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Archimedes and Liu Hui on Circles and Spheres

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Abstract

This article describes the mystery of a long lost codex of Archimedes that resurfaced briefly at the turn of the last century by Johan Ludwig Heiberg. Long enough for the Danish historian of mathematics Heiberg to identify, photograph and eventually transcribe "The Method" and several other works by Archimedes of considerable mathematical interest. In 1879 Heiberg completed his dissertation, *Quaestiones Archimedeae,* devoted to Archimedes' life, works, and transmission of his texts.

Keywords: Archimedes, Ephodos, Method, Johan Ludwig Heiberg.

Resumen. *Arquímedes y Hui Liu en torno a círculos y esferas.*

Este artículo describe el misterio de un códice de Arquímedes perdido hace mucho tiempo que reapareció brevemente a principios del siglo pasado de la mano de Johan Ludwig Heiberg. Tiempo suficiente para que el historiador danés de las matemáticas Heiberg pudiese identificar, fotografiar y, finalmente, transcribir "El Método" y varias otras obras de Arquímedes de interés matemático considerable. En 1879 Heiberg completó su tesis doctoral, Quaestiones Archimedeae, dedicado a la vida de Arquímedes, las obras, y la transmisión de sus textos.

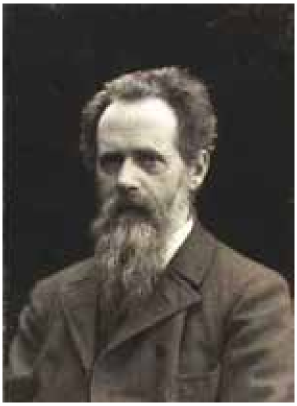
Palabras clave: Arquímedes, ephodos, método, Johan Ludwig Heiberg.

This story begins with a mystery—the mystery of a long lost codex of Archimedes that resurfaced briefly at the turn of the last century, long enough for the Danish historian of mathematics Johan Ludwig Heiberg to identify, photograph and eventually transcribe "The Method" and several other works by Archimedes of considerable mathematical interest. Heiberg was from a wealthy family. The son of a doctor, Heiberg studied classical philosophy and was professor of the subject at the University of Copenhagen. In 1879 he completed his dissertation, *Quaestiones Archimedeae*, devoted to Archimedes' life, works, and transmission of his texts. He subsequently published editions of Euclid's *Opera* (1883-1895), Apollonius' *Conics* (1890-1893), and the complete works of the Danish philosopher Søren Kierkegaard. But his life's greatest achievement was reconstruction of the text of Archimedes' *Ephodos* (The Method), discovered in Constantinople in 1906.[1]

Because of his thorough knowledge of Archimedes, Heiberg was able to decipher most of the barely-legible palimpsest, aided by his good friend H.G. Zeuthen, a mathematician and historian of classical Greek mathematics. Oddly, Heiberg wrote little on the subject from a purely mathematical point of view—his major interest was the transmission of mathematical texts—their transmission and preservation—as well as the remarkable contents of the long-lost palimpsest. The details of the rediscovery of this remarkable work are recounted in the recent book by Reviel Netz and William Noel, *The Archimedes Codex*.[2]

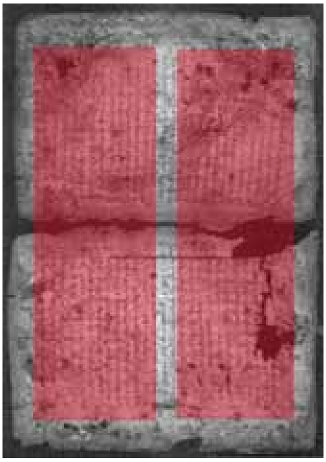
1 This long lost work has been almost completely obliterated, transformed into a palimpsest from which the original Archimedes text was scraped away and upon which a medieval prayer book was copied, written literally on top of the Archimedes manuscript. As for the title of the codex that interests us here—*The Method*—there is a problem with this usual translation that dates to Heiberg's edition of the text. Eberhard Knobloch, in his "Commentary on Bowen 2003," *Centaurus* **50** (1-2) (2008): 205, points out that: "The underlying Greek notion for 'procedure' is 'ephodos' that is the same notion used by Archimedes in his famous letter to Eratosthenes. In both cases, 'ephodos' must not be translated by 'method'. Whatever (sic) Archimedes spoke of the method we nowadays call 'mechanical method' in this letter he used the word 'tropos'." As C.M. Taisbak puts this, *ephodos* "does not mean 'Method' in the modern sense of the word, but rather 'Approach'. As it emerges from Archimedes' foreword, a title 'Entering Mechanical Problems by the Back Door' would much better cover his attitude. After all, he knew that this was heuristics, not deduction," *Historia Mathematica* Mailing List Archive: 23 Jun 1999 19:32:38 +0200 [http://sunsite.utk.edu/math_archives/.http/hypermail/historia/jun99/0149.html]. My own reading of "Ephodos," however, translates this as "attack"—its meaning in both ancient and modern Greek—not random attack but systematic, methodical, careful attack—just the sort a mathematician like Archimedes might wage upon a particularly difficult and challenging problem.

