SUPERATOMIC BOOLEAN ALGEBRAS CONSTRUCTED FROM STRONGLY UNBOUNDED FUNCTIONS

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Abstract. Using Koszmider's strongly unbounded functions, we show the following consistency result:

Suppose that κ, λ are infinite cardinals such that $\kappa^{+++} \le \lambda$, $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, and η is an ordinal with $\kappa^+ \le \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$. Then, in some cardinal-preserving generic extension there is a superatomic Boolean algebra $\mathbb{B}$ such that $\mathrm{ht}(\mathbb{B}) = \eta + 1$, $\mathrm{wd}_\alpha(\mathbb{B}) = \kappa$ for every $\alpha < \eta$ and $\mathrm{wd}_\eta(\mathbb{B}) = \lambda$ (i.e. there is a locally compact scattered space with cardinal sequence $\langle\kappa\rangle_\eta {}^\frown \langle\lambda\rangle$).

Especially, $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_3\rangle$ and $\langle\omega_1\rangle_{\omega_2} {}^\frown \langle\omega_4\rangle$ can be cardinal sequences of superatomic Boolean algebras.

1. Introduction

A Boolean algebra $\mathcal{B}$ is *superatomic* iff every homomorphic image of $\mathcal{B}$ is atomic. Under Stone duality, homomorphic images of a Boolean algebra $\mathcal{A}$ correspond to closed subspaces of its Stone space $S(\mathcal{A})$, and atoms of $\mathcal{A}$ correspond to isolated points of $S(\mathcal{A})$. Thus $\mathcal{B}$ is superatomic iff its dual space $S(\mathcal{B})$ is *scattered*, i.e. every non-empty (closed) subspace has some isolated point.

For every Boolean algebra $\mathcal{A}$, let $\mathcal{I}(\mathcal{A})$ be the ideal generated by the atoms of $\mathcal{A}$. Define, by induction on α, the α^{th} *Cantor-Bendixson ideal* $\mathcal{J}_\alpha(\mathcal{A})$, and the α^{th} *Cantor-Bendixson derivative* $\mathcal{A}^{(\alpha)}$ of $\mathcal{A}$ as follows. If $\mathcal{J}_\alpha(\mathcal{A})$ has been defined, put $\mathcal{A}^{(\alpha)} = \mathcal{A}/\mathcal{J}_\alpha(\mathcal{A})$ and let $\pi_\alpha : \mathcal{A} \longrightarrow \mathcal{A}^{(\alpha)}$ be the canonical map. Define $\mathcal{J}_0(\mathcal{A}) = \{0_\mathcal{A}\}$, $\mathcal{J}_{\alpha+1}(\mathcal{A}) = \pi_\alpha^{-1}[\mathcal{I}(\mathcal{A}^{(\alpha)})]$, and for α a limit $\mathcal{J}_\alpha(\mathcal{A}) = \bigcup\{\mathcal{J}_{\alpha'}(\mathcal{A}) : \alpha' < \alpha\}$. It is easy to see that the sequence of the ideals $\mathcal{J}_\alpha(\mathcal{A})$ is increasing. And it is a well-known fact that a non-trivial Boolean algebra $\mathcal{A}$ is superatomic iff there is an ordinal α such that $\mathcal{A} = \mathcal{J}_\alpha(\mathcal{A})$ (see [4, Proposition 17.8]).

Assume that $\mathcal{B}$ is a superatomic Boolean algebra. The *height* of $\mathcal{B}$, $ht(\mathcal{B})$, is the least ordinal δ such that $\mathcal{B} = \mathcal{J}_\delta(\mathcal{B})$. This ordinal δ is always a successor ordinal. Then, we define the *reduced height* of B, $ht^-(\mathcal{B})$, as the least ordinal δ such that $\mathcal{B} = \mathcal{J}_{\delta+1}(\mathcal{B})$. It is well-known that if $ht^-(\mathcal{B}) = \delta$, then $\mathcal{J}_{\delta+1}(\mathcal{B}) \setminus \mathcal{J}_\delta(\mathcal{B})$ is a finite set. For each $\alpha < ht^-(\mathcal{B})$ let $wd_\alpha(\mathcal{B}) = |\mathcal{J}_{\alpha+1}(\mathcal{B}) \setminus \mathcal{J}_\alpha(\mathcal{B})|$, the

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number of atoms in $\mathcal{B}/\mathcal{J}_\alpha(\mathcal{B})$. The *cardinal sequence* of $\mathcal{B}$, $CS(\mathcal{B})$, is the sequence $\langle wd_\alpha(\mathcal{B}) : \alpha < ht^-(\mathcal{B})\rangle$.

Let us turn now our attention from Boolean algebras to topological spaces for a moment. Given a scattered space X, define, by induction on α, the α^{th} *Cantor-Bendixson derivative* X^α of X as follows: $X^0 = X$, $X^\alpha = \bigcap_{\beta<\alpha} X^\beta$ for α a limit, and $X^{\alpha+1} = X^\alpha \setminus I(X^\alpha)$, where $I(Y)$ denotes the set of isolated points of a space Y. The set $I_\alpha(X) = X^\alpha \setminus X^{\alpha+1}$ is the α^{th} *Cantor-Bendixson level* of X. The *reduced height* of X, $ht^-(X)$, is the least ordinal δ such that X^δ is finite (and so $X^{\delta+1} = \emptyset$). For $\alpha < ht^-(X)$ let $wd_\alpha(X) = |I_\alpha(X)|$. The *cardinal sequence* of X, $CS(X)$, is defined as $\langle wd_\alpha(X) : \alpha < ht^-(X)\rangle$.

It is well-known that if $\mathcal{B}$ is a superatomic Boolean algebra, then the dual space of $\mathcal{B}^{(\alpha)}$ is $(S(\mathcal{B}))^{(\alpha)}$ (see [4, Construction 17.7]). So $ht^-(\mathcal{B}) = ht^-(S(\mathcal{B}))$, and $wd_\alpha(\mathcal{B}) = wd_\alpha(S(\mathcal{B}))$ for each $\alpha < ht^-(\mathcal{B})$, that is, $\mathcal{B}$ and $S(\mathcal{B})$ have the same cardinal sequences.

In this paper we consider the following problem: given a sequence $\mathbf{s}$ of infinite cardinals, construct a superatomic Boolean algebra having $\mathbf{s}$ as its cardinal sequence.

For basic facts and results on superatomic Boolean algebras and cardinal sequences we refer the reader to [4] and [8]. We shall use the notation $\langle\kappa\rangle_\alpha$ to denote the constant κ-valued sequence of length α. Let us denote the concatenation of two sequences f and g by $f{}^\frown g$. If η is an ordinal we denote by $\mathcal{C}(\eta)$ the family of all cardinal sequences of superatomic Boolean algebras whose reduced height is η.

Definition 1. If κ, λ are infinite cardinals and η is an ordinal, we say that a superatomic Boolean algebra $\mathcal{B}$ is a (κ, η, λ)-*Boolean algebra* iff $CS(\mathcal{B}) = \langle\kappa\rangle_\eta {}^\frown \langle\lambda\rangle$, i.e. if $\mathrm{ht}(\mathcal{B}) = \eta + 1$, $\mathrm{wd}_\alpha(\mathcal{B}) = \kappa$ for each $\alpha < \eta$ and $\mathrm{wd}_\eta(\mathcal{B}) = \lambda$.

An $(\omega, \omega_1, \omega_2)$-Boolean algebra is called a *very thin-thick Boolean algebra*. And, for an infinite cardinal κ, a $(\kappa, \kappa^+, \kappa^{++})$-Boolean algebra is called a κ-*very thin-thick Boolean algebra*.

By using the combinatorial notion of the *new* Δ *property (NDP)* of a function, it was proved by Roitman that the existence of an $(\omega, \omega_1, \omega_2)$-Boolean algebra is consistent with ZFC (see [7] and [8]). It is worth to mention that [7] was the first paper in which such a special function was used to guarantee the chain condition of a certain poset. Roitman's result was generalized in [3], where for every infinite regular cardinal κ, it was proved that the existence of a $(\kappa, \kappa^+, \kappa^{++})$-Boolean algebra is consistent with ZFC. Then, our aim here is to prove the following stronger result.

Theorem 2. *Assume that κ, λ are infinite cardinals such that $\kappa^{+++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$. Then for each ordinal η with $\kappa^+ \leq \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$, in some cardinal-preserving generic extension there is a (κ, η, λ)-Boolean algebra, i.e. $\langle\kappa\rangle_\eta {}^\frown \langle\lambda\rangle \in \mathcal{C}(\eta + 1)$.*

Corollary 3. *The existence of an* $(\omega, \omega_1, \omega_3)$*-Boolean algebra is consistent with ZFC. An* $(\omega_1, \omega_2, \omega_4)$*-Boolean algebra may also exist.*

In order to prove Theorem 2, we shall use the main result of [5].

Definition 4. Assume that κ, λ are infinite cardinals such that κ is regular and $\kappa < \lambda$. We say that a function $F : [\lambda]^2 \longrightarrow \kappa^+$ is a κ^+*-strongly unbounded function on* λ iff for every ordinal $\delta < \kappa^+$, every cardinal $\nu < \kappa$ and every family $A \subseteq [\lambda]^\nu$ of pairwise disjoint sets with $|A| = \kappa^+$, there are different $a, b \in A$ such that $F\{\alpha, \beta\} > \delta$ for every $\alpha \in a$ and $\beta \in b$.

The following result was proved in [5].

