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LOCAL AND NONLOCAL WEIGHTED p-LAPLACIAN EVOLUTION EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract

In this paper we study existence and uniqueness of solutions to the local diffusion equation with Neumann boundary conditions and a bounded nonhomogeneous diffusion coefficient $g \geq 0$,

$$\begin{cases} u_t = \operatorname{div}\left(g|\nabla u|^{p-2}\nabla u\right) & \text{in }]0,T[\times\Omega, \\ g|\nabla u|^{p-2}\nabla u\cdot\eta = 0 & \text{on }]0,T[\times\partial\Omega, \end{cases}$$

for $1 \leq p < \infty$. We show that a nonlocal counterpart of this diffusion problem is

$$u_t(t,x) = \int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy \quad \text{in }]0,T[\times\Omega,$$

where the diffusion coefficient has been reinterpreted by means of the values of g at the point $\frac{x+y}{2}$ in the integral operator. The fact that $g \geq 0$ is allowed to vanish in a set of positive measure involves subtle difficulties, specially in the case $p = 1$.

1. Introduction

We consider the p-Laplacian evolution equation with homogeneous Neumann boundary conditions and a bounded nonhomogeneous diffusion coefficient $g \geq 0$, that is

$$N_p^g(u_0)\begin{cases} u_t = \operatorname{div}\left(g|\nabla u|^{p-2}\nabla u\right) & \text{in }]0,T[\times\Omega, \\ g|\nabla u|^{p-2}\nabla u\cdot\eta = 0 & \text{on }]0,T[\times\partial\Omega, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, η is the unit outward normal on $\partial\Omega$,

$$g \text{ can vanish in a subset of } \Omega \text{ of positive measure}$$

and

$$1 \leq p < +\infty.$$

We will see that a nonlocal counterpart of this problem is the following nonlocal nonlinear diffusion problem

$$P_p^{J,g}(u_0) \begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y) g\left(\frac{x+y}{2}\right) \\ \qquad\qquad \times |u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))\, dy, \\ u(x,0) = u_0(x), \end{cases}$$

where the kernel J satisfies

(HJ): $J \colon \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(z)\, dz = 1$,

and where the diffusion coefficient appears measuring the values on intermediate points in the integral operator $g\left(\frac{x+y}{2}\right)$. Note that the form in which the diffusion coefficient appears in the nonlocal problem involves nice symmetry properties as well as a precise behavior under scalings of the kernel J. In fact, we will prove that solutions of the nonlocal problem converge to solutions of the local one when the kernel J is suitable rescaled in relation to the size of its support. It is at this point where we need the choice of the point where g is evaluated, $\frac{x+y}{2}$. Note that simpler choices like $g(x)$ or $g(y)$ will not give the right limit under scaling. We want to remark that, for this convergence result, the fact that g can vanish in a subset of Ω of positive measure turns the whole issue more involved that previous known results for homogeneous diffusion ($g = 1$) since the nonlocal problem, in contrast with what happens in general for the local one, takes into account the part of the domain where the diffusion coefficient g is null, that is, this part of the domain plays a role in the nonlocal diffusion case.

For the homogeneous diffusion $g = 1$, the operator in the local problem is given by

$$\operatorname{div}\left(g|\nabla u|^{p-2}\nabla u\right) = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Delta_p u,$$

that is, the well-studied p-Laplacian of u (see for instance, [**47**], [**48**]), while the study of the nonlocal problem has been done in [**6**] where, moreover, it is proved that suitable rescaled nonlocal problems converge to the local one.

Although the study of the existence of solutions of the nonlocal problem with non-homogeneous diffusion coefficient $g \geq 0$ is somehow easy after the results in [**6**], for the local problem we have to face new technical difficulties due to the lost of coercivity of the associated functional in the usual Sobolev spaces (this happens even if $g > 0$ a.e. but not greater than a positive constant). These difficulties are overcome, for $p > 1$, by using weighted Sobolev spaces involving g with appropriate hypothesis on it, let us say, g is taken in the Muckenhoupt's A_p class.

The case $p = 1$ is somehow different. We need to work in weighted BV spaces (that is, weighted bounded variation spaces), an issue that forces us to introduce some delicate results from measure theory. The local problem for $g = 1$ with $p = 1$, that is, the Neumann problem for the total variation flow, was studied in [**3**] (see also [**4**]), motivated by problems in image processing. This PDE appears when one uses the steepest descent method to minimize the total variation, a method introduced by L. I. Rudin, S. Osher and E. Fatemi [**43**] in the context of image denoising and reconstruction. The use of weighted total variational functionals in image processing began with the seminal work of V. Caselles, R. Kimmel and G. Sapiro ([**25**], [**26**]) on geodesic active contours. Also in the the unpublished paper [**46**] the weighted total variational functionals in image processing was considered (see also [**29**]). Until the recent paper of V. Caselles, G. Facciolo and E. Meinhardt [**24**], it was always supposed that the weight g is positive. In [**24**] it is admitted that g can be null in a set of positive measure. Here, this is the possibility considered.

To finish this introduction, let us briefly introduce some references for the prototype of nonlocal problem considered along this work. Nonlocal evolution equations of the form $u_t(t,x) = (J * u - u)(t,x) = \int_{\mathbb{R}^N} J(x-y)u(t,y)\,dy - u(t,x)$, and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [**39**], if $u(t,x)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(t,y)\,dy = (J*u)(t,x)$ is the rate at which individuals are arriving at position x from all other places and $-u(t,x) = -\int_{\mathbb{R}^N} J(y-x)u(t,x)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies the equation $u_t = J * u - u$. Nonlocal diffusion equations have been recently widely studied and have connections with probability theory (for example, Levy processes are related to the fractional Laplacian), see [**5**], [**6**], [**7**], [**12**], [**13**], [**22**], [**21**], [**23**], [**27**], [**28**], [**32**], [**33**], [**34**], [**39**], [**44**], [**45**] and references therein. Concerning inhomogeneous nonlocal diffusion we quote [**31**] and [**35**] where the authors study the nonlocal analogous to the linear equation $u_t = \Delta(g^2 u)$ in the whole $\mathbb{R}^N$.

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we prove existence and uniqueness for the nonlocal problem with $p > 1$. Section 3 deal with the local problem for $p > 1$ and in Section 4 we show the convergence of the nonlocal problems to the local problem for $p > 1$. In Sections 5, 6 and 7 we deal with analogous questions for $p = 1$. We prefer to present the results for $p = 1$ in separate sections since, as we have mentioned, in this case the use of weighted BV spaces introduces technical differences that we want to highlight.

2. Existence and uniqueness of solutions for the nonlocal problem. The case $p > 1$

Let us begin this section by collecting some preliminaries and notations that will be used in the rest of the paper. We denote by J_0 and P_0 the following sets of functions,

$$J_0 = \{j\colon \mathbb{R} \to [0, +\infty], \text{ convex and lower semi-continuos with } j(0){=}0\},$$

$$P_0 = \{q \in C^\infty(\mathbb{R}) : 0 \le q' \le 1, \text{ supp}(q') \text{ is compact, and } 0 \notin \text{supp}(q)\}\,.$$

In [**15**] the following relation for $u, v \in L^1(\Omega)$ is defined,

$$u \ll v \text{ if and only if } \int_\Omega j(u)\,dx \le \int_\Omega j(v)\,dx \quad \text{for all } j \in J_0,$$

and the following facts are proved.

Proposition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$.*

- (i) *For any $u, v \in L^1(\Omega)$, if $\int_\Omega uq(u) \le \int_\Omega vq(u)$ for all $q \in P_0$, then $u \ll v$.*
- (ii) *If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_r \le \|v\|_r$ for any $r \in [1, +\infty]$.*
- (iii) *If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.*

Solutions of the nonlocal problem $P_p^{J,g}(u_0)$ will be understood according to the following definition.

Definition 2.2. Let $p > 1$. A *solution* of the problem $P_p^{J,g}(u_0)$ in $[0, T]$ is a function $u \in W^{1,1}(0, T; L^1(\Omega))$ which satisfies $u(0, x) = u_0(x)$ *a.e.*

$x \in \Omega$ and

$$u_t(t,x) = \int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy$$

a.e. in $]0,T[\times\Omega$.

Following [**6**] we can solve the evolution problem $P_p^{J,g}(u_0)$. This is done by using Nonlinear Semigroup Theory. We introduce in $L^1(\Omega)$ the following operator associated with our problem.

Definition 2.3. Let J satisfies (HJ), $g \in L^\infty(\mathbb{R}^N)$, $g \geq 0$ a.e., and $1 < p < +\infty$. We define in $L^1(\Omega)$ the operator $B_p^{J,g}$ by

$$B_p^{J,g}u(x) = -\int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \quad x \in \Omega.$$

It is easy to see that,

(1) $B_p^{J,g}$ is positively homogeneous of degree $p-1$,
(2) $L^{p-1}(\Omega) \subset \mathrm{Dom}(B_p^{J,g})$, if $p > 2$,
(3) for $1 < p \leq 2$, $\mathrm{Dom}(B_p^{J,g}) = L^1(\Omega)$ and $B_p^{J,g}$ is closed in $L^1(\Omega) \times L^1(\Omega)$.

Moreover, we have the following monotonicity lemma, whose proof is straightforward.

Lemma 2.4. *Let $T\colon \mathbb{R} \to \mathbb{R}$ a nondecreasing function. Then,*

(i) *for every $u, v \in L^p(\Omega)$ such that $T(u-v) \in L^p(\Omega)$, it holds*

(2.1)

$$\int_\Omega (B_p^{J,g}u(x) - B_p^{J,g}v(x))T(u(x)-v(x))\,dx$$
$$= \frac{1}{2}\int_\Omega\int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)(T(u(y)-v(y))-T(u(x)-v(x)))$$
$$\times\big(|u(y)-u(x)|^{p-2}(u(y)-u(x))-|v(y)-v(x)|^{p-2}(v(y)-v(x))\big)\,dy\,dx.$$

(ii) *Moreover, if T is bounded,* (2.1) *holds for $u, v \in \mathrm{Dom}(B_p^{J,g})$.*

Following the technique of the proof of [**6**, Theorem 2.4] we have that $B_p^{J,g}$ is completely accretive and verifies the range condition $L^p(\Omega) \subset \mathrm{Ran}(I + B_p^{J,g})$. In short, this means that for any $\phi \in L^p(\Omega)$ there is a unique solution of the problem $u + B_p^{J,g}u = \phi$ and the resolvent $(I + B_p^{J,g})^{-1}$ is a contraction in $L^q(\Omega)$ for all $1 \leq q \leq +\infty$.

Theorem 2.5. *The operator $B_p^{J,g}$ is completely accretive and verifies the range condition*

$$L^p(\Omega) \subset \mathrm{Ran}(I + B_p^{J,g}). \tag{2.2}$$

As a consequence we get the following existence and uniqueness theorem for the evolution problem.

