Publicacions Matemàtiques, Vol **40** (1996), 85–94.

ESTIMATES OF SOLUTIONS OF IMPULSIVE PARABOLIC EQUATIONS AND APPLICATIONS TO THE POPULATION DYNAMICS

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Abstract

A theorem on estimates of solutions of impulsive parabolic equations by means of solutions of impulsive ordinary differential equations is proved. An application to the population dynamics is given.

Introduction

The impulsive differential equations can be successfully used for mathematical simulation of processes and phenomena which are subject to short-term perturbations during their evolution. The duration of the perturbations is negligible in comparison with the duration of the process considered, and they can be thought of as momentary. The theory of impulsive ordinary differential equations started with the pioneer paper of V. Mil'man and A. Myshkis [**17**] and it was an object of intensive investigations during the last three decades. Detailed bibliographical information can be found in the monographs [**2**], [**8**]-[**10**], [**16**].

The theory of impulsive partial differential equations (PDE) marked its beginning with the paper [**14**]. The impulsive PDE provide natural framework for mathematical simulation of many processes and phenomena in theoretical physics, population dynamics, bio-technologies, chemistry, impulse technique and economics. We would like to note the applications of the impulsive PDE in the quantum mechanics. In 1992 it was introduced a model of impulsive moving mirror [**19**], [**20**] presented by the apparatus of the impulsive PDE.

It must be pointed out that this theory is an object of many lectures delivered at international meetings. Note the lectures of C. Y. Chan, L. Ke [**12**] delivered at the First International Conference on Dynamic

Systems and Applications, 1993, Atlanta, USA and S. Ahmad, M. Rama Mohana Rao [**1**] and D. Bainov, E. Minchev [**7**] delivered at the Second International Conference on Dynamic Systems and Applications, 1995, Atlanta, USA.

At the present time the theory of impulsive PDE undergoes rapid development [**3**]-[**6**], [**11**]-[**15**], [**19**], [**20**].

In this paper we give estimates of the solutions of impulsive parabolic equations and consider their applications to a model in the population dynamics. The estimates obtained can be used successfully in the qualitative theory of the impulsive parabolic equations. These estimates can be applied for obtaining of sufficient conditions for stability of the solutions of the equations investigated as well as they are useful for the entire formulation of the fundamental theory of the impulsive parabolic equations.

2. Preliminary notes

First we propose two models describing processes in the population dynamics.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. Suppose that

$$0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$$

are given numbers such that $\lim_{k\to\infty} t_k = +\infty$.

We define

$$\begin{aligned}
E &= \Big\{(t,x) \in \mathbb{R}^{1+n} : t \in \mathbb{R}_+,\, x \in \overline{\Omega}\Big\}, \quad \mathbb{R}_+ = [0,+\infty),\\
\Gamma_k &= \Big\{(t,x) \in E : t \in (t_k, t_{k+1}),\, x \in \Omega\Big\}, \quad k = 0,1,\dots;\\
\Gamma &= \cup_{k=0}^{\infty} \Gamma_k,\\
B_k &= \Big\{(t,x) \in E : t \in (t_k, t_{k+1}),\, x \in \partial\Omega\Big\}, \quad k = 0,1,\dots.
\end{aligned}$$

Let $C_{\mathrm{imp}}[E,\mathbb{R}]$ be the class of all functions $u : E \to \mathbb{R}$ such that:

(i) The functions $u|_{\Gamma_k \cup B_k}$, $k = 0,1,\dots$, are continuous.

(ii) For each k, $k = 1,2,\dots$, $t = t_k$, there exists

$$\lim_{\substack{(s,q)\to(t,x)\\ s<t}} u(s,q) = u(t^-,x), \quad x \in \overline{\Omega}.$$

(iii) For each k, $k = 0,1,\dots$, $t = t_k$, there exists

$$\lim_{\substack{(s,q)\to(t,x)\\ s>t}} u(s,q) = u(t^+,x), \quad x \in \overline{\Omega},$$

and $u(t,x) = u(t^+,x)$, $x \in \overline{\Omega}$.