Koszmider's Theorem . *If* κ, λ *are infinite cardinals such that* $\kappa^{+++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$ *and* $2^\kappa = \kappa^+$*, then there is a* κ*-closed and cardinal-preserving partial order that forces the existence of a* κ^+*- strongly unbounded function on* λ*.*

So, in order to prove Theorem 2 it is enough to show the following result.

Theorem 5. *Assume that* κ, λ *are infinite cardinals with* $\kappa^{+++} \leq \lambda$ *and* $\kappa^{<\kappa} = \kappa$*, and* η *is an ordinal with* $\kappa^+ \leq \eta < \kappa^{++}$ *and* $\mathrm{cf}(\eta) = \kappa^+$*. Assume that there is a* κ^+*- strongly unbounded function on* λ*. Then, there is a cardinal-preserving partial order that forces the existence of a* (κ, η, λ)*-Boolean algebra.*

In [3], [6], [7] and in many other papers, the authors proved the existence of certain superatomic Boolean algebras in such a way that instead of constructing the algebras directly, they actually produced certain "graded posets" which guaranteed the existence of the wanted superatomic Boolean algebras. From these constructions, Bagaria, [1], extracted the following notion and proved the Lemma 7 below which was implicitly used in many earlier papers.

Definition 6 ([1])**.** Given a sequence $\mathfrak{s} = \langle \kappa_\alpha : \alpha < \delta \rangle$ of infinite cardinals, we say that a poset $\langle T, \prec \rangle$ is an $\mathfrak{s}$*-poset* iff the following conditions are satisfied:

(1) $T = \bigcup\{T_\alpha : \alpha < \delta\}$ where $T_\alpha = \{\alpha\} \times \kappa_\alpha$ for each $\alpha < \delta$.
(2) For each $s \in T_\alpha$ and $t \in T_\beta$, if $s \prec t$ then $\alpha < \beta$.
(3) For every $\{s, t\} \in \left[T\right]^2$ there is a finite subset $\mathrm{i}\{s, t\}$ of T such that for each $u \in T$:

$$(u \preceq s \wedge u \preceq t) \text{ iff } u \preceq v \text{ for some } v \in \mathrm{i}\{s, t\}.$$

(4) For $\alpha < \beta < \delta$, if $t \in T_\beta$ then the set $\{s \in T_\alpha : s \prec t\}$ is infinite.

Lemma 7 ([1, Lemma 1])**.** *If there is an* $\mathfrak{s}$*-poset then there is a superatomic Boolean algebra with cardinal sequence* $\mathfrak{s}$*.*

Actually, if $\mathcal{T} = \langle T, \prec \rangle$ is an $\mathfrak{s}$-poset, we write $U_{\mathcal{T}}(x) = \{y \in T : y \preceq x\}$ for $x \in T$, and we denote by $X_{\mathcal{T}}$ the topological space on T whose subbase is the family

$$\{U_{\mathcal{T}}(x), T \setminus U_{\mathcal{T}}(x) : x \in T\}, \tag{1}$$

then $X_{\mathcal{T}}$ is a locally compact, Hausdorff, scattered space whose cardinal sequence is $\mathfrak{s}$, and so the clopen algebra of the one-point compactification of $X_{\mathcal{T}}$ is the required superatomic Boolean algebra with cardinal sequence $\mathfrak{s}$.

So, to prove Theorem 5 it will be enough to show that $\langle\kappa\rangle_\eta{}^\frown\langle\lambda\rangle$-posets may exist for κ, η and λ as above.

The organization of this paper is as follows. In Section 2, we shall prove Theorem 5 for the special case in which $\kappa = \omega$ and $\lambda \geq \omega_3$, generalizing in this way the result proved by Roitman in [7]. In Section 3, we shall define the combinatorial notions that make the proof of Theorem 5 work. And in Section 4, we shall present the proof of Theorem 5.

2. Generalization of Roitman's Theorem

In this section, our aim is to prove the following result.

Theorem 8. *Let λ be a cardinal with $\lambda \geq \omega_3$. Assume that there is an ω_1-strongly unbounded function on λ. Then, in some cardinal-preserving generic extension for each ordinal η with $\omega_1 \leq \eta < \omega_2$ and $\mathrm{cf}(\eta) = \omega_1$ there is an (ω, η, λ)-Boolean algebra.*

The theorem above is a bit stronger than Theorem 5 for $\kappa = \omega$, because the generic extension does not depend on η. However, as we will see, its proof is much simpler than the proof of the general case.

By Lemma 7, it is enough to construct a c.c.c. poset $\mathcal{P}$ such that in $V^{\mathcal{P}}$ for each $\eta < \omega_2$ with $\mathrm{cf}(\eta) = \omega_1$ there is an $\langle\omega\rangle_\eta{}^\frown\langle\lambda\rangle$-poset.

For $\eta = \omega_1$ it is straightforward to obtain a suitable $\mathcal{P}$: all we need is to plug Kosmider's strongly unbounded function into the original argument of Roitman. For $\omega_1 < \eta < \omega_2$ this simple approach does not work, but we can use the "stepping-up" method of Er-rhaimini and Veličkovic from [2]. Using this method, it will be enough to construct a single $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$-poset (with some extra properties) to obtain $\langle\omega\rangle_\eta{}^\frown\langle\lambda\rangle$-posets for each $\eta < \omega_2$ with $\mathrm{cf}(\eta) = \omega_1$.

To start with, we adapt the notion of a skeleton introduced in [2] to the cardinal sequences we are considering.

Definition 9. Assume that $\mathcal{T} = \langle T, \prec\rangle$ is an $\mathfrak{s}$-poset such that $\mathfrak{s}$ is a cardinal sequence of the form $\langle\kappa\rangle_\mu{}^\frown\langle\lambda\rangle$ where κ, λ are infinite cardinals with $\kappa < \lambda$ and μ is a non-zero ordinal. Let i be the infimum function associated with $\mathcal{T}$. Then:

(a) For $\gamma < \mu$ we say that T_γ, the γ^{th}-level of $\mathcal{T}$, is a *bone level* iff the following holds:
   (1) $i\{s, t\} = \emptyset$ for every $s, t \in T_\gamma$ with $s \neq t$.
   (2) If $x \in T_{\gamma+1}$ and $y \prec x$ then there is a $z \in T_\gamma$ with $y \preceq z \prec x$.

(b) We say that $\mathcal{T}$ is a *μ-skeleton* iff T_γ is a bone level of $\mathcal{T}$ for each $\gamma < \mu$.

The next statement can be proved by a straightforward modification of the proof of [2, Theorem 2.8].

Theorem 10. *Let κ, λ be infinite cardinals. If there is a $\langle\kappa\rangle_{\kappa^+} {}^\frown\langle\lambda\rangle$-poset which is a κ^+-skeleton, then for each $\eta < \kappa^{++}$ with $cf(\eta) = \kappa^+$ there is a $\langle\kappa\rangle_{\eta} {}^\frown\langle\lambda\rangle$-poset.*

So, to get Theorem 8 it is enough to prove the following result.

Theorem 11. *Let λ be a cardinal with $\lambda \geq \omega_3$. Assume that there is an ω_1-strongly unbounded function on λ. Then, in some c.c.c. generic extension there is an $\langle\omega\rangle_{\omega_1} {}^\frown\langle\lambda\rangle$-poset which is an ω_1-skeleton.*

Let $F : [\lambda]^2 \longrightarrow \omega_1$ be an ω_1- strongly unbounded function on λ. In order to prove Theorem 11, we shall define a c.c.c. forcing notion $\mathcal{P} = \langle P, \leq\rangle$ that adjoins an $\mathfrak{s}$-poset $\mathcal{T} = \langle T, \preceq\rangle$ which is an ω_1-skeleton, where $\mathfrak{s}$ is the cardinal sequence $\langle\omega\rangle_{\omega_1} {}^\frown\langle\lambda\rangle$.

First, we define the underlying set of the required $\mathfrak{s}$-poset.

Definition 12.
(a) We put $T = \bigcup\{T_\alpha : \alpha \leq \omega_1\}$ where $T_\alpha = \{\alpha\} \times \omega$ for $\alpha < \omega_1$ and $T_{\omega_1} = \{\omega_1\} \times \lambda$.
(b) If $s = (\alpha, \nu) \in T$, we write $\pi(s) = \alpha$ and $\xi(s) = \nu$.

Definition 13. We define the poset $\mathcal{P} = \langle P, \leq\rangle$ as follows.

(a) We say that $p = \langle X, \preceq, i\rangle \in P$ iff the following conditions hold:
- (P1) X is a finite subset of T.
- (P2) $\preceq$ is a partial order on X such that $s \prec t$ implies $\pi(s) < \pi(t)$.
- (P3) $i : [X]^2 \longrightarrow [X]^{<\omega}$ is an infimum function, that is, a function such that for every $\{s, t\} \in [X]^2$ we have:
$$\forall x \in X([x \preceq s \wedge x \preceq t] \text{ iff } x \preceq v \text{ for some } v \in i\{s, t\}).$$
- (P4) If $s, t \in X \cap T_{\omega_1}$ and $v \in i\{s, t\}$, then $\pi(v) \in F\{\xi(s), \xi(t)\}$.
- (P5) If $s, t \in X$ with $\pi(s) = \pi(t) < \omega_1$, then $i\{s, t\} = \emptyset$.
- (P6) If $s, t \in X$, $s \prec t$ and $\pi(t) = \alpha + 1$, then there is a $u \in X$ such that $s \preceq u \prec t$ and $\pi(u) = \alpha$.