Theorem 2.6. *Assume $p > 1$. Let $T > 0$ and $u_0 \in L^1(\Omega)$. Then, there exists a unique mild solution u of*

$$\begin{cases} u'(t) + B_p^{J,g} u(t) = 0, & t \in]0, T[, \\ u(0) = u_0. \end{cases} \tag{2.3}$$

Moreover,

(1) *If $u_0 \in L^p(\Omega)$, the unique mild solution u of (2.3) is a solution of $P_p^{J,g}(u_0)$ in the sense of Definition 2.2. If $1 < p \le 2$, this is true for any $u_0 \in L^1(\Omega)$.*

(2) *Let $u_{i0} \in L^1(\Omega)$, $i = 1, 2$, and u_i a solution in $[0, T]$ of $P_p^{J,g}(u_{i0})$, $i = 1, 2$. Then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \le \int_\Omega (u_{10} - u_{20})^+ \quad \textit{for every } t \in]0, T[.$$

Moreover, for $q \in [1, +\infty]$, if $u_{i0} \in L^q(\Omega)$, $i = 1, 2$, then

$$\|u_1(t) - u_2(t)\|_{L^q(\Omega)} \le \|u_{10} - u_{20}\|_{L^q(\Omega)} \quad \textit{for every } t \in]0, T[.$$

Proof: As a consequence of Theorem 2.5 we get the existence of mild solution of (2.3) (see [**16**] and [**15**]). On the other hand, $u(t)$ is a solution of $P_p^{J,g}(u_0)$ if and only if $u(t)$ is a strong solution of the abstract Cauchy problem (2.3). Now, due to the complete accretivity of $B_p^{J,g}$ and the range condition (2.2), $u(t)$ is a strong solution (see [**15**]). Moreover, in the case $1 < p \le 2$, since $\mathrm{Dom}(B_p^{J,g}) = L^1(\Omega)$ and $B_p^{J,g}$ is closed in $L^1(\Omega) \times L^1(\Omega)$, the result holds for L^1-data. Finally, the contraction principle is a consequence of the general Nonlinear Semigroup Theory. □

3. The local problem for $p > 1$

We consider now the local evolution equation with homogeneous Neumann boundary conditions

$$N_p^g(u_0) \begin{cases} u_t = \mathrm{div}\left(g|Du|^{p-2}Du\right) & \text{in }]0, T[\times \Omega, \\ g|Du|^{p-2}Du \cdot \eta = 0 & \text{on }]0, T[\times \partial\Omega, \\ u(\cdot, 0) = u_0 \in L^1(\Omega) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth domain, η is the unit outward normal on $\partial\Omega$ and g verifies

$$\begin{cases} g \in L^\infty(\Omega), \\ g > 0 \text{ a.e. in } S, \\ g = 0 \text{ a.e. in } \Omega \setminus S, \end{cases} \tag{3.1}$$

being S a smooth domain contained in Ω, and

$$g^{\frac{1}{1-p}} \in L^1(S). \tag{3.2}$$

We will work in the following weighted Sobolev space.

Definition 3.1. Set $W^{1,p}_{g,S}(\Omega)$ the space of functions $u \in L^p(\Omega)$ such that the distributional derivatives in S, $\frac{\partial u}{\partial x_i}$, satisfy

$$g^{1/p} \frac{\partial u}{\partial x_i} \in L^p(S), \quad i = 1, 2, \ldots, N.$$

This space $W^{1,p}_{g,S}(\Omega)$ endowed with the norm

$$\|u\|_{W^{1,p}_{g,S}(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx + \int_S |Du(x)|^p g(x) \, dx \right)^{\frac{1}{p}}$$

is a Banach space.

Let us recall that w is a weight in the Muckenhoupt's A_p-class, or an A_p-weight, if w is a nonnegative, locally (Lebesgue) integrable function in $\mathbb{R}^N$ such that

$$\sup \left(\frac{1}{|B|} \int_B w(x) \, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{\frac{1}{1-p}} \, dx \right)^{p-1} = c_{w,p} < \infty,$$

where the supremum is taken over all ball B in $\mathbb{R}^N$.

We also assume that

(3.3) there exists a weight function g_0 in the Muckenhoupt's A_p-class such that $g_0 = g$ in S.

This hypothesis implies (3.2) since S is bounded. Moreover, under this hypothesis, functions in $W^{1,p}_{g,S}(\Omega) \cap L^\infty(\Omega)$ can be approximated in the $\| \cdot \|_{W^{1,p}_{g,S}(\Omega)}$-norm by smooth functions (see [**30**], [**37**], [**41**], [**42**] and references therein for related topics). Indeed, we have the following result.

Lemma 3.2. *For any $u \in W^{1,p}_{g,S}(\Omega) \cap L^\infty(\Omega)$ there exists $\varphi_n \in C^\infty(\Omega)$ such that $\varphi_n \to u$ in $W^{1,p}_{g,S}(\Omega)$.*

Proof: Given $u \in W^{1,p}_{g,S}(\Omega)\cap L^\infty(\Omega)$, by the results in [**30**], $u_{|S}$ can be extended to a function $\tilde{u} \in W^{1,p}_{g_0,\mathbb{R}^N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $||\tilde{u}||_{W^{1,p}_{g_0,\mathbb{R}^N}(\mathbb{R}^N)} \leq K||u||_{W^{1,p}_{g,S}(S)}$ and $||\tilde{u}||_{L^\infty(\mathbb{R}^N)} \leq ||u||_{L^\infty(S)}$, where K is independent of u. Now, by the results of [**42**], $\tilde{u}$ can be approximated in the $W^{1,p}_{g_0,\mathbb{R}^N}(\mathbb{R}^N)$-norm by C^∞ functions $\tilde{\varphi}_n$ that are uniformly bounded in L^∞. On the other hand, $u_{|\Omega\setminus S}$ can be approximated in the L^p-norm by smooth functions $\hat{\varphi}_n$ uniformly bounded in L^∞. Therefore, we can find φ_n such that

$$\varphi_n = \begin{cases} \tilde{\varphi}_n & \text{in } S, \\ \hat{\varphi}_n & \text{in } \Omega \setminus \left(S + B(0, \frac{1}{n})\right), \end{cases}$$

and in such a way that φ_n is smooth and uniformly L^∞-bounded. We conclude that $\varphi_n \to u$ in $W^{1,p}_{g,S}(\Omega)$. □

We use the following concept of solution for problem $N^g_p(u_0)$.

Definition 3.3. A function $u \in W^{1,1}(0,T;L^1(\Omega))$ is an entropy solution of problem $N^g_p(u_0)$ in $]0,T[$ if $u(0) = u_0$, $T_k(u(t)) \in W^{1,p}_{g,S}(\Omega)$ for every $k > 0$ and

$$\int_\Omega u'(t)T_k(u(t)-\phi)\,dx + \int_S g(x)|Du(t)|^{p-2}Du(t)\cdot D(T_k(u(t)-\phi))\,dx \leq 0,$$

for every $\phi \in W^{1,p}_{g,S}(\Omega)\cap L^\infty(\Omega)$ and all $k > 0$. Here $T_k(r)$ is the classical truncature function $T_k(r) = \sup\{\inf\{r,k\},-k\}$.

To get the existence of entropy solutions of problem $N^g_p(u_0)$ we use again the Nonlinear Semigroups Theory, so we start with the study of the elliptic problem

$$E^g_p(f) \begin{cases} u - \operatorname{div}\left(g|Du|^{p-2}Du\right) = f & \text{in } \Omega, \\ g|Du|^{p-2}Du\cdot\eta = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us introduce the following operator related to the local problem.

Definition 3.4. For $p > 1$ and g satisfying (3.1) and (3.3), we define the operator B^g_p in $L^1(\Omega)$ by the following rule: $(u,\hat{u}) \in B^g_p$ if and only if $u \in W^{1,p}_{g,S}(\Omega) \cap L^\infty(\Omega)$, $\hat{u} \in L^1(\Omega)$ and

$$\int_S g(x)|Du|^{p-2}Du\cdot Dv\,dx = \int_\Omega \hat{u}(x)v(x)\,dx \quad \forall\, v \in W^{1,p}_{g,S}(\Omega)\cap L^\infty(\Omega).$$

Proposition 3.5. *Assume g satisfies* (3.1) *and* (3.3). *Then the operator B^g_p is completely accretive and satisfies the range condition $L^\infty(\Omega) \subset R(I + B^g_p)$.*

Proof: Given $(u_i, v_i) \in B_p^g$, $i = 1, 2$, for any $q \in C^\infty(\mathbb{R})$, $0 \le q' \le 1$, supp(q') compact, $0 \notin$ supp(q), we have that

$$\int_\Omega (v_1 - v_2) q(u_1) - u_2) = \int_S g q'(u_1 - u_2) \left(|Du_1|^{p-2} Du_1 - |Du_2|^{p-2} Du_2 \right) \cdot D(u_1 - u_2) \ge 0,$$

from where it follows that B_p^g is a completely accretive operator (see [**15**]).

Let $n \in \mathbb{N}$. By the results in [**9**], given $f \in L^\infty(\Omega)$ there exists a unique $u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega g(x) |Du_n|^{p-2} Du_n \cdot Dv + \frac{1}{n} \int_\Omega |Du_n|^{p-2} Du_n \cdot Dv = \int_\Omega (f - u_n) v \tag{3.4}$$

for every $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, $u_n \ll f$ for every $n \in \mathbb{N}$, which implies that

$$\|u_n\|_q \le \|f\|_q \quad \text{for every } n \in \mathbb{N}, \text{ and all } 1 \le q \le \infty. \tag{3.5}$$

Taking $v = u_n$ as test function in (3.4), we get

$$\int_\Omega g(x) |Du_n|^p \, dx + \frac{1}{n} \int_\Omega |Du_n|^p \, dx \le \int_\Omega (f - u_n) u_n \, dx \quad \text{for every } n \in \mathbb{N}. \tag{3.6}$$

From (3.5), taking a subsequence if necessary, we have there exists $u \in L^\infty(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega). \tag{3.7}$$

On the other hand, by (3.5) and (3.6), we get

$$\int_\Omega g(x) |Du_n|^p \, dx + \frac{1}{n} \int_\Omega |Du_n|^p \, dx \le M \quad \text{for every } n \in \mathbb{N}.$$

Then, by Hölder's inequality we have

$$\left| \frac{1}{n} \int_\Omega |Du_n|^{p-2} Du_n \cdot Dv \right| \le \frac{M^{\frac{1}{p'}}}{n^{\frac{1}{p}}} \|Dv\|_p \quad \forall\, n \in \mathbb{N}, \tag{3.8}$$

$$\|g^{\frac{1}{p}} |Du_n|\|_{L^p(\Omega)} \le M^{\frac{1}{p}} \qquad \forall\, n \in \mathbb{N} \tag{3.9}$$

and

$$\|g^{\frac{1}{p'}} |Du_n|^{p-2} Du_n\|_{L^{p'}(\Omega, \mathbb{R}^N)} \le M^{\frac{1}{p'}} \qquad \forall\, n \in \mathbb{N}. \tag{3.10}$$

From (3.9), taking a subsequence if necessary, we have that

$$g^{\frac{1}{p}} \frac{\partial u_n}{\partial x_i} \rightharpoonup w_i \quad \text{weakly in } L^p(\Omega), \quad i = 1, \ldots, N. \tag{3.11}$$

Given $\varphi \in \mathcal{D}(S)$, by (3.2), we have $g^{-\frac{1}{p}}\varphi \in L^{p'}(S)$. Then, having in mind (3.11), we obtain

$$\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle = -\int_S u \frac{\partial \varphi}{\partial x_i}\, dx = -\lim_{n\to\infty} \int_S u_n \frac{\partial \varphi}{\partial x_i}\, dx = \lim_{n\to\infty} \int_S \frac{\partial u_n}{\partial x_i} \varphi\, dx$$

$$= \lim_{n\to\infty} \int_S g^{\frac{1}{p}} \frac{\partial u_n}{\partial x_i} g^{-\frac{1}{p}} \varphi\, dx = \int_S w_i g^{-\frac{1}{p}} \varphi\, dx.$$

Consequently, we get

$$\frac{\partial u}{\partial x_i} = w_i g^{-\frac{1}{p}} \quad \text{in } \mathcal{D}'(S), \quad i = 1, \ldots, N.$$

Hence, since $w_i \in L^p(\Omega)$ and $g^{-\frac{1}{p}} \in L^{p'}(S)$, we obtain that $\frac{\partial u}{\partial x_i} \in L^1(S)$, and $u \in W^{1,1}(S)$. Moreover, since $g \in L^\infty(\mathbb{R}^N)$,

$$g^{\frac{1}{p}} \frac{\partial u}{\partial x_i} = w_i \in L^p(S), \quad i = 1, \ldots, N. \tag{3.12}$$

Therefore $u \in W^{1,p}_{g,S}(\Omega)$. Moreover, by (3.11) and (3.12), we have

$$g^{\frac{1}{p}} Du_n \rightharpoonup g^{\frac{1}{p}} Du \quad \text{weakly in } L^p(S, \mathbb{R}^N). \tag{3.13}$$

