *yangma* below), but how to prove it?



The argument runs something like this in the case of the above diagrams. The volume of two *qiandu* is clearly *abh*. If one *qiandu* is equal to one *yangma* (the square based pyramid) and one *bienao*, and if the *bienao* is ½ a *yangma*, then the volume of the *yangma* must be $(1/3)abh$. The proof that this is so proceeds by an "infinite descent argument." The *yangma* above, if bisected at half its height, can be broken down into the volumes C and D, where the volume of C, equivalent to four *qiandu*, is exactly twice the volume of the two *qiandu* of D. Thus, to prove that the *yangma* is twice the *bienao*, it sufficies to show that the two remaining *yangma* volumes E are exactly twice the volume with respect to the two *bienao* F. But it is easy to see that these volumes are similar to the initial *yangma* and *bienao*. By bisecting the volumes E and F, Liu Hui reiterates the same argument, but with components half as large each time. As Jean-Claude Martzloff puts it in his discussion of Liu Hui's method, "Thus, he obtains more and more portions of the initial *yangma* which are themselves in the desired proportion. The remaining parts decrease constantly and, after passing to the limit, he concludes that Y=2B," [Martzloff 1997, pp. 284-285; see also Lloyd 1996, pp. 152-156, and Wu 2000, p. 60-61, and Chemla and Guo 2004, pp. 396-398]. In this case, the process *ad infinitium* serves to establish the proposition, that since in all cases the ratio of the *yangma* to *bienao* is 2:1, this suffices to establish the general proposition that the volume of the pyramid is (1/3)*abh*.

Liu Hui and Approximations for the Value of Pi

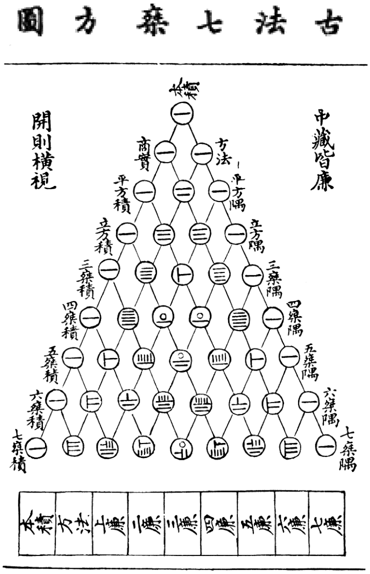
One last example will suffice here to illustrate how ancient Chinese mathematicians approached the non-terminating approximation of certain ratios, namely of the diameter to the circumference of the circle. Liu Hui notes that the traditional value was exactly the perimeter of the hexagon inscribed in a circle of unit diameter, and thus fell short of what the actual value of the circumference of the circle should be. To get a closer approximation, Liu Hui considered successively larger inscribed regular polygons, increasing from the 6-sided to 12-sided to 24-sided all the way up to the 192-sided polygon for which he sowed that the ratio of circumference to the diameter of a circle must fall between 3.14 64/625 and 3.14 169/625.

Here, the diagram on the right from the 永樂大典 *Yongle dadian* (Yongle Encyclopedia, 1403 CE) [Chapter 16,344, p. 8a] shows a figure accompanying Liu Hui's commentary in the *Nine Chapters,* which shows how closely Liu Hui's thinking was to the familiar Greek approach to the approximation of *pi* taken by Archimedes. As Liu Hui explains the argument in question: "The finer we cut the segments, the less will be the loss. Cut further and further until unable to cut further. Then [the polygon] will coincide with the circle and there will be no more any loss," [translated by Wu Wen-Tsun 2000, p. 64].

Wu Wen-tsun and the Mechanization of Mathematical Proofs

It is generally agreed that the "golden age" of ancient Chinese mathematics was reached in the Song and Yuan dynasties. A good example of the power and generality of the methods achieved in this period is that of Yang Hui (ca. 1238-1298) of the late Southern Song Dynasty. In commenting on the works of one of Yang Hui's forerunners, Jia Xian, Wu Wen-Tsun explains:

> According to Yang Hui's works, there already occurred in Jia's time some diagrams bearing the name of *Root-Extraction Basic Diagrams*. Such diagrams are actually the same as the so-called Pascalian Triangle of 17th century. Thus, it seems that Jia had freed himself from geometrical considerations and, with the aid of the root-extraction basic diagrams, had discovered his methods directly from a generalization of the *arithmetized* Root-Extraction Shu of his ancestors [Wu 2000, p. 22; note that *Shu* means "method"].