2.1. Impulsive single species model. Suppose that $f_0 : \mathbb{R} \to \mathbb{R}$, $u_0 : \overline{\Omega} \to \mathbb{R}$, $g : \{t_k\}_{k=1}^{\infty} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are given functions.

Consider the reaction-diffusion equation (see [**18**])

$$u_t(t,x) = \varkappa \Delta u(t,x) + u(t,x) f_0(u(t,x)) + c_0, \quad (t,x) \in \Gamma, \tag{1}$$

subject to the initial condition

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \tag{2}$$

the boundary condition

$$u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega \tag{3}$$

and the impulses at fixed moments

$$u(t_k,x) = u(t_k^-,x) + g(t_k,x,u(t_k^-,x)), \quad x \in \overline{\Omega},\ k = 1,2,\dots . \tag{4}$$

The initial-boundary value problem (IBVP) (1)-(4) describes a single species population in a bounded environment. The function $u(t,x)$ represents the population density at the point $x \in \overline{\Omega}$ and time $t \geq 0$, $f_0(u)$ is the specific growth rate of u, $\varkappa > 0$ is the diffusion coefficient, $c_0 \geq 0$ is a constant. Condition (4) describes instantaneous changes in the population density due to phenomena as: harvesting, disasters, immigration, emigration, etc. Particularly, the case $g(t_k,x,u(t_k^-,x)) < 0$ corresponds to instantaneous harvesting of a plant population ($c_0 = 0$) at times t_k, $k = 1,2,\dots$, while the case $g(t_k,x,u(t_k^-,x)) > 0$ describes heavy immigration of a human population ($c_0 > 0$).

2.2. Impulsive predator-prey system. Suppose that $v_0^{(1)}$, $v_0^{(2)}$: $\overline{\Omega} \to \mathbb{R}$, $g^{(1)}$, $g^{(2)} : \{t_k\}_{k=1}^{\infty} \times \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

Consider a system describing predator-prey interaction

$$v_t^{(1)} = \Delta v^{(1)} + v^{(1)}(\tilde{a} - v^{(1)} - \tilde{c} v^{(2)}) \text{ in } \Gamma, \tag{5}$$

$$v_t^{(2)} = \Delta v^{(2)} + v^{(2)}(-\tilde{d} + \tilde{e} v^{(1)} - v^{(2)}) \text{ in } \Gamma, \tag{6}$$

$$v^{(1)}(0,x) = v_0^{(1)}(x), \quad v^{(2)}(0,x) = v_0^{(2)}(x), \quad x \in \overline{\Omega}, \tag{7}$$

$$v^{(1)}(t,x) = v^{(2)}(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega, \tag{8}$$

$$v^{(1)}(t_k,x) = v^{(1)}(t_k^-,x) + g^{(1)}(t_k,x,v^{(1)}(t_k^-,x),v^{(2)}(t_k^-,x)), \tag{9}$$

$$v^{(2)}(t_k,x) = v^{(2)}(t_k^-,x) + g^{(2)}(t_k,x,v^{(1)}(t_k^-,x),v^{(2)}(t_k^-,x)), \tag{10}$$

$$x \in \overline{\Omega}, \quad k = 1,2,\dots .$$

In the IBVP (5)-(10) $v^{(1)}(t,x)$ denotes the population density of a prey and $v^{(2)}(t,x)$ that of a predator; $\tilde{a}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$ are positive constants. Conditions (9) and (10) represent instantaneous changes in the population density of the prey and the predator, respectively. For example, the case when $g^{(1)} \equiv 0$ and $g^{(2)} < 0$ describes killing of the predators by hunters at the moments t_k, $k = 1, 2, \dots$. Other possible situation is $g^{(1)} > 0$, $g^{(2)} \equiv 0$ which corresponds to heavy immigration of the prey due to human interference.

Motivated by the above models we consider initial boundary value problem for impulsive nonlinear parabolic equations.

