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## INTEGRABILITY OF A LINEAR CENTER PERTURBED BY A FOURTH DEGREE HOMOGENEOUS POLYNOMIAL\*

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*Abstract*

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In this work we study the integrability of a two-dimensional autonomous system in the plane with linear part of center type and non-linear part given by homogeneous polynomials of fourth degree. We give sufficient conditions for integrability in polar coordinates. Finally we establish a conjecture about the independence of the two classes of parameters which appear in the system; if this conjecture is true the integrable cases found will be the only possible ones.

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### 1. Introduction

We consider the system

$$(1.1) \quad \begin{aligned} \dot{x} &= -y + X_s(x, y), \\ \dot{y} &= x + Y_s(x, y), \end{aligned}$$

where  $X_s(x, y)$  and  $Y_s(x, y)$  are homogeneous polynomials of degree  $s$ , with  $s \geq 2$ .

The aim of this paper is to find the integrable cases of system (1.1) when  $s = 4$  (see Theorem 1). The integrable cases for quadratic systems,  $s = 2$ , and cubic homogeneous systems,  $s = 3$ , have been studied by several authors: Bautin [1], Chavarriga [2], Coppel [5], Frommer [6], Kapteyn [7], Lloyd [8], Lunkevich and Sibirskii [9], Schlomiuk [11] and Żoladek [15]. Poincaré [10] developed an important technique for the general solution of these problems. It consists in finding a formal power series of the form

$$(1.2) \quad H(x, y) = \sum_{n=2}^{\infty} H_n(x, y),$$

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where  $H_2(x, y) = \frac{(x^2+y^2)}{2}$ , and for each  $n$ ,  $H_n(x, y)$  is a homogeneous polynomial of degree  $n$ , so that the derivative of  $H$  along the solutions of system (1.1) satisfies

$$\dot{H} = \sum_{k=2}^{\infty} V_{2k}(x^2 + y^2)^k,$$

where  $V_{2k}$  are real numbers called *Lyapunov constants*. The vanishing of all Lyapunov constants is a necessary condition for the integrability of the system (1.1); in this case the series  $H(x, y)$  would be a first integral of the system if it converges. It is not known if the series converges when the Lyapunov constants vanish. On the other hand, it is not possible in general to express this first integral (if it exists) by means of elementary functions.

Lyapunov constants are polynomials in the variables given by the coefficients of the polynomials  $X_s(x, y)$  and  $Y_s(x, y)$ . Thus the ideal generated by  $V_{2k}$  has a finite number of generators by Hilbert's Theorem. We denote by  $M(s)$  the minimum number of generators of such an ideal. It is well known that the number of small amplitudes limit cycles around the origin is at least  $M(s)$  (see for instance [12]). The vanishing of these generators is a sufficient condition for the vanishing of all Lyapunov constants and for the integrability of the system.

In Section 2 we give without proof some known results which are used in the proof of Theorem 1. In Section 3 we prove Theorem 1. This theorem characterizes the integrable cases by means of polar coordinates for  $s = 4$ . Finally, we give an appendix on the computation of the Lyapunov constants for  $s = 4$ .

## 2. Some preliminary results

In the study of this problem we use polar coordinates. In Lemma 1 we give the expression of system (1.1) in polar coordinates. In Proposition 1 we give the expression of series (1.2) in these coordinates.

**Lemma 1.** *In polar coordinates  $x = r \cos(\varphi)$ ,  $y = r \sin(\varphi)$  we can write system (1.1) as*

$$(2.1) \quad \begin{aligned} \dot{r} &= P_s(\varphi)r^s, \\ \dot{\varphi} &= 1 + Q_s(\varphi)r^{s-1}, \end{aligned}$$

where  $P_s(\varphi)$  and  $Q_s(\varphi)$  are trigonometric polynomials of the form

$$\begin{aligned} P_s(\varphi) &= R_{s+1} \cos((s+1)\varphi + \varphi_{s+1}) + R_{s-1} \cos((s-1)\varphi + \varphi_{s-1}) \\ &\quad + \cdots + \begin{cases} R_1 \cos(\varphi + \varphi_1) & \text{if } s \text{ is even;} \\ R_0 & \text{if } s \text{ is odd;} \end{cases} \\ Q_s(\varphi) &= -R_{s+1} \sin((s+1)\varphi + \varphi_{s+1}) + r_{s-1} \sin((s-1)\varphi + \bar{\varphi}_{s-1}) \\ &\quad + \cdots + \begin{cases} r_1 \sin(\varphi + \bar{\varphi}_1) & \text{if } s \text{ is even;} \\ r_0 & \text{if } s \text{ is odd;} \end{cases} \end{aligned}$$

being  $R_s, r_s, \varphi_s, \bar{\varphi}_s$  arbitrary coefficients.

**Proposition 1.** In polar coordinates series (1.2) for system (1.1)

is  $H(r, \varphi) = \sum_{m=0}^{\infty} \bar{H}_m(\varphi) r^{m(s-1)+2}$  where  $\bar{H}_0(\varphi) = \frac{1}{2}$  and  $\bar{H}_m(\varphi)$ ,  $m = 0, 1, \dots$ , are homogeneous trigonometric polynomials of degree  $m(s-1) + 2$ , satisfying the differential equations

$$(2.2) \quad \begin{aligned} \frac{d\bar{H}_{m+1}}{d\varphi} + (m(s-1) + 2) \bar{H}_m P_s(\varphi) + \frac{d\bar{H}_m}{d\varphi} Q_s(\varphi) \\ = \begin{cases} 0 & \text{if } (m+1)(s-1) + 2 \text{ is odd,} \\ V_{(m+1)(s-1)+2} & \text{if } (m+1)(s-1) + 2 \text{ is even,} \end{cases} \end{aligned}$$

where  $V_{(m+1)(s-1)+2}$ ,  $m = 0, 1, \dots$ , are the Lyapunov constants.

Lemma 1 and Proposition 1 are proved in [2].

In particular, for  $s = 4$  system (1.1) takes the form

$$(2.3) \quad \begin{aligned} \dot{r} &= P_4(\varphi) r^4, \\ \dot{\varphi} &= 1 + Q_4(\varphi) r^3, \end{aligned}$$

where

$$(2.4) \quad \begin{aligned} P_4(\varphi) &= R_5 \cos(5\varphi + \varphi_5) + R_3 \cos(3\varphi + \varphi_3) + R_1 \cos(\varphi + \varphi_1), \\ Q_4(\varphi) &= -R_5 \sin(5\varphi + \varphi_5) + r_3 \sin(3\varphi + \bar{\varphi}_3) + r_1 \sin(\varphi + \bar{\varphi}_1). \end{aligned}$$

In this case the evaluation of  $\dot{H}(r, \varphi)$  from system (2.3) yields

$$\begin{aligned} \frac{d\bar{H}_{m+1}}{d\varphi} + (3m+2) \bar{H}_m P_4(\varphi) + \frac{d\bar{H}_m}{d\varphi} Q_4(\varphi) &= \bar{V}_{m+1} \\ &= \begin{cases} 0 & \text{if } 3(m+1) + 2 \text{ is odd;} \\ V_{3(m+1)+2} & \text{if } 3(m+1) + 2 \text{ is even;} \end{cases} \end{aligned}$$

for  $m = 0, 1, \dots$ , with  $\bar{H}_0(\varphi) = \frac{1}{2}$  and  $\bar{H}_m(\varphi) = H_{3m+2}(\varphi)$ .

In Proposition 2 we will give the general form of the symmetric integrable systems. In Proposition 3 we will give a class of integrable systems which have an integrant factor given by a quadratic polynomial in the variable  $r^{s-1}$  whose coefficients are functions in  $\varphi$ .