By (3.10), taking a subsequence if necessary, there exists $\mathbf{z} \in L^{p'}(\Omega, \mathbb{R}^N)$ such that

$$g^{\frac{1}{p'}} |Du_n|^{p-2} Du_n \rightharpoonup \mathbf{z} \quad \text{weakly in } L^{p'}(\Omega, \mathbb{R}^N). \tag{3.14}$$

Given $v \in W^{1,p}(\Omega)$, taking limit in (3.4) and having in mind (3.7), (3.8) and (3.14), we obtain

$$\int_\Omega g(x)^{\frac{1}{p}} \mathbf{z} \cdot Dv\, dx = \int_\Omega (f - u) v\, dx. \tag{3.15}$$

Setting $v = u_n$ in (3.15), using (3.7) and (3.13), and taking limit we get

$$\int_S g(x)^{\frac{1}{p}} \mathbf{z} \cdot Du\, dx = \int_\Omega (f - u) u\, dx. \tag{3.16}$$

Then, by Minty-Browder's method, it is easy to see that $g^{\frac{1}{p}}\mathbf{z} = g|Du|^{p-2}Du$ a.e. in S. Therefore, by (3.15),

$$\int_S g(x)|Du|^{p-2}Du \cdot Dv\, dx = \int_\Omega (f-u)v\, dx \quad \forall\, v \in W^{1,p}(\Omega). \tag{3.17}$$

Now, using the fact that g in S is the restriction of a weight of Muckenhoupt's A_p-class, by Lemma 3.2, any $v \in W^{1,p}_{g,S}(\Omega) \cap L^\infty(\Omega)$ can be approximated by smooth functions and then, from (3.17), we obtain

$$\int_S g(x)|Du|^{p-2}Du \cdot Dv\, dx = \int_\Omega (f-u)v\, dx \quad \forall\, v \in W^{1,p}_{g,S}(\Omega) \cap L^\infty(\Omega). \tag{3.18}$$

Therefore, $(u, f-u) \in B^g_p$, and consequently, $f \in R(I + B^g_p)$. □

As we are considering a weight g that is strictly positive in S and the corresponding integrals that involve g take place in S, we can follow the arguments of [**9**], with minor modifications, to obtain the following characterization of the closure $\mathcal{B}^g_p$ of the operator B^g_p in $L^1(\Omega) \times L^1(\Omega)$.

Proposition 3.6. *The closure of B^g_p in $L^1(\Omega)\times L^1(\Omega)$ is given by $(u,v)\in \mathcal{B}^g_p$ if $u, v \in L^1(\Omega)$, $T_k(u) \in W^{1,p}_{g,S}(\Omega)$ and*

$$\int_S g(x)|Du|^{p-2}Du \cdot D(T_k(u-\phi))\, dx \le \int_\Omega vT_k(u-\phi)\, dx,$$

for every $\phi \in W^{1,p}_{g,S}(\Omega) \cap L^\infty(\Omega)$ and all $k > 0$.

Theorem 3.7. *For any $u_0 \in L^1(\Omega)$ and any $T > 0$, the problem $N^g_p(u_0)$ has a unique entropy solution in $]0, T[$. Moreover, an L^1-contraction principle holds for such solutions.*

Proof: As a consequence of Proposition 3.5 the operator $\mathcal{B}^g_p$ is m-completely accretive in $L^1(\Omega)$. On the other hand, it is easy to see that $\overline{D\,(B^g_p)}^{L^1(\Omega)} = L^1(\Omega)$. Therefore, using the Nonlinear Semigroup Theory (see [**36**] and [**16**]), for any $u_0 \in L^1(\Omega)$, the abstract Cauchy problem associated to $N^g_p(u_0)$ has a unique mild solution given by the exponential formula $v(t) = \mathrm{e}^{-t\mathcal{B}^g_p}u_0$. Moreover, as the operator is homogeneous of degree $p-1$, this solution is the unique strong solution of such abstract problem (see [**15**]). Now, by Proposition 3.6, the concept of strong solution and the concept of entropy solution of $N^g_p(u_0)$ coincide. The contraction principle follows by the Nonlinear Semigroup Theory. □

Remark 3.8. Observe that, in fact, a solution u of $E^g_p(f)$ satisfies

$$u = f \quad \text{a.e. in } \Omega \setminus S$$

and $u_{|_S}$ is a solution of

$$\begin{cases} u - \operatorname{div}\left(g|Du|^{p-2}Du\right) = f & \text{in } S, \\ g|Du|^{p-2}Du \cdot \eta = 0 & \text{on } \partial S. \end{cases}$$

We can think that g is a space-depending diffusion coefficient such that it has broken its diffusivity to 0 in some parts, so $u = f$ where there is no diffusivity. For the parabolic problem u must be equal to the initial condition in places where g vanishes. However, if $|\Omega \setminus S| > 0$, when dealing with the nonlocal problem, it is not true that in general $u = f$ in $\Omega \setminus S$, even if $\operatorname{supp}(J)$ is very "small", there exists $\mathcal{O} \subset \Omega \setminus S$ with $|\mathcal{O}| > 0$ where u may differ from f. So, the part where $g = 0$ plays a role in the nonlocal problem. Nevertheless, in the next section we will see that, under rescaling, solutions to the nonlocal problems converge to solutions to the local one.

4. Convergence of the nonlocal problems to the local problem. The case $p > 1$

Our main goal in this section is to show that the problem $N_p^g(u_0)$ can be approximated by suitable nonlocal Neumann problems of the form $P_p^{J,g}(u_0)$.

Let us now give the reescaling procedure. For given $p > 1$ and J, we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right),$$

where $C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p \, dz$ is a normalizing constant.

Associated to these kernels we solve $P_p^{J,g}(u_0)$ with $J_{p,\varepsilon}$ instead of J with the same initial condition u_0 and we obtain a solution $u_\varepsilon(t,x)$. Our main concern in this section is to show that u_ε converge to u as $\varepsilon \to 0$, being u a solution of $N_p^g(u_0)$.

First, let us perform a formal calculation in one space dimension just to convince the reader that the convergence result is correct. Let $g(x)$ and $u(x)$ be smooth functions and consider

$$A_\varepsilon(u) = \frac{1}{\varepsilon^{p+1}} \int_{\mathbb{R}} J\left(\frac{x-y}{\varepsilon}\right) g\left(\frac{x+y}{2}\right) |u(y)-u(x)|^{p-2}(u(y)-u(x))\, dy.$$

Changing variables, $y = x - \varepsilon z$, we get

$$A_\varepsilon(u) = \frac{1}{\varepsilon^p} \int_{\mathbb{R}} J(z) g\Big(x - \frac{\varepsilon z}{2}\Big) |u(x-\varepsilon z)-u(x)|^{p-2}(u(x-\varepsilon z)-u(x))\, dz. \tag{4.1}$$

Now, we expand in powers of ε to obtain

$$|u(x-\varepsilon z)-u(x)|^{p-2} = \varepsilon^{p-2}\left|u'(x)z - \frac{u''(x)}{2}\varepsilon z^2 + O(\varepsilon^2)\right|^{p-2}$$
$$= \varepsilon^{p-2}|u'(x)|^{p-2}|z|^{p-2}$$
$$-\varepsilon^{p-1}(p-2)|u'(x)z|^{p-4}u'(x)z\frac{u''(x)}{2}z^2 + O(\varepsilon^p),$$

and

$$u(x-\varepsilon z)-u(x) = -\varepsilon u'(x)z + \frac{u''(x)}{2}\varepsilon^2 z^2 + O(\varepsilon^3),$$

on the other hand, since g is smooth,

$$g\left(x-\frac{\varepsilon z}{2}\right) = g(x) - g'(x)\frac{\varepsilon z}{2} + O(\varepsilon^2).$$

Hence, (4.1) becomes

$$A_\varepsilon(u) = -\frac{1}{\varepsilon}\int_{\mathbb{R}} J(z)|z|^{p-2}z\,dz\,\left[g(x)|u'(x)|^{p-2}u'(x)\right]$$
$$+\frac{1}{2}\int_{\mathbb{R}} J(z)|z|^p\,dz\big[g(x)\big((p-2)|u'(x)|^{p-2}u''(x)+|u'(x)|^{p-2}u''(x)\big)\big]$$
$$+\frac{1}{2}\int_{\mathbb{R}} J(z)|z|^p\,dz\big[g'(x)\big(|u'(x)|^{p-2}u'(x)\big)\big] + O(\varepsilon).$$

Using that J is radially symmetric, the first integral vanishes and therefore,

$$\lim_{\varepsilon\to 0} A_\varepsilon(u) = C\left(g(x)|u'(x)|^{p-2}u'(x)\right)',$$

where the constant C is given by $C = \frac{1}{2}\int_{\mathbb{R}} J(z)|z|^p\,dz$.

To do this formal calculation rigorous we need to obtain the following result which is a variant of [**6**, Proposition 3.2(1.i)]. From now on, we denote by $\overline{f}$ the extension by zero outside Ω of a function $f \in L^p(\Omega)$.

Proposition 4.1. *Let $1 < q < +\infty$. Let $\rho\colon \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non-identically zero, and $\rho_n(x) := n^N\rho(nx)$. Let S an open set, $S \subset \Omega$, and let $l \in L^\infty(\mathbb{R}^N)$ such that*

$$l(x) = \begin{cases} l(x) > 0 & \text{a.e. in } S,\\ 0 & \text{a.e. in } \mathbb{R}^N\setminus S.\end{cases} \tag{4.2}$$

Let us also assume that l satisfies

$$l^{\frac{1}{1-q}} \in L^1_{\mathrm{loc}}(S). \tag{4.3}$$

Let $\{f_n\}$ *be a sequence of functions in* $L^q(\Omega)$ *such that*

$$\int_\Omega\int_\Omega \rho_n(y-x)l\left(\frac{x+y}{2}\right)|f_n(y)-f_n(x)|^q\,dx\,dy \le M\frac{1}{n^q} \tag{4.4}$$

and $\{f_n\}$ *is weakly convergent in* $L^q(S)$ *to* f.

Then, $l^{1/q}|\nabla f| \in L^q(S)$, $|\nabla f| \in L^1_{\mathrm{loc}}(S)$, *and moreover*

$$\lim_n \left[(\rho(z))^{1/q}(l(w))^{1/q}\chi_\Omega\left(w+\frac{1}{2n}\right)\chi_\Omega\left(w-\frac{1}{2n}z\right) \times\frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n}\right] = (\rho(z))^{1/q}h(w,z)$$

weakly in $L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)$, *with*

$$(\rho(z))^{1/q}h(w,z) = (\rho(z))^{1/q}(l(w))^{1/q}z\cdot\nabla f(w) \quad \text{in } S\times\mathbb{R}^N,$$

and

$$(\rho(z))^{1/q}h(w,z) = 0 \quad \text{in } (\mathbb{R}^N\setminus\overline{\Omega})\times\mathbb{R}^N.$$

Proof: Making the change of variables $y = x+\frac{1}{n}z$, $x = w-\frac{1}{2n}z$, we rewrite (4.4) as

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho(z)l(w)\chi_\Omega^\times\left(w\pm\frac{1}{2n}z\right)\left|\frac{\overline{f}_n(w+\frac{1}{2n}z)-\overline{f}_n(w-\frac{1}{2n}z)}{1/n}\right|^q dw\,dz \le M,$$

where we use the notation $\chi_\Omega^\times\left(w\pm\frac{1}{2n}z\right) = \chi_\Omega\left(w+\frac{1}{2n}z\right)\chi_\Omega\left(w-\frac{1}{2n}z\right)$. Therefore, up to a subsequence,

$$(\rho(z))^{1/q}(l(w))^{1/q}\chi_\Omega^\times\left(w\pm\frac{1}{2n}z\right)\frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n} \rightharpoonup (\rho(z))^{1/q}h(w,z) \tag{4.5}$$

weakly in $L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)$, and $(\rho(z))^{1/q}h(w,z) = 0$ in $(\mathbb{R}^N\setminus\overline{\Omega})\times\mathbb{R}^N$.