(b) If $\langle X, \preceq, i\rangle, \langle X', \preceq', i'\rangle \in P$ we put $\langle X', \preceq', i'\rangle \leq \langle X, \preceq, i\rangle$ iff $X \subseteq X'$, $\preceq = \preceq' \cap (X \times X)$ and $i \subseteq i'$.

We will need condition (P4) in order to show that $\mathcal{P}$ is c.c.c.

Lemma 14. *Assume that $p = \langle X, \preceq, i\rangle \in P$, $t \in X$, $\alpha < \pi(t)$ and $n < \omega$. Then, there is a $p' = \langle X', \preceq', i'\rangle \in P$ with $p' \leq p$ and there is an $s \in X' \setminus X$ with $\pi(s) = \alpha$ and $\xi(s) > n$ such that, for every $x \in X$, $s \preceq' x$ iff $t \preceq' x$.*

Proof. Let $L = \{\alpha\} \cup \{\xi : \alpha < \xi < \pi(t) \wedge \exists j < \omega\, \xi + j = \pi(t)\}$. Let $\alpha = \alpha_0, \ldots, \alpha_\ell$ be the increasing enumeration of L. Since X is finite, we can pick an $s_j \in T_{\alpha_j} \setminus X$ with $\xi(s_j) > n$ for $j \leq \ell$. Let $X' = X \cup \{s_j : j \leq \ell\}$ and let
$$\prec' = \prec \cup \{(s_j, y) : j \leq l, t \preceq y\} \cup \{(s_j, s_k) : j < k \leq \ell\}.$$
Now, we put $i'\{x, y\} = i\{x, y\}$ if $x, y \in X$, $i'\{s_j, y\} = \{s_j\}$ if $t \preceq y$, $i'\{s_j, s_k\} = s_{\min(j,k)}$, and $i'\{s_j, y\} = \emptyset$ otherwise. Clearly, $\langle X', \preceq', i'\rangle$ is as required. □

Lemma 15. *If $\mathcal{P}$ preserves cardinals, then $\mathcal{P}$ adjoins an $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$-poset which is an ω_1-skeleton.*

Proof. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter. We put $p = \langle X_p, \preceq_p, i_p\rangle$ for $p \in \mathcal{G}$. By Lemma 10 and standard density arguments, we have

$$T = \bigcup\{X_p : p \in \mathcal{G}\}, \tag{2}$$

and taking

$$\preceq = \bigcup\{\preceq_p : p \in \mathcal{G}\}, \tag{3}$$

the poset $\langle T, \preceq\rangle$ is an $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$-poset. Especially, Lemma 14 ensures that $\langle T, \preceq\rangle$ satisfies (4) in Definition 6. Properties (P5) and (P6) guarantee that $\langle T, \preceq\rangle$ is an ω_1-skeleton. □

Now, we prove the key lemma for showing that $\mathcal{P}$ adjoins the required poset.

Lemma 16. *$\mathcal{P}$ is c.c.c.*

Proof. Assume that $R = \langle r_\nu : \nu < \omega_1\rangle \subseteq P$ with $r_\nu \neq r_\mu$ for $\nu < \mu < \omega_1$. For $\nu < \omega_1$, write $r_\nu = \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$ and put $L_\nu = \pi[X_\nu]$. By the Δ-System Lemma, we may suppose that the set $\{X_\nu : \nu < \omega_1\}$ forms a Δ-system with root X^*. By thinning out R again if necessary, we may assume that $\{L_\nu : \nu < \omega_1\}$ forms a Δ-system with root L^* in such a way that $X_\nu \cap T_\alpha = X_\mu \cap T_\alpha$ for every $\alpha \in L^* \setminus \{\omega_1\}$ and $\nu < \mu < \omega_1$. Without loss of generality, we may assume that $\omega_1 \in L^*$. Since $\beta \setminus \alpha$ is a countable set for $\alpha, \beta \in L^*$ with $\alpha < \beta < \omega_1$, we may suppose that $L^* \setminus \{\omega_1\}$ is an initial segment of L_ν for every $\nu < \omega_1$. Of course, this may require a further thinning out of R. Now, we put $Z_\nu = X_\nu \cap T_{\omega_1}$ for $\nu < \omega_1$. Without loss of generality, we may assume that the domains of the forcing conditions of R have the same size and that there is a natural number $n > 0$ with $|Z_\nu \setminus X^*| = |Z_\mu \setminus X^*| = n$ for $\nu < \mu < \omega_1$. We consider in T_{ω_1} the well-order induced by λ. Then, by thinning out R again if necessary, we may assume that for every $\{\nu, \mu\} \in [\omega_1]^2$ there is an order-preserving bijection $h = h_{\nu,\mu} : L_\nu \longrightarrow L_\mu$ with $h \restriction L^* = L^*$ that lifts to an isomorphism of X_ν with X_μ satisfying the following:

(A) For every $\alpha \in L_\nu \setminus \{\omega_1\}$, $h(\alpha, \xi) = (h(\alpha), \xi)$.
(B) h is the identity on X^*.
(C) For every $i < n$, if x is the i^{th}-element in $Z_\nu \setminus X^*$ and y is the i^{th}-element in $Z_\mu \setminus X^*$, then $h(x) = y$.
(D) For every $x, y \in X_\nu$, $x \preceq_\nu y$ iff $h(x) \preceq_\mu h(y)$.
(E) For every $\{x, y\} \in [X_\nu]^2$, $h[i_\nu\{x, y\}] = i_\mu\{h(x), h(y)\}$.

Now, we deduce from condition (P4) and the fact that R is uncountable that if $\{x, y\} \in [X^*]^2$ then $i_\nu\{x, y\} \subseteq X^*$ for every $\nu < \omega_1$. So if $\{x, y\} \in [X^*]^2$, then $i_\nu\{x, y\} = i_\mu\{x, y\}$ for $\nu < \mu < \omega_1$.

Let $\delta = \max(L^* \setminus \{\omega_1\})$. Since F is an ω_1-strongly unbounded function on λ, there are ordinals ν, μ with $\nu < \mu < \omega_1$ such that if we put $a = \{\xi \in \lambda : (\omega_1, \xi) \in Z_\nu \setminus X^*\}$ and $a' = \{\xi \in \lambda : (\omega_1, \xi) \in Z_\mu \setminus X^*\}$, then $F\{\xi, \xi'\} > \delta$ for every $\xi \in a$ and every $\xi' \in a'$. Our purpose is to prove that r_ν and r_μ are compatible in $\mathcal{P}$. We put $p = r_\nu$ and $q = r_\mu$. And we write $p = \langle X_p, \preceq_p, \mathrm{i}_p \rangle$ and $q = \langle X_q, \preceq_q, \mathrm{i}_q \rangle$. Then, we define the extension $r = \langle X_r, \preceq_r, i_r \rangle$ of p and q as follows. We put $X_r = X_p \cup X_q$. We define $\preceq_r = \preceq_p \cup \preceq_q$. Note that $\preceq_r$ is a partial order on X_r, because $L^* \setminus \{\omega_1\}$ is an initial segment of $\pi[X_p]$ and $\pi[X_q]$. Now, we define the infimum function i_r. Assume that $\{x, y\} \in [X_r]^2$. We put $i_r\{x, y\} = i_p\{x, y\}$ if $x, y \in X_p$, and $i_r\{x, y\} = i_q\{x, y\}$ if $x, y \in X_q$. Suppose that $x \in X_p \setminus X_q$ and $y \in X_q \setminus X_p$. Note that x, y are not comparable in $\langle X_r, \preceq_r \rangle$ and there is no $u \in (X_p \cup X_q) \setminus X^*$ such that $u \preceq_r x, y$. Then, we define $i_r\{x, y\} = \{u \in X^* : u \prec_r x, y\}$. It is easy to check that $r \in P$, and so $r \leq p, q$. □

After finishing the proof of Theorem 5 for $\kappa = \omega$, try to prove it for $\kappa = \omega_1$. So, assume that $2^\omega = \omega_1$, $\omega_4 \leq \lambda$, and there is an ω_2-strongly unbounded function on λ. We want to find $\langle \omega_1 \rangle_\eta {}^\frown \langle \lambda \rangle$-posets for each ordinal $\eta < \omega_3$ with $\mathrm{cf}(\eta) = \omega_2$ in some cardinal-preserving generic extension. Since the "stepping-up" method of Er-rhaimini and Veličkovic worked for $\kappa = \omega$, it is natural to try to apply Theorem 10 for the case $\kappa = \omega_1$. That is, we can try to find a cardinal-preserving generic extension that contains an $\langle \omega_1 \rangle_{\omega_2} {}^\frown \langle \lambda \rangle$-poset which is an ω_2-skeleton. For this, first we should consider the forcing construction given in [3, Section 4] to add an $\langle \omega_1 \rangle_{\omega_2} {}^\frown \langle \omega_3 \rangle$-poset, and then try to extend this construction to add the required ω_2-skeleton. However, the construction from [3] is σ-complete and requires that CH holds in the ground model. Then, the following results show that the forcing construction of an $\langle \omega_1 \rangle_{\omega_2} {}^\frown \langle \lambda \rangle$-poset which is an ω_2-skeleton is quite hopeless, at least by using the standard forcing from [3].

If X is the topological space associated with a skeleton and $x \in X$, we denote by $t(x, X)$ the tightness of x in X. Also, if A is a subset of points of X we denote by A' the set of all points $x \in X$ such that x is an accumulation point of A.