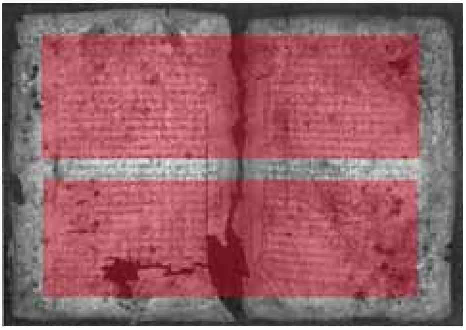
2 Reviel Netz and William Noel, *The Archimedes Codex. How a medieval prayer book is revealing the true genius of antiquity's greatest scientist*. Cambridge, MA: Da Capo Press, 2007.



J.L. Heiberg (1854-1928) J.L. Heiberg in 1918

The basic story, in very brief outline, is as follows. A Byzantine Greek copied an earlier Archimedes manuscript onto parchment sometime in the 10th century, possibly in Constantinople where Leo the Mathematician transcribed many ancient texts into minuscule. On April 14, 1229, Ioannes (John) Myronas, probably working in Jerusalem, finished the palimpsest version of a *Euchologion*, or prayer book, using the parchment from which the text of Archimedes had been scraped away; eventually, the prayer book was moved to the Monastery of Saint Sabas, near Jerusalem.





The original Archimedes codex, copied in miniscule, in two vertical columns, as illustrated on the left; the *Euchologion* would have been created by rotating the original folio page 90° to create both a verso and recto page for the prayer book. The Archimedes text is now split horizontally, half on the left, half on the right.

In the mid-nineteenth century the prayer book was transferred to the library of the Greek Patriarch in Jerusalem, the Metochion. It was at this point that the German scholar, Constantine Tischendorf (1815-1874), saw it on one of his travels to Greek monastic libraries, and he mentioned it in 1846.[3] He also took one sheet as a souvenir (see below)!

In 1876 this was sold to the Cambridge University Library, but it was only identified much later as an Archimedes manuscript page by Nigel Wilson, in 1971.[4] Even before the Archimedes palimpsest was rediscovered, Tohru Sato used his analysis of this one page from the Cambridge University Library to reconstruct Propositions 14 and 18 in the *Method*, to which we shall return momentarily.[5]

It was in Jerusalem that the Greek scholar Athanasios Papadopoulos-Kerameus catalogued and identified the palimpsest as being in part a work of Archimedes—something Tischendorf had apparently missed (in fact, in the latter part of the nineteenth century Papadopoulos-Kerameus catalogued nearly 900 manuscripts in the Metochion's collection). This is how Heiberg originally learned of the Archimedes palimpsest.

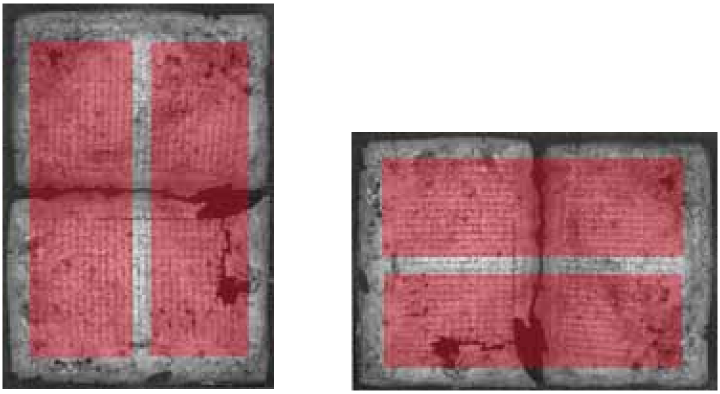
To make a palimpsest, one begins by unbinding a bound parchment manuscript, and then cutting the parchment in half down the middle, separating the formerly verso (left) and recto (right) portions of the parchment. These are then rotated ninety degrees, stacked with other divided pages of the original manuscript, and folded into folios, producing right and left pages for the new manuscript that are one-fourth the size of the original

3 For more on Tischendorf, see Matthew Black and Robert Davidson, *Constantin von Tischendorf and the Greek New Testament,* Glasgow: University of Glasgow Press, 1981.

4 Netz and Noel 2007, p. 130. The page was taken from the palimpsest between folios 2 and 3.

5 Tohru Sato "A Reconstruction of *The Method* Proposition 17, and the Development of Archimedes' Thought on Quadrature—Why did Archimedes not notice the internal connection in the problems dealt with in many of his works?" *Historia Scientiarum*, 31 (1986): 61-86; p. 74.

parchment. Note that the top of the original codex page is now on the recto side of the new page, and the bottom half is on the verso, and the middle part of the original text is lost in the gutter of the new binding. Worse yet, when the palimpsest sheets are grouped into folios, the top and bottom of the original codex page may be separated by many pages of other pages stacked on top of them in comprising the folio, and the recto of a later page will run onto the verso of an earlier page of the palimpsest. Thus reconstructing the original text involves a kind of jig-saw puzzle reconfiguration.



The original parchment manuscript of the Archimedes codex as copied into miniscule is represented on the left; after rotating these folio pages 90°, these were stacked with other leaves from the original manuscript to form the folios of the Ecologion, making a jigsaw of the original codex.