If $\varphi\in C_c^\infty(\Omega)$, $\mathrm{supp}(\varphi)\subset S$, taking

$$\hat{\varphi} = \begin{cases} \dfrac{\varphi}{l^{1/q}} & \text{in } S, \\ 0 & \text{otherwise,} \end{cases}$$

which is an $L^{q'}$-function since $l^{\frac{1}{1-q}} \in L^1_{\rm loc}(S)$, and $\psi \in C^\infty_c(\mathbb{R}^N)$, by (4.5), we obtain

$$\int_{\mathbb{R}^N}\int_\Omega (\rho(z))^{1/q}(l(w))^{1/q}\chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) \times \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n}\hat{\varphi}(w)\,dw\psi(z)\,dz \to \int_{\mathbb{R}^N}\int_\Omega (\rho(z))^{1/q}\,h(w,z)\hat{\varphi}(w)\,dw\psi(z)\,dz.$$

That is,

$$\int_{\mathbb{R}^N}\int_S (\rho(z))^{1/q}\chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) \times \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n}\varphi(w)\,dw\psi(z)\,dz \to \int_{\mathbb{R}^N}\int_S (\rho(z))^{1/q}\,(l(w))^{-1/q}h(w,z)\varphi(w)\,dw\psi(z)\,dz. \tag{4.6}$$

Now, for n large enough, $\rho(z)^{1/q}\chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) = \rho(z)^{1/q}$ for all $z \in \mathbb{R}^N$ and all $w \in \operatorname{supp}(\varphi)$, therefore

$$\begin{aligned}\int_{\mathbb{R}^N}\int_S (\rho(z))^{1/q}\chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right)\frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n}\varphi(w)\,dw\psi(z)\,dz \\ = \int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_S \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n(w-\frac{1}{2n}z)}{1/n}\varphi(w)\,dw\psi(z)\,dz \\ = -\int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_S f_n(w)\frac{\overline{\varphi}(w+\frac{1}{2n}z)-\overline{\varphi}\left(w-\frac{1}{2n}z\right)}{1/n}\,dw\psi(z)\,dz.\end{aligned}$$

Then, passing to the limit, on account of (4.6), we get

$$\int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_S (l(w))^{-1/q}h(w,z)\varphi(w)\,dw\psi(z)\,dz = -\int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_S f(w)\,z\cdot\nabla\varphi(w)\,dw\psi(z)\,dz.$$

Consequently,

$$\int_S (l(w))^{-1/q} h(w,z)\varphi(w)\,dw = -\int_S f(w)\, z\cdot\nabla\varphi(w)\,dw \quad \forall\, z\in \mathrm{int}(\mathrm{supp}(J)).$$

From here, for s small,

$$\int_S (l(w))^{-1/q} h(w, se_i)\varphi(w)\,dw = -\int_S f(w)\, s\frac{\partial}{\partial w_i}\varphi(w)\,dw,$$

which implies, since S is open, $|\nabla f| \in L^1_{\mathrm{loc}}(S)$ (using that $l^{\frac{1}{1-q}} \in L^1_{\mathrm{loc}}(S)$ together with Hölder's inequality), $l^{1/q}|\nabla f| \in L^q(S)$ and $(\rho(z))^{1/q} h(w,z) = (\rho(z))^{1/q}(l(w))^{1/q} z\cdot\nabla f(w)$ in $S\times\mathbb{R}^N$. □

Proposition 4.2. *Assume $p>1$, J satisfies (HJ), and g satisfies* (3.1) *and* (3.3). *Then, for any $\phi\in L^\infty(\Omega)$, we have that*

$$(4.7)\qquad \left(I + B_p^{J_{p,\varepsilon},g}\right)^{-1}\phi \rightharpoonup \left(I + B_p^g\right)^{-1}\phi \quad \textit{weakly in } L^p(\Omega) \textit{ as } \varepsilon\to 0.$$

Proof: For $\varepsilon > 0$, let $u_\varepsilon = \left(I + B_p^{J_{p,\varepsilon},g}\right)^{-1}\phi$. Then, $u_\varepsilon \ll \phi$, and, by changing variables,

$$(4.8)\quad \int_\Omega \phi(x)v(x)\,dx - \int_\Omega u_\varepsilon(x)v(x)\,dx$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z)g(w)\chi_\Omega^\times\left(w\pm\frac{\varepsilon}{2}z\right)\left|\frac{\overline{u}_\varepsilon(w+\frac{\varepsilon}{2}z)-\overline{u}_\varepsilon(w-\frac{\varepsilon}{2}z)}{\varepsilon}\right|^{p-2}$$
$$\times \frac{\overline{u}_\varepsilon(w+\frac{\varepsilon}{2}z)-\overline{u}_\varepsilon(w-\frac{\varepsilon}{2}z)}{\varepsilon}\,\frac{\overline{v}(w+\frac{\varepsilon}{2}z)-\overline{v}(w-\frac{\varepsilon}{2}z)}{\varepsilon}\,dw\,dz,$$

where $\chi_\Omega^\times\left(w\pm\frac{\varepsilon}{2}z\right) = \chi_\Omega\left(w+\frac{\varepsilon}{2}z\right)\chi_\Omega\left(w-\frac{\varepsilon}{2}z\right)$.

Let us see that there exists a sequence $\varepsilon_n\to 0$ such that $u_{\varepsilon_n}\rightharpoonup u$ weakly in $L^p(\Omega)$, $u\in W^{1,p}_g(\Omega)\cap L^\infty(\Omega)$, a solution of

$$\int_\Omega uv + \int_S g\,|\nabla u|^{p-2}\nabla u\cdot\nabla v = \int_\Omega \phi v \quad \text{for every } v\in W^{1,p}(\Omega)\cap L^\infty(\Omega),$$

that is, $u = \left(I+B_p^g\right)^{-1}\phi$.

Since $u_\varepsilon\ll\phi$, there exists a sequence $\varepsilon_n\to 0$ such that

$$(4.9)\qquad u_{\varepsilon_n}\rightharpoonup u, \quad \text{weakly in } L^p(\Omega), \quad u\ll\phi.$$

Observe that also $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)} \le \|\phi\|_{L^\infty(\Omega)}$. Taking $\varepsilon = \varepsilon_n$ and $v = u_{\varepsilon_n}$ in (4.8), we get

$$\int_\Omega \int_\Omega \frac{1}{2} \frac{C_{J,p}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right) g\left(\frac{x+y}{2}\right) \left|\frac{u_{\varepsilon_n}(y) - u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\, dy \tag{4.10}$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z) g(w) \chi_\Omega^\times \left(w \pm \frac{\varepsilon_n}{2} z\right)$$

$$\times \left|\frac{\overline{u}_{\varepsilon_n}(w + \frac{\varepsilon_n}{2} z) - \overline{u}_\varepsilon(w - \frac{\varepsilon_n}{2} z)}{\varepsilon_n}\right|^p dw\, dz \le M.$$

Therefore, by Proposition 4.1, $u \in W_g^{1,p}(\Omega)$ and

$$\left(\frac{C_{J,p}}{2} J(z)\right)^{1/p} (g(w))^{1/p} \chi_\Omega^\times \left(w \pm \frac{\varepsilon}{2} z\right) \frac{\overline{u}_{\varepsilon_n}(w + \frac{\varepsilon_n}{2} z) - \overline{u}_{\varepsilon_n}(w - \frac{\varepsilon_n}{2} z)}{\varepsilon_n} \tag{4.11}$$

$$\rightharpoonup \left(\frac{C_{J,p}}{2} J(z)\right)^{1/p} h(w,z)$$

weakly in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ with $(J(z))^{1/p} h(w,z) = (J(z))^{1/p} (g(w))^{1/p} z \cdot \nabla u(w)$ in $S \times \mathbb{R}^N$ and $(J(z))^{1/p} h(w,z) = 0$ in $(\mathbb{R}^N \setminus \overline{\Omega}) \times \mathbb{R}^N$. Moreover, we can also assume that

$$J(z)^{1/p'} g(w)^{1/p'} \chi_\Omega^\times \left(w \pm \frac{\varepsilon_n}{2} z\right) \left|\frac{\overline{u}_{\varepsilon_n}(w + \frac{\varepsilon_n}{2} z) - \overline{u}_{\varepsilon_n}(w - \frac{\varepsilon_n}{2} z)}{\varepsilon_n}\right|^{p-2}$$

$$\times \frac{\overline{u}_{\varepsilon_n}(w + \frac{\varepsilon_n}{2} z) - \overline{u}_{\varepsilon_n}(w - \frac{\varepsilon_n}{2} z)}{\varepsilon_n} \rightharpoonup J(z)^{1/p'} \chi(w,z)$$

weakly in $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$, with $J(z)^{1/p'} \chi(w,z) = 0$ in $(\mathbb{R}^N \setminus \overline{\Omega}) \times \mathbb{R}^N$. Therefore, passing to the limit in (4.8) for $\varepsilon = \varepsilon_n$, we get

$$\int_\Omega uv + \int_{\mathbb{R}^N} \int_S \frac{C_{J,p}}{2} J(z) g(w)^{1/p} \chi(w,z)\, z \cdot \nabla v(w)\, dw\, dz = \int_\Omega \phi v \tag{4.12}$$

for every v smooth and, by approximation, for every $v \in W_{g,S}^{1,p}(\Omega)$. Proceeding now in a similar way to the proof of [**6**, Proposition 3.3] we get that, for every $v \in W_{g,S}^{1,p}(\Omega)$,

$$\int_{\mathbb{R}^N} \int_S \frac{C_{J,p}}{2} J(z) g(w)^{1/p} \chi(w,z) z \cdot \nabla v(x)\, dw\, dz \tag{4.13}$$

$$= \int_S g\, |\nabla u|^{p-2}\, \nabla u \cdot \nabla v. \quad \square$$

Theorem 4.3. *Assume $p > 1$, J satisfies (HJ) and $J(x) \geq J(y)$ if $|x| \leq |y|$, and g satisfies* (3.1) *and* (3.3). *Assume also g is lower semi-continuous. Then, for any $\phi \in L^\infty(\Omega)$, we have that*

$$(4.14) \qquad \left(I + B_p^{J_{p,\varepsilon},g}\right)^{-1} \phi \to \left(I + B_p^g\right)^{-1} \phi \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

Proof: For each $m \in \mathbb{N}$, let the open sets $S_m = \{x \in \Omega : \operatorname{dist}(x, \partial S) > 1/m\}$. We have that $S = \cup_m S_m$, there exists $\alpha_m > 0$ such that $g(x) \geq \alpha_m > 0$ for every $x \in S_m$, and there exists a finite number of balls B_i covering S_m, with $B_i \subset S_{m+1}$.

Let ε_n a subsequence converging to 0. We can suppose that such sequence, or a subsequence if necessary, satisfies (4.10), then, in each ball B_i,

$$\int_{B_i}\int_{B_i} \frac{1}{2}\frac{C_{J,p}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right) g\left(\frac{x+y}{2}\right) \left|\frac{u_{\varepsilon_n}(y) - u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dy \leq M,$$

and also

$$\int_{B_i}\int_{B_i} \frac{1}{2}\frac{C_{J,p}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right) \left|\frac{u_{\varepsilon_n}(y) - u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dy \leq M/\alpha_{m+1}.$$

Therefore, by [**6**, Proposition 3.2(2.i)] (see also [**17**, Theorem 4]), taking into account (4.7),

$$\left(I + B_p^{J_{p,\varepsilon_n},g}\right)^{-1} \phi \to \left(I + B_p^g\right)^{-1} \phi \quad \text{a.e. in } \Omega.$$

Now, since in fact $\left\{\left(I + B_p^{J_{p,\varepsilon_n},g}\right)^{-1} \phi\right\}$ is bounded in $L^\infty(\Omega)$ the result follows. □

From the above theorem, by standard results of the Nonlinear Semigroup Theory (see [**20**] and [**16**]), we obtain the following result.

