Publicacions Matemàtiques, Vol **39** (1995), 367–377.

ADJUNCTION OF n-EQUIVALENCES AND TRIAD CONNECTIVITY

P. J. Witbooi

Abstract

We prove a new adjunction theorem for n-equivalences. This theorem enables us to produce a simple geometric version of proof of the triad connectivity theorem of Blakers and Massey. An important intermediate step is a study of the collapsing map $S \vee X \to S$, S being a sphere.

Since the invention of the concept of quasifibration by Dold and Thom, their basic globalization theorem [**7**, Satz 2.2] has been applied widely. Some of the applications make use of an adjunction theorem of Hardie [**9**, Theorem 0.2]. This adjunction theorem, when applicable, is more convenient to use than the original theorem of Dold and Thom. The work [**10**] of May provides a new approach to quasifibrations. In Part I of this article we prove an adjunction theorem, Theorem 7, which merges the work of Hardie and that of May. As an application of this theorem, in Part II we provide a quite straightforward geometric proof of the triad connectivity theorem of Blakers and Massey [**4**].

Following the original proof due to Blakers and Massey, several alternative proofs have emerged. Moore [**11**] gives a proof based on the Serre spectral sequence of a fibration, and Namioka's proof [**12**] uses the Hurewicz isomorphism theorem. There are several other versions of the proof such as in [**6**], [**8**], and [**14**], which do not make use of homology. In these three citations the proofs follow by geometric arguments. Using algebraic methods of a completely different nature, Brown and Loday [**3**] give yet another proof of the triad connectivity theorem (in fact they prove a stronger result, the homotopy excision theorem of [**5**]). In cases where homology is not used in the arguments, we can deduce simple alternative proofs of the Freudenthal suspension theorem and the Hurewicz isomorphism theorems as shown in [**8**] and [**14**].

A special case of the homotopy excision theorem follows from Theorem 13, and the more general homotopy excision theorem can be deduced from our result by making use of CW-approximation. We do not include this latter step here since the arguments are similar to those in [**8**]. Our method also supports proofs of the suspension theorem and the Hurewicz theorems. The merit of the proof of Theorem 13 is its simplicity.

PART I: Adjunction of k-equivalences.

Definition 1. Let k be a positive integer and let $p : (X, A) \to (Y, B)$ be a map.

(a) The map $p : X \to Y$ of spaces is said to be a 0-*equivalence* if it induces a surjective function on path components. For $k > 0$, the map p is said to be a k-*equivalence* if it induces a bijection on path components, and if, for every $x \in X$, the function $p_* : \pi_r(X, x) \to \pi_r(Y, x')$ with $x' = p(x)$, is bijective whenever $r < k$ and surjective for $r = k$.

(b) The map $p : (X, A) \to (Y, B)$ of pairs of spaces is said to be a 0-*equivalence* if condition (1) below holds. If also condition (2) holds, then the map of pairs is said to be a k-*equivalence* ($k > 0$).

(1) $\mathrm{Im}[\pi_0(A) \to \pi_0(X)] = p_*^{-1}\,\mathrm{Im}[\pi_0(B) \to \pi_0(Y)]$.

(2) For every $a \in A$, and $b = p(a)$, the function $p_* : \pi_r(X, A, a) \to \pi_r(Y, B, b)$ is bijective whenever $r < k$ and surjective for $r = k$.

(c) A map of spaces or of pairs of spaces is said to be a *weak equivalence* if it is a k-equivalence for all $k > 0$.

Definition 1(a) appears in textbooks such as [**8**] or [**14**], while Definition 1(b) is due to May [**10**]. For Definition 1(b), the condition (1) on path components is automatically fulfilled if $A = p^{-1}(B)$. We note that the composition of two k-equivalences is again a k-equivalence. The proof of the following proposition is an easy exercise on the level of set theory and we omit it.