**Proposition 2.** *In the following two cases system (2.1) is integrable (in the sense that all its Lyapunov constants vanish):*

- (i)  $(s+1)P_s(\varphi) + \frac{dQ_s(\varphi)}{d\varphi} = 0$ .
- (ii)  $P_s(\varphi)$  and  $Q_s(\varphi)$  are of the form

$$\begin{aligned} P_s(\varphi) &= R_{s+1} \sin(s+1)\omega + R_{s-1} \sin(s-1)\omega \\ &\quad + \cdots + \begin{cases} R_1 \sin \omega & \text{if } s \text{ is even;} \\ R_2 \sin 2\omega & \text{if } s \text{ is odd;} \end{cases} \\ Q_s(\varphi) &= R_{s+1} \cos(s+1)\omega + r_{s-1} \cos(s-1)\omega \\ &\quad + \cdots + \begin{cases} r_1 \cos \omega & \text{if } s \text{ is even;} \\ r_2 \cos 2\omega + r_0 & \text{if } s \text{ is odd;} \end{cases} \end{aligned}$$

where  $\omega = \varphi + \varphi_0$  and  $\varphi_0$  and the coefficients  $R_j$  and  $r_j$  are arbitrary.

Systems satisfying (i) have null divergence and those satisfying (ii) have a certain resonance between the angular parameters  $\varphi_j, \bar{\varphi}_j$ .

**Proposition 3.** *For  $s \in \mathbb{N}$  with  $s \geq 2$  and arbitrary  $k_1, k_2, \varphi_0 \in \mathbb{R}$  system (2.1) with*

$$\begin{aligned} (2.5) \quad P_s(\varphi) &= 2(-k_1 \cos^{s-2}(\varphi + \varphi_0) \sin^3(\varphi + \varphi_0) \\ &\quad + k_2 \sin^{s-2}(\varphi + \varphi_0) \cos^3(\varphi + \varphi_0)), \\ Q_s(\varphi) &= (k_1 \cos^{s-1}(\varphi + \varphi_0) - k_2 \sin^{s-1}(\varphi + \varphi_0)) \cos 2(\varphi + \varphi_0), \end{aligned}$$

is integrable.

In cartesian coordinates  $x = r \cos(\varphi + \varphi_0)$  and  $y = r \sin(\varphi + \varphi_0)$  we can write system (2.5) in the form

$$(2.6) \quad \begin{aligned} \dot{x} &= -y - k_1 x^{s-1} y + k_2 y^{s-2} (2x^2 - y^2), \\ \dot{y} &= x + k_1 x^{s-2} (x^2 - 2y^2) + k_2 x y^{s-1}, \end{aligned}$$

with  $s \geq 2$ . We note that the origin is a center for system (2.4).

Proposition 2 is proved in [2] and Proposition 3 is proved in [3].

In order to establish the cases of Theorem 1, we have used a certain simplification expressed in the form of a conjecture. This conjecture simplifies the great number of factors that appear in the different Lyapunov constants. This assumption, which we think to be always satisfied, is stated as Conjecture 1.

**Conjecture 1.** *A necessary condition for all the Lyapunov constants of system (2.3) are zero is that the angular parameters  $(\varphi_5, \varphi_3, \bar{\varphi}_3, \varphi_1, \bar{\varphi}_1)$  and the radial ones  $(R_5, R_3, r_3, R_1, r_1)$  must be independent.*

With reference to the number of small amplitude limit cycles around the origin that appear when we perturbed system (2.3) inside the same class of systems, we note that in the third integrable case of Theorem 1 (Case 10 in its proof) the number of relations of parameters is seven. So if Conjecture 1 is true, we think that the number of small amplitude limit cycles is at least seven.

The Lyapunov constants were obtained using the computer algebra system Mathematica.

### 3. The main result

**Theorem 1.** *System (2.3) is integrable in the following cases:*

- (i)  $\bar{\varphi}_1 = \varphi_1, \bar{\varphi}_3 = \varphi_3, 5R_3 + 3r_3 = 0$  and  $5R_1 + r_1 = 0$ .
- (ii)  $\bar{\varphi}_1 = \varphi_1, \bar{\varphi}_3 = \varphi_3, \varphi_5 = 5\varphi_1$  and  $\varphi_3 = 3\varphi_1$ .
- (iii)  $\bar{\varphi}_1 = \varphi_1, \bar{\varphi}_3 = \varphi_3, \varphi_5 = 2\varphi_1 + \varphi_3, r_3 = 3R_3, r_1 = 2R_1, R_5 = R_3,$   
and  $|R_1| = 2|R_3|$ .
- (iv)  $\bar{\varphi}_1 = \varphi_1, \bar{\varphi}_3 = \varphi_3, R_5 = 0$  and
 
$$\begin{cases} \text{(iv.1)} & R_3r_1 - 3r_3R_1 = 0, \\ \text{(iv.2)} & r_1 = 3R_1 \text{ and } r_3 = -3R_3, \\ \text{(iv.3)} & r_1 = R_1 = |3R_3| \text{ and } r_3 = -3R_3. \end{cases}$$

This theorem is independent on Conjecture 1, but if Conjecture 1 is true then system (2.3) will be integrable *only* in the cases given by Theorem 1.

*Proof of Theorem 1:* The first not zero Lyapunov constant is

$$V_8 = -\frac{1}{2}(R_1r_1 \sin(\varphi_1 - \bar{\varphi}_1) + R_3r_3 \sin(\varphi_3 - \bar{\varphi}_3)).$$

In particular the previous constant vanishes when  $\bar{\varphi}_1 = \varphi_1$  and  $\bar{\varphi}_3 = \varphi_3$ . On the other hand, if Conjecture 1 is true, we arrive at the same condition as above. With this assumption the next not zero Lyapunov constant is  $V_{14}$  given by

$$\begin{aligned} 40V_{14} = & 5((R_3r_1 - 3r_3R_1)(r_1 - 3R_1)(r_1 - R_1)) \sin(3\varphi_1 - \varphi_3) \\ & + (3(5r_1^2 - 6r_1R_1 - 11R_1^2)r_3 + (11r_1^2 - 74r_1R_1 + 75R_1^2)R_3) \\ & \quad R_5 \sin(2\varphi_1 + \varphi_3 - \varphi_5) \\ & + ((15r_1 - 51R_1)r_3^2 + 22(r_1 - R_1)r_3R_3 - (21r_1 - 25R_1)R_3^2) \\ & \quad R_5 \sin(\varphi_1 - 2\varphi_3 + \varphi_5). \end{aligned}$$

The simultaneous vanishing of three factors of  $V_{14}$  respect to the radicals parameters with arbitrary angular parameters leads to the following cases:

1.  $5R_1 + r_1 = 0$  and  $5R_3 + 3r_3 = 0$ ;
2.  $R_1 = r_1 = 0$ ;
3.  $R_3 = r_3 = 0$ ;
4.  $R_5 = 0$ ,  $R_3r_1 - 3r_3R_1 = 0$ ,  $R_1^2 + r_1^2 \neq 0$  and  $R_3^2 + r_3^2 \neq 0$ ;
5.  $R_5 = 0$  and  $r_1 - 3R_1 = 0$ ;
6.  $R_5 = 0$  and  $r_1 - R_1 = 0$ .

If we impose that the values of the angular parameters are not arbitrary, the possible dependence relations between them are the following:

7.  $\varphi_3 - 3\varphi_1 = 0$  and  $\varphi_5 - 5\varphi_1 = 0$ ;
8.  $\varphi_3 - 3\varphi_1 = 0$ ;
9.  $\varphi_1 - 2\varphi_3 + \varphi_5 = 0$ ;
10.  $2\varphi_1 + \varphi_3 - \varphi_5 = 0$ ;
11.  $\varphi_5 - 5\varphi_1 = 0$ ;
12.  $\varphi_5 - 3\varphi_3 + 4\varphi_1 = 0$ .