For example, over the critical Proposition 14 of the *Method*, the scribe wrote a prayer for the dead. But to read the entire proposition 14, beginning in column 1 of folio 110 recto of the palimpsest, it would then be necessary to turn the codex 90 degrees to read the Archimedes text, which eventually disappears into the gutter of the palimpsest, reappearing five folios earlier, on 105 verso, but again, the first few lines would be hidden in the gutter of the palimpsest, lines that Heiberg was unable to see. The next continuation of the Archimedes text appears on folio 158.[6]

Comparable in some interesting ways to the Archimedes palimpsest is the oldest actual work of mathematics that currently survives from ancient China, the 算數書 *Suan shu shu*. 彭浩 Peng Hao and other Chinese scholars who have studied this text simply refer to it as a *Book on Arithmetic*.[7] But the problems it treats—68 in all on nearly 200 bamboo

6 For details supplied by William Noel, see Netz and Noel 2007, p. 125.

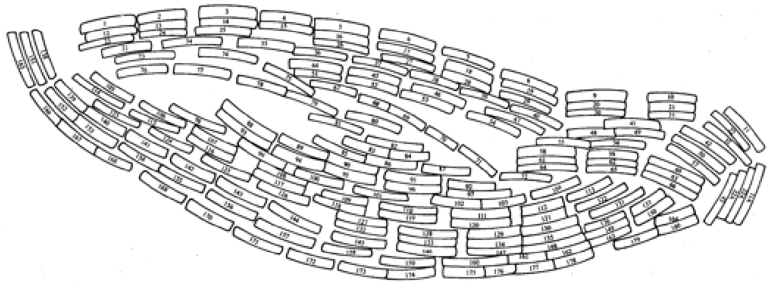
7 彭浩 Peng Hao, trans. and ed., 张家山漢简 «算数書» 註釋 *Zhangjiashan hanjian «Suan shu shu» zhushi* (Commentary and Explanation of the *Suan shu shu* on bamboo strips from Zhangjiashan), Beijing: Science Press, 2002.

slips—include more than arithmetic, notably geometry, and so the title is problematic. On the back of what is taken to be the sixth bamboo strip comprising the book are three characters, 算數書 *Suan shu shu*. Christopher Cullen, Director of the Needham Research Institute for study of the history of East Asian science, technology and medicine in Cambridge, England, treats 算數 *suan shu* as one word meaning "computation," and therefore has called this a "Book on Reckoning" in his translation with commentary.[8] But I prefer to translate each of the characters separately as "A Book on Numbers and Computations." Note the bamboo slip carries the ancient seal character for 筭 *suan*, but in virtually all modern editions of this work this character is rendered by the modern form of the character, 算 *suan*.[9]

Like other documents preserved on silk, bamboo and bronze, the *Suan shu shu* constitutes an artifact from the time it was written. When archaeologists excavating the tomb of an ancient Chinese nobleman at a Western Han Dynasty site near Zhangjiashan, in Jiangling county, Hubei Province, discovered a number of books on bamboo strips in December and January of 1983-1984, these included works on legal statutes, military practice, and medicine. Among these was a previously unknown mathematical work on some 200 bamboo strips, the 算數書 *Suan shu shu*, or *Book of Numbers and Computations*. Based upon other works found in the tomb, especially a copy of the 二年律令 *Er Nian Lü Ling* (Statutes of the Second Year of the Lü Reign), archaeologists have dated the tomb to ca. 186 BCE. When found in tomb 247 at Zhangjiashan, the individual bamboo strips constituting the book were found strewn on the floor of the tomb, and the first challenge facing archaeologists after deciphering the characters on the strips, sometimes faded or illegible, was to rearrange the individual strips to reconstitute the original book itself. This was not unlike the challenge facing the conservators and editors of the Archimedes palimpsest, who also had to tackle the initial problem of reconstituting the original mathematical text of the *Method*.

8 Christopher Cullen, *The Suan shu shu* 筭數書 '*Writings on Reckoning*': A translation of a Chinese mathematical collection of the second century BC, with explanatory commentary, and an edition of the Chinese text. Cambridge, England: Needham Research Institute Working Papers, No. 1, 2004. See: http://www.nri.org.uk/SuanshushuC.Cullen2004.pdf.

9 "算數書 *Suan shu shu* (A Book on Numbers and Computations). English Translation with Commentary," *Archive for History of Exact Sciences,* 62 (2008): 91-178.



This diagram shows the relative placement of individual bamboo strips from the *Suan shu shu* as they were discovered by archaeologists excavating the tomb in which the book was found. From Peng Hao 2002.

As he approached the Archimedes palimpsest, Heiberg photographed the pages from the palimpsest that interested him, mainly those corresponding to the *Method*. More than a century later, these now serve to document how the palimpsest has deteriorated dramatically over the past century, due to careless—one might say derelict—neglect to preserve it, allowing mold to eat away at the parchment due to overly-damp conditions. The *Euchologion* was further compromised when its Parisian owner, in an attempt to enhance its value probably sometime in the 1940s, paid a skilled forger to paint illuminated panels over several pages that had no such ornamentation when Heiberg saw the palimpsest in Istanbul at the turn of the century.