Proposition 2. *Suppose that we have a commutative triangle of maps of spaces as below. If α is a $(k-1)$-equivalence and δ is a k-equivalence, then β is a k-equivalence.*

$$\begin{tikzcd} A \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

By **Top**^{2} we shall mean the category of which the objects are maps of spaces, that is to say we take the morphisms of **Top** as objects. The morphisms in **Top**^{2} from an object q to an object p, is a pair of maps (g, f) such that $f \circ q = p \circ g$, see Diagram (A) below.

We note that the mapping path fibration construction is an endofunctor of **Top**^{2}. We observe also that, given a **Top**^{2}-morphism $(g, f) : q \to p$, then for every point $b \in B$, there is an induced map of the homotopy fibre of q over b into the homotopy fibre of p over $f(b)$. If every such induced map is a k-equivalence, then the **Top**^{2}-morphism (g, f) is said to be a *k-equivalence of homotopy fibres*.

If $(g, f) : q \to p$ is a **Top**^{2}-morphism, note also that there is an obvious map $r : Z_g \to Z_f$ from the mapping cylinder of g into the mapping cylinder of f.

(A)
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ B \arrow[r, "f"'] & Y \end{tikzcd}$$

Proposition 3. *Consider the commutative Diagram (A).*

(a) *The following conditions are equivalent.*

(1) The map $r : (Z_g, A) \to (Z_f, B)$ of mapping cylinders is a k-equivalence.

(2) The **Top**^{2}*-morphism $(g, f) : q \to p$ is a $(k-1)$-equivalence of homotopy fibres.*

(b) *If the maps g and f are inclusions of subspaces, then the following condition, (3), is equivalent to condition (2) in (a) above:*

(3) The map $p : (X, A) \to (Y, B)$ is a k-equivalence.

(c) *Suppose that g and f are inclusion maps, and that $A \to B$ is a k-equivalence.*

Then $p : (X, A) \to (Y, B)$ is a k-equivalence if and only if $X \to Y$ is a k-equivalence.

Proof: (c) This follows easily by the five-lemma applied to the ladder formed by the homotopy sequences of the pairs (X, A) and (Y, B), together with the homomorphisms arising from the map $(X, A) \to (Y, B)$.

(b) We consider the mapping path fibration factorization of p and q, respectively.

$$X \xrightarrow{p_0} W \xrightarrow{p_1} Y \qquad A \xrightarrow{q_0} E \xrightarrow{q_1} B.$$

In each case it is a homotopy equivalence followed by a fibration. The maps p_0 and q_0 are embeddings and we regard them as inclusions. Let

$F = q_1^{-1}(*)$, $F' = p_1^{-1}(*)$ and $E' = p_1^{-1}(B)$. Then $E \subset E'$, $F \subset F'$, and $q_1(x) = p_1(x)$ for every $x \in E$.

We first assume that (3) holds. Then $p_1 : (W, E) \to (Y, B)$ is a k-equivalence. Since p_1 is a fibration, the map $p_1 : (W, E') \to (Y, B)$ is a weak equivalence. Thus the injection $(W, E) \to (W, E')$ is a k-equivalence. Similar to the argument for (c), it follows that the inclusion $E \to E'$ is a $(k-1)$-equivalence.

The pull-back of p_1 over the inclusion $B \subset Y$, is (a fibration and thus) a weak equivalence $(E', F') \to (B, *)$. Similarly, $q_1 : (E, F) \to (B, *)$ is a weak equivalence. Thus the injection $(E, F) \to (E', F')$ is a weak equivalence. Since the inclusion $E \subset E'$ is a $(k-1)$-equivalence, it follows that also the inclusion $F \subset F'$ is a $(k-1)$-equivalence. Therefore condition (2) of (a) follows. This proves one of the implications claimed in (b). The other implication claimed in (b) can be proved by simply reversing the previous argument.

(a) This follows by (b). ■

The theorem which we quote without proof below, appears (in a stronger form) in a paper by May [**10**, Theorem 1.2], and results from a reworking of the fundamental theory of quasifibrations in the pioneering paper [**7**] by Dold and Thom.