*Case 1:* In this case it is easy to see that  $5P_4 + Q'_4 = 0$ , where  $' = \frac{d}{d\varphi}$ , and the divergence of the vector field defined by system (2.3) is zero. So the system is integrable and all Lyapunov constants are zero (see Proposition 2).

*Case 2:* If  $r_1 = R_1 = 0$  then the first not zero Lyapunov constant after  $V_{14}$  is  $V_{26}$ , that is

$$8400V_{26} = ((5R_3 + 3r_3)(3R_3 - r_3)(3R_3 - 5r_3)(R_3 - 3r_3)(29R_3 - 89r_3)) \\ R_5^3 \sin(5\varphi_3 - 3\varphi_5).$$

The vanishing of the first factor corresponds to Case 1. The situation when  $R_3 = 0$  and  $r_3 = 0$  corresponds to a degenerate case of zero divergence. The vanishing of the rest of the factors of  $V_{26}$  allows to express  $r_3$  in function of  $R_3$ . If we introduce these relations in the next non-zero constant  $V_{32}$  (see Appendix 2), we can find  $R_5$  as a function of  $R_3$ . Finally by substituting them in the next non-zero Lyapunov constant  $V_{38}$  (see Appendix 2), we can see that this constant never vanishes. Therefore there is no possible integrable cases. The vanishing of  $R_5$  will be seen later. The vanishing of  $\sin(5\varphi_3 - 3\varphi_5)$  corresponds to Case 7.

*Case 3:* If  $r_3 = R_3 = 0$  then the first non-zero Lyapunov constant after  $V_{14}$  is  $V_{20}$ , that is

$$320V_{20} = ((r_1 + 5R_1)(5r_1 - 3R_1)(2r_1 - 3R_1)(r_1 - 3R_1)(r_1 - 6R_1)) \\ R_5 \sin(5\varphi_1 - \varphi_5).$$

The vanishing of the first factor corresponds to Case 1. The situation when  $R_1 = 0$  and  $r_1 = 0$  corresponds to a degenerate case of zero divergence. The vanishing of the rest of the factors of  $V_{20}$  allows to express  $r_1$  in function of  $R_1$ . If we introduce these relations in the next non-zero constant  $V_{26}$  (see Appendix 3), we can find  $R_5$  as a function of  $R_1$ . Finally by substituting them in the next non-zero Lyapunov constant  $V_{32}$  (see Appendix 3), we can see that this constant does not vanish. Therefore there is no possible integrable cases. The vanishing of  $R_5$  will be seen later. The vanishing of  $\sin(5\varphi_1 - \varphi_5)$  corresponds to Case 7.

We now consider the case  $R_5 = 0$ . Then

$$V_{14} = \frac{1}{8}(R_3 r_1 - 3r_3 R_1)(r_1 - 3R_1)(r_1 - R_1) \sin(3\varphi_1 - \varphi_3).$$

If we impose that  $V_{14} = 0$  we have four possibilities. The vanishing of  $\sin(3\varphi_1 - \varphi_3)$  corresponds to Case 7. The others correspond to Cases 4, 5 and 6 which we will be studied.

*Case 4:* In this case  $R_5 = 0$  and  $R_3 r_1 - 3r_3 R_1 = 0$ . If  $r_1 \neq 0$  and  $r_3 \neq 0$ , then  $P_4(\varphi) = \lambda Q'_4(\varphi)$  where  $\lambda = R_1/r_1 = R_3/3r_3$ . System (2.3) takes the form

$$\begin{aligned} \dot{r} &= \lambda r^4 Q'_4(\varphi), \\ \dot{\varphi} &= 1 + r^3 Q(\varphi). \end{aligned}$$

The previous system is integrable and its first integral is

$$H(r, \varphi) = (1 + (1 + 3\lambda)Q(\varphi)r^3) r^{-(3+\frac{1}{\lambda})}.$$

If  $r_1 = 0$  or  $r_3 = 0$  implies  $\dot{\varphi} = 1$  and system (2.3) is trivially integrable.

*Case 5:* In this case  $R_5 = 0$  and  $r_1 - 3R_1 = 0$ . We can suppose  $r_1 = 3R_1 \neq 0$ , because  $r_1 = R_1 = 0$  corresponds to a particular case of Case 7. Then

$$\begin{aligned} 64V_{20} &= 27(R_3 - r_3)^2(r_3 + 3R_3)R_1^3 \sin(3\varphi_1 - \varphi_3), \\ 17920V_{26} &= 3(5283r_3^2 - 4932r_3R_3 - 28943R_3^2 - 245232R_1^2) \\ &\quad (R_3 - r_3)^2(r_3 + 3R_3)R_1^3 \sin(3\varphi_1 - \varphi_3). \end{aligned}$$

The simultaneous vanishing of the two previous Lyapunov constants give rise to two possible cases either  $r_3 + 3R_3 = 0$  or  $r_3 - R_3 = 0$ . The last one corresponds to a particular case of  $R_3 r_1 - 3r_3 R_1 = 0$  which has been

studied in Case 4. In the first case, that is  $r_1 = 3R_1$  and  $r_3 = -3R_3$ , system (2.3) reduces to

$$\begin{aligned}\dot{r} &= r^4 (R_3 \cos(3\varphi + \varphi_3) + R_1 \cos(\varphi + \varphi_1)), \\ \dot{\varphi} &= 1 + r^3 (-3R_3 \sin(3\varphi + \varphi_3) + 3R_1 \sin(\varphi + \varphi_1)).\end{aligned}$$

If we make the change  $R = r^3$  the system takes the form

$$\begin{aligned}\dot{R} &= 3R^2 (R_3 \cos(3\varphi + \varphi_3) + R_1 \cos(\varphi + \varphi_1)), \\ \dot{\varphi} &= 1 + 3R (-R_3 \sin(3\varphi + \varphi_3) + R_1 \sin(\varphi + \varphi_1)),\end{aligned}$$

which corresponds to an integrable case of a quadratic system with linear part of center type, see for instance [2].

*Case 6:* In this case  $R_5 = 0$  and  $r_1 - R_1 = 0$ . We can suppose  $r_1 = R_1 \neq 0$ , because  $r_1 = R_1 = 0$  corresponds to a particular case of Case 7. Then

$$\begin{aligned}64V_{20} &= (R_3 - 3r_3)(-20r_1^2 + 17r_3^2 - 10R_3r_3 - 3R_3^2) \sin(3\varphi_1 - \varphi_3), \\ 23040V_{26} &= (799200r_1^4 + (2238104R_3^2 + 421824r_3R_3 - 699048r_3^2)r_1^2 \\ &\quad + 18603r_3^4 - 24450r_3^3R_3 - 66784r_3^2R_3^2 + 44402r_3R_3^3 + 13029R_3^4) \\ &\quad (R_3 - 3r_3)r_1^3 \sin(3\varphi_1 - \varphi_3).\end{aligned}$$

If  $R_3 = 3r_3$  the above constants are zero and Case 6 corresponds to a particular case of  $R_3r_1 - 3r_3R_1 = 0$  studied in Case 4. If the second factor of  $V_{20}$  is zero, we have

$$(3.1) \quad 20r_1^2 - 17r_3^2 + 10R_3r_3 + 3R_3^2 = 0.$$

By substituting the expression obtained for  $r_1$  from (3.1) in  $V_{26}$  we get

$$\begin{aligned}38400V_{26} &= -(r_3 + 3R_3)(3r_3 - R_3)^2(1019r_3^2 - 326r_3R_3 - 273R_3^2) \\ &\quad r_1^3 \sin(3\varphi_1 - \varphi_3).\end{aligned}$$

From the vanishing of the previous constant we obtain three possible cases:  $R_3 = -3r_3$  (already studied),  $1019r_3^2 - 326r_3R_3 - 273R_3^2 = 0$  and  $r_3 = -3R_3$ .