Suppose that $M[n]$ be the class of all matrices $A = [a_{ij}]_{1 \le i,j \le n}$ with real entries. Let $f : \Gamma \times \mathbb{R} \times \mathbb{R}^n \times M[n] \to \mathbb{R}$, $\varphi : \mathbb{R}_+ \times \partial\Omega \to \mathbb{R}$, $u_0 : \overline{\Omega} \to \mathbb{R}$, $g : \{t_k\}_{k=1}^{\infty} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are given functions.

Consider the initial-boundary value problem

(11) $u_t(t,x) = f(t,x,u(t,x),u_x(t,x),u_{xx}(t,x)), \quad (t,x) \in \Gamma,$

(12) $u(0,x) = u_0(x), \quad x \in \overline{\Omega},$

(13) $u(t,x) = \varphi(t,x), \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega,$

(14) $u(t_k,x) = u(t_k^-,x) + g(t_k,x,u(t_k^-,x)), \quad x \in \overline{\Omega},\ k = 1,2,\dots,$

where $u_x = (u_{x_1}, \dots, u_{x_n})$, $u_{xx} = [u_{x_i x_j}]_{1 \le i,j \le n}$.

Definition 1. A function $u : E \to \mathbb{R}$ is a solution of the IBVP (11)-(14) if:

(i) $u \in C_{\mathrm{imp}}[E,\mathbb{R}]$, there exist continuous partial derivatives $u_t(t,x)$, $u_x(t,x)$, $u_{xx}(t,x)$ for $(t,x) \in \Gamma$ and u satisfies (11) on Γ,

(ii) u satisfies (12)-(14).

Definition 2. A function $f : \Gamma \times \mathbb{R} \times \mathbb{R}^n \times M[n] \to \mathbb{R}$ is said to be elliptic at Γ if for each point $(t,x) \in \Gamma$ and any Q, $S \in M[n]$ the quadratic form

$$\sum_{i,j=1}^{n} (Q_{ij} - S_{ij})\lambda_i \lambda_j \le 0$$

for arbitrary vector $\lambda \in \mathbb{R}^n$ implies

$$f(t,x,u,P,Q) \le f(t,x,u,P,S)$$

for fixed $(t,x,u,P) \in \Gamma \times \mathbb{R} \times \mathbb{R}^n$.

We introduce the following assumption:

H1. The function f is elliptic at Γ.

3. Main results

Theorem 1. *Let the following conditions hold:*

1. Assumption H1 is fulfilled.

2. There exist functions $f_1,\ f_2 \in C((\mathbb{R}_+ \setminus \{t_k\}_{k=1}^{\infty}) \times \mathbb{R}, \mathbb{R})$ *such that*

$$f_1(t,p) \le f(t,x,p,0,0) \le f_2(t,p) \tag{15}$$

for $(t,x) \in \Gamma$, $p \in \mathbb{R}$.

3. There exist functions $g_1,\ g_2 \in C(\{t_k\}_{k=1}^{\infty} \times \mathbb{R}, \mathbb{R})$ *such that*

$$g_1(t_k,p) \le g(t_k,x,p) \le g_2(t_k,p), \tag{16}$$

$x \in \overline{\Omega}$, $p \in \mathbb{R}$, $k = 1,2,\ldots$.

4. The functions $p + g_1(t_k,p)$ *and* $p + g_2(t_k,p)$ *are nondecreasing on* $\mathbb{R}$ *for each* k, $k = 1,2,\ldots$.

5. There exist functions $r(t)$ *and* $\gamma(t)$ *which are minimal and maximal solutions of the problems*

$$\begin{aligned} &r'(t) = f_1(t,r(t)), \quad t \ne t_k,\\ &r(0) = r_0,\\ &r(t_k) = r(t_k^-) + g_1(t_k, r(t_k^-)), \quad k = 1,2,\ldots \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\gamma'(t) = f_2(t,\gamma(t)), \quad t \ne t_k,\\ &\gamma(0) = \gamma_0,\\ &\gamma(t_k) = \gamma(t_k^-) + g_2(t_k, \gamma(t_k^-)), \quad k = 1,2,\ldots, \end{aligned} \tag{18}$$

respectively, where

$$r_0 \le u_0(x) \le \gamma_0, \quad x \in \overline{\Omega}, \tag{19}$$

$$r(t) \le \varphi(t,x) \le \gamma(t), \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega. \tag{20}$$