But this was not all. The fullest generalization of algebraic methods in ancient Chinese methods came with the "tian yuan" or "heavenly element" method. It was the mathematician Zhu Shijie who generalized this method to enable the solution of simultaneous sets of equations in as many as four unknowns. As Wu Wen-tsun describes the method, he also accounts for the reason why it was limited to equations in no more than four unknowns:

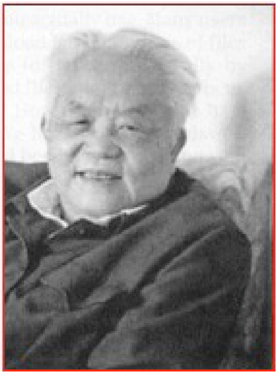
The Chinese version of the Pascal triangle shown on the left is from the四元玉鑑 *Si Yuan Yu Jian* (Jade Mirror of the Four Unknowns, 1303) of 朱世杰 Zhu Shijie (1260-1320) [Guo and Guo 2006, vol. 1, p. 32].

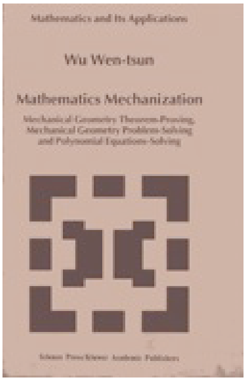
> For the actual computation at the time of Zhu one had to place the coefficients of various kinds of terms of polynomials in counting rods at definite positions of the counting board. This limited the method to at most four equations in four unknowns and only quite simple ones can be so treated. However, that the method of Zhu enjoys a general character which can be applied to arbitrary systems of equations is quite clear [Wu

> 2000, p. 26].[2]

In his approach to the mechanization of mathematics, Wu found a useful parallel in the methods of Descartes, what he calls "Descartes' Program," which he compares with Chinese mathematics as follows:

> In fact … it seems clear that Descartes had the attitude of emphasizing on geometry problem-solving by means of equations-solving rather than geometry theorem-proving, just in the same spirit of our ancestors… In a word, it may be said that Chinese ancient mathematics in the main were developed along the way as indicated in Descartes' Program, and conversely, Descartes' Program may be considered as an overview of the way of developments of Chinese ancient mathematics… [Wu 2000, p. 32-33].





2 The limitation arises due to the delegation of unknowns to the four cardinal regions of the counting board, each corresponding to an unknown x,y,z,w (bottom, left, top, right), schematically, as follows:

⋮	⋮	⋮		⋮	⋮	⋮	⋮
....y^3u^2	y^2u^2	yu^2		u^2	zu^2	z^2u^2	z^3u^2....
....y^3u	y^2u	yu		u	zu	z^2u	z^3u....
					$\boxed{yz}$		
....y^3	y^2	y			z	z^2	z^3....
			$\boxed{xu}$				
....xy^3	xy^2	xy		x	xz	xz^2	xz^3....
....x^2y^3	x^2y^2	x^2y		x^2	x^2z	x^2z^2	x^2z^3....
....x^3y^3	x^3y^2	x^3y		x^3	x^3z	x^3z^2	x^3z^3....
⋮	⋮	⋮		⋮	⋮	⋮	⋮

Wu also acknowledges that the method is subject to "great defects and required clarification." In fact, one of the main objectives of his book on *Mathematics Mechanization* was to "give a solid mathematical foundation of methods influenced by that of Zhu" [Wu 2000, p. 26].

When the *Notices of the AMS* announced the winners of the 2006 Shaw Prize in October of 2006, the citation read as follows:

> In the 1970s Wu turned his attention to questions of computation, in particular the search for effective methods of automatic machine proofs in geometry. In 1977 Wu introduced a powerful mechanical method, based on Ritt's concept of characteristic sets. This transforms a problem in elementary geometry into an algebraic statement about polynomials that lends itself to effective computation. This method of Wu completely revolutionized the field, effectively provoking a paradigm shift. Before Wu the dominant approach had been the use of AI search methods, which proved a computational dead end.
>
> By introducing sophisticated mathematical ideas Wu opened a whole new approach that has proved extremely effective on a wide range of problems, not just in elementary geometry… Under his leadership mathematics mechanization has expanded in recent years into a rapidly growing discipline, encompassing research in computational algebraic geometry, symbolic computation, computer theorem proving, and coding theory [*AMS Notices* 2006, pp. 1054-55].

The great success of Wu Wen-tsun's interest in mechanizing mathematics was to discover a means of translating geometric problems into algebraic equivalents subject to algorithmic solutions. This had long been a hallmark of the Chinese mathematical mind-set, so to speak, from antiquity to the present, and in the present, it has been the transformation of those ancient mathematical procedures by Wu Wen-tsun that has led to new and profound methods of proof on the very same terms that rule in internet ontology. By translating the terms of a problem from the continuous space of geometry to its algebraic equivalent, and then subjecting the latter to a suitable algorithmic interpretation, computers can then be programmed to provide computational verification. This means of proceeding algorithmically was one of the great strengths of ancient Chinese mathematics, and as Wu Wen-tsun has often acknowledged, it was in the methods of his predecessors that he found inspiration for the very modern applications of those methods in his own work on mechanical problem solving. In this sense, therefore, Chinese mathematicians ancient and modern have long operated and in very productive ways continue to work in a highly digital world, where algorithmic thinking is a particularly successful approach to conceptualizing and solving mathematical problems.

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