Theorem 4.4. *Let $p > 1$. Assume J satisfies (HJ) and $J(x) \geq J(y)$ if $|x| \leq |y|$, and g satisfies* (3.1) *and* (3.3). *Let $T > 0$ and $u_0 \in L^p(\Omega)$. Let u_ε the unique solution of $P_p^{J_{p,\varepsilon},g}(u_0)$ and u the unique entropy solution of $N_p^g(u_0)$. Then*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^p(\Omega)} = 0.$$

Proof: Since $B_p^{J,g}$ and $\mathcal{B}_p^g \cap (L^p(\Omega) \times L^p(\Omega))$ are m-completely accretive in $L^p(\Omega)$, to get the result it is enough to see that $\left(I + B_p^{J_{p,\varepsilon},g}\right)^{-1} \phi \to \left(I + B_p^g\right)^{-1} \phi$ in $L^p(\Omega)$ as $\varepsilon \to 0$ for any $\phi \in L^\infty(\Omega)$, which follows by Theorem 4.3. □

5. Existence and uniqueness of solutions for the nonlocal problem. The case $p = 1$

This section deals with the existence and uniqueness of solutions for the nonlocal problem

$$P_1^{J,g}(u_0)\begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)\frac{u(t,y)-u(t,x)}{|u(t,y)-u(t,x)|}\,dy, \\ u(x,0) = u_0(x). \end{cases}$$

First, let us introduce what will be understood as a solution.

Definition 5.1. A *solution* of $P_1^{J,g}(u_0)$ in $[0,T]$ is a function $u \in W^{1,1}(0,T;L^1(\Omega))$ which satisfies $u(0,x) = u_0(x)$ a.e. $x \in \Omega$ and

$$u_t(t,x) = \int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)h(t,x,y)\,dy \quad \text{a.e. in }]0,T[\times\Omega,$$

for some $h \in L^\infty(0,T;L^\infty(\Omega\times\Omega))$ with $\|h\|_\infty \le 1$ such that $h(t,x,y) = -h(t,y,x)$ and

$$J(x-y)g\left(\frac{x+y}{2}\right)h(t,x,y) \in J(x-y)g\left(\frac{x+y}{2}\right)\operatorname{sign}(u(t,y)-u(t,x)).$$

Here $\operatorname{sign}(\cdot)$ is the multivalued function given by

$$\operatorname{sign}(r) = \begin{cases} -1 & \text{if } r < 0, \\ [-1,1] & \text{if } r = 0, \\ 1 & \text{if } r > 0. \end{cases}$$

As in the case $p > 1$, to prove the existence and uniqueness of solutions of $P_1^J(u_0)$ we use the Nonlinear Semigroup Theory, so we start introducing the following operator in $L^1(\Omega)$.

Definition 5.2. Let J satisfies (HJ), $g \in L^\infty(\mathbb{R}^N)$, $g \ge 0$ a.e. We define the operator $B_1^{J,g}$ in $L^1(\Omega)\times L^1(\Omega)$ by $\hat{u} \in B_1^{J,g}u$ if and only if $u, \hat{u} \in L^1(\Omega)$, there exists $h \in L^\infty(\Omega\times\Omega)$, $h(x,y) = -h(y,x)$ for almost all $(x,y) \in \Omega\times\Omega$, $\|h\|_\infty \le 1$,

$$\hat{u}(x) = -\int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)h(x,y)\,dy, \quad \text{a.e. } x \in \Omega$$

and

$$J(x-y)g\left(\frac{x+y}{2}\right)h(x,y) \in J(x-y)g\left(\frac{x+y}{2}\right)\operatorname{sign}(u(y)-u(x)), \tag{5.1}$$

a.e. $(x,y) \in \Omega\times\Omega$.

Remark 5.3. (1) It is not difficult to see that (5.1) is equivalent to

$$-\int_\Omega\int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)h(x,y)\,dyu(x)\,dx$$
$$=\frac{1}{2}\int_\Omega\int_\Omega J(x-y)g\left(\frac{x+y}{2}\right)|u(y)-u(x)|\,dy\,dx.$$

(2) $L^1(\Omega)=\mathrm{Dom}(B_1^{J,g})$ and $B_1^{J,g}$ is closed in $L^1(\Omega)\times L^1(\Omega)$.

(3) $B_1^{J,g}$ is positively homogeneous of degree zero, that is, if $\hat{u}\in B_1^{J,g}u$ and $\lambda>0$ then $\lambda\hat{u}\in B_1^{J,g}(\lambda u)$.

Following the same ideas than in the proof of [**6**, Theorem 2.9] we have the following result.

Theorem 5.4. *The operator $B_1^{J,g}$ is completely accretive and satisfies $L^\infty(\Omega)\subset\mathrm{Ran}(I+B_1^{J,g})$.*

Theorem 5.5. *For every initial datum $u_0\in L^1(\Omega)$ and any $T>0$ the problem $P_1^{J,g}(u_0)$ has a unique solution in $(0,T)$ and, moreover, an L^1-contraction principle holds for such solutions.*

Proof: As a consequence of the above results, we have that the abstract Cauchy problem

$$\begin{cases} u'(t)+B_1^{J,g}u(t)\ni 0, & t\in]0,T[,\\ u(0)=u_0 \end{cases}\tag{5.2}$$

has a unique mild solution u for every initial datum $u_0\in L^1(\Omega)$ and $T>0$ (see [**16**]). Moreover, due to the complete accretivity and the homogeneity of the operator $B_1^{J,g}$, the mild solution of (5.2) is a strong solution ([**15**]) and, so, a solution of $P_1^{J,g}(u_0)$. □

6. The local problem for $p=1$

Let $\Omega\subset\mathbb{R}^N$ a bounded domain and $0\le g\in L^\infty(\Omega)$. In this section we are interested in the following local diffusion equation with homogeneous Neumann boundary condition,

$$N_1^g(u_0)\begin{cases} u_t=\mathrm{div}\left(g\dfrac{Du}{|Du|}\right) & \text{in }]0,T[\times\Omega,\\ g\dfrac{Du}{|Du|}\cdot\eta=0 & \text{on }]0,T[\times\partial\Omega,\\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases}$$

where η is the unit outward normal on $\partial\Omega$.

Due to the linear growth condition on the Lagrangian, the natural energy space to study problem $N_1^g(u_0)$ is the space of functions of bounded variation. Let us recall several facts concerning functions of bounded variation (for further information concerning functions of bounded variation we refer to [**38**], [**51**] or [**2**]).

A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \ldots, \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i \varphi \, dx = -\int_\Omega \varphi \, d\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$|Du| = \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), \ |\varphi(x)| \leq 1 \text{ for } x \in \Omega \right\}. \tag{6.1}$$

The space $BV(\Omega)$ is endowed with the norm $\|u\|_{BV} = \|u\|_{L^1(\Omega)} + |Du|$. For $u \in BV(\Omega)$, the gradient Du is a Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \, \mathcal{L}^N$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$.

We shall need several results from [**10**] (see also [**4**]). Following [**10**], let

$$X_p(\Omega) = \left\{ \mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) : \operatorname{div}(\mathbf{z}) \in L^p(\Omega) \right\}, \quad 1 \leq p \leq N.$$

If $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$ we define the functional $(\mathbf{z}, Dw)$: $C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle = -\int_\Omega w \, \varphi \operatorname{div}(\mathbf{z}) \, dx - \int_\Omega w \, \mathbf{z} \cdot \nabla \varphi \, dx.$$

Then $(\mathbf{z}, Dw)$ is a Radon measure in Ω,

$$\int_\Omega (\mathbf{z}, Dw) = \int_\Omega \mathbf{z} \cdot \nabla w \, dx \quad \forall \, w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$$

and

$$\left| \int_B (\mathbf{z}, Dw) \right| \leq \int_B |(\mathbf{z}, Dw)| \leq \|\mathbf{z}\|_\infty \int_B |Dw|$$

for any Borel set $B \subseteq \Omega$.

In [**10**], a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z} \in X_p(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator γ: $X_p(\Omega) \to L^\infty(\partial\Omega)$ such that $\|\gamma(\mathbf{z})\|_\infty \leq \|\mathbf{z}\|_\infty$ and $\gamma(\mathbf{z})(x) =$

$\mathbf{z}(x) \cdot \nu(x)$ for all $x \in \partial\Omega$ if $\mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^N)$. We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z}, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[\mathbf{z}, \nu]$ and the measure $(\mathbf{z}, Dw)$, for $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, is established:

$$\int_\Omega w \operatorname{div}(\mathbf{z})\, dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w \, d\mathcal{H}^{N-1}. \tag{6.2}$$

To define the differential operator $\operatorname{div}\left(g \frac{Du}{|Du|}\right)$ we need to recall the concept of total variation with respect to an anisotropy (see [**1**], [**14**] and [**24**]). We say that a function $\phi \colon \Omega \times \mathbb{R}^N \to [0, \infty)$ is a *metric integrand* if ϕ is a Borel function satisfying the conditions

$$\text{for a.e. } x \in \Omega, \text{ the map } \xi \in \mathbb{R}^N \to \phi(x, \xi) \text{ is convex,} \tag{6.3}$$

$$\phi(x, t\xi) = |t|\, \phi(x, \xi) \quad \forall\, x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \quad \forall\, t \in \mathbb{R}, \tag{6.4}$$

and there exists a constant $\Gamma > 0$ such that

$$0 \le \phi(x, \xi) \le \Gamma \|\xi\| \quad \forall\, x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N.$$

Recall that the polar function $\phi^0 \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ of ϕ defined by

$$\phi^0(x, \xi^*) = \sup\{\langle \xi^*, \xi \rangle : \xi \in \mathbb{R}^N,\, \phi(x, \xi) \le 1\}.$$

Let

$$\mathcal{K}_\phi(\Omega) := \left\{ \mathbf{z} \in X_\infty(\Omega) : \phi^0(x, \mathbf{z}(x)) \le 1 \text{ for a.e. } x \in \Omega,\, [\mathbf{z}, \nu] = 0 \right\}.$$

Definition 6.1 ([**24**]). Let $u \in L^1(\Omega)$. We define the *ϕ-total variation* of u in Ω as

$$\int_\Omega |Du|_\phi := \sup \left\{ \int_\Omega u \operatorname{div} \mathbf{z}\, dx : \mathbf{z} \in \mathcal{K}_\phi(\Omega) \right\}.$$

We set

$$BV_\phi(\Omega) := \left\{ u \in L^1(\Omega) : \int_\Omega |Du|_\phi < \infty \right\}.$$

From the definition it follows that $u \in L^1(\Omega) \to \int_\Omega |Du|_\phi$ is a lower-semicontinuous functional with respect to the L^1-convergence.

It is easy to see that if $u \in BV(\Omega)$, then

$$\int_\Omega |Du|_\phi \le \Gamma \int_\Omega |Du|.$$

Moreover, if ϕ is *coercive* in Ω, that is, there exist $\lambda > 0$ such that $\lambda \|\xi\| \le \phi(x, \xi)$ for all $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, and continuous in second variable, in [**1**] it is proved that $BV_\phi(\Omega) = BV(\Omega)$ and

$$\lambda \int_\Omega |Du| \le \int_\Omega |Du|_\phi \le \Gamma \int_\Omega |Du|.$$

In [**14**] (see also [**24**]) the following result is proved.

Proposition 6.2. *Given a metric integrand ϕ, let*

$$J_\phi(u) := \begin{cases} \displaystyle\int_\Omega \phi(x, \nabla u(x))\, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus W^{1,1}(\Omega). \end{cases}$$

Let $\overline{J_\phi}$ be the relaxed functional, that is,

$$\overline{J_\phi}(u) := \inf \left\{ \liminf_{n\to\infty} J_\phi(u_n) \colon u_n \to u \text{ in } L^1(\Omega),\ u_n \in W^{1,1}(\Omega) \right\}.$$

Then, for every $u \in BV_\phi(\Omega)$, we have

$$\overline{J_\phi}(u) = \int_\Omega |Du|_\phi.$$

Hence, for every $u \in BV_\phi(\Omega)$, there exists a sequence $u_n \in W^{1,1}(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and

$$\int_\Omega \phi(x, \nabla u_n(x))\, dx \to \int_\Omega |Du|_\phi.$$

In particular, $BV_\phi(\Omega)$ is the finiteness domain of $\overline{J_\phi}$.