Then for any solution u *of the IBVP (11)-(14) we have that*

$$r(t) \le u(t,x) \le \gamma(t) \ \textit{on}\ E. \tag{21}$$

Proof: Let $T_0 > 0$, $E_{T_0} = [0,T_0] \times \overline{\Omega}$ and there exists a positive integer m such that $t_m < T_0 < t_{m+1}$. We prove that

$$r(t) \le u(t,x) \le \gamma(t) \text{ on } E_{T_0}. \tag{22}$$

There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the solution $\gamma(\cdot;\varepsilon)$ of the problem

$$\begin{aligned} &\gamma'(t;\varepsilon) = f_2(t,\gamma(t;\varepsilon)) + \varepsilon, \quad t \neq t_k, \\ &\gamma(0;\varepsilon) = \gamma(0) + \varepsilon, \\ &\gamma(t_k;\varepsilon) = \gamma(t_k^-;\varepsilon) + g_2(t_k,\gamma(t_k^-;\varepsilon)) + \varepsilon, \quad k = 1,2,\dots,m, \end{aligned} \tag{23}$$

is defined on $[0,T_0]$ and $\lim_{\varepsilon\to 0}\gamma(t;\varepsilon) = \gamma(t)$, uniformly on $[0,T_0]$. We prove that

$$u(t,x) < \gamma(t;\varepsilon) \text{ on } E_{T_0}. \tag{24}$$

Suppose (24) is not true. Then the set $Z = \{t \in [0,T_0]$: there exists $x \in \overline{\Omega}$ such that $u(t,x) \geq \gamma(t;\varepsilon)\}$ is non-empty. Defining $\tilde{t} = \inf Z$. It follows from (19) and (20) that $\tilde{t} > 0$ and there exists a point $\tilde{x} \in \Omega$ such that:

$$\begin{aligned} &u(t,x) < \gamma(t;\varepsilon), \quad (t,x) \in [0,\tilde{t}) \times \overline{\Omega}, \\ &u(\tilde{t},\tilde{x}) = \gamma(\tilde{t};\varepsilon). \end{aligned} \tag{25}$$

There are two cases to be distinguished:

Case 1. $(\tilde{t},\tilde{x}) \in \Gamma$. Then we have

$$\begin{aligned} &u_t(\tilde{t},\tilde{x}) \geq \gamma'(\tilde{t};\varepsilon), \\ &u_x(\tilde{t},\tilde{x}) = 0, \\ &\sum_{i,j=1}^{n} u_{x_i x_j}(\tilde{t},\tilde{x})\lambda_i\lambda_j \leq 0, \quad \lambda \in \mathbb{R}^n. \end{aligned}$$

From H1 and (15) we obtain that

$$\begin{aligned} 0 &\leq u_t(\tilde{t},\tilde{x}) - \gamma'(\tilde{t};\varepsilon) \\ &\leq f(\tilde{t},\tilde{x},u(\tilde{t},\tilde{x}),0,0) - f_2(\tilde{t},\gamma(\tilde{t};\varepsilon)) - \varepsilon < 0, \end{aligned}$$

which is a contradiction.

Case 2. $\tilde{t} = t_k$ for some k, $1 \leq k \leq m$. Then we have from (25) that

$$\begin{aligned} &u(t_k^-,\tilde{x}) \leq \gamma(t_k^-;\varepsilon), \\ &u(t_k,\tilde{x}) = \gamma(t_k;\varepsilon). \end{aligned} \tag{26}$$

From (16), (26) and Condition 4 of the theorem we conclude that

$$\begin{aligned} 0 &= u(t_k,\tilde{x}) - \gamma(t_k;\varepsilon) \\ &= u(t_k^-,\tilde{x}) + g(t_k,\tilde{x},u(t_k^-,\tilde{x})) - \gamma(t_k^-;\varepsilon) - g_2(t_k,\gamma(t_k^-;\varepsilon)) - \varepsilon \\ &\leq u(t_k^-,\tilde{x}) + g_2(t_k,u(t_k^-,\tilde{x})) - \gamma(t_k^-;\varepsilon) - g_2(t_k,\gamma(t_k^-;\varepsilon)) - \varepsilon < 0, \end{aligned}$$

which is a contradiction.