What is amazing is that Heiberg was able to read as much of the original text as he did. Apart from what he could readily see for himself, he had only the photographs and a magnifying glass to aid him. However, high-resolution laser technology has now made it possible to produce higher resolution composite images of the Archimedes palimpsest, enabling scholars to recover more information than Heiberg could make out a century earlier. Above all, the new imaging techniques have revealed the diagrams, as close to Archimedes' originals as we are likely to get, and as Reviel Netz argues, the mathematics for Archimedes was literally *in* the diagrams (see below).[10]

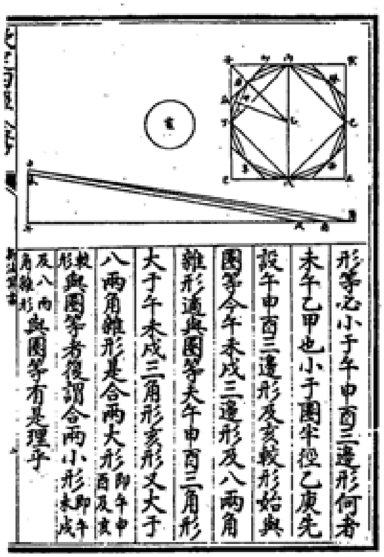
10 Netz and Noel 2007, p. 108.

Amazingly, it appears that Heiberg paid little attention to the diagrams, which were drawn by Zeuthen. Nevertheless, according to Netz, ancient mathematicians thought in terms of diagrams, not in terms of text.[11] But before turning to the actual diagrams, and Archimedean arguments concerning circles and spheres, it will be helpful to know something about what the Chinese knew of Archimedes.

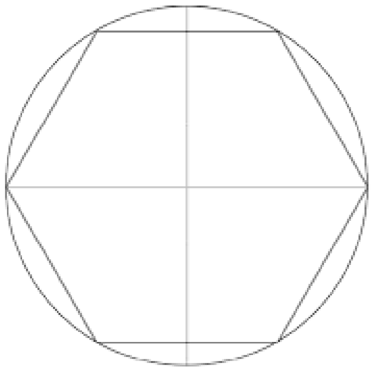
Archimedes was known in China, thanks to Jesuit missionaries who used mathematics and science in hopes of persuading the educated elite of the superiority of Western Christianity.[12] A complete translation of Archimedes' short treatise on finding the area of the circle was translated into Chinese as the 测量圈椅 *Ce liang quan yi* (On the Measurement of the Circle) and printed in 1635.

11 Netz, in Netz and Noel 2008, p. 132.

12 See Joseph W. Dauben, "Chinese Mathematics," the section on "Matteo Ricci and Xu Guanqi, 'Prefaces' to the First Chinese Edition of Euclid's *Elements* (1607)," in Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*, Princeton, N.J.: Princeton University Press, 2007, pp.



形等必小于午申酉三邊形何者未午乙甲也小于圜半徑乙庚先設午申酉三邊形及亥較形始與圜等今午未戌三邊形及八兩角雜形適與圜等夫午申酉三角形大于午未戌三角形亥形又大于八兩角雜形是合兩大形（即午申酉及亥）較形與圜等者復謂合兩小形（即午未戌）及八兩角雜形與圜等有是理乎



On the left is a page from the Chinese translation of Archimedes' 测量圈椅 Ce liang quan yi (On the Measurement of the Circle) (1635); on the right, given a circle 1 unit in diameter, the radius will be ½ unit, and the perimeter of the inscribed hexagon will be 3, clearly less than the circumference of the circle. Thus the ratio of the diameter to the circumference of the circle must be greater than 3.

How did Archimedes determine the value of *pi* or the ratio of the diameter to the circumference of a circle? A preliminary survey of the diagram of a circle of unit diameter makes it clear from the perimeter of an inscribed hexagon, whose perimeter is exactly 3, that the hexagon falls far short of what must be a larger value for the circumference of the circumscribed circle—but exactly how much larger than 3 is the ratio of circumference to diameter? Again, even a casual inspection of a 12-sided dodecagon inscribed in the circle in place of the hexagon (as above) will provide a much better approximation of the ratio of circumference to diameter than 3.

Considering this problem from the point of view of approximating the area of the circle, Euclid proves that the area is equivalent to the area of the triangle whose height is the radius and whose base is the circumference of the circle (see diagram above left). When Euclid in *Elements* XII.2 proves that :"Circles are to one another as the squares on the diameters," he demonstrates that the area of the circle can be exhausted by a series of successively higher-order polygons, beginning with the square, the octagon, and then progressively doubling the number of sides of the inscribed polygons. By considering regular polygons of increasingly-many sides, Archimedes shows that their areas approximate as closely as one may wish—the area of the circle. But application of this so-called method of "exhaustion" is really a misnomer, since the area is never completely exhausted.[13]

Archimedes, in Proposition 1of "Measurement of a Circle," adopts one of the most powerful arguments in the arsenal of ancient Greek mathematics—a counterfactual argument that proceeds as follows:

> Let ABCD be the given circle, and K the area of the triangle described (of height equal to the radius and base equal to the circumference of the circle). Then, if the circle is not equal to K, it must be either greater or less.14

As Geoffrey Lloyd puts it, "The Greek preference for the method of exhaustion is thus evidence *both* of their demand for rigour *and* of their avoidance of infinite processes wherever possible."[15] What needs to be added is that although the Greeks did seek to avoid actually infinite processes, they were nevertheless prepared and willing to consider "potentially infinite" processes—which could stop after any arbitrary level of accuracy had been reached, the process was exhausted, or the calculator had simply grown tired of the process.