Moreover, if $u \in BV_\phi(\Omega) \cap L^q(\Omega)$ $(1 \le q < \infty)$, then we can find a sequence $u_n \in W^{1,1}(\Omega) \cap L^q(\Omega)$ such that $u_n \to u$ in $L^q(\Omega)$.

In [**24**], the generalized Green's formula of Anzellotti (6.2) (see [**10**]) is extended to the case in which the function belongs to $BV_\phi(\Omega)$. Given $u \in BV_\phi(\Omega) \cap L^{p'}(\Omega)$ and $\mathbf{z} \in X_p(\Omega)$, we define the functional $(\mathbf{z}, Du) \colon \mathcal{D}(\Omega) \to \mathbb{R}$ as

$$\langle (\mathbf{z}, Du), \varphi \rangle := -\int_\Omega u\varphi \operatorname{div}(\mathbf{z})\, dx - \int_\Omega u\mathbf{z} \cdot \nabla\varphi\, dx.$$

For $1 \le p \le \infty$, we denote,

$$\mathcal{A}_{p,\phi}(\Omega) := \left\{ \mathbf{z} \in X_p(\Omega) : \|\phi^0(x, \mathbf{z}(x))\|_{L^\infty(\Omega)} < \infty \right\}.$$

The following result can be proved as in [**10**] (see also [**4**]).

Proposition 6.3. *Assume ϕ is a metric integrand. If $u \in BV_\phi(\Omega) \cap L^{p'}(\Omega)$ and $\mathbf{z} \in \mathcal{A}_{p,\phi}(\Omega)$, then $(\mathbf{z}, Du)$ is a Radon measure in Ω and*

$$\left| \int_\Omega (\mathbf{z}, Du) \right| \le \|\phi_g^0(\cdot, \mathbf{z}(\cdot))\|_{L^\infty(\Omega)} \int_\Omega |Du|_\phi.$$

Moreover, if $[\mathbf{z}, \nu] = 0$ on $\partial\Omega$, the following Green's formula holds,

$$\int_\Omega u \operatorname{div}(\mathbf{z})\, dx + \int_\Omega (\mathbf{z}, Du) = 0. \tag{6.5}$$

As a consequence of Green's formula (6.5), we have

$$\int_\Omega |Du|_\phi := \sup\left\{ \int_\Omega (\mathbf{z}, Du) : \mathbf{z} \in \mathcal{K}_\phi(\Omega) \right\}.$$

A particular case, interesting for our purposes, is when $g\colon \Omega \to [0,\infty)$ is a bounded Borel function and we consider the metric integrand $\phi_g : \Omega \times \mathbb{R}^N \to [0,+\infty]$ defined by $\phi_g(x,\xi) := g(x)\|\xi\|$. Then (see [**1**])

$$\phi_g^0(x,\xi^*) = \begin{cases} 0 & \text{if } g(x) = 0,\ \xi^* = 0, \\ +\infty & \text{if } g(x) = 0,\ \xi^* \neq 0, \\ \dfrac{\|\xi^*\|}{g(x)} & \text{if } g(x) > 0,\ \xi^* \in \mathbb{R}^N. \end{cases}$$

Consequently,

$$\mathcal{K}_g(\Omega) := \mathcal{K}_{\phi_g}(\Omega) = \{\mathbf{z} \in X_\infty(\Omega) : \|\mathbf{z}(x)\| \leq g(x) \text{ for a.e. } x \in \Omega,\ [\mathbf{z},\nu] = 0\}.$$

In this particular case we will use the notation

$$BV_g(\Omega) := \left\{ u \in L^1(\Omega) : \int_\Omega |Du|_g < \infty \right\},$$

where

$$\int_\Omega |Du|_g := \sup\left\{ \int_\Omega u \operatorname{div} \mathbf{z}\, dx : \mathbf{z} \in \mathcal{K}_g(\Omega) \right\} = \sup\left\{ \int_\Omega (\mathbf{z}, Du) : \mathbf{z} \in \mathcal{K}_g(\Omega) \right\}.$$

We define the energy functional $\Phi_g\colon L^2(\Omega) \to [0,+\infty]$, associated with the problem $N_1^g(u_0)$, by

$$\Phi_g(u) := \begin{cases} \displaystyle\int_\Omega |Du|_g & \text{if } u \in BV_g(\Omega) \cap L^2(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV_g(\Omega). \end{cases}$$

We have that Φ_g is convex and lower semi-continuous. Therefore, the subdifferential $\partial\Phi_g$ of Φ_g, i.e. the operator in $L^2(\Omega)$ defined by

$$\text{(6.6)} \quad v \in \partial\Phi_g(u) \iff \Phi_g(w) - \Phi_g(u) \geq \int_\Omega v(w-u)\, dx, \quad \forall\, w \in L^2(\Omega)$$

is a maximal monotone operator in $L^2(\Omega)$. Consequently, the existence and uniqueness of a solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Phi_g(u(t)) \ni 0 & t \in]0,\infty[\\ u(0) = u_0 & u_0 \in L^2(\Omega) \end{cases} \tag{6.7}$$

follows immediately from the Nonlinear Semigroup Theory (see [**19**]). Now, to get the full strength of the abstract result derived from Semigroup Theory we need to characterize $\partial\Phi_g$.

Lemma 6.4. *The following assertions are equivalent:*

(a) $(u, v) \in \partial\Phi_g$;

(b)

$$u \in L^2(\Omega) \cap BV_g(\Omega), \quad v \in L^2(\Omega), \tag{6.8}$$

$$\exists\, \mathbf{z} \in X(\Omega)_2,\ \|\mathbf{z}(x)\| \le g(x), \quad \text{a.e. } x \in \Omega \text{ such that } v = -\operatorname{div}(\mathbf{z}) \text{ in } \mathcal{D}'(\Omega), \tag{6.9}$$

and

$$\int_\Omega (z, Du) = \int_\Omega |Du|_g, \tag{6.10}$$

$$[\mathbf{z}, \nu] = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.; \tag{6.11}$$

(c) (6.8) *and* (6.9) *hold, and*

$$\int_\Omega (w-u)v\,dx \le \int_\Omega \mathbf{z}\cdot\nabla w\,dx - \int_\Omega |Du|_g, \quad \forall\, w \in W^{1,1}(\Omega) \cap L^2(\Omega); \tag{6.12}$$

(d) (6.8) *and* (6.9) *hold, and*

$$\int_\Omega (w-u)v\,dx \le \int_\Omega (\mathbf{z}, Dw) - \int_\Omega |Du|_g \quad \forall\, w \in L^2(\Omega) \cap BV_g(\Omega). \tag{6.13}$$

Proof: First let us see the equivalence of (a) and (b). This follows working as in the proof of Proposition 1.10 in [**4**]. If we denote by

$$\widetilde{\Phi_g}(v) := \sup\left\{ \frac{\int_\Omega wv\,dx}{\Phi_g(w)} : w \in L^2(\Omega) \right\},$$

since Φ_g is positive homogeneous of degree 1, by Theorem 1.8 in [**4**], we have

$$(u, v) \in \partial\Phi_g \iff \widetilde{\Phi_g}(v) \le 1, \quad \int_\Omega vu\,dx = \Phi_g(u). \tag{6.14}$$

Let us define for $v \in L^2(\Omega)$

$$\Psi_g(v) := \begin{cases} \inf \left\{ \|\phi_g^0(\cdot, \mathbf{z}(\cdot))\|_{L^\infty(\Omega)} : \mathbf{z} \in \mathcal{C}(v) \right\} & \text{if } \mathcal{C}(v) \neq \emptyset \\ +\infty & \text{if } \mathcal{C}(v) = \emptyset, \end{cases}$$

where

$$\mathcal{C}(v) := \{ \mathbf{z} \in \mathcal{A}_{2,\phi_g}(\Omega) : v = -\operatorname{div}(\mathbf{z}) \text{ in } \mathcal{D}'(\Omega), \\ [\mathbf{z}, \nu] = 0 \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega \}.$$

We claim that

$$\Psi_g = \widetilde{\Phi_g}. \tag{6.15}$$

Let $v \in L^2(\Omega)$. If $\Psi_g(v) = +\infty$, then $\widetilde{\Phi_g}(v) \leq \Psi_g(v)$. Then, we may assume $\Psi_g(v) < \infty$. Let $\mathbf{z} \in \mathcal{C}(v)$ such that $\|\phi_g^0(\cdot, \mathbf{z}(\cdot))\|_{L^\infty(\Omega)} < \infty$. By Proposition 6.3, for any $u \in BV_g(\Omega) \cap L^2(\Omega)$ we have

$$\int_\Omega uv \, dx = \int_\Omega (\mathbf{z}, Du) \leq \|\phi_g^0(\cdot, \mathbf{z}(\cdot))\|_{L^\infty(\Omega)} \int_\Omega |Du|_\phi.$$

Taking supremums in u we obtain $\widetilde{\Phi_g}(v) \leq \|\phi_g^0(\cdot, \mathbf{z}(\cdot))\|_{L^\infty(\Omega)}$. Now, taking infimums in $\mathbf{z}$ we obtain $\widetilde{\Phi_g}(v) \leq \Psi_g(v)$. To prove the opposite inequality, let $\mathcal{D} := \left\{ -\operatorname{div}(\mathbf{z}) : \mathbf{z} \in \mathcal{C}(v),\, v \in L^2(\Omega) \right\}$. Then, for $u \in BV_g(\Omega) \cap L^2(\Omega)$, we have

$$\widetilde{\Psi_g}(u) := \sup \left\{ \frac{\int_\Omega uw \, dx}{\Psi_g(w)} : w \in L^2(\Omega) \right\} \geq \sup \left\{ \frac{\int_\Omega uw \, dx}{\Psi_g(w)} : w \in \mathcal{D} \right\}$$

$$\geq \sup \left\{ \frac{-\int_\Omega u \operatorname{div}(\mathbf{z}) \, dx}{\|\phi_g^0(\cdot, \mathbf{z}(\cdot))\|_{L^\infty(\Omega)}} : \mathbf{z} \in \mathcal{C}(w),\, w \in L^2(\Omega) \right\} = \Phi_g(u).$$

Hence, $\Psi_g(u) \leq \widetilde{\Phi_g}(u)$, and (6.15) holds.

By (6.14) and (6.15), it follows the equivalence between (a) and (b). To obtain (d) from (b) is sufficient to multiply both terms of the equation $v = -\operatorname{div}(\mathbf{z})$ by $w - u$, for $w \in L^2(\Omega) \cap BV_g(\Omega)$ and to use Green's formula (6.5). It is clear that (d) implies (c). To prove that (b) follows from (d), we chose $w = u$ in (6.13) and having in mind Proposition 6.3 and (6.9), we obtain that

$$\int_\Omega |Du|_g \leq \int_\Omega (\mathbf{z}, Du) \leq \|\phi_g^0(\cdot, \mathbf{z}(\cdot))\|_{L^\infty(\Omega)} \int_\Omega |Du|_\phi \leq \int_\Omega |Du|_g,$$

from where (6.10) follows. To obtain (6.11), we choose $w = u \pm \varphi$ in (6.13) with $\varphi \in BV(\Omega) \cap C^\infty(\Omega) \cap W^{1,1}(\Omega)$ and we get

$$\pm \int_\Omega \varphi v \, dx \leq \pm \int_\Omega \mathbf{z} \cdot \nabla \varphi \, dx,$$

from where it follows that

$$\int_\Omega \varphi \, \mathrm{div}(\mathbf{z}) \, dx + \int_\Omega \mathbf{z} \cdot \nabla \varphi \, dx = 0.$$

Then, having in mind the definition of the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ given in [**10**], we get

$$[\mathbf{z}, \nu] = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

In order to prove that (c) implies (d), let $w \in L^2(\Omega) \cap BV_g(\Omega)$. By Proposition 6.2, there exists a sequence $w_n \in W^{1,1}(\Omega) \cap L^2(\Omega)$ such that

$$w_n \to w \quad \text{in } L^2(\Omega) \quad \text{and} \quad \int_\Omega g(x)|\nabla w_n(x)| \, dx \to \int_\Omega |Dw|_g.$$

Using w_n as test function in (6.12), we have

$$\int_\Omega (w_n - u) v \, dx \leq \int_\Omega \mathbf{z} \cdot \nabla w_n \, dx - \int_\Omega |Du|_g. \tag{6.16}$$

Now, by Lemma 13.2 in [**24**], we have

$$\int_\Omega \mathbf{z} \cdot \nabla w_n \, dx \to \int_\Omega (\mathbf{z}, Dw).$$

Therefore, taking limit as $n \to +\infty$ in (6.16), we get (6.13). □

Definition 6.5. We say that $u \in C([0,T]; L^2(\Omega))$ is a solution of problem $N_1^g(u_0)$ in $[0,T] \times \Omega$ if $u \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$ $u(t) \in BV_g(\Omega)$ for almost all $t \in]0,T[$, $u(0) = u_0$, and there exists $\mathbf{z} \in L^\infty(]0,T[\times\Omega; \mathbb{R}^N)$, $\|\mathbf{z}(t,x)\| \leq g(x)$, a.e. $(t,x) \in]0,T[\times\Omega$ such that $[\mathbf{z}(t), \nu] = 0$ in $\partial\Omega$ a.e. $t \in]0,T[$, satisfying

$$u_t = \mathrm{div}(\mathbf{z}) \quad \text{in} \quad \mathcal{D}'(]0,T[\times\Omega)$$

and

$$\int_\Omega (u(t) - w) u_t \, dx \leq \int_\Omega (\mathbf{z}, Dw) - \int_\Omega |Du(t)|_g$$

for all $w \in L^2(\Omega) \cap BV_g(\Omega)$ an a.e. $t \in [0,T]$.