Hence Z is empty and (24) follows. Since $\lim_{\varepsilon\to 0}\gamma(t;\varepsilon) = \gamma(t)$ uniformly on $[0, T_0]$ we conclude that

$$u(t,x) \le \gamma(t) \text{ on } E_{T_0}.$$

Analogously we can prove that

$$r(t) \le u(t,x) \text{ on } E_{T_0}.$$

Since $T_0 > 0$ was arbitrary we get the estimates (21). ■

Remark 1. Existence and uniqueness results for impulsive parabolic equations are considered in [**5**], [**11**] and [**15**].

4. Effective estimates of the population density in the impulsive single species model

Particular interest for the mathematical biology is the special case of IBVP (1)-(4) when $pf_0(p) = p(a - bp)$, $a > 0$, $b > 0$ are constants and $c_0 = 0$. Then the IBVP (1)-(4) describing single species model takes on the form

(27) $u_t(t,x) = \varkappa\Delta u(t,x) + u(t,x)(a - bu(t,x)), \quad (t,x) \in \Gamma,$

(28) $u(0,x) = u_0(x), \quad x \in \overline{\Omega},$

(29) $u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega,$

(30) $u(t_k,x) = u(t_k^-,x) + g(t_k,x,u(t_k^-,x)), \quad x \in \overline{\Omega},\ k = 1,2,\ldots.$

Suppose that $g(t_k,x,p) \le g_2(t_k,p) = \beta_k p$, $\beta_k > -1$, $x \in \overline{\Omega}$, $k = 1,2,\ldots$, and

$$\prod_{s<t_k\le t}\left(\frac{1}{1+\beta_k}\right) \ge Le^{-\beta(t-s)},$$

where $L > 0$, $\beta > 0$ are constants, $\gamma_0 = \max_{x\in\overline{\Omega}} u_0(x) > 0$. Let u be a solution of the IBVP (27)-(30). Then we consider the problem

$$\begin{aligned}
&\gamma'(t) = \gamma(t)(a - b\gamma(t)), \quad t \ne t_k,\\
&\gamma(0) = \gamma_0,\\
&\gamma(t_k) = \gamma(t_k^-) + \beta_k\gamma(t_k^-), \quad k = 1,2,\ldots.
\end{aligned}$$

We substitute $\rho(t) = \frac{1}{\gamma(t)}$ and obtain the problem

$$\begin{aligned}
&\rho'(t) = -a\rho(t) + b, \quad t \neq t_k, \\
&\rho(0) = \rho_0 = \frac{1}{\gamma_0}, \\
&\rho(t_k) = \frac{1}{1+\beta_k}\rho(t_k^-), \quad k = 1, 2, \dots .
\end{aligned}$$

Then we have

$$\begin{aligned}
\rho(t) &= \rho_0 \prod_{0<t_k\leq t} \left(\frac{1}{1+\beta_k}\right) e^{-at} + \int_0^t \prod_{s<t_k\leq t} \left(\frac{1}{1+\beta_k}\right) e^{-a(t-s)} b\, ds \\
&\geq L\rho_0 e^{-(a+\beta)t} + Lb \int_0^t e^{-\beta(t-s)} e^{-a(t-s)}\, ds \\
&= \left(L\rho_0 - \frac{Lb}{a+\beta}\right) e^{-(a+\beta)t} + \frac{Lb}{a+\beta}.
\end{aligned}$$

Therefore

$$\gamma(t) \leq \left[L\left(\frac{1}{\gamma_0} - \frac{b}{a+\beta}\right) e^{-(a+\beta)t} + \frac{Lb}{a+\beta}\right]^{-1}$$

and by Theorem 1 we conclude that

$$u(t,x) \leq \left[L\left(\frac{1}{\gamma_0} - \frac{b}{a+\beta}\right) e^{-(a+\beta)t} + \frac{Lb}{a+\beta}\right]^{-1}.$$

In the case without impulsive perturbations we have the above inequality with $L = 1$ and $\beta = 0$.