Knowing how the Greeks treated the problem of finding the area of the circle, how did ancient Chinese mathematicians approach this same problem? Here a diagram drawn according to procedures to determine the area of the circle as given in the Chinese mathematical classic, the *Nine Chapters on the Art of Mathematics*, through a commentary of the 3rd-century mathematician Liu Hui and reconstructed by a later editor of the text, Dai Zhen, about 1773, speaks for itself:

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13 Euclid XII.2, in T.L. Heath, *The Thirteen Books of Euclid's Elements. Translated from the Text of Heiberg, with Introduction and Commentary,* New York: Dover, 1956, vol. 3, p. 371. See also G.E.R. Lloyd, "Finite and Infinite in Greece and China," *Chinese Science* 13 (1996): 11-34, esp. pp. 20-21, and G.E.R. Lloyd, *Adversaries and Authorities. Investigations into Ancient Greek and Chinese Science*, Cambridge, England: Cambridge University Press, 1996), pp. 149-150.

14 By then appealing to both inscribed and circumscribed regular polygons, Archimedes shows that neither of these alternatives is possible, QED, the area of the circle must equal *K*. See Thomas L. Heath, ed., *The Method of Archimedes, recently discovered by Heiberg. A Supplement to The Works of Archimedes 1897*, Cambridge, England: Cambridge University Press, 1912, p. 91.

15 G.E.R. Lloyd, *Chinese Science* (1996), p. 21; *Adversaries and Authorities* (1996), p. 150.

Dai Zhen's diagram, as reproduced in Joseph Needham and Wang Ling, *Science and Civilisation in China*, vol. 3, Cambridge, England: Cambridge University Press, 1958. P. 29. Note that Needham transliterates Zhen's name as "Tai Chen."

Again, as for Archimedes, we don't have the original diagram and so must rely on various reconstructions (as above, by Dai Zhen), but as Review Netz would say, in this case the diagram literally speaks for itself and conveys the essence of the mathematical thought in question.

Note that in the Chinese diagram, however, the vertices are not lettered, but areas are identified by color—the Chinese characters in the various areas specify different colors, red, yellow and blue[16] —and the text itself refers to terms relative to the triangles, but in Chinese there is no term for triangle; reference is only made to the colored areas, and to their sides and in the case of right triangles, their hypotenuses, respectively.[17] Nevertheless, the gist of Liu Hui's argument as he gives it in the *Nine Chapters* relies on successively finer approximations of inscribed polygons, as is immediately clear upon inspection of the diagram. From the sides of the polygons, Liu Hui can compute the area of the circle with

16 For an example of how Chinese mathematicians color-coded their diagrams in the course of their proofs, see the cover illustration of Christopher Cullen's "Learning from Liu Hui? A Different Way to Do Mathematics," *Notices of the AMS*, 49(7)(August, 2002), pp. 783-790. The illustration in question is a hand-colored version of the 弦圖 *xian tu* (hypotenuse diagram) from a copy of the Zhou bi suan jing xxx

17 Lisa Raphals, "When is a Triangle Not a Triangle?" *Ex/Change. Newsletter of Centre for Cross-Cultural Studies* (City University of Hong Kong) 5 (September, 2002), pp. 9-11.

increasing degrees of accuracy. For example, in the case of the inscribed polygon of 96 sides, he computes the ratio of circumference to diameter of 100 as 314 64/624.

However, and this should be stressed because it is an important conceptual difference between Greek and Chinese thinking on these matters, Liu Hui always speaks of the *lü* of the diameter and circumference, in what he calls the "precise rate," 50 and 157. He did not think in terms of a specific number like *pi* but of a pair of numbers relating diameter and circumference. Another Chinese mathematician, Li Chunfeng, speaks of the *lü* of diameter to circumference as 7 and 22, and Zu Chongzhi takes the 密 *mi lü* (meaning "more accurate rate") to be 113 and 355.[18]

Returning now to the Archimedes codex, the really exciting discovery thanks to the text recovered through the applications of modern technology and computer imaging is Archimedes' *Method* and his determination of the volume of the sphere. This is Archimedes' most famous result, established in his treatise "On the Sphere and Cylinder," wherein he shows that the volume and surface area of a sphere are 2/3 of the volume and of the total surface area of a circumscribed cylinder, respectively. Archimedes considered this his greatest discovery, and the corresponding diagram, which said it all, was according to ancient accounts, engraved on his tombstone.[19]

What is of such importance in the *Method* is that Archimedes describes how he went about finding his results, which he outlined in a letter to Eratosthenes. In part, here is what Archimedes says:

> If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube.[20]

He adds that he discovered this theorem by a certain "mechanical method," as he had many others of his published works. The gist of the proof is to take various plane cuts or sections that are in proportion to one another and show that they are in "equilibrium" or balance; knowing the solution, Archimedes can then derive a geometric proof applying the method of exhaustion to establish the known result.