By Lemma 6.4, the concept of solution for problem $N_1^g(u_0)$ coincides with the concept of strong solution for the abstract Cauchy problem (6.7). Then, since we know that problem (6.7) has a unique strong solution for any initial data in $L^2(\Omega)$, we have the following existence and uniqueness result.

Theorem 6.6. *Let $g\colon \Omega \to [0,\infty)$ is a bounded Borel function. For any initial data $u_0 \in L^2(\Omega)$ there exists a unique solution u of the problem $N_1^g(u_0)$ in $[0,T]\times\Omega$ for every $T>0$. Moreover if u and v are solutions of $N_1^g(u_0)$ corresponding to the initial conditions $u_0, v_0 \in L^2(\Omega)$, then*

$$\|u(t)-v(t)\|_{L^2(\Omega)} \le \|u_0 - v_0\|_{L^2(\Omega)} \quad \textit{for any} \quad t>0.$$

7. Convergence of the nonlocal problems to the local problem. The case $p = 1$

Similarly to the case $p>1$, in order to do the rescaling, we need a variant of [**6**, Proposition 3.2(1.ii)].

Proposition 7.1. *Let $\rho\colon \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non-identically zero, and $\rho_n(x) := n^N \rho(nx)$. Let S an open set, $S\subset\Omega$, and let $l \in L^\infty(\mathbb{R}^N)$ such that*

$$l(x) = \begin{cases} l(x) > 0 & \textit{in } S, \\ 0 & \textit{in } \mathbb{R}^N \setminus S. \end{cases}$$

Let us also assume that l satisfies

$$l \in C(S). \tag{7.1}$$

Let $\{f_n\}$ be a sequence of functions in $L^1(\Omega)$ such that

$$\int_\Omega \int_\Omega \rho_n(y-x)\, l\left(\frac{x+y}{2}\right) |f_n(y) - f_n(x)|\, dx\, dy \le M\frac{1}{n} \tag{7.2}$$

and $\{f_n\}$ is weakly convergent in $L^1(S)$ to f.

Then, $l\frac{\partial f}{\partial w_j}$ is a bounded Radon measure in S, $j = 1,\ldots,N$, and moreover

$$\lim_n \left[\rho(z) l(w) \chi_\Omega\left(w + \frac{1}{2n}\right) \chi_\Omega\left(w - \frac{1}{2n}z\right) \times \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right) - \overline{f}_n\left(w - \frac{1}{2n}z\right)}{1/n}\right] = \mu(w,z)$$

weakly as measures with

$$\mu(w,z) = \rho(z) l(w) z\cdot\nabla f(w) \quad \textit{in } S\times\mathbb{R}^N,$$

and

$$\mu(w,z) = 0 \quad \textit{in } (\mathbb{R}^N\setminus\overline{\Omega})\times\mathbb{R}^N.$$

Proof: Making the change of variables $y = x + \frac{1}{n}z$, $x = w - \frac{1}{2n}z$, we rewrite (7.2) as

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho(z) l(w) \chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) \left| \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right) - \overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n} \right| dw\, dz \leq M,$$

where $\chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) = \chi_\Omega\left(w + \frac{1}{2n}z\right)\chi_\Omega\left(w - \frac{1}{2n}z\right)$. Therefore, up to a subsequence,

$$\rho(z) l(w) \chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right) - \overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n} \rightharpoonup \mu(w,z) \tag{7.3}$$

as measures and $\mu(w,z) = 0$ in $(\mathbb{R}^N \setminus \overline{\Omega}) \times \mathbb{R}^N$. If $\varphi \in C_c^\infty(\Omega)$, $\operatorname{supp}(\varphi) \subset S$, taking

$$\hat{\varphi} = \begin{cases} \dfrac{\varphi}{l} & \text{in } S, \\ 0 & \text{otherwise,} \end{cases}$$

and $\psi \in C_c^\infty(\mathbb{R}^N)$, by (7.3) and [**2**, Proposition 1.62], we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_\Omega \rho(z) l(w) \chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) \\ \times \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right) - \overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n} \hat{\varphi}(w)\, dw \psi(z)\, dz \\ \to \int_{\mathbb{R}^N}\int_\Omega \hat{\varphi}(w)\psi(z)\, d\mu(w,z).\end{aligned}$$

That is,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_S \rho(z) \chi_\Omega^\times\left(w \pm \frac{1}{2n}z\right) \\ \times \frac{\overline{f}_n\left(w+\frac{1}{2n}z\right) - \overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n} \varphi(w)\, dw \psi(z)\, dz \\ \to \int_{\mathbb{R}^N}\int_S \frac{1}{l(w)}\varphi(w)\psi(z)\, d\mu(w,z).\end{aligned} \tag{7.4}$$

Now, for n large enough,

$$\int_{\mathbb{R}^N}\int_S \rho(z)\chi_\Omega^\times\left(w\pm\frac{1}{2n}z\right)\frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n\left(w-\frac{1}{2n}z\right)}{1/n}\varphi(w)\,dw\psi(z)\,dz$$

$$=\int_{\mathbb{R}^N}\rho(z)\int_S\frac{\overline{f}_n\left(w+\frac{1}{2n}z\right)-\overline{f}_n(w-\frac{1}{2n}z)}{1/n}\varphi(w)\,dw\psi(z)\,dz$$

$$=-\int_{\mathbb{R}^N}\rho(z)\int_S f_n(w)\frac{\overline{\varphi}(w+\frac{1}{2n}z)-\overline{\varphi}\left(w-\frac{1}{2n}z\right)}{1/n}\,dw\psi(z)\,dz.$$

Then, passing to the limit, on account of (7.4), we get

$$\int_{\mathbb{R}^N}\int_S\frac{1}{l(w)}\varphi(w)\psi(z)\,d\mu(w,z) = -\int_{\mathbb{R}^N}\int_S\rho(z)f(w)\,z\cdot\nabla\varphi(w)\psi(z)\,dw\,dz. \tag{7.5}$$

Now, applying the disintegration theorem (Theorem 2.28 in [**2**]) to the measure μ, we get that if $\pi\colon S\times\mathbb{R}^N\to\mathbb{R}^N$ is the projection on the first factor and $\nu=\pi_\#|\mu|$, then there exists a Radon measures μ_w in $\mathbb{R}^N$ such that $w\mapsto\mu_w$ is ν-measurable, $|\mu_w|(\mathbb{R}^N)\le 1$ ν-a.e. in S and, for any $h\in L^1(S\times\mathbb{R}^N,|\mu|)$,

$$h(w,\cdot)\in L^1(\mathbb{R}^N,|\mu_w|)\quad \nu\text{-a.e. in } w\in S,$$

$$w\mapsto\int_{\mathbb{R}^N}h(w,z)d\mu_w(z)\in L^1(S,\nu)$$

and

$$\int_{S\times\mathbb{R}^N}h(w,z)\,d\mu(w,z)=\int_S\left(\int_{\mathbb{R}^N}h(w,z)\,d\mu_w(z)\right)\,d\nu(w). \tag{7.6}$$

From (7.5) and (7.6), we get, for $\varphi\in C_c^\infty(S)$ and $\psi\in C_c^\infty(\mathbb{R}^N)$,

$$\int_S\left(\int_{\mathbb{R}^N}\psi(z)\,d\mu_w(z)\right)\frac{1}{l(w)}\varphi(w)\,d\nu(w)=\left\langle\sum_{i=1}^N\int_{\mathbb{R}^N}\rho(z)z_i\psi(z)\,dz\frac{\partial f}{\partial w_i},\varphi\right\rangle.$$

Hence, as measures,

$$\sum_{i=1}^N\int_{\mathbb{R}^N}\rho(z)z_i\psi(z)\,dz\frac{\partial f}{\partial w_i}=\int_{\mathbb{R}^N}\psi(z)\,d\mu_w(z)\frac{1}{l}\nu\quad\text{in } S,$$

and therefore

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} \rho(z) z_i \psi(z)\, dz\, l \frac{\partial f}{\partial w_i} = \int_{\mathbb{R}^N} \psi(z)\, d\mu_w(z)\, \nu \quad \text{in } S.$$

Let now $\tilde{\psi} \in C_c^\infty(\mathbb{R}^N)$ be a radial function such that $\tilde{\psi} = 1$ in $\operatorname{supp}(\rho)$. Taking $\psi(z) = \tilde{\psi}(z) z_j$ in the above expression and having in mind that

$$\int_{\mathbb{R}^N} \rho(z) z_i z_j \tilde{\psi}(z)\, dz = 0 \quad \text{if } i \neq j,$$

we get

$$\int_{\mathbb{R}^N} \rho(z) {z_j}^2 dz\, l \frac{\partial f}{\partial w_j} = \int_{\mathbb{R}^N} \tilde{\psi}(z) z_j\, d\mu_w(z) \nu \quad \text{in } S.$$

Since $\nu \in M_b(S)$ and $w \mapsto \int_{\mathbb{R}^N} \tilde{\psi}(z) z_j\, d\mu_w(z) \in L^1(S, \nu)$, we obtain that $l \frac{\partial f}{\partial w_j}$ is a bounded Radon measure in S. Going back to (7.6), we get

$$\mu(w, z) = l(w) \sum_{i=1}^{N} \frac{\partial f}{\partial w_i}(x) \cdot \rho(z) z_i \mathcal{L}^N(z). \qquad \square$$

For the proof our next results we need the following assumptions: we assume that $g \in L^\infty(\mathbb{R}^N)$ is such that

$$(7.7) \qquad g(x) = \begin{cases} g(x) > 0 & \text{a.e. in } S \subset \Omega,\ S \text{ an open set}, \\ 0 & \text{a.e. in } \mathbb{R}^N \setminus S, \end{cases}$$

$$(7.8) \qquad g \in C(S).$$

Let us now proceed with the rescaling. Set

$$J_{1,\varepsilon}(x) := \frac{C_{J,1}}{\varepsilon^{1+N}} J\left(\frac{x}{\varepsilon}\right), \quad \text{with} \quad \frac{1}{C_{J,1}} := \frac{1}{2} \int_{\mathbb{R}^N} J(z) |z_N|\, dz.$$

Theorem 7.2. *Assume J satisfies (HJ) and $J(x) \geq J(y)$ if $|x| \leq |y|$ and that g satisfies* (7.7) *and* (7.8). *For any $\phi \in L^\infty(\Omega)$, we have*

$$\left(I + B_1^{J_{1,\varepsilon}, g}\right)^{-1} \phi \to (I + \partial \Phi_g)^{-1} \phi \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0.$$