Let  $1019r_3^2 - 326r_3R_3 - 273R_3^2 = 0$ . From (3.1) we obtain  $6520r_1^2 + 4648r_3^2 + 1752R_3^2 = 0$ , which implies  $r_1 = r_3 = R_3 = 0$ . Finally let  $r_3 = -3R_3$ . By replacing the value of  $r_3$  in (3.1) we obtain  $r_1^2 = 9R_3^2$ . So  $r_1 = 3|R_3|$ , and system (2.3) reduces to

$$\begin{aligned}\dot{r} &= r^4 (R_3 \cos(3\varphi + \varphi_3) + 3|R_3| \cos(\varphi + \varphi_1)), \\ \dot{\varphi} &= 1 + r^3 (-3R_3 \sin(3\varphi + \varphi_3) + 3|R_3| \sin(\varphi + \varphi_1)).\end{aligned}$$



If we make the same change  $R = r^3$ , then we can write the system as

$$\begin{aligned}\dot{R} &= 3R^2 (R_3 \cos(3\varphi + \varphi_3) + 3|R_3| \cos(\varphi + \varphi_1)), \\ \dot{\varphi} &= 1 + 3R (-R_3 \sin(3\varphi + \varphi_3) + |R_3| \sin(\varphi + \varphi_1)),\end{aligned}$$

which corresponds to an integrable case of a quadratic system with linear part of center type, see for instance [2].

*Case 7:* We call this case *resonance angles*. Then system (2.3) is integrable (see Proposition 2).

*Case 8:* Let  $\varphi_3 - 3\varphi_1 = 0$ . We also assume that  $5\varphi_1 - \varphi_5 \neq 0$  (the opposite case has been studied in Case 7). Then a term that appears in  $V_{26}$  and which must be zero independently on the rest is

$$\frac{1}{8400} ((5R_3 + 3r_3)(3R_3 - r_3)(3R_3 - 5r_3)(R_3 - 3r_3)(29R_3 - 89r_3)) R_5^3 \sin(15\varphi_1 - 3\varphi_5).$$

From the vanishing of the factors of the previous expression and their substitution in the constants  $V_{14}$ ,  $V_{20}$  and the remaining terms of  $V_{26}$ , see Appendix 1, we obtain cases already studied.

*Case 9:* Let  $\varphi_1 - 2\varphi_3 + \varphi_5 = 0$ . We also assume  $3\varphi_1 - \varphi_3 \neq 0$  (the opposite case has been studied in Case 7). Then a term that appears in  $V_{20}$  and which must be zero independently on the rest is

$$\frac{1}{320} ((r_1 + 5R_1)(5r_1 - 3R_1)(2r_1 - 3R_1)(r_1 - 3R_1)(r_1 - 6R_1)) R_5 \sin(6\varphi_1 - 2\varphi_3).$$

From the vanishing of the factors of the previous expression and their substitution into the constants  $V_{14}$ ,  $V_{26}$  and the remaining terms of  $V_{20}$ , see Appendix 1, we obtain cases already studied.

*Case 10:* Let  $2\varphi_1 + \varphi_3 - \varphi_5 = 0$ . Then

$$\begin{aligned}40V_{14} &= ((5(R_3r_1 - 3r_3R_1)(r_1 - 3R_1)(r_1 - R_1)) \\ &\quad + ((15r_1 - 51R_1)r_3^2 + 22(r_1 - R_1)r_3R_3 - (21r_1 - 25R_1)R_3^2) R_5) \\ &\quad \sin(3\varphi_1 - \varphi_3).\end{aligned}$$

From the vanishing of the previous constant (we can assume that  $3\varphi_1 - \varphi_3 \neq 0$ , the opposite case has been studied in Case 7) we obtain  $R_5$  as a function of the remaining radial parameters. By substituting this value in  $V_{20}$  and  $V_{26}$  and imposing that these Lyapunov constants to be zero, we obtain a system of three equations since  $V_{26}$  has two terms with independent trigonometric part, see Appendix 1. By solving this system we obtain some cases already studied, and a new solution given by  $R_1 = \frac{r_1}{2}$ ,  $R_5 = R_3$ ,  $r_3 = 3R_3$  and  $R_3 = \left| \frac{r_1}{4} \right|$ . These relations give rise to a new integrable case (see Proposition 3). System (2.3) takes the form

$$(3.2) \quad \begin{aligned} \dot{r} &= r^4 R_5 (\cos(5\varphi + 2\varphi_1 + \varphi_3) + \cos(3\varphi + \varphi_3) \pm 2 \cos(\varphi + \varphi_1)), \\ \dot{\varphi} &= 1 + r^3 R_5 (-\sin(5\varphi + 2\varphi_1 + \varphi_3) + 3 \sin(3\varphi + \varphi_3) \pm 4 \sin(\varphi + \varphi_1)). \end{aligned}$$

For  $R_3 = -\frac{r_1}{4}$  system (3.2) becomes

$$\begin{aligned} \dot{r} &= 2r^4 (-k_1 \sin(\varphi + \varphi_0) + k_2 \cos(\varphi + \varphi_0)) \sin^2(\varphi + \varphi_0) \cos^2(\varphi + \varphi_0), \\ \dot{\varphi} &= 1 + r^3 (k_1 \cos^3(\varphi + \varphi_0) - k_2 \sin^3(\varphi + \varphi_0)) \cos(2(\varphi + \varphi_0)), \end{aligned}$$

where  $\varphi_0 = \frac{\varphi_1 + \varphi_3}{4}$ ,  $k_1 = -8 \sin \left[ \frac{3\varphi_1 - \varphi_3}{4} \right]$  and  $k_2 = -8 \cos \left[ \frac{3\varphi_1 - \varphi_3}{4} \right]$ .

For  $R_3 = \frac{r_1}{4}$  we make the change  $\varphi = \omega + \pi/4$  and system (3.2) becomes

$$\begin{aligned} \dot{r} &= 2r^4 (-k_1 \sin(\omega + \omega_0) + k_2 \cos(\omega + \omega_0)) \sin^2(\omega + \omega_0) \cos^2(\omega + \omega_0), \\ \dot{\omega} &= 1 + r^3 (k_1 \cos^3(\omega + \omega_0) - k_2 \sin^3(\omega + \omega_0)) \cos(2(\omega + \omega_0)), \end{aligned}$$

where  $\omega_0 = \frac{\varphi_1 + \varphi_3}{4}$ ,  $k_1 = 8 \sin \left[ \frac{3\varphi_1 - \varphi_3}{4} + \frac{\pi}{4} \right]$  and  $k_2 = 8 \cos \left[ \frac{3\varphi_1 - \varphi_3}{4} + \frac{\pi}{4} \right]$ .

*Cases 11 and 12:* In these cases we do not find anything, except the integrable cases mentioned before.

### Appendix 1

Lyapunov constants  $V_{14}$ ,  $V_{20}$  and  $V_{26}$  if  $\varphi_1 = \overline{\varphi}_1$  and  $\varphi_3 = \overline{\varphi}_3$ :

$$\begin{aligned}
40V_{14} &= 5((r_1 - 3R_1)(r_1 - R_1)(-3R_1r_3 + r_1R_3)) \sin(3\varphi_1 - \varphi_3) \\
&+ (15r_1^2r_3 - 18r_1R_1r_3 - 33R_1^2r_3 + 11r_1^2R_3 - 74r_1R_1R_3 + 75R_1^2R_3) \\
&\quad R_5 \sin(2\varphi_1 + \varphi_3 - \varphi_5) \\
&+ (15r_1r_3^2 - 51R_1r_3^2 + 22r_1r_3R_3 - 22R_1r_3R_3 - 21r_1R_3^2 + 25R_1R_3^2) \\
&\quad R_5 \sin(\varphi_1 - 2\varphi_3 + \varphi_5).
\end{aligned}$$