18 See 李儼Li Yan and杜石然 Du Shiran, *A Concise History of Chinese Mathematics*, John N. Crossley and Anthony W.-C. Lun, trans., Oxford: Clarendon Press: 198, p. 83; Karine Chemla and Guo Shuchun, 九章算術 *Les Neuf chapitres. Le Classique mathématique de la Chine ancienne et ses commentaires*, Paris: Dunod, 2004, p. 64.

19 As mentioned by both Plutarch and Cicero; for details, see T.L. Heath, *The Works of Archimedes*, Cambridge, England: Cambridge University Press, 1912, p. xviii.

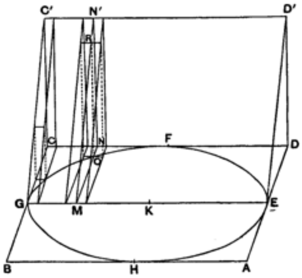
20 T.L. Heath, ed., *The Method*, p. 12.

But there is one theorem in the *Method* that is different—the geometrical derivation of Proposition 14 that so excited Reviel Netz—because a large part of this text could not be read by Heiberg. Proposition 14 establishes the volume of the sphere by examining the proportional relations between cuts of infinitesimal lamina; Proposition 15 then establishes this result rigorously using the method of exhaustion.

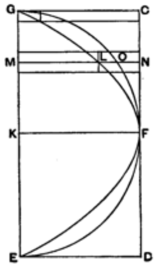
THE METHOD 45

We may supply the missing proof as follows*.

In the accompanying figure are represented (1) the first



element-prism circumscribed to the portion of the cylinder, (2) two element-prisms adjacent to the ordinate *OM*, of which that on the left is circumscribed and that on the right (equal to the other) inscribed, (3) the corresponding element-prisms forming part of the prism cut off (*CC′GEDD′*) which is ¼ of the original prism.



In the second figure are shown element-rectangles circumscribed and inscribed to the auxiliary parabola, which rectangles correspond exactly to the circumscribed and inscribed element-prisms represented in the first figure (the length of *GM* is the same in both figures, and the breadths of the element-rectangles are the same as the heights of the element-prisms);

* It is right to mention that this has already been done by Th. Reinach in his version of the treatise ("Un Traité de Géométrie inédit d'Archimède" in *Revue générale des sciences pures et appliquées*, 30 Nov. and 15 Dec. 1907); but I prefer my own statement of the proof.

The diagram for Theorem 14, as given in T.L. Heath, *The Method of Archimedes* (1912), p. 45.

Archimedes was dissatisfied with his Theorem 14, and there were probably at least two reasons for this—he did not regard results based on mechanical procedures as described in the *Method* as mathematically rigorous; and the use of indivisibles, as they appeared in Theorem 14, were equally suspicious as they involved the paradoxical notion of the infinite. Since the time of Zeno and Democritus, the paradoxes of the infinite had weighed heavily on Greek philosophy and mathematics, hence the method of exhaustion and its actual "limit avoidance," precluding any need to appeal to an actually infinite number of cases.

Nevertheless, we know that it is impossible to determine the volume of a pyramid, for example, without recourse to infinitary arguments, and likewise, to square the circle or find the volume of a sphere or the horse-shoe ("fingernail-like shape" as Reviel Netz calls it).[21] In Proposition 14, Archimedes depended on the application of an infinitary argument—one that set up a complex series of proportional lines which Archimedes could then show held for *any* plane or cut one might choose to make—even though there were an infinite number of such possible cuts to be made.

Liu Hui also considered the problem of finding a formula for the volume of the sphere, given its diameter. In Chapter 4 of the 九章算術 *Jiuzhang suanshu*, the *Nine Chapters*, the diameter d of a sphere of volume V is given as d = ^{3}Ö(16/9)V. This result was known empirically, as a comment on this passage states: "a copper cube of diameter 1 *cun* weight 16 ounces, while a copper ball of the same diameter weight 9 ounces: this is the origin of the ratio 16:9."[22]

Liu Hui explains this result as follows in his commentary. Consider a circle inscribed in a square. The circle is ¾ the area of the square. Now consider the cylinder inscribed in a cube. The ratio of the volumes must again be 3:4 (think of any plane cut through the cube parallel to the base; the ratio is always the same). If we assume the sphere inscribed in the cylinder id ¾ the volume of the cylinder, then the volume of the sphere is ¾(¾)d^3 or 9/16 of the volume of the cube; hence d = ^{3}Ö(16/9)V.[23]

But Liu Hui knows that this is not quite right, and tries to get a more precise result. He notes that the formula for the volume of the sphere as ¾ the volume of the cylinder would be exact if one considered *not* the sphere and cylinder, but another object with a volume less than the cylinder, which he called a 牟合方蓋 *mouhe fangai*, two "inverted umbrellas."[24]

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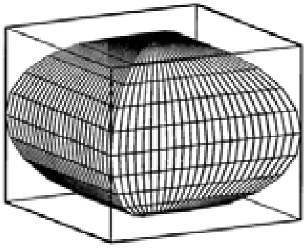
21 See Netz and Noel 2008. p. 189. A more detailed account of the mathematics in the Archimedes codex is given by Reviel Netz, Ken Saito, and Natalie Tchernetska, "A New Reading of *Method* Proposition 14: Preliminary Evidence from the Archimedes Palimpsest (Part 1)," *Sciamvs* 2 (2001): 9-29; and Reviel Netz, Ken Saito, and Natalie Tchernetska, "A New Reading of *Method* Proposition 14: Preliminary Evidence from the Archimedes Palimpsest (Part 2)," *Sciamvs* 3 (2002): 109-125.