Theorem 4. *Let Y be a space with open subspaces B_1 and B_2 such that $Y = B_1 \cup B_2$. Let $B_0 = B_1 \cap B_2$. Let $p : X \to Y$ be a map and let $A_i = p^{-1}(B_i)$, $i = 0, 1, 2$.*

If for each $j = 1, 2$, the map $(A_j, A_0) \to (B_j, B_0)$ is a k-equivalence, then for each $j = 1, 2$, the map $(X, A_j) \to (Y, B_j)$ is a k-equivalence.

From Theorem 4 we deduce an adjunction theorem, Theorem 7, generalizing the adjunction theorem for quasifibrations [**9**, Theorem 0.2] of Hardie. We consider the commutative Diagram (B) in **Top** to be a cotriad in the category **Top**^{2}.

(B)
$$\begin{tikzcd}
E_1 \arrow[d, "p_1"'] & E_0 \arrow[l, "g_1"'] \arrow[r, "g_2"] \arrow[d, "p_0"] & E_2 \arrow[d, "p_2"] \\
B_1 & B_0 \arrow[l, "f_1"] \arrow[r, "f_2"'] & B_2
\end{tikzcd}$$

The push-out of this **Top**^{2}-cotriad is a map $p : E \to B$, where E and B are the spaces obtained as the push-outs (in **Top**) of the cotriads sitting in the top row and bottom row of Diagram (B). There is a similar map of double mapping cylinders. We denote this map by $p' : E' \to B'$, and refer to it as the double mapping cylinder of the **Top**^{2}-cotriad. Firstly we note the following fact regarding the natural map $B' \to B$ between the double mapping cylinder and the push-out of a **Top**-cotriad,

(C)
$$B_1 \xleftarrow{f_1} B_0 \xrightarrow{f_2} B_2.$$

Proposition 5 is a weaker form of the result [**2**, 7.5.4 on p. 275] in the book of Brown. A proof in an axiomatic setting appears in Baues's book [**1**]. We state it without proof.

Proposition 5. *If in the* **Top***-cotriad of Diagram (C), f_1 is a cofibration, then the natural map $B' \to B$ is a homotopy equivalence.*

The homotopy fibres approach to the study of **Top**^{2}-cotriads can be observed in the work of Puppe [**13**]. Theorem 6 supplements and generalizes Puppe's work.

Theorem 6. *Suppose that in Diagram (B), for each $j = 1, 2$, the* **Top**^{2}*-morphism (g_j, f_j) is a k-equivalence of homotopy fibres, and g_1 and f_1 are cofibrations.*

Then for each $j = 1, 2$, the **Top**^{2}*-morphism $p_i \to p'$ to the map of double mapping cylinders is a k-equivalence of homotopy fibres.*

Proof: The double mapping cylinder B' has open subsets V_0, V_1 and V_2 satisfying the following three conditions.

(1) $V_0 = V_1 \cap V_2$ and $V_1 \cup V_2 = B'$.
(2) For the pull-back $q_i : U_i \to V_i$ of p' over the inclusion $V_i \subset B'$, there are homotopy equivalences $\epsilon_i : U_i \to E_i$ and $\beta_i : V_i \to B_i$, such that for each $i = 0, 1, 2$, $p_i \circ \epsilon_i = \beta_i \circ q_i$.
(3) The **Top**^{2}-morphisms (ϵ_i, β_i), resulting from (2) above, fits into a commutative diagram in **Top**^{2} as shown below.

(D)
$$\begin{tikzcd}
q_1 \arrow[d, "{(\epsilon_1, B_1)}"'] & q_0 \arrow[l] \arrow[r] \arrow[d, "{(\epsilon_0, \beta_0)}"'] & q_2 \arrow[d, "{(\epsilon_2, B_2)}"'] \\
p_1 & p_0 \arrow[l] \arrow[r] & p_2
\end{tikzcd}$$

Due to the homotopy equivalences of (2) and the conditions of the theorem, it follows that for each $j = 1, 2$, the **Top**^{2}-morphism $q_0 \to q_j$ is a k-equivalence of homotopy fibres. By Proposition 3(a) then, each map $(U_j, U_0) \to (V_j, V_0)$ is a $(k+1)$-equivalence. Thus by Theorem 4, each map $(E', U_j) \to (B', V_j)$ is a $(k+1)$-equivalence. Again by Proposition 3(a), each **Top**^{2}-morphism $q_j \to p'$ is a k-equivalence of homotopy fibres. ■