Proof: Given $\varepsilon > 0$, we set $u_\varepsilon = \left(I + B_1^{J_{1,\varepsilon}, g}\right)^{-1} \phi$. Then, there exists $h_\epsilon \in L^\infty(\Omega \times \Omega)$, $h_\varepsilon(x, y) = -h_\varepsilon(y, x)$ for almost all $x, y \in \Omega$, $\|h_\varepsilon\|_\infty \leq 1$,

$$J\left(\frac{x-y}{\varepsilon}\right) g\left(\frac{x+y}{2}\right) h_\varepsilon(x, y) \in J\left(\frac{x-y}{\varepsilon}\right) g\left(\frac{x+y}{2}\right) \operatorname{sign}(u_\varepsilon(y) - u_\varepsilon(x))$$
$$\text{a.e. } x, y \in \Omega$$

and

$$-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_\Omega J\left(\frac{x-y}{\varepsilon}\right)g\left(\frac{x+y}{2}\right)h_\varepsilon(x,y)\,dy=\phi(x)-u_\varepsilon(x) \quad \text{a.e. } x\in\Omega. \tag{7.9}$$

Observe that

$$\begin{aligned}-\frac{C_{J,1}}{\varepsilon^{1+N}}&\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right)g\left(\frac{x+y}{2}\right)h_\varepsilon(x,y)\,dy u_\varepsilon(x)\,dx\\ &=\frac{C_{J,1}}{\varepsilon^{1+N}}\frac{1}{2}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right)g\left(\frac{x+y}{2}\right)|u_\varepsilon(y)-u_\varepsilon(x)|\,dy\,dx.\end{aligned} \tag{7.10}$$

By (7.9), we can write

$$\begin{aligned}&\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right)g\left(\frac{x+y}{2}\right)h_\varepsilon(x,y)(v(y)-v(x))\,dx\,dy\\ &\qquad=-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right)g\left(\frac{x+y}{2}\right)h_\varepsilon(x,y)\,dy v(x)\,dx\\ &\qquad=\int_\Omega(\phi(x)-u_\varepsilon(x))v(x)\,dx, \qquad \forall\, v\in L^\infty(\Omega).\end{aligned} \tag{7.11}$$

Since $u_\varepsilon \ll \phi$, there exists a sequence $\varepsilon_n\to 0$ such that

$$u_{\varepsilon_n}\rightharpoonup u \quad \text{weakly in } L^1(\Omega), \quad u\ll\phi.$$

Observe that $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}\le\|\phi\|_{L^\infty(\Omega)}$. Hence taking $\varepsilon=\varepsilon_n$ and $v=u_{\varepsilon_n}$ in (7.11), changing variables and having in mind (7.10), we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,1}}{2}J(z)g(w)\chi_\Omega^\times\left(x\pm\frac{\varepsilon_n}{2}z\right)\left|\frac{\overline{u}_{\varepsilon_n}\left(w+\frac{\varepsilon_n}{2}z\right)-\overline{u}_{\varepsilon_n}\left(w-\frac{\varepsilon_n}{2}z\right)}{\varepsilon_n}\right|\,dw\,dz\\ &\quad=\int_\Omega\int_\Omega\frac{1}{2}\frac{C_{J,1}}{\varepsilon_n{}^N}J\left(\frac{x-y}{\varepsilon_n}\right)g\left(\frac{x+y}{2}\right)\left|\frac{u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|\,dx\,dy\\ &\quad=\int_\Omega(\phi(x)-u_{\varepsilon_n}(x))u_{\varepsilon_n}(x)\,dx\le M \qquad \forall\, n\in\mathbb{N}.\end{aligned}$$

Therefore, by Proposition 7.1, $g\frac{\partial u}{\partial w_j}$ is a bounded Radon measure in S, $j=1,\ldots,N$,

$$\frac{C_{J,1}}{2}J(z)g(w)\chi_\Omega^\times\left(w\pm\frac{\varepsilon_n}{2}z\right)\frac{\overline{u}_{\varepsilon_n}\left(w+\frac{\varepsilon_n}{2}z\right)-\overline{u}_{\varepsilon_n}\left(w-\frac{\varepsilon_n}{2}z\right)}{\varepsilon_n}\rightharpoonup\mu(w,z) \tag{7.12}$$

weakly as measures with

$$\mu(w,z) = \frac{C_{J,1}}{2} J(z)g(w)z\cdot Du(w) \quad \text{in } S\times\mathbb{R}^N,$$

and

$$\mu(w,z) = 0 \quad \text{in } (\mathbb{R}^N\setminus\overline{\Omega})\times\mathbb{R}^N.$$

And by [**6**, Proposition 3.2(2.ii)] (see also [**17**, Theorem 4]) $u_{\varepsilon_n} \to u$ strongly in $L^1(\Omega)$.

Moreover, we can also assume that

$$J(z)\chi_\Omega^\times\left(w\pm\frac{\varepsilon_n}{2}z\right)g(w)\overline{h}_{\varepsilon_n}\left(w-\frac{\varepsilon_n}{2}z, w+\frac{\varepsilon_n}{2}z\right) \rightharpoonup \Lambda(w,z) \tag{7.13}$$

weakly* in $L^\infty(\mathbb{R}^N)\times L^\infty(\mathbb{R}^N)$, and $|\Lambda(w,z)|\le g(w)J(z)$ almost everywhere in $\mathbb{R}^N\times\mathbb{R}^N$. Changing variables and having in mind (7.11), we can write

$$\begin{aligned}\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & J(z)\chi_\Omega^\times\left(w\pm\frac{\varepsilon_n}{2}z\right)g(w)\overline{h}_{\varepsilon_n}\left(w-\frac{\varepsilon}{2}z, w+\frac{\varepsilon_n}{2}z\right)\\ &\times\frac{\overline{v}(w+\frac{\varepsilon_n}{2}z)-\overline{v}(w-\frac{\varepsilon_n}{2}z)}{\varepsilon_n}\,dz\,dw = \int_\Omega(\phi(x)-u_{\varepsilon_n}(x))v(x)\,dx\\ &\qquad\qquad\forall\, v\in L^\infty(\Omega).\end{aligned} \tag{7.14}$$

By (7.13), passing to the limit in (7.14), we get

$$\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_S \Lambda(w,z)z\cdot\nabla v(w)\,dw\,dz = \int_\Omega(\phi-u)v \quad \forall\, v \text{ smooth}, \tag{7.15}$$

and, by approximation, $\forall\, v\in L^\infty(\Omega)\cap W^{1,1}(\Omega)$. We set $\zeta=(\zeta_1,\dots,\zeta_N)$, the vector field defined by

$$\zeta_j(w) := \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(w,z)z_j\,dz, \quad j=1,\dots,N.$$

Then, $\zeta\in L^\infty(\Omega,\mathbb{R}^N)$, and from (7.15), $-\operatorname{div}(\zeta)=\phi-u$ in $\mathcal{D}'(\Omega)$.

Given $\xi\in\mathbb{R}^N\setminus\{0\}$, let R_ξ be the rotation such that $R_\xi^t(\xi)=\mathbf{e}_1|\xi|$. If we make the change of variables $z=R_\xi(y)$, we obtain

$$\begin{aligned}\zeta(x)\cdot\xi &= \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,z)z\cdot\xi\,dz = \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,R_\xi(y))R_\xi(y)\cdot\xi\,dy\\ &= \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,R_\xi(y))y_1|\xi|\,dy.\end{aligned}$$

On the other hand, since J is a radial function and $\Lambda(w,z) \le g(w)J(z)$ almost everywhere, we obtain, ${C_{J,1}}^{-1} = \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_1|\,dz$ and

$$|\zeta(w)\cdot\xi| \le \frac{C_{J,1}}{2}\int_{\mathbb{R}^N} g(w)J(y)|y_1|\,dy|\xi| = g(w)|\xi| \quad \text{a.e. } w \in \mathbb{R}^N.$$

Therefore, $\|\zeta(w)\|_{l^2(N)} \le g(w)$ a.e. $w \in \mathbb{R}^N$.

Since $u \in L^\infty(\Omega)$, $u \in BV_g(\Omega)$ and $\int_\Omega |Du|_g \le |gDu|(S)$, by Lemma 6.4, to finish the proof we only need to show that

$$\int_\Omega (\rho - u)(\phi - u) \le \int_S \zeta\cdot\nabla\rho - |gDu|(S) \quad \forall\, \rho \in W^{1,1}(\Omega). \tag{7.16}$$

Given $\rho \in W^{1,1}(\Omega)$, taking $v = \rho - u_{\varepsilon_n}$ in (7.14), we get

$$\begin{aligned}
&\int_\Omega (\phi(x) - u_{\varepsilon_n}(x))(\rho(x) - u_{\varepsilon_n}(x))\,dx \\
&\quad = \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)g(w)\chi_\Omega^\times\left(w \pm \frac{\varepsilon_n}{2}z\right)\overline{h}_{\varepsilon_n}\left(w - \frac{\varepsilon_n}{2}z, w + \frac{\varepsilon_n}{2}z\right)\times \\
&\qquad \times\left(\frac{\overline{\rho}\big(w+\frac{\varepsilon_n}{2}z\big) - \rho\big(w-\frac{\varepsilon_n}{2}z\big)}{\varepsilon_n} - \frac{\overline{u}_{\varepsilon_n}\big(w+\frac{\varepsilon_n}{2}z\big) - \overline{u}_{\varepsilon_n}\big(w-\frac{\varepsilon_n}{2}z\big)}{\varepsilon_n}\right)dz\,dw.
\end{aligned} \tag{7.17}$$

Having in mind (7.12) and (7.13) and taking limit in (7.17) as $n \to \infty$, we obtain that

$$\begin{aligned}
\int_\Omega (\rho - u)(\phi - u)\,dx &\le \frac{C_{J,1}}{2}\int_S\int_{\mathbb{R}^N} \Lambda(w,z)z\cdot\nabla\rho(w)\,dz\,dw \\
&\quad - \frac{C_{J,1}}{2}\int_S\int_{\mathbb{R}^N} |g(w)J(z)z\cdot Du|\,dz\,dw \\
&= \int_S \zeta\cdot\nabla\rho - \frac{C_{J,1}}{2}\int_S\int_{\mathbb{R}^N} |g(w)J(z)z\cdot Du|\,dz\,dw.
\end{aligned}$$

Now, for every $w \in S$ such that the Radon-Nikodym derivative $\frac{gDu}{|gDu|}(w) \neq 0$, let R_w be the rotation such that $R_w^t\left(\frac{gDu}{|gDu|}(w)\right) = \mathbf{e}_1\left|\frac{gDu}{|gDu|}(w)\right|$. Then, since J is a radial function and $\left|\frac{gDu}{|gDu|}(w)\right| = 1\,|gDu|$-a.e. in S, if we make the change of variables $y = R_w(z)$, we have

$$\begin{aligned} &\frac{C_{J,1}}{2}\int_S\int_{\mathbb{R}^N} |g(w)J(z)z\cdot Du|\,dz\,dw \\ &= \frac{C_{J,1}}{2}\int_S\int_{\mathbb{R}^N} J(z)\left|z\cdot \frac{gDu}{|gDu|}(w)\right|\,dz\,d|gDu|(w) \\ &= \frac{C_{J,1}}{2}\int_S\int_{\mathbb{R}^N} J(y)\,|y_1|\,\,dy\,d|gDu|(w) = \int_S |gDu|. \end{aligned}$$

Therefore (7.16) holds. □

As a consequence of Theorem 7.2 and the Nonlinear Semigroup Theory (see [**20**]), we have the following convergence result.

Theorem 7.3. *Assume J satisfies (HJ) and $J(x) \geq J(y)$ if $|x| \leq |y|$, and g satisfies* (7.7) *and* (7.8). *Let $T > 0$ and $u_0 \in L^2(\Omega)$. Let u_ε the unique solution of $P_1^{J_1,\varepsilon,g}(u_0)$ and u the unique solution of $N_1^g(u_0)$. Then*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^1(\Omega)} = 0. \tag{7.18}$$

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