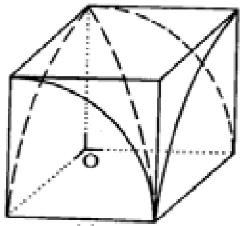
22 Liu Hui, commentary on the *Nine Chapters*, in 錢寶琮 Qian Baocong, ed., 算經十書 *Suan jing shi shu* (Ten mathematical classics), Beijing: Zhonghua Shuju, 1963, vol. 1, p. 156.

23 Liu Hui, in Qian 1963, vol. 1, pp. 155-156.

24 There is no agreement among historians of Chinese mathematics as to exactly how this phrase should be translated. Donald Wagner prefers Li Yan's interpretation (Li Yan, 中國古代數學史料 *Zhongguo gudai shuxue shiliao* (Historical materials on ancient Chinese mathematics), Shanghai: Shanghai Kexue Jishu Chubanshe, cited from second ed., 1963, p. 59; 1st ed. Shanghai 1954) of the phrase *mouhe fanggai*, adding that "An anonymous referee of this article for *Chinese science* suggested an alternative interpretation: 'a combination of a pair of covers on a common square base', *mou* meaning 'double' and *he* having its usual meaning, 'to combine'. This still leaves open the question of what sort of 'covers' these might be. The late Prof. Kurt Vogel pointed out to me the similarity of the geometric form under consideration here with the ancient bronze or ceramic vessel type called a *fang* 鈁.



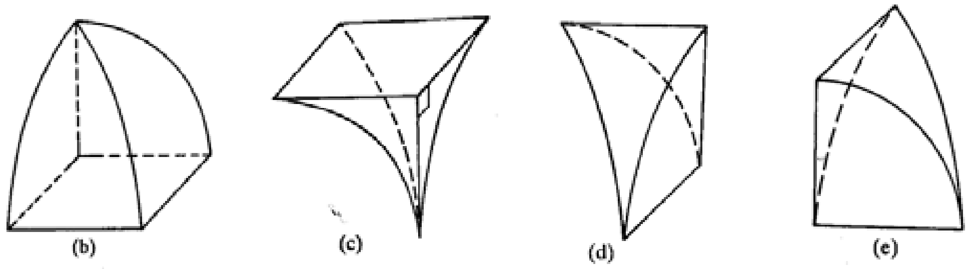
This in fact is the bicylinder—the same figure contemplated by Archimedes formed by the intersection of two cylinders at right angles to each other. Unfortunately, Liu Hui realized that he could not solve the problem of finding the exact volume of this figure, but two centuries later, the mathematician Zu Xuan managed to do so by considering 1/8 of the cross section of the cube containing the bicylinder (eight of these together would constitute the entire bicylinder, but since this is a symmetric figure, it suffices to consider the case shown here in figure (a):



(a)

Zu Xuan dissected the cube using the two cylindrical cuts, obtaining four pieces, one of which he called the "internal" piece (b) and three "external pieces" (c), (d), and (e):

In the phrase *mouhe fanggai* the character 方 *fang* could be a loan character for this *fang*鈁, and the phrase could be translated, 'a combined pair of *fang* 鈁-covers'. Bai Shangshu («九章算術»住釋 «*Jiu zhang suanshu*» *zhushi* (Annotated edition of *Jiu zhang suanshu*), Beijing: Kexue Chubanshe, 1983: 123) interprets the cover as an 'umbrella' (*san* 傘). Therefore Crossley's translation of Li Yan and Du Shiran (Li and Du, *Chinese mathematics: A concise history*, trs. John N. Crossley and Anthony W.-C. Lun, Oxford: Clarendon Press, 1987: 74, 85) translates *mouhe fanggai* as 'two square umbrellas'. J.-C. Martzloff (*Histoire des mathématiques chinoises*, Paris: Masson, 1988: 270) interprets gai 蓋 as 'vault' (in the architectural sense) rather than 'cover', so that the phrase could be translated, 'a double vault'." See Donald Wagner, "Liu Hui and Zu Gengzhi on the Volume of a Sphere," *Chinese science* 3 (1978): 59-79. See also the modified web-site version of this paper, from which the above has been quoted: http://www.staff.hum.ku.dk/dbwagner/Sphere/Sphere.html. Among other changes, the web version of Wagner's paper uses pinyin transliterations rather than the Wade-Giles transliterations to be found in the *Chinese Science* version of this paper.