Proposition 17. *Assume that $\mathcal{T} = \langle T, \prec \rangle$ is a μ-skeleton, $\alpha < \mu$ and $x \in I_{\alpha+1}(X_\mathcal{T})$. Then, $t(x, X_\mathcal{T}) = \omega$.*

Proof. Assume that $A \subseteq T$ and $x \in A'$. We can assume that $a \prec x$ for each $a \in A$.

Let

$$U = \{u \in I_\alpha(X_\mathcal{T}) : u \prec x \wedge \exists a_u \in A \; a_u \preceq u\}. \tag{4}$$

Since $y \prec x$ iff $y \preceq u$ for some $u \prec x$ with $u \in I_\alpha(X_\mathcal{T})$, the set U is infinite.

Pick $V \in [U]^\omega$, and put $B = \{a_v : v \in V\}$. We claim that $x \in B'$. Indeed, if $y \prec x$ then there is a $u \in I_\alpha(X_\mathcal{T})$ such that $y \preceq u \prec x$. So $|\{b \in B : b \preceq y\}| \leq 1$. Hence $y \notin B'$. However, B has an accumulation point because $B \subseteq U_\mathcal{T}(x)$ and $U_\mathcal{T}(x)$ is compact in $X_\mathcal{T}$. So, B should converge to x. □

Corollary 18. *If $\mathcal{T}$ is a μ-skeleton, then $\mu \le |I_0(X_\mathcal{T})|^\omega$. Especially, under CH an $\langle\omega_1\rangle_{\omega_2}{}^\frown\langle\lambda\rangle$-poset can not be an ω_2-skeleton.*

Thus, we are unable to use Theorem 10 to prove Theorem 5 even for $\kappa = \omega_1$. Instead of this stepping-up method, in the next two sections we will construct $\langle\omega_1\rangle_\eta{}^\frown\langle\lambda\rangle$-posets directly using the method of orbits from [6]. This method was used to construct by forcing $\langle\omega_1\rangle_\eta$-posets for $\omega_2 \le \eta < \omega_3$. It is not difficult to get an $\langle\omega_1\rangle_{\omega_2}$-poset by means of countable "approximations" of the required poset. However, for $\omega_2 \le \eta < \omega_3$ we need the notion of orbit and a much more involved forcing to obtain $\langle\omega_1\rangle_\eta$-posets (see [6]).

3. Combinatorial notions

In this section, we define the combinatorial notions that will be used in the proof of Theorem 5.

If α, β are ordinals with $\alpha \le \beta$ let

$$[\alpha, \beta) = \{\gamma : \alpha \le \gamma < \beta\}. \tag{5}$$

Definition 19.
(a) We say that I is an *ordinal interval* iff there are ordinals α and β with $\alpha \le \beta$ and $I = [\alpha, \beta)$. Then, we write $I^- = \alpha$ and $I^+ = \beta$.
(b) Assume that $I = [\alpha, \beta)$ is an ordinal interval. If β is a limit ordinal, let $\mathrm{E}(I) = \{\varepsilon^I_\nu : \nu < \mathrm{cf}(\beta)\}$ be a cofinal closed subset of I having order type $\mathrm{cf}(\beta)$ with $\alpha = \varepsilon^I_0$, and then put

$$\mathcal{E}(I) = \{[\varepsilon^I_\nu, \varepsilon^I_{\nu+1}) : \nu < \mathrm{cf}\,(\beta)\}. \tag{6}$$

If $\beta = \beta' + 1$ is a successor ordinal, put $\mathrm{E}(I) = \{\alpha, \beta'\}$ and

$$\mathcal{E}(I) = \{[\alpha, \beta'), \{\beta'\}\}. \tag{7}$$

(c) If κ is an infinite cardinal and η is an ordinal with $\kappa^+ \le \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$, we define $\mathbb{I}_\eta = \bigcup\{\mathcal{I}_n : n < \omega\}$ where:

$$\mathcal{I}_0 = \{[0, \eta)\} \text{ and } \mathcal{I}_{n+1} = \bigcup\{\mathcal{E}(I) : I \in \mathcal{I}_n\}. \tag{8}$$

Note that $\mathbb{I}_\eta$ is a cofinal tree of intervals in the sense defined in [6]. So, the following conditions are satisfied:

(i) For every $I, J \in \mathbb{I}_\eta$, $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$.
(ii) If I, J are different elements of $\mathbb{I}_\eta$ with $I \subseteq J$ and J^+ is a limit, then $I^+ < J^+$.
(iii) $\mathcal{I}_n$ partitions $[0, \eta)$ for each $n < \omega$.
(iv) $\mathcal{I}_{n+1}$ refines $\mathcal{I}_n$ for each $n < \omega$.
(v) For every $\alpha < \eta$ there is an $I \in \mathbb{I}_\eta$ such that $I^- = \alpha$.

Definition 20.
(a) For each $\alpha < \eta$ and $n < \omega$ we define $\mathrm{I}(\alpha, n)$ as the unique interval $I \in \mathcal{I}_n$ such that $\alpha \in I$.

(b) For each $\alpha < \eta$ we define $n(\alpha)$ as the least natural number n such that there is an interval $I \in \mathcal{I}_n$ with $I^- = \alpha$.

Note that if $n(\alpha) = k$, then for every $m \geq k$ we have $I(\alpha, m)^- = \alpha$.
The following notion will be essential in our forcing construction.

Definition 21. Assume that $\alpha < \eta$. If $m < \mathrm{n}(\alpha)$, we put $o_m(\alpha) = \mathrm{E}(\mathrm{I}(\alpha, m)) \cap \alpha$. Then, we define the *orbit* of α (with respect to $\mathbb{I}_\eta$) as

$$o(\alpha) = \bigcup \{o_m(\alpha) : m < \mathrm{n}(\alpha)\}. \tag{9}$$

For basic facts on orbits and trees of intervals, we refer the reader to [6, Section 1]. In particular, we have $|o(\alpha)| \leq \kappa$ for every $\alpha < \eta$.
We write $E([0, \eta)) = \{\varepsilon_\nu : \nu < \kappa^+\}$.

Claim 22. *$o(\varepsilon_\nu) = \{\varepsilon_\zeta : \zeta < \nu\}$ for $\nu < \kappa^+$.*

Proof. Clearly $I(\varepsilon_\nu, 0) = [0, \eta)$ and $I(\varepsilon_\nu, 1) = [\varepsilon_\nu, \varepsilon_{\nu+1})$. So $n(\varepsilon_\nu) = 1$. Thus $o(\varepsilon_\nu) = o_0(\varepsilon_\nu) = E(I(\varepsilon_\nu, 0)) \cap \varepsilon_\nu = E([0, \eta)) \cap \varepsilon_\nu = \{\varepsilon_\zeta : \zeta < \nu\}$. □

For $\alpha < \beta < \eta$ let

$$j(\alpha, \beta) = \max\{j : I(\alpha, j) = I(\beta, j)\}, \tag{10}$$

and put

$$J(\alpha, \beta) = I(\alpha, j(\alpha, \beta) + 1). \tag{11}$$

For $\alpha < \eta$ let

$$J(\alpha, \eta) = I(\alpha, 1). \tag{12}$$

Claim 23. *If $\varepsilon_\zeta \leq \alpha < \varepsilon_{\zeta+1} \leq \beta \leq \eta$, then $J(\alpha, \beta) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$.*

Proof. For $\beta = \eta$, $J(\alpha, \beta) = I(\alpha, 1) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$.
Now assume that $\beta < \eta$. Since $I(\alpha, 0) = I(\beta, 0) = [0, \eta)$, but $I(\alpha, 1) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$ and $I(\beta, 1) = [\varepsilon_\xi, \varepsilon_{\xi+1})$ for some ε_ξ with $\varepsilon_{\zeta+1} \leq \varepsilon_\xi$, we have $j(\alpha, \beta) = 0$ and so $J(\alpha, \beta) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$. □

4. Proof of the Main Theorem

In order to prove Theorem 5, suppose that κ, λ are infinite cardinals with $\kappa^{+++} \leq \lambda$ and $\kappa^{<\kappa} = \kappa$, η is an ordinal with $\kappa^+ \leq \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$, and there is a κ^+-strongly unbounded function on λ. We will use a refinement of the arguments given in [6] and [3, Section 4].

First, we define the underlying set of our construction.

Definition 24.
(a) We put $T = \bigcup\{T_\alpha : \alpha \leq \eta\}$ where $T_\alpha = \{\alpha\} \times \kappa$ for every $\alpha < \eta$ and $T_\eta = \{\eta\} \times \lambda$.
(b) We write $T_{<\eta} = T \setminus T_\eta$.

Definition 25.
(a) We put $\mathbb{I} = \mathbb{I}_\eta$.
(b) We define $E = E([0,\eta)) = \{\varepsilon_\nu : \nu < \kappa^+\}$.

Since there is a κ^+-strongly unbounded function on λ and $\mathrm{cf}(\eta) = \kappa^+$ there is a function $F : \left[\lambda\right]^2 \longrightarrow E$ such that the following condition holds:

($\star$) For every ordinal $\gamma < \eta$ and every family $A \subseteq [\lambda]^{<\kappa}$ of pairwise disjoint sets with $|A| = \kappa^+$, there are different $a, b \in A$ such that $F\{\alpha, \beta\} > \gamma$ for every $\alpha \in a$ and $\beta \in b$.

The following notion will be used in our forcing construction.

Definition 26. Let $\Lambda \in \mathbb{I}$ and $\{s,t\} \in [T]^2$ with $\pi(s) < \pi(t)$. We say that Λ *isolates s from t* iff $\Lambda^- < \pi(s) < \Lambda^+$ and $\Lambda^+ \leq \pi(t)$.