$$\begin{aligned}
144000V_{20} &= 25(-1620r_1^4R_1r_3 + 4050r_1^3R_1^2r_3 + 137700r_1^2R_1^3r_3 \\
&\quad - 541350r_1R_1^4r_3 + 406620R_1^5r_3 - 4320r_1^2R_1r_3^3 + 20790r_1R_1^2r_3^3 \\
&\quad - 21060R_1^3r_3^3 + 540r_1^5R_3 - 1350r_1^4R_1R_3 - 45900r_1^3R_1^2R_3 \\
&+ 180450r_1^2R_1^3R_3 - 135540r_1R_1^4R_3 + 1440r_1^3r_3^2R_3 - 2520r_1^2R_1r_3^2R_3 \\
&- 9540r_1R_1^2r_3^2R_3 + 14850R_1^3r_3^2R_3 - 1470r_1^3r_3R_3^2 + 17340r_1^2R_1r_3R_3^2 \\
&- 56280r_1R_1^2r_3R_3^2 + 40320R_1^3r_3R_3^2 - 3940r_1^3R_3^3 + 17110r_1^2R_1R_3^3 \\
&- 13440r_1R_1^2R_3^3 - 135r_1^3r_3R_5^2 - 16659r_1^2R_1r_3R_5^2 + 73359r_1R_1^2r_3R_5^2 \\
&\quad - 69957R_1^3r_3R_5^2 + 5283r_1^3R_3R_5^2 - 23913r_1^2R_1R_3R_5^2 \\
&\quad + 26469r_1R_1^2R_3R_5^2 - 3375R_1^3R_3R_5^2) \sin(3\varphi_1 - \varphi_3) \\
&+ 90(-2005r_1r_3^3 + 2872R_1r_3^3 - 414r_1r_3^2R_3 + 4539R_1r_3^2R_3 \\
&\quad + 2947r_1r_3R_3^2 - 6646R_1r_3R_3^2 - 848r_1R_3^3 \\
&\quad + 1475R_1R_3^3) R_5^2 \sin(\varphi_1 + 3\varphi_3 - 2\varphi_5) \\
&+ 450((r_1 - 6R_1)(2r_1 - 3R_1)(5r_1 - 3R_1)(-r_1 + 3R_1) \\
&\quad (r_1 + 5R_1)) R_5 \sin(5\varphi_1 - \varphi_5) \\
&\quad + R_5(63000r_1^4r_3 - 133200r_1^3R_1r_3 - 3105000r_1^2R_1^2r_3 \\
&+ 3596400r_1R_1^3r_3 + 7268400R_1^4r_3 + 108000r_1^2r_3^3 - 145125r_1R_1r_3^3 \\
&- 313200R_1^2r_3^3 + 32400r_1^4R_3 - 414000r_1^3R_1R_3 - 1351800r_1^2R_1^2R_3 \\
&\quad + 15145200r_1R_1^3R_3 - 16335000R_1^4R_3 - 55125r_1^2r_3^2R_3 \\
&\quad - 420975r_1R_1r_3^2R_3 + 941175R_1^2r_3^2R_3 - 396375r_1^2r_3R_3^2 \\
&\quad + 919875r_1R_1r_3R_3^2 + 150825R_1^2r_3R_3^2 - 189700r_1^2R_3^3 \\
&\quad + 1546225r_1R_1R_3^3 - 1680000R_1^2R_3^3 + 68715r_1^2r_3R_5^2 \\
&\quad - 111618r_1R_1r_3R_5^2 - 109053R_1^2r_3R_5^2 + 50661r_1^2R_3R_5^2 \\
&- 343134r_1R_1R_3R_5^2 + 629325R_1^2R_3R_5^2) \sin(2\varphi_1 + \varphi_3 - \varphi_5) \\
&\quad + R_5(108000r_1^3r_3^2 - 661500r_1^2R_1r_3^2 - 2166750r_1R_1^2r_3^2
\end{aligned}$$

$$\begin{aligned}
& +10277550R_1^3r_3^2 + 63000r_1r_3^4 - 267525R_1r_3^4 + 117000r_1^3r_3R_3 \\
& - 213300r_1^2R_1r_3R_3 - 4360500r_1R_1^2r_3R_3 + 4516200R_1^3r_3R_3 \\
& + 77175r_1r_3^3R_3 - 48375R_1r_3^3R_3 - 259650r_1^3R_3^2 + 976050r_1^2R_1R_3^2 \\
& + 3284100r_1R_1^2R_3^2 - 5040000R_1^3R_3^2 - 466875r_1r_3^2R_3^2 \\
& + 1394325R_1r_3^2R_3^2 - 481775r_1r_3R_3^3 + 457775R_1r_3R_3^3 \\
& + 498075r_1R_3^4 - 605000R_1R_3^4 + 73440r_1r_3^2R_5^2 \\
& - 304776R_1r_3^2R_5^2 + 127872r_1r_3R_3R_5^2 - 155952R_1r_3R_3R_5^2 \\
& - 71136r_1R_3^2R_5^2 + 185400R_1R_3^2R_5^2) \sin(\varphi_1 - 2\varphi_3 + \varphi_5).
\end{aligned}$$

$$\begin{aligned}
& 1995840000V_{26} = 2475R_5 (31500r_1^7 - 229320r_1^6R_1 - 3226860r_1^5R_1^2 \\
& + 22267980r_1^4R_1^3 + 38922660r_1^3R_1^4 - 393996960r_1^2R_1^5 + 601983900r_1R_1^6 \\
& - 232186500R_1^7 + 176400r_1^5r_3^2 - 1408950r_1^4R_1r_3^2 + 34316631r_1^3R_1^2r_3^2 \\
& - 145310598r_1^2R_1^3r_3^2 + 193311819r_1R_1^4r_3^2 - 71737362R_1^5r_3^2 \\
& + 223110r_1^5r_3R_3 - 20756259r_1^4R_1r_3R_3 + 120925167r_1^3R_1^2r_3R_3 \\
& - 243611151r_1^2R_1^3r_3R_3 + 12120243r_1R_1^4r_3R_3 + 294793050R_1^5r_3R_3 \\
& - 725446r_1^5R_3^2 - 4407647r_1^4R_1R_3^2 + 95407266r_1^3R_1^2R_3^2 \\
& - 448139203r_1^2R_1^3R_3^2 + 926142250r_1R_1^4R_3^2 - 620151000R_1^5R_3^2 \\
& + 217545r_1^5R_5^2 - 1023621r_1^4R_1R_5^2 - 3370902r_1^3R_1^2R_5^2 \\
& + 25845846r_1^2R_1^3R_5^2 - 45661203r_1R_1^4R_5^2 + 23776335R_1^5R_5^2) \sin(5\varphi_1 - \varphi_5) \\
& + 900 (1376595r_1^2r_3^4 - 22590630r_1R_1r_3^4 + 58656123R_1^2r_3^4 \\
& + 17168520r_1^2r_3^3R_3 - 177991152r_1R_1r_3^3R_3 + 367268520R_1^2r_3^3R_3 \\
& + 15459842r_1^2r_3^2R_3^2 - 53348860r_1R_1r_3^2R_3^2 - 123298102R_1^2r_3^2R_3^2 \\
& - 27940120r_1^2r_3R_3^3 + 275319488r_1R_1r_3R_3^3 - 274457320R_1^2r_3R_3^3 \\
& + 6854907r_1^2R_3^4 - 107758510r_1R_1R_3^4 \\
& + 119730875R_1^2R_3^4) R_5^2 \sin(2\varphi_1 - 4\varphi_3 + 2\varphi_5) \\
& + 9900 (36225r_1^4r_3^3 - 414900r_1^3R_1r_3^3 + 5560362r_1^2R_1^2r_3^3 \\
& - 20524968r_1R_1^3r_3^3 + 18080001R_1^4r_3^3 + 95775r_1^4r_3^2R_3 \\
& - 5749548r_1^3R_1r_3^2R_3 + 48599826r_1^2R_1^2r_3^2R_3 - 135196320r_1R_1^3r_3^2R_3 \\
& + 134036907R_1^4r_3^2R_3 + 430523r_1^4r_3R_3^2 - 9397688r_1^3R_1r_3R_3^2 \\
& + 29371338r_1^2R_1^2r_3R_3^2 - 38087228r_1R_1^3r_3R_3^2 + 17244575R_1^4r_3R_3^2 \\
& - 67159r_1^4R_3^3 + 7174896r_1^3R_1R_3^3 - 36221622r_1^2R_1^2R_3^3 \\
& + 74045100r_1R_1^3R_3^3 - 49529375R_1^4R_3^3) R_5 \sin(4\varphi_1 - 3\varphi_3 + \varphi_5)
\end{aligned}$$