Theorem 7. *Suppose that for the commutative Diagram (B) we have a subset T of B_0 such that the inclusion $T \subset B_0$ is a surjection of path components.*

Suppose that for each $x \in T$, there are subsets $F_0^x \subset {p_0}^{-1}(x)$ and $F_j^x \subset {p_j}^{-1}(f_j(x))$. We assume that $F_j^x = F_j^y$ whenever $f_j(x) = f_j(y)$. It is also assumed that for each $j = 1, 2$, $g_j(F_0^x) \subset F_j^x$ so that there is an induced map $h_j^x : F_0^x \to F_j^x$.

Suppose further that for each $x \in S$, the following conditions hold ($k \geq 0$):

(1) *$p_0 : (E_0, F_0^x) \to (B_0, x)$ is a k-equivalence,*
(2) *$p_j : (E_j, F_j^x) \to (B_j, f_j(x))$ is a $(k+1)$-equivalence for each $j = 1, 2$,*
(3) *$h_j^x : F_0^x \to F_j^x$ is a k-equivalence for each $j = 1, 2$.*

Then for each $j = 1, 2$, the map of double mapping cylinders is a $(k+1)$-equivalence, $p' : (E', E_j) \to (B', B_j)$.

Proof: Fix any $x \in S$ and $j \in \{1, 2\}$. In Diagram (E) below, H_0^x and H_j^x are, respectively, the fibres of the mapping path fibration of p_0 and p_j over x and $f_j(x)$. The vertical arrows are inclusions. The map β is the induced map due to functoriality of the mapping path fibration construction.

(E)
$$\begin{tikzcd} F_0^x \arrow[r, "h_j^x"] \arrow[d, "\alpha"'] & F_j^x \arrow[d, "\alpha'"'] \\ H_0^x \arrow[r, "\beta"'] & H_j^x \end{tikzcd}$$

Due to condition (1), α is a $(k-1)$-equivalence and due to (2), α' is a k-equivalence. In view of condition (3), it follows that $\alpha' \circ h_j^x$ is a k-equivalence. We put $\delta = \alpha' \circ h_j^x$ and apply Proposition 2, by which β is a k-equivalence. Since $T \subset B_0$ is a surjection of path components, it follows that the $\mathbf{Top}^2$-morphism (g_j, f_j) is a k-equivalence of homotopy fibres (for both values of j). Thus by Theorem 6, for both values of j, the $\mathbf{Top}^2$-morphism $p_j \to p'$ is a k-equivalence of homotopy fibres. Our result follows from Proposition 3(b). ■

Corollary 8. *We assume the conditions of Theorem 7 together with the requirement that f_1 and g_1 are cofibrations.*

Then for each $j = 1, 2$, the map of push-outs is a $(k+1)$-equivalence $p : (E, E_j) \to (B, B_j)$.

Proof: We deduce this result from Theorem 7 as follows. By Proposition 5, the canonical map $E' \to E$ is a homotopy equivalence, and hence a weak equivalence. By the five-lemma applied to the ladder formed by the homotopy sequences of the pairs (E', E_j) and (E, E_j) and the homomorphisms arising from the map $(E', E_j) \to (E, E_j)$, it follows that the map of pairs is a weak equivalence.

Similarly, $(B', B_j) \to (B, B_j)$ is a weak equivalence. The assertion now follows from Theorem 7. ■

PART II: Relative homeomorphisms.

In the sequel, we assume spaces to have a base point, denoted by the symbol $*$. The definition of k-equivalence remains as for free spaces.

Definition 9. Let $p : (X, A) \to (Y, B)$ be a map of pairs of spaces. Then p induces maps p_1 and p_2 as in Diagram (F). The map of pairs p is said to be a *relative homeomorphism* if Diagram (F) is a push-out square.

(F)
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "p_1"'] & X \arrow[