Definition 27. We define the poset $\mathcal{P} = \langle P, \leq\rangle$ as follows.

(a) We say that $p = \langle X, \preceq, i\rangle \in P$ iff the following conditions hold:
  (P1) $X \in [T]^{<\kappa}$.
  (P2) $\preceq$ is a partial order on X such that $s \prec t$ implies $\pi(s) < \pi(t)$.
  (P3) $\mathrm{i} : \left[X\right]^2 \longrightarrow X \cup \{\mathrm{undef}\}$ is an infimum function, that is, a function such that for every $\{s,t\} \in \left[X\right]^2$ we have:

$$\forall x \in X([x \preceq s \wedge x \preceq t] \text{ iff } x \preceq \mathrm{i}\{s,t\}).$$

  (P4) If $s, t \in X$ are compatible but not comparable in $\langle X, \preceq\rangle$, $v = i\{s,t\}$ and $\pi(s) = \alpha_1$, $\pi(t) = \alpha_2$ and $\pi(v) = \beta$, we have:
    (a) If $\alpha_1, \alpha_2 < \eta$, then $\beta \in o(\alpha_1) \cap o(\alpha_2)$.
    (b) If $\alpha_1 < \eta$ and $\alpha_2 = \eta$, then $\beta \in o(\alpha_1) \cap E$.
    (c) If $\alpha_1 = \eta$ and $\alpha_2 < \eta$, then $\beta \in o(\alpha_2) \cap E$.
    (d) If $\alpha_1 = \alpha_2 = \eta$, then $\beta \in F\{\xi(s), \xi(t)\} \cap E$.
  (P5) If $s, t \in X$ with $s \preceq t$ and $\Lambda = J(\pi(s), \pi(t))$ isolates s from t, then there is a $u \in X$ such that $s \preceq u \preceq t$ and $\pi(u) = \Lambda^+$.

(b) If $\langle X, \preceq, \mathrm{i}\rangle, \langle X', \preceq', \mathrm{i}'\rangle \in P$, we put $\langle X', \preceq', \mathrm{i}'\rangle \leq \langle X, \preceq, \mathrm{i}\rangle$ iff $X \subseteq X'$, $\preceq = \preceq' \cap (X \times X)$ and $\mathrm{i} \subseteq \mathrm{i}'$.

Lemma 28. *Assume that $p = \langle X, \preceq, \mathrm{i}\rangle \in P$, $t \in X$, $\alpha < \pi(t)$ and $\nu < \kappa$. Then, there is a $p' = \langle X', \preceq', \mathrm{i}'\rangle \in P$ with $p' \leq p$ and there is an $s \in X' \setminus X$ with $\pi(s) = \alpha$ and $\xi(s) > \nu$ such that, for every $x \in X$, $s \preceq' x$ iff $t \preceq' x$.*

Proof. Since $|X| < \kappa$, we can take an $s \in T_\alpha \setminus X$ with $\xi(s) > \nu$. Let $\{I_0, \ldots, I_n\}$ be the list of all the intervals in $\mathbb{I}$ that isolate s from t in such a way that

$I_0^+ > I_1^+ > \cdots > I_n^+$. Put $\gamma_i = I_i^+$ for $i \leq n$. We take points $c_i \in T \setminus X$ with $\pi(c_i) = \gamma_i$ for $i \leq n$. Let $X' = X \cup \{s\} \cup \{c_i : i \leq n\}$ and let

$$\prec' = \prec \cup \{\langle s, c_i\rangle : i \leq n\} \cup \{\langle s, y\rangle : t \preceq y\} \cup \{\langle c_j, c_i\rangle : i < j\} \cup \{\langle c_i, y\rangle : i \leq n, t \preceq y\}.$$

Note that, for $z \in X'$ and $y \in \{s\} \cup \{c_i : i \leq n\}$, either z and y are comparable or they are incompatible with respect to $\preceq'$. So, the definition of i' is clear.

Finally, observe that p' satisfies (P5) because if $x \prec' y$ with $x \in \{s\} \cup \{c_i : i \leq n\}$, $y \in X'$ and $J(\pi(x), \pi(y))$ isolates x from y then either $J(\pi(x), \pi(y)) = I_k$ for some $0 \leq k \leq n$ or $J(\pi(x), \pi(y)) = J(\pi(t), \pi(y))$. But if $J(\pi(x), \pi(y)) = I_k$, then c_k witnesses (P5) for x and y; and if $J(\pi(x), \pi(y)) = J(\pi(t), \pi(y))$, we are done by condition (P5) for p. □

Definition 29. For $p \in P$ we write $p = \langle X_p, \preceq_p, i_p\rangle$, $Y_p = X_p \cap T_{<\eta}$ and $Z_p = X_p \cap T_\eta$.

Lemma 30. *If $\mathcal{P}$ preserves cardinals, then forcing with $\mathcal{P}$ adjoins a (κ, η, λ)-Boolean algebra.*

Proof. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter. Then

$$T = \bigcup\{X_p : p \in \mathcal{G}\}, \tag{13}$$

and taking

$$\preceq = \bigcup\{\preceq_p : p \in \mathcal{G}\} \tag{14}$$

the poset $\langle T, \preceq\rangle$ is a $\langle\kappa\rangle_\eta{}^\frown\langle\lambda\rangle$-poset. Especially, Lemma 28 guarantees that $\langle T, \prec\rangle$ satisfies (4) from Definition 6. So, by Lemma 7, in $V[\mathcal{G}]$ there is a (κ, η, λ)-Boolean algebra. □

To complete our proof we should check that forcing with P preserves cardinals. It is straightforward that $\mathcal{P}$ is κ-closed. The burden of our proof is to verify the following statement, which completes the proof of Theorem 5.

Lemma 31. *$\mathcal{P}$ has the κ^+-chain condition.*

We need to consider the partial order introduced in [6].

Definition 32. We define the subposet $\mathcal{P}_\eta = \langle P_\eta, \leq_\eta\rangle$ of $\mathcal{P}$ as follows. We put

$$P_\eta = \{p \in P : X_p \subseteq \eta \times \kappa\}, \tag{15}$$

and we let $\leq_\eta = \leq \restriction P_\eta$.

The poset $\mathcal{P}_\eta$ was defined in [6, Definition 2.1], and it was proved that $\mathcal{P}_\eta$ satisfies the κ^+-chain condition. In [6, Lemmas 2.5 and 2.6] it was shown that every set $R \in \left[P_\eta\right]^{\kappa^+}$ has a linked subset of size κ^+. Actually, a stronger statement was proved, and we will use that statement to prove Lemma 31. However, before doing so, we need some preparation.

Definition 33. Suppose that $g : A \longrightarrow B$ is a bijection, where $A, B \in \left[T\right]^{<\kappa}$. We say that g is *adequate* iff the following conditions hold:

(1) $g[A \cap T_{<\eta}] = B \cap T_{<\eta}$ and $g[A \cap T_\eta] = B \cap T_\eta$.
(2) For every $s, t \in A$, $\pi(s) < \pi(t)$ iff $\pi(g(s)) < \pi(g(t))$.
(3) For every $s = \langle \alpha, \nu \rangle \in A \cap T_{<\eta}$, $g(\alpha, \nu) = (\beta, \zeta)$ implies $\nu = \zeta$.
(4) For every $s, t \in A \cap T_\eta$, $\xi(s) < \xi(t)$ iff $\xi(g(s)) < \xi(g(t))$.

For $A, B \subseteq T_{<\eta}$, this definition is just [6, Definition 2.2].

Definition 34. A set $Z \subseteq P$ is *separated* iff the following conditions are satisfied:

(1) $\{X_p : p \in Z\}$ forms a Δ-system with root X.
(2) For each $\alpha < \eta$, either $X_p \cap T_\alpha = X \cap T_\alpha$ for every $p \in Z$, or there is at most one $p \in Z$ such that $X_p \cap T_\alpha \neq \emptyset$.
(3) For every $p, q \in Z$ there is an adequate bijection $h_{p,q} : X_p \longrightarrow X_q$ which satisfies the following:
    (a) For any $s \in X$, $h_{p,q}(s) = s$.
    (b) If $s, t \in X_p$, then $s \prec_p t$ iff $h_{p,q}(s) \prec_q h_{p,q}(t)$.
    (c) If $s, t \in X_p$, then $h_{p,q}(\mathrm{i}_p\{s, t\}) = \mathrm{i}_q\{h_{p,q}(s), h_{p,q}(t)\}$.

For $Z \subseteq P_\eta$, this definition is just [6, Definition 2.3].

Lemma 35. *Assume that $Z \in \left[P\right]^{\kappa^+}$ is separated and X is the root of the Δ-system $\{X_p : p \in Z\}$. If s, t are compatible but not comparable in $p \in Z$ and $s \in X \cap T_{<\eta}$, then $\mathrm{i}_p\{s, t\} \in X$.*

Proof. Assume that s, t are compatible but not comparable in $p \in Z$ and $s \in X \cap T_{<\eta}$. Assume that $\mathrm{i}_p\{s, t\} \notin X$. Then since

$$\{\mathrm{i}_q\{s, h_{p,q}(t)\} : q \in Z\} = \{h_{p,q}(\mathrm{i}_p\{s, t\}) : q \in Z\}, \tag{16}$$

the elements of $\{\mathrm{i}_q\{s, h_{p,q}(t)\} : q \in Z\}$ are all different. But this is impossible, because $\pi(\mathrm{i}_q\{s, h_{p,q}(t)\}) \in o(s)$ for all $q \in Z$ and $|o(s)| \leq \kappa$. □

In [6, Lemmas 2.5 and 2.6], as we explain in the Appendix of this paper, actually the following statement was proved.