$$\begin{aligned}
& +45R_5^2 (188010900r_1^3r_3^3 + 277148475r_1^2R_1r_3^3 - 11275089480r_1R_1^2r_3^3 \\
& \quad + 16481281185R_1^3r_3^3 + 127378350r_1r_3^5 - 224994825R_1r_3^5 \\
& +406138275r_1^3r_3^2R_3 - 1976826630r_1^2R_1r_3^2R_3 + 1650294105r_1R_1^2r_3^2R_3 \\
& \quad + 11618192850R_1^3r_3^2R_3 - 106129650r_1r_3^4R_3 - 144600225R_1r_3^4R_3 \\
& -40820670r_1^3r_3R_3^2 - 2528927865r_1^2R_1r_3R_3^2 + 26140686450r_1R_1^2r_3R_3^2 \\
& \quad - 40014039075R_1^3r_3R_3^2 - 833488975r_1r_3^3R_3^2 + 1699300075R_1r_3^3R_3^2 \\
& \quad - 119309545r_1^3R_3^3 + 1977099700r_1^2R_1R_3^3 - 13674395475r_1R_1^2R_3^3 \\
& \quad + 15412320000R_1^3R_3^3 + 129133125r_1r_3^2R_3^3 + 881613975R_1r_3^2R_3^3 \\
& \quad + 866625925r_1r_3R_3^4 - 2044436350R_1r_3R_3^4 - 267013175r_1R_3^5 \\
& \quad + 478403750R_1R_3^5 + 141583815r_1r_3^3R_5^2 - 213503121R_1r_3^3R_5^2 \\
& +61169427r_1r_3^2R_3R_5^2 - 290758437R_1r_3^2R_3R_5^2 - 167905851r_1r_3R_3R_5^2 \\
& \quad + 513742653R_1r_3R_3R_5^2 + 45109089r_1R_3^3R_5^2 \\
& \quad - 127607175R_1R_3^3R_5^2) \sin(\varphi_1 + 3\varphi_3 - 2\varphi_5) \\
& +R_5(-2058210000r_1^5r_3^2 + 13831884000r_1^4R_1r_3^2 + 210252588975r_1^3R_1^2r_3^2 \\
& \quad - 1372000268925r_1^2R_1^3r_3^2 - 1923091386975r_1R_1^4r_3^2 \\
& \quad + 12382675328925R_1^5r_3^2 - 4241160000r_1^3r_3^4 + 29529299250r_1^2R_1r_3^4 \\
& \quad + 113916777975r_1R_1^2r_3^4 - 615399800400R_1^3r_3^4 - 919957500r_1r_3^6 \\
& \quad + 4478206500R_1r_3^6 - 1564893000r_1^5r_3R_3 + 19971928350r_1^4R_1r_3R_3 \\
& \quad + 108674779950r_1^3R_1^2r_3R_3 + 167108074650r_1^2R_1^3r_3R_3 \\
& \quad - 6084308542950r_1R_1^4r_3R_3 + 5474234205000R_1^5r_3R_3 \\
& \quad - 2452254750r_1^3r_3^3R_3 + 14151396600r_1^2R_1r_3^3R_3 \\
& +90500362425r_1R_1^2r_3^3R_3 + 72915274575R_1^3r_3^3R_3 - 759888000r_1r_3^5R_3 \\
& \quad - 1032793875R_1r_3^5R_3 + 4201631775r_1^5R_3^2 + 5173227675r_1^4R_1R_3^2 \\
& \quad - 728892620775r_1^3R_1^2R_3^2 + 2780678258325r_1^2R_1^3R_3^2 \\
& +1510872165000r_1R_1^4R_3^2 - 4778061750000R_1^5R_3^2 + 48134738025r_1^3r_3^2R_3^2 \\
& \quad - 297515651400r_1^2R_1r_3^2R_3^2 - 400254737850r_1R_1^2r_3^2R_3^2 \\
& +2437146129825R_1^3r_3^2R_3^2 + 17095807125r_1r_3^4R_3^2 - 63625541625R_1r_3^4R_3^2 \\
& \quad + 38843573175r_1^3r_3R_3^3 - 220177744875r_1^2R_1r_3R_3^3 \\
& \quad - 514855484550r_1R_1^2r_3R_3^3 + 649385055000R_1^3r_3R_3^3 \\
& \quad + 18862131375r_1r_3^3R_3^3 - 14658559875R_1r_3^3R_3^3 - 67676355450r_1^3R_3^4 \\
& +344964715425r_1^2R_1R_3^4 + 460610865000r_1R_1^2R_3^4 - 972798750000R_1^3R_3^4 \\
& \quad - 73138105875r_1r_3^2R_3^4 + 206310721125R_1r_3^2R_3^4 - 67495697875r_1r_3R_3^5
\end{aligned}$$