Acknowledgements. The authors express their deep gratitude to the referee for his valuable advices and helpful suggestions.

The present investigation was partially supported by the Bulgarian Ministry of Education, Science and Technologies under grant MM-422.

References

1. S. AHMAD AND M. RAMA MOHANA RAO, Stability of reaction-diffusion equations with impulsive effects, Proceedings of Dynamic Systems and Applications (to appear).

2. D. Bainov and V. Covachev, "*Impulsive Differential Equations with a Small Parameter*," World Scientific, Singapore, 1994.
3. D. Bainov, Z. Kamont, and E. Minchev, Difference methods for impulsive differential-functional equations, *Appl. Numer. Math.* **16** (1995), 401–416.
4. D. Bainov, Z. Kamont and E. Minchev, First order impulsive partial differential inequalities, *Internat. J. Theoret. Phys.* **33(6)** (1994), 1341–1358.
5. D. Bainov, Z. Kamont and E. Minchev, Initial-boundary value problems for impulsive parabolic functional-differential equations, in "*Boundary Value Problems for Functional Differential Equations*," ed. J. Henderson, World Scientific, Singapore, 1995, pp. 35–45.
6. D. Bainov, Z. Kamont and E. Minchev, Periodic boundary value problem for impulsive hyperbolic partial differential equations of first order, *Appl. Math. Comput.* **68** (1995), 95–104.
7. D. Bainov and E. Minchev, Trends in the theory of impulsive partial differential equations, *Nonlinear World* (to appear).
8. D. Bainov and P. Simeonov, "*Systems with Impulse Effect: Stability, Theory and Applications*," Ellis Horwood, Chichester, 1989.
9. D. Bainov and P. Simeonov, "*Theory of Impulsive Differential Equations: Asymptotic Properties of the Solutions*," World Scientific, Singapore, 1995.
10. D. Bainov and P. Simeonov, "*Theory of Impulsive Differential Equations: Periodic Solutions and Applications*," Longman, Harlow, 1993.
11. L. Byszewski, Impulsive degenerate nonlinear parabolic functional-differential inequalities, *J. Math. Anal. Appl.* **164** (1992), 549–559.
12. C. Y. Chan and L. Ke, Remarks on impulsive quenching problems, *Proceedings of Dynamic Systems and Applications* **1** (1994), 59–62.
13. C. Y. Chan, L. Ke and A. Vatsala, Impulsive quenching for reaction-diffusion equations, *Nonlinear Anal.* **22(11)** (1994), 1323–1328.
14. L. H. Erbe, H. I. Freedman, X. Z. Liu and J. H. Wu, Comparison principles for impulsive parabolic equations with applications to models of single species growth, *J. Austral. Math. Soc. Ser. B* **32** (1991), 382–400.
15. V. Gupta, Parabolic Equations with Impulse Effect: a Semigroup Approach, Ph. D. Thesis, Kanpur, India, 1994.

16. V. Lakshmikantham, D. Bainov and P. Simeonov, "*Theory of Impulsive Differential Equations*," World Scientific, Singapore, 1989.
17. V. Mil'man and A. Myshkis, On the stability of motion in the presence of impulses, (in Russian), *Sib. Math. J.* **1(2)** (1960), 233–237.
18. J. Murray, "*Lectures on Nonlinear-Differential-Equations Models in Biology*," Clarendon Press, Oxford, 1977.
19. G. Petrov, Impulsive moving mirror model and impulsive differential equations in Banach space, Communications of the Joint Institute for Nuclear Research, preprint E2–92–276, Dubna, Russia, 1992.
20. G. Petrov, Impulsive moving mirror model in a Schrödinger picture with impulse effect in a Banach space, Communications of the Joint Institute for Nuclear Research, preprint E2–92–272, Dubna, Russia, 1992.

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Primera versió rebuda el 2 de Març de 1995,
darrera versió rebuda el 2 de Setembre de 1995