Proposition 36. *For each subset $R \in \left[P_\eta\right]^{\kappa^+}$ there is a separated subset $Z \in \left[R\right]^{\kappa^+}$ and an ordinal $\gamma < \eta$ such that every $p, q \in Z$ have a common extension $r \in P_\eta$ such that the following holds:*

(R1) $\sup\ \pi[X_r \setminus (X_p \cup X_q)] < \gamma$.
(R2) (a) $y \prec_r s$ iff $y \prec_r h_{p,q}(s)$ for each $s \in X_p$ and $y \in X_r \setminus (X_p \cup X_q)$,
    (b) $s \prec_r y$ iff $h_{p,q}(s) \prec_r y$ for each $s \in X_p$ and $y \in X_r \setminus (X_p \cup X_q)$,
    (c) if $s \prec_r y$ for $s \in X_p \cup X_q$ and $y \in X_r \setminus (X_p \cup X_q)$, then there is a $w \in X_p \cap X_q$ with $s \preceq_r w \prec_r y$,

(d) for $s \in X_p \setminus X_q$ and $t \in X_q \setminus X_p$,

$$s \prec_r t \text{ iff } \exists u \in X_p \cap X_q \text{ such that } s \prec_p u \prec_q t,$$
$$t \prec_r s \text{ iff } \exists u \in X_p \cap X_q \text{ such that } t \prec_q u \prec_p s. \tag{17}$$

After this preparation, we are ready to prove Lemma 31.

Proof of Lemma 31. We will argue in the following way. Assume that $R = \langle r_\nu : \nu < \kappa^+ \rangle \subseteq P$, where $r_\nu = \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu \rangle$. For each $\nu < \kappa^+$ we will "push down" r_ν into P_η, more precisely, we will construct an isomorphic copy $r'_\nu \in P_\eta$ of r_ν. Using Proposition 36 we can find a separated subfamily $\{r'_\nu : \nu \in K\}$ of size κ^+ and an ordinal $\gamma < \eta$ such that for each $\nu, \mu \in K$ with $\nu \neq \mu$ there is a condition $r'_{\nu,\mu} \in P_\eta$ such that $r'_{\nu,\mu} \leq_\eta r'_\nu, r'_\mu$ and (R1)–(R2) hold, especially

$$\sup \pi[X'_{\nu,\mu} \setminus (X'_\nu \cup X'_\mu)] < \gamma. \tag{18}$$

Let X be the root of $\{X_\nu : \nu < \kappa^+\}$, $Y = X \setminus T_\eta$ and $\gamma_0 = \max(\gamma,\ \sup \pi[Y])$. Since F is κ^+-strongly unbounded, there are $\nu, \mu \in K$ with $\nu < \mu$ such that

$$\forall s \in (X_\nu \setminus X_\mu) \cap T_\eta \quad \forall t \in (X_\mu \setminus X_\nu) \cap T_\eta \quad F\{\xi(s), \xi(t)\} > \gamma_0. \tag{19}$$

Then we will be able to "pull back" $r' = r'_{\nu,\mu}$ into P to get a condition $r = r_{\nu,\mu}$ which is a common extension of r_ν and r_μ. Let us remark that r will not be an isomorphic copy of r', rather r will be a "homomorphic image" of r'.

Now we carry out our plan.

Since $\kappa^{<\kappa} = \kappa$, by thinning out our sequence we can assume that R itself is a separated set. So $\{X_r : r \in R\}$ forms a Δ-system with kernel $\bar{X}$. We write $\bar{Y} = \bar{X} \cap T_{<\eta}$ and $\bar{Z} = \bar{X} \cap T_\eta$.

Recall that $E = E([0, \eta)) = \{\varepsilon_\zeta : \zeta < \kappa^+\}$ is a closed unbounded subset of η.

Fix $\nu < \kappa^+$. Write $Y_\nu = X_\nu \cap T_{<\eta}$ and $Z_\nu = X_\nu \cap T_\eta$. Pick a limit ordinal $\zeta(\nu) < \kappa^+$ such that:

(i) $\sup(\pi[Y_\nu]) < \varepsilon_{\zeta(\nu)}$,
(ii) $\zeta(\mu) < \zeta(\nu)$ for $\mu < \nu$.

Let $\theta = \mathrm{tp}(\xi[Z_\nu])$ and $\alpha = \varepsilon_{\zeta(\nu)}$. We put $Z'_\nu = \{\langle \alpha, \xi \rangle : \xi < \theta\}$. Clearly, $Z'_\nu \subseteq T_{\varepsilon_{\zeta(\nu)}}$ and $\mathrm{tp}(\xi[Z'_\nu]) = \mathrm{tp}(\xi[Z_\nu])$. We consider in Z'_ν and Z_ν the well-orderings induced by κ and λ respectively. Put $X'_\nu = Y_\nu \cup Z'_\nu$, and let $g_\nu : X'_\nu \longrightarrow X_\nu$ be the natural bijection, i.e. $g_\nu \restriction Y_\nu = id$ and $g_\nu(s) = t$ if for some $\xi < \mathrm{tp}(\xi[Z_\nu])$ s is the ξ-element in Z'_ν and t is the ξ-element in Z_ν.

Let $\bar{Z}'_\nu = g_\nu^{-1}\bar{Z}$. We define the condition $r'_\nu = \langle X'_\nu, \preceq'_\nu, \mathrm{i}'_\nu \rangle \in P_\eta$ as follows: for $s, t \in X'_\nu$ with $s \neq t$ we put

$$s \prec'_\nu t \text{ iff } g_\nu(s) \prec_\nu g_\nu(t), \tag{20}$$

and

$$\mathrm{i}'_\nu\{s, t\} = \mathrm{i}_\nu\{g_\nu(s), g_\nu(t)\}. \tag{21}$$

Claim 37. $r'_\nu \in P_\eta$.

Proof. (P1), (P2) and (P3) are clear because g_ν is an isomorphism between $r'_\nu = \langle X'_\nu, \preceq'_\nu, \mathrm{i}'_\nu\rangle$ and $r_\nu = \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$, moreover $\pi(s) < \pi(t)$ iff $\pi(g_\nu(s)) < \pi(g_\nu(t))$.
(P4) Since $X'_\nu \subseteq T_{<\eta}$ we should check just (a). So assume that $s', t' \in X'_\nu$ are compatible but not comparable in $\langle X'_\nu, \leq'_\nu\rangle$ and $v' = \mathrm{i}'_\nu\{s', t'\}$. Put $s = g_\nu(s')$, $t = g_\nu(t')$. Since $g_\nu \restriction Y_\nu = id$, we can assume that $\{s', t'\} \notin \left[Y_\nu\right]^2$, e.g. $s' \in Z'_\nu$ and so $s \in Z_\nu$.

First observe that $v' \in Y_\nu$, so $v' = g_\nu(v')$.

If $t' \in Y_\nu$, then $t' = g_\nu(t')$, and $v' = \mathrm{i}_\nu\{s, t'\}$. By applying (P4)(c) in r_ν for s and t' we obtain

$$\pi(v') \in E \cap o(\pi(t')) \subseteq E \cap \varepsilon_{\zeta(\nu)} \cap o(\pi(t')) = o(\pi(s')) \cap o(\pi(t')) \tag{22}$$

because $o(\pi(s')) = E \cap \varepsilon_{\zeta(\nu)}$ by Claim 22.

If $t' \in Z'_\nu$, then $t = g_\nu(t') \in Z_\nu \subseteq T_\eta$. Since $v' = i'_\nu\{s', t'\} = i_\nu\{s, t\}$, applying (P4)(d) in r_ν for s and t we obtain

$$\pi(v') \in F\{\xi(s), \xi(t)\} \cap E \cap \varepsilon_{\zeta(\nu)} \subseteq E \cap \varepsilon_{\zeta(\nu)} = o(\pi(s')) \cap o(\pi(t'))$$

because $o(\pi(s')) = o(\pi(t')) = E \cap \varepsilon_{\zeta(\nu)}$ by Claim 22.
(P5) Assume that $s', t' \in X'_\nu$, $s' \prec'_\nu t'$ and $\Lambda = J(\pi(s'), \pi(t'))$ isolates s' from t'. Then $s' \in Y_\nu$, so $g_\nu(s') = s'$. Since $g_\nu \restriction Y_\nu = id$, we can assume that $\{s', t'\} \notin \left[Y_\nu\right]^2$, i.e. $t' \in Z'_\nu$.

Write $t = g_\nu(t')$. Since $\pi(t') = \varepsilon_{\zeta(\nu)} \in E$, by Claim 23, $J(\pi(s'), \pi(t')) = J(\pi(s'), \pi(t)) = [\varepsilon_\zeta, \varepsilon_{\zeta+1}) = I(\pi(s'), 1)$, where $\varepsilon_\zeta \leq \pi(s') < \varepsilon_{\zeta+1}$. Applying (P5) in r_ν for s' and t, we obtain a $v \in Y_\nu$ such that $\pi(v) = \Lambda^+$ and $s' \prec_\nu v \prec_\nu t$. Then $g_\nu(v) = v$, so $s' \prec'_\nu v \prec'_\nu t'$, which was to be proved. □

Now applying Proposition 36 to the family $\{r'_\nu : \nu < \kappa^+\}$, there are $K \in \left[\kappa^+\right]^{\kappa^+}$ and $\gamma < \eta$ such that $\{r'_\nu : \nu \in K\}$ is separated and for every $\nu, \mu \in K$ with $\nu \neq \mu$ there is a common extension $r' \in P_\eta$ of r'_ν and r'_μ such that (R1)-(R2) hold. Let $\gamma_0 = \max(\gamma,\ \sup \pi[\bar{Y}])$. Recall that $\bar{Y}$ is the root of the Δ-system $\{Y_\nu : \nu \in \kappa^+\}$. For $\nu < \mu < \kappa^+$ we denote by $h'_{\nu,\mu}$ the adequate bijection $h_{r'_\nu, r'_\mu}$.