$$\begin{aligned}
& +63496446250R_1r_3R_5^5 + 69710238750r_1R_3^6 - 85032062500R_1R_3^6 \\
& \quad -4606085925r_1^3r_3^2R_5^2 + 8490640320r_1^2R_1r_3^2R_5^2 \\
& \quad +225636825015r_1R_1^2r_3^2R_5^2 - 705982117950R_1^3r_3^2R_5^2 \\
& +4002884325r_1r_3^4R_5^2 - 15080254380R_1r_3^4R_5^2 - 8093798865r_1^3r_3R_3R_5^2 \\
& \quad -14054083065r_1^2R_1r_3R_3R_5^2 + 312683482305r_1R_1^2r_3R_3R_5^2 \\
& \quad -330863491575R_1^3r_3R_3R_5^2 - 5443565715r_1r_3^3R_3R_5^2 \\
& +32781602745R_1r_3^3R_3R_5^2 + 6281712270r_1^3R_3^2R_5^2 + 7011577665r_1^2R_1R_3^2R_5^2 \\
& \quad -213859335600r_1R_1^2R_3^2R_5^2 + 390057418125R_1^3R_3^2R_5^2 \\
& \quad -4308252165r_1r_3^2R_3^2R_5^2 - 47037249315R_1r_3^2R_3^2R_5^2 \\
& \quad +46035049455r_1r_3R_3^3R_5^2 - 56178436725R_1r_3R_3^3R_5^2 \\
& \quad -20091786300r_1R_3^4R_5^2 + 41674516875R_1R_3^4R_5^2 - 1148827860r_1r_3^2R_3^4 \\
& +4807068444R_1r_3^2R_3^4 - 1761106968r_1r_3R_3R_3^4 + 3028199688R_1r_3R_3R_3^4 \\
& \quad + 1552507884r_1R_3^2R_3^4 - 1823480100R_1R_3^2R_3^4) \sin(\varphi_1 - 2\varphi_3 + \varphi_5) \\
& \quad +346500 ((r_1 - 3R_1)(r_1 - R_1)(3R_1r_3 - r_1R_3)(-135r_1^3r_3 \\
& \quad +402r_1^2R_1r_3 - 25893r_1R_1^2r_3 + 36390R_1^3r_3 - 750r_1^3R_3 \\
& +36349r_1^2R_1R_3 - 155106r_1R_1^2R_3 + 285455R_1^3R_3)) \sin(6\varphi_1 - 2\varphi_3) \\
& \quad +237600 ((3r_3 - R_3)(-5r_3 + 3R_3)(-r_3 + 3R_3)(3r_3 + 5R_3) \\
& \quad (-89r_3 + 29R_3)) R_5^2 \sin(5\varphi_3 - 3\varphi_5) \\
& +25 (16839900r_1^6R_1r_3 + 7858620r_1^5R_1^2r_3 - 4663529640r_1^4R_1^3r_3 \\
& \quad +13708052820r_1^3R_1^4r_3 + 172100784240r_1^2R_1^5r_3 \\
& -691301096640r_1R_1^6r_3 + 518438774700R_1^7r_3 + 110394900r_1^4R_1r_3^3 \\
& -424258560r_1^3R_1^2r_3^3 - 9581368500r_1^2R_1^3r_3^3 + 44697104100r_1R_1^4r_3^3 \\
& -42068475900R_1^5r_3^3 + 89812800r_1^2R_1r_3^5 - 491243940r_1R_1^2r_3^5 \\
& +594809325R_1^3r_3^5 - 5613300r_1^7R_3 - 2619540r_1^6R_1R_3 \\
& +1554509880r_1^5R_1^2R_3 - 4569350940r_1^4R_1^3R_3 - 57366928080r_1^3R_1^4R_3 \\
& +230433698880r_1^2R_1^5R_3 - 172812924900r_1R_1^6R_3 - 36798300r_1^5r_3^2R_3 \\
& +62672940r_1^4R_1r_3^2R_3 + 2442498300r_1^3R_1^2r_3^2R_3 - 791671320r_1^2R_1^3r_3^2R_3 \\
& -28404883980r_1R_1^4r_3^2R_3 + 33535244160R_1^5r_3^2R_3 - 29937600r_1^3r_3^4R_3 \\
& -37992240r_1^2R_1r_3^4R_3 + 698120775r_1R_1^2r_3^4R_3 - 948808080R_1^3r_3^4R_3 \\
& +26248860r_1^5r_3R_3^2 - 722690100r_1^4R_1r_3R_3^2 + 3366982080r_1^3R_1^2r_3R_3^2 \\
& \quad +15849244620r_1^2R_1^3r_3R_3^2 - 89956761840r_1R_1^4r_3R_3^2 \\
& +72090364500R_1^5r_3R_3^2 + 67246740r_1^3r_3^3R_3^2 - 973454625r_1^2R_1r_3^3R_3^2 \\
& +3727249020r_1R_1^2r_3^3R_3^2 - 3430541565R_1^3r_3^3R_3^2 + 324373500r_1^5R_3^3
\end{aligned}$$

$$\begin{aligned}
& -2689812180r_1^4R_1R_3^3 - 568891620r_1^3R_1^2R_3^3 + 26259449040r_1^2R_1^3R_3^3 \\
& -24030121500r_1R_1^4R_3^3 + 224885925r_1^3r_3^2R_3^3 - 534330720r_1^2R_1r_3^2R_3^3 \\
& -1209966615r_1R_1^2r_3^2R_3^3 + 2212376760R_1^3r_3^2R_3^3 - 200887500r_1^3r_3R_3^4 \\
& \quad + 2384299665r_1^2R_1r_3R_3^4 - 7784636640r_1R_1^2r_3R_3^4 \\
& \quad + 5582808000R_1^3r_3R_3^4 - 533268725r_1^3R_3^5 + 2349059240r_1^2R_1R_3^5 \\
& \quad - 1860936000r_1R_1^2R_3^5 + 7543800r_1^5r_3R_3^5 + 361375047r_1^4R_1r_3R_3^5 \\
& \quad - 940143699r_1^3R_1^2r_3R_3^5 - 31309275195r_1^2R_1^3r_3R_3^5 \\
& + 129987690327r_1R_1^4r_3R_3^5 - 119364276240R_1^5r_3R_3^5 + 39313350r_1^3r_3^2R_3^5 \\
& - 2210262228r_1^2R_1r_3^2R_3^5 + 8944718139r_1R_1^2r_3^2R_3^5 - 8152816131R_1^3r_3^2R_3^5 \\
& - 113416974r_1^5R_3R_3^5 + 170430183r_1^4R_1R_3R_3^5 + 9968595615r_1^3R_1^2R_3R_3^5 \\
& \quad - 41819430609r_1^2R_1^3R_3R_3^5 + 43642223955r_1R_1^4R_3R_3^5 \\
& \quad - 4894931250R_1^5R_3R_3^5 + 239127876r_1^3r_3^2R_3R_3^5 \\
& \quad - 6348916944r_1^2R_1r_3^2R_3R_3^5 + 8965782621r_1R_1^2r_3^2R_3R_3^5 \\
& \quad + 11323304793R_1^3r_3^2R_3R_3^5 + 148481577r_1^3r_3R_3^2R_3^5 \\
& \quad - 1670198787r_1^2R_1r_3R_3^2R_3^5 + 15377567166r_1R_1^2r_3R_3^2R_3^5 \\
& - 23931769950R_1^3r_3R_3^2R_3^5 + 995250213r_1^3R_3^3R_3^5 - 2676696849r_1^2R_1R_3^3R_3^5 \\
& \quad - 2146582350r_1R_1^2R_3^3R_3^5 + 5665140000R_1^3R_3^3R_3^5 + 8690355r_1^3r_3R_3^4 \\
& \quad + 473339457r_1^2R_1r_3R_3^4 - 2427142887r_1R_1^2r_3R_3^4 \\
& \quad + 2868219531R_1^3r_3R_3^4 - 142118739r_1^3R_3R_3^4 + 762823359r_1^2R_1R_3R_3^4 \\
& \quad - 1144504377r_1R_1^2R_3R_3^4 + 360595125R_1^3R_3R_3^4) \sin(3\varphi_1 - \varphi_3) \\
& + R_5 (-919957500r_1^6r_3 + 2884167000r_1^5R_1r_3 + 130419679500r_1^4R_1^2r_3 \\
& \quad - 209395692000r_1^3R_1^3r_3 - 4046142820500r_1^2R_1^4r_3 \\
& \quad + 4456353429000r_1R_1^5r_3 + 9210309322500R_1^6r_3 - 4054050000r_1^4r_3^3 \\
& \quad + 8518182750r_1^3R_1r_3^3 + 227511882675r_1^2R_1^2r_3^3 - 238260928950r_1R_1^3r_3^3 \\
& \quad - 725514324975R_1^4r_3^3 - 2245320000r_1^2r_3^5 + 3523074750r_1R_1r_3^5 \\
& \quad + 7655265450R_1^2r_3^5 - 228541500r_1^6R_3 + 6781401000r_1^5R_1R_3 \\
& + 48681121500r_1^4R_1^2R_3 - 757941624000r_1^3R_1^3R_3 - 1382213200500r_1^2R_1^4R_3 \\
& + 19012697055000r_1R_1^5R_3 - 20662791187500R_1^6R_3 + 2202180750r_1^4r_3^2R_3 \\
& \quad + 50964502050r_1^3R_1r_3^2R_3 - 331994225475r_1^2R_1^2r_3^2R_3 \\
& \quad - 425366295750r_1R_1^3r_3^2R_3 + 1216448894925R_1^4r_3^2R_3 \\
& + 4220646750r_1^2r_3^4R_3 + 4866982200r_1R_1r_3^4R_3 - 31276991700R_1^2r_3^4R_3 \\
& \quad + 34489414575r_1^4r_3R_3^2 - 151947259200r_1^3R_1r_3R_3^2 \\
& \quad - 864582278175r_1^2R_1^2r_3R_3^2 + 2890713493800r_1R_1^3r_3R_3^2
\end{aligned}$$