Now, looking at any horizontal cross section through the cube, Zhu Xuan examined the relation between the one "internal" cross sectional area (b) and the three "external" cross sectional areas (c), (d), and (e), and discovered that the area relations between the sections remained the same wherever the cross section might be taken, pointing out that "this is true whatever the height."[25]

The crucial application of a "Cavallieri" principle then appears as follows:

The stacked 棋 *qi* (blocks) form the volumes, the *shi* of the areas being identical, the volumes cannot differ from one another.[26]

And what of the 棋 *qi* here? Is this really an infinitesimal lamina, as the 18th-century commentator Li Huan has edited the text at this point to read *mi* rather than *qi*? And is Donald Wagner right in agreeing that this should indeed be read as *mi* rather than

25 Zhu Xuan (Zu Gengzhi), quoted from Wagner 1978.

26 Wagner quotes Zu Gengzhi as follows:

疊棋成立積
緣冪勢既同
則積不容異

If blocks are piled up to form volumes,
And corresponding areas are equal,
Then the volumes cannot be unequal.

Here, instead of *qi* in the first line, the 18th-century commentator Li Huang edited the text to read 冪 *mi*, amending the text to mean an infinitesimal laminal surface or plane cut. As Wagner remarks, "Though this statement might be understood in any number of ways, it is clear from the context that it is in fact a statement of Cavallieri's Theorem," and this seems a reasonable conclusion as I reads this passage as well. See Wagner 1978.

qi, which would correct the text to mean an infinitesimal laminal surface or plane cut? Despite such questions, it seems clear that the gist of the argument Zu Xuan is making is a version of Cavallieri's principle.

Conclusion

In conclusion, how are we to account for this extraordinary example of what appears to be near simultaneous discovery—or should we say creation—of the same mathematical results and techniques by mathematicians working in disparate locations and cultures, but with the same intents and goals in mind? Both Archimedes and Liu Hui devoted their lives to mathematics. Liu Hui even realized there were problems he could not solve, but hoped one day they would be.

This now sets the stage for a comparison of the mathematics considered here, east versus west. Geoffrey Lloyd has suggested that one way to view the differences that distinguish Greek and Chinese thought is in terms of essential cultural differences. By entitling his book comparing the two cultures as *Adversaries and Authorities*, Lloyd in a nutshell characterizes what he takes to be distinctive differences between science in ancient Greece and China. In short, if the examples may be the same, if you will, universal, in that we are dealing with right triangles, circles, inscribed polygons, spheres—whose mathematical properties may indeed by universal, especially if idealized in the same way, whether in Athens or Xi'an, one of the capitals of ancient China—the *contexts* in which those universals are considered indeed may differ greatly from ancient Greece to ancient China.

Lloyd sees the Greeks as developing their axiomatic approach to proof, including the paragon of their mathematical method with respect to the infinite in the method of exhaustion and the use of indirect proofs—namely the reliance on *reductio ad absurdum* methods—as growing out of the adversarial experiences of the Greek city states, the law courts, and democratic arguments necessary to convince political adversaries of the acceptability, the legitimacy of a given argument. The Chinese political context placed greater emphasis on authoritarian rule, and hence, Lloyd argues, there resulted a different sort of argumentation. Reliance on authority in turn impeded the progress of mathematics in China, or so this argument goes, whereas argumentation served to advance the subject in Greece.

But more to the point, I think, Chinese mathematicians were certainly willing to criticize bad or poor results. The interest in finding better and better approximations of the value of *pi*, or algorithms for extracting square and cube roots, are examples to consider. But Chinese mathematicians had little patience for a clever argument for the sake of argument; there are no examples of counter-factual reasoning in China, and when confronted by *reductio ad absurdum* premises, the Chinese reaction is "why begin by assuming something you know to be false?"[27]

It may be that the Chinese were more practically minded than the Greeks, and not interested in clever sophistry for the sake of argument, but closer to what constitutes the difference between the character of reasoning east and west, rather than authorities and adversaries, *Yin* and *Yang*, I would emphasize the difference between consensus in China versus contrariness in ancient Greece, a comparison aptly reflected in the recent comparative study of Geoffrey Lloyd and Nathan Sivin, *The Way and the Word (Dao and* λόγος*). Science and Medicine in Early China and Greece*.[28] The differences between the 道 *dao* and λόγος, consensus and argument, whether they be the result of sociological, political, or even psychological differences between east and west, is an open question for more detailed and serious investigation.

However, what is apparent from comparison of the mathematics of Archimedes and Liu Hui is that wherever the human mind may confront mathematics, and specific mathematical cases like circles or spheres—the goal is the same—to establish results, discover relationships, and to provide arguments not only for their plausibility, but for their general validity, laws if you will, governing all circles and all spheres, whether they be in ancient Greece or China, or the contemporary mathematics classroom. As Plato understood, mathematics is one of the most remarkable and enduring achievements of the human mind.

27 For example, see the interesting if controversial study by Alfred Bloom, *The Linguistic Shaping of Thought*, Hillsdale, N.J.: Erlbaum, 1981; Chad Hansen, *Language and Logic in Ancient China*, Ann Arbor, MI: University of Michigan Press, 1983; Robert K. Logan, T*he Alphabet Effect. A Media Ecology Understanding of the Making of Western Civilization*, Ann Arbor, MI: University of Michigan Press, 1983; and A.C. Graham, *Disputers of the Tao: Philosophical Argument in Ancient China*, LaSalle, IL: Open Court, 1989.

28 Geoffrey Lloyd and Nathan Sivin, *The Way and the Word (Dao and λόγος). Science and Medicine in Early China and Greece*, New Haven: Yale University Press, 2002.