Since F satisfies $(\star)$, there are $\nu, \mu \in K$ with $\nu \neq \mu$ such that for each $s \in (Z_\nu \setminus Z_\mu)$ and $t \in (Z_\mu \setminus Z_\nu)$ we have

$$F\{\xi(s), \xi(t)\} > \gamma_0. \tag{23}$$

We show that the conditions r_ν and r_μ have a common extension $r = \langle X, \preceq, \mathrm{i}\rangle \in P$.

Consider a condition $r' = \langle X', \preceq', \mathrm{i}'\rangle$ which is a common extension of r'_ν and r'_μ and satisfies (R1)–(R2). We define the condition $r = \langle X, \preceq, \mathrm{i}\rangle$ as follows. Let

$$X = (X' \setminus (Z'_\nu \cup Z'_\mu)) \cup (Z_\nu \cup Z_\mu). \tag{24}$$

Write $U = X' \setminus (Z'_\nu \cup Z'_\mu) = X \setminus (Z_\nu \cup Z_\mu)$ and $V = X' \setminus (X'_\nu \cup X'_\mu)$. Clearly, $V \subseteq U$. We define the function $h : X' \longrightarrow X$ as follows:

$$h = g_\nu \cup g_\mu \cup (id \restriction U). \tag{25}$$

Then h is well-defined, h is onto, $h \restriction X' \setminus (\bar{Z}'_\nu \cup \bar{Z}'_\mu)$ is injective, and $h[\bar{Z}'_\nu] = h[\bar{Z}'_\mu] = \bar{Z}$.

Now, if $s, t \in X$ we put

$$s \prec t \text{ iff there is a } t' \in X' \text{ with } h(t') = t \text{ and } s \prec' t'. \tag{26}$$

Finally, we define the meet function i on $\big[X\big]^2$ as follows:

$$\mathrm{i}\{s,t\} = \max_{\prec'}\{\mathrm{i}'\{s',t'\} : \ h(s') = s \text{ and } h(t') = t\}. \tag{27}$$

We will prove in the following claim that the definition of the function i is meaningful. Then, the proof of Lemma 31 will be complete as soon as we verify that $r \in P$ and $r \le r_\nu, r_\mu$.

Claim 38. i *is well-defined by (27), moreover* $\mathrm{i} \supseteq \mathrm{i}_\nu \cup \mathrm{i}_\mu$.

Proof. We need to verify that the maximum in (27) does exist when we define $\mathrm{i}\{s,t\}$. So, suppose that $\{s,t\} \in [X]^2$.

If $\{s,t\} \in \big[X \setminus \bar{Z}\big]^2$ then there is exactly one pair (s',t') such that $h(s') = s$ and $h(t') = t$, and hence there is no problem in (27). So if $\{s,t\} \in \big[X_\nu\big]^2$ then $\mathrm{i}\{s,t\} = \mathrm{i}'\{s',t'\} = \mathrm{i}_\nu\{s,t\}$ by the construction of r'_ν. If $\{s,t\} \in \big[X_\mu\big]^2$ proceeding similarly we obtain $\mathrm{i}\{s,t\} = \mathrm{i}'\{s',t'\} = \mathrm{i}_\mu\{s,t\}$.

So we can assume that e.g. $s \in \bar{Z}$. Then $h^{-1}(s) = \{s', s''\}$ for some $s' \in \bar{Z}'_\nu$ and $s'' \in \bar{Z}'_\mu$.

First assume that $t \notin \bar{Z}$, so there is exactly one $t' \in X'$ with $h(t') = t$. We distinguish the following cases.

Case 1. $t \in V$.

Note that since $t \in V$, $t = t'$. We show that $\mathrm{i}'\{s',t\} = \mathrm{i}'\{s'',t\}$.

Let $v = \mathrm{i}'\{s',t\}$. Assume that $v \in X'_\nu \cup X'_\mu$. Then, by (R2)(c), $v \prec' t$ and $t \in V$ imply that there is a $w \in \bar{Y} = X'_\nu \cap X'_\mu$ such that $v \preceq' w \prec' t$. Thus $v = \mathrm{i}'\{s',w\}$ and $\mathrm{i}'\{s',w\} = \mathrm{i}'_\nu\{s',w\} = \mathrm{i}_\nu\{s,w\} \in \bar{Y}$ by Lemma 35 for $w \in \bar{Y}$. Clearly, $v \prec' t, s''$. Hence $v \preceq' \mathrm{i}'\{s'',t\}$.

Now assume that $v \in V$. Then $v \prec' s'$ implies $v \prec' h'_{\nu,\mu}(s') = s''$ by (R2)(a). So $v \prec' t, s''$, thus $\mathrm{i}'\{s',t\} \preceq' \mathrm{i}'\{s'',t\}$.

So, in both cases $\mathrm{i}'\{s',t\} \preceq' \mathrm{i}'\{s'',t\}$. But s' and s'' are symmetrical, hence $\mathrm{i}'\{s'',t\} \preceq' \mathrm{i}'\{s',t\}$, and so we are done.

Case 2. $t \in X_\nu \setminus \bar{Z}$.

We show that in this case $\mathrm{i}'\{s'',t'\} \preceq' \mathrm{i}'\{s',t'\}$.

Let $v = \mathrm{i}'\{s'',t'\}$. If $v \in V$, then $v \prec' s''$ and $h'_{\nu,\mu}(s') = s''$ imply $v \prec' s'$ by (R2)(a). Thus $v \preceq' s', t'$, and so $v \preceq' \mathrm{i}'\{s',t'\}$.

Now assume that $v \in X'_\nu \cup X'_\mu$. Note that if $v \in \bar{Y} = X'_\nu \cap X'_\mu$, then $v \prec' s'$, and so $v \prec' \mathrm{i}'\{s',t'\}$. We show that $v \in \bar{Y}$. For this, assume that $v \in (X'_\nu \cup X'_\mu) \setminus \bar{Y}$. Without loss of generality, we may suppose that $v \in X'_\nu \setminus X'_\mu$. Then, by (R2)(d), there is a $w \in \bar{Y}$ such that $v \prec' w \prec' s''$. Thus $v = \mathrm{i}'\{w, t'\} = \mathrm{i}'_\nu\{w, t'\} \in \bar{Y}$ by Lemma 35.

Moreover, $\{s, t\} \in \left[X_\nu\right]^2$ and $\mathrm{i}\{s, t\} = \mathrm{i}'\{s', t'\} = \mathrm{i}_\nu\{s, t\}$ because $g_\nu(s') = h(s') = s$ and $g_\nu(t') = h(t') = t$.

Case 3. $t \in X_\mu \setminus \bar{Z}$.

Proceeding as in Case 2, we can show that $\mathrm{i}'\{s', t'\} \preceq' \mathrm{i}'\{s'', t'\} = \mathrm{i}_\mu\{s, t\}$.

Finally, assume that $t \in \bar{Z}$. Then $h^{-1}(t) = \{t', t''\}$ for some $t' \in \bar{Z}'_\nu$ and $t'' \in \bar{Z}'_\mu$.

Note that by Cases (2) and (3),

$$\mathrm{i}'\{s'', t'\} \preceq' \mathrm{i}'\{s', t'\} \text{ and } \mathrm{i}'\{s', t''\} \preceq' \mathrm{i}'\{s'', t''\}.$$

Since $\mathrm{i}'\{s', t'\} = \mathrm{i}_\nu\{s, t\} = \mathrm{i}_\mu\{s, t\} = \mathrm{i}'\{s'', t''\}$ by the construction of r'_ν and r'_μ, we have

$$\mathrm{i}'\{s', t'\} = \mathrm{i}'\{s'', t''\} = \max_{\prec'}(\mathrm{i}'\{s', t'\}, \mathrm{i}'\{s'', t'\}, \mathrm{i}'\{s', t''\}, \mathrm{i}'\{s'', t''\}). \tag{28}$$

Moreover, in this case $\{s, t\} \in \left[X_\nu\right]^2 \cap \left[X_\mu\right]^2$ and we have just proved that $\mathrm{i}\{s, t\} = \mathrm{i}_\nu\{s, t\} = \mathrm{i}_\mu\{s, t\}$. □

By Claim 38 above, r is well-defined. Since $\mathrm{i} \supseteq \mathrm{i}_\nu \cup \mathrm{i}_\mu$, it is easy to check that if $r \in P$ then $r \leq r_\nu, r_\mu$. So, the following claim completes the verification of the chain condition.

Claim 39. *$r \in P$.*

Proof. (P1) and (P2) are clear.

(P3) Assume that $\{s, t\} \in [X]^2$. Without loss of generality, we may assume that s, t are compatible but not comparable in $\langle X, \preceq \rangle$. Note that by (26), (27) and condition (P3) for r', we have $\mathrm{i}\{s, t\} \prec s, t$. So, we have to show that if $v \prec s, t$ then $v \preceq \mathrm{i}\{s, t\}$.