$$\begin{aligned}
& -572024722500R_1^4r_3R_3^2 + 20500037550r_1^2r_3^3R_3^2 \\
& -49479662700r_1R_1r_3^3R_3^2 - 19379780550R_1^2r_3^3R_3^2 + 14229762525r_1^4R_3^3 \\
& -244978014600r_1^3R_1R_3^3 + 632738742075r_1^2R_1^2R_3^3 \\
& +1386330907500r_1R_1^3R_3^3 - 2417530500000R_1^4R_3^3 - 7980247800r_1^2r_3^2R_3^3 \\
& -70150101300r_1R_1r_3^2R_3^3 + 151108935300R_1^2r_3^2R_3^3 - 53409883950r_1^2r_3R_3^4 \\
& +127889399550r_1R_1r_3R_3^4 + 15480487500R_1^2r_3R_3^4 - 24927916550r_1^2R_3^5 \\
& +211221387500r_1R_1R_3^5 - 232617000000R_1^2R_3^5 + 1453308075r_1^4r_3R_3^5 \\
& -42985052190r_1^3R_1r_3R_3^5 + 242789114880r_1^2R_1^2r_3R_3^5 \\
& -171680012730r_1R_1^3r_3R_3^5 - 404427233475R_1^4r_3R_3^5 \\
& -1421290800r_1^2r_3^3R_3^5 + 3652068060r_1R_1r_3^3R_3^5 - 3585857040R_1^2r_3^3R_3^5 \\
& +1543409055r_1^4R_3R_3^5 - 36386475210r_1^3R_1R_3R_3^5 \\
& +338814678060r_1^2R_1^2R_3R_3^5 - 1182414401550r_1R_1^3R_3R_3^5 \\
& +1446537268125R_1^4R_3R_3^5 - 2616459570r_1^2r_3^2R_3R_3^5 \\
& +14166828720r_1R_1r_3^2R_3R_3^5 - 26265773610R_1^2r_3^2R_3R_3^5 \\
& +12524885820r_1^2r_3R_3^2R_3^5 - 5917042980r_1R_1r_3R_3^2R_3^5 \\
& -30720899700R_1^2r_3R_3^2R_3^5 + 9427256550r_1^2R_3^3R_3^5 \\
& -70208058600r_1R_1R_3^3R_3^5 + 119035248750R_1^2R_3^3R_3^5 - 1130948460r_1^2r_3R_3^4 \\
& +1929611592r_1R_1r_3R_3^4 + 2916332532R_1^2r_3R_3^4 - 804688884r_1^2R_3R_3^4 \\
& + 6239285496r_1R_1R_3R_3^4 - 7028745300R_1^2R_3R_3^4) \sin(2\varphi_1 + \varphi_3 - \varphi_5) \\
& +2475 (-104580r_1^4r_3^2 - 20131965r_1^3R_1r_3^2 + 59842161r_1^2R_1^2r_3^2 \\
& +6015393r_1R_1^3r_3^2 - 86403753R_1^4r_3^2 - 1872885r_1^4r_3R_3 \\
& -3885144r_1^3R_1r_3R_3 + 77226918r_1^2R_1^2r_3R_3 - 246205968r_1R_1^3r_3R_3 \\
& +273837735R_1^4r_3R_3 - 1461101r_1^4R_3^2 + 14956237r_1^3R_1R_3^2 \\
& -75810827r_1^2R_1^2R_3^2 + 235432455r_1R_1^3R_3^2 \\
& - 201316500R_1^4R_3^2) R_3^2 \sin(4\varphi_1 + 2\varphi_3 - 2\varphi_5).
\end{aligned}$$

## Appendix 2

Lyapunov constants  $V_{32}$  and  $V_{38}$  if  $\varphi_1 = \bar{\varphi}_1$ ,  $\varphi_3 = \bar{\varphi}_3$  and  $R_1 = r_1 = 0$ :

$$\begin{aligned}
3386880000V_{32} &= (3r_3 + 5R_3)R_5^3 (-1459292625r_3^6 \\
& +8614578420r_3^5R_3 - 3561439455r_3^4R_3^2 - 34418277320r_3^3R_3^3 \\
& +41410717145r_3^2R_3^4 - 16050420540r_3R_3^5 + 2034246375R_3^6 \\
& -2170849005r_3^4R_5^2 + 6255604188r_3^3R_3R_5^2 - 6059843478r_3^2R_3^2R_5^2
\end{aligned}$$



$$+ 2315564748r_3R_3^3R_5^2 - 303704613R_3^4R_5^2) \sin(5\varphi_3 - 3\varphi_5).$$

$$\begin{aligned} 1482030950400000V_{38} = & (3r_3 + 5R_3)R_5^3 (-1149145404553125r_3^8 \\ & + 8848194134872050r_3^7R_3 + 1185862616782200r_3^6R_3^2 \\ & - 96387033484384650r_3^5R_3^3 + 58538665263406650r_3^4R_3^4 \\ & + 332980469705167550r_3^3R_3^5 - 413020341318321800r_3^2R_3^6 \\ & + 161016094351964250r_3R_3^7 - 20445404262373125R_3^8 \\ & + 2891909356127550r_3^6R_5^2 - 42487871210003790r_3^5R_3R_5^2 \\ & + 152493752666073780r_3^4R_3^2R_5^2 - 176828521900465980r_3^3R_3^3R_5^2 \\ & + 96197776031891430r_3^2R_3^4R_5^2 - 25839946288107990r_3R_3^5R_5^2 \\ & + 2766024379211400R_3^6R_5^2 - 3621290735420280r_3^4R_5^4 \\ & + 14619453338344608r_3^3R_3R_5^4 - 13917796242684048r_3^2R_3^2R_5^4 \\ & + 5015027246100768r_3R_3^3R_5^4 - 622727965831608R_3^4R_5^4) \sin(5\varphi_3 - 3\varphi_5). \end{aligned}$$

### Appendix 3

Lyapunov constants  $V_{26}$  and  $V_{32}$  if  $\varphi_1 = \bar{\varphi}_1$ ,  $\varphi_3 = \bar{\varphi}_3$  and  $R_3 = r_3 = 0$ :

$$\begin{aligned} 268800V_{26} = & ((r_1 + 5R_1)R_5(10500r_1^6 - 128940r_1^5R_1 - 430920r_1^4R_1^2 \\ & + 9577260r_1^3R_1^3 - 34912080r_1^2R_1^4 + 43228080r_1R_1^5 - 15479100R_1^6 \\ & + 72515r_1^4R_5^2 - 703782r_1^3R_1R_5^2 + 2395276r_1^2R_1^2R_5^2 \\ & - 3361098r_1R_1^3R_5^2 + 1585089R_1^4R_5^2) \sin(5\varphi_1 - \varphi_5). \end{aligned}$$

$$\begin{aligned} 2483712000V_{32} = & ((r_1 + 5R_1)R_5(97020000r_1^8 - 1217046600r_1^7R_1 \\ & - 19884803400r_1^6R_1^2 + 277869853800r_1^5R_1^3 + 222079611600r_1^4R_1^4 \\ & - 12326128630200r_1^3R_1^5 + 46086997987800r_1^2R_1^6 \\ & - 57154152825000r_1R_1^7 + 20495280960000R_1^8 + 1020766985r_1^6R_5^2 \\ & + 1114175216r_1^5R_1R_5^2 - 223447537033r_1^4R_1^2R_5^2 \\ & + 1715463721944r_1^3R_1^3R_5^2 - 5101631600457r_1^2R_1^4R_5^2 \\ & + 6576435520200r_1R_1^5R_5^2 - 2941989081255R_1^6R_5^2 + 1712647422R_1^4R_5^4 \\ & - 17633067984r_1^3R_1R_5^4 + 64359273276r_1^2R_1^2R_5^4 \\ & - 97959123360r_1R_1^3R_5^4 + 53197740630R_1^4R_5^4) \sin(5\varphi_1 - \varphi_5). \end{aligned}$$

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