Assume that $v \prec s, t$. Then, $v \in U$ and there are $s', t' \in X'$ such that $h(s') = s$, $h(t') = t$ and $v \prec' s', t'$. By (P3) for r', $v \preceq' \mathrm{i}'\{s', t'\}$. Now as $v, \mathrm{i}'\{s', t'\}, \mathrm{i}\{s, t\} \in U$ and $h \restriction U = id$, we infer from (27) that $v \preceq' \mathrm{i}'\{s', t'\} \preceq' \mathrm{i}\{s, t\}$ and hence $v \preceq \mathrm{i}\{s, t\}$.

(P4) Assume that $s, t \in X$ are compatible but not comparable in $\langle X, \preceq \rangle$. Let $v = \mathrm{i}\{s, t\}$.

(a) In this case $\pi(s), \pi(t) < \eta$. Then $s, t \in X \setminus (Z_\nu \cup Z_\mu) = U$, so $h(s) = s$ and $h(t) = t$. Thus $\mathrm{i}\{s, t\} = \mathrm{i}'\{s, t\}$. Hence, it follows from condition (P4)(a) for r' that $\pi(\mathrm{i}\{s, t\}) \in o(s) \cap o(t)$.

(b) In this case $\pi(s) < \eta$ and $\pi(t) = \eta$. Then $s \in X \setminus (Z_\nu \cup Z_\mu) = U$ and $t \in Z_\nu \cup Z_\mu$.

By (27) and Claim 38, there is a $t^* \in Z'_\nu \cup Z'_\mu$ such that $h(t^*) = t$ and $\mathrm{i}\{s, t\} = \mathrm{i}'\{s, t^*\}$.

Now, applying (P4)(a) for r', we infer that $\pi(v) \in o(s) \cap o(t^*)$. Since $\pi(t^*) \in E$, we have $o(t^*) \subseteq E$ by Claim 22. Then we deduce that $\pi(v) \in o(s) \cap E$, which was to be proved.
(c) The same as (b).
(d) In this case $\pi(s) = \pi(t) = \eta$. If $\{s,t\} \in \left[Z_\nu\right]^2$ then $\mathrm{i}\{s,t\} = \mathrm{i}_\nu\{s,t\}$, and by (P4)(d) for r_ν, we deduce that $\pi(\mathrm{i}\{s,t\}) \in F\{\xi(s),\xi(t)\} \cap E$. A parallel argument works if $s,t \in Z_\mu$.

So we can assume that $s \in Z_\nu \setminus Z_\mu$ and $t \in Z_\mu \setminus Z_\nu$. Note that there are a unique $s' \in Z'_\nu$ with $h(s') = s$ and a unique $t' \in Z'_\mu$ with $h(t') = t$. Then, $v = \mathrm{i}\{s,t\} = \mathrm{i}'\{s',t'\} \in U$. Hence either $v \in V$, or $v \in X_\nu \cup X_\mu$ and in this case there is a $w \in X_\nu \cap X_\mu$ with $v \preceq' w$ by (R2)(d).

In both cases $\pi(v) \leq \gamma_0$. Note that, applying (P4)(a) in r' for s', t' and $v = \mathrm{i}'\{s',t'\}$, we obtain $\pi(v) \in o(s') \cap o(t')$. Since $\pi(s'),\pi(t') \in E$ we have $o(s') \cup o(t') \subseteq E$ by Claim 22. Thus $\pi(v) \in E$. And since $\pi(v) \leq \gamma_0$, we have $\pi(v) \in F\{\xi(s),\xi(t)\} \cap E$, which was to be proved.
(P5) Assume that $s,t \in X$, $s \prec t$ and $\Lambda = J(\pi(s),\pi(t))$ isolates s from t. Then $s \notin T_\eta$, so $h(s) = s$.

If $t \notin T_\eta$ then $h(t) = t$, so we are done because r' satisfies (P5).

Assume that $t \in T_\eta$. As $s \prec t$, there is a $t' \in T_{\varepsilon_{\zeta(\nu)}} \cup T_{\varepsilon_{\zeta(\mu)}}$ such that $h(t') = t$ and $s \prec' t'$. Since $\pi(t') \in E$, by Claim 23 we have $J(\pi(s),\pi(t')) = I(\pi(s),1) = J(\pi(s),\pi(t))$. Applying (P5) in r' for s and t', we obtain a $v \in X'$ such that $s \prec' v \preceq' t'$ and $\pi(v') = \Lambda^+$. But as $\zeta(\nu),\zeta(\mu)$ are limit ordinals, we have $v \prec' t'$, and hence $v \in X' \setminus (Z'_\nu \cup Z'_\mu) = U$. Then $h(v) = v$, so $s \prec v \prec t$, which was to be proved. $\square$

Hence we have proved that $\mathcal{P}$ satisfies the κ^+-chain condition, which completes the proof of Theorem 5. $\square$

5. Appendix

We explain in detail how Proposition 36 was proved in [6].

Definition 40. Assume that $Z \subseteq P_\eta$ is a separated set and $\bar{X}$ is the root of $\{X_p : p \in Z\}$.

(a) For every $n \in \omega$ and every $I \in \mathcal{I}_n$ with $\mathrm{cf}(I^+) = \kappa^+$, we define $\xi(I) =$ the least ordinal γ such that $\varepsilon^I_\gamma \supseteq \pi[\bar{X}] \cap I$ and we put $\gamma(I) = \varepsilon^I_{\xi(I)+\kappa}$.

(b) For every $\alpha < \eta$, if there is an $n < \omega$ and an interval $I \in \mathcal{I}_n$ with $\mathrm{cf}(I^+) = \kappa^+$ such that $\alpha \in I$ and $\gamma(I) \leq \alpha$, we consider the least natural number k with this property and write $I(\alpha) = I(\alpha,k)$. Otherwise, we write $I(\alpha) = \{\alpha\}$.

(c) We say that Z is *pairwise equivalent* iff for every $p,q \in Z$ and every $s \in X_p$, $I(\pi(s)) = I(\pi(h_{p,q}(s)))$.

In [6], the following two lemmas were proved:

Lemma 41 ([6, Lemma 2.5]). *Every set in* $\left[P_\eta\right]^{\kappa^+}$ *has a pairwise equivalent subset of size* κ^+.

Lemma 42 ([6, Lemma 2.6]). *A pairwise equivalent set* $Z \subseteq P_\eta$ *of size* κ^+ *is linked.*

To get Proposition 36 we explain that the proof of [6, Lemma 2.6] actually gives the following statement:

If $Z \subseteq P_\eta$ *is a pairwise equivalent set of size* κ^+, *then there is an ordinal* $\gamma < \eta$ *such that every* $p, q \in Z$ *have a common extension* $r \in P_\eta$ *satisfying (R1)–(R2).*

As above, we denote by $\bar{X}$ the root of $\{X_p : p \in Z\}$. Assume that $p, q \in Z$ with $p \neq q$. First observe that the ordering $\prec_r$ is defined in [6, Definition 2.4]. For this, adequate bijections $g_1 : X_r \setminus (X_p \cup X_q) \longrightarrow X_p \setminus \bar{X}$ and $g_2 : X_r \setminus (X_p \cup X_q) \longrightarrow X_q \setminus \bar{X}$ are considered in such a way that $g_2 = h_{p,q} \circ g_1$. Then since $g_2 = h_{p,q} \circ g_1$, [6, Definition 2.4](b) and (c) imply (R2)(a) and [6, Definition 2.4](d) and (f) imply (R2)(b). Also, (R2)(c) follows directly from [6, Definition 2.4](d) and (f), and (R2)(d) is just [6, Definition 2.4](e) and (g). So, we have verified (R2).

To check (R1), i.e. to get the right γ we need a bit more work. Let

$$\mathcal{J} = \{I(\pi(s)) : s \in X_p\} \tag{29}$$

where $p \in Z$. Since Z is pairwise equivalent, $\mathcal{J}$ does not depend on the choice of $p \in Z$. For every $I \in \mathbb{I}_\eta$ with $\mathrm{cf}(I^+) = \kappa^+$ we can choose a set $D(I) \in [E(I) \cap \gamma(I)]^\kappa$ unbounded in $\gamma(I)$. We claim that

$$\gamma = \sup(\bigcup\{D(I) : I \in \mathcal{J}\}) + 1 \tag{30}$$

works.

First observe that $\gamma < \eta$, because $\mathrm{cf}(\eta) = \kappa^+$, $|\mathcal{J}| < \kappa$ and $|D(I)| = \kappa$ for any $I \in \mathcal{J}$.

Now assume that $p, q \in Z$ with $p \neq q$. Write $L_p = \pi[X_p]$, $L_q = \pi[X_q]$ and $\bar{L} = \pi[\bar{X}]$. Let $\{\alpha_\xi : \xi < \delta\}$ and $\{\alpha'_\xi : \xi < \delta\}$ be the strictly increasing enumerations of $L_p \setminus \bar{L}$ and $L_q \setminus \bar{L}$ respectively. In the proof of [6, Lemma 2.6], for each $\xi < \delta$ an element $\beta_\xi \in D(I(\alpha_\xi)) = D(I(\alpha'_\xi))$ was chosen, and then a condition $r \leq_\eta p, q$ was constructed in such a way that $X_r = X_p \cup X_q \cup Y$ where $Y \cap (X_p \cup X_q) = \emptyset$ and $\pi[Y] = \{\beta_\xi : \xi < \delta\}$. Then since $\{\beta_\xi : \xi < \delta\} \subseteq \bigcup\{D(I) : I \in \mathcal{J}\}$, we infer that

$$\sup \pi[X_r \setminus (X_p \cup X_q)] = \sup \pi[Y] < \gamma, \tag{31}$$

which was to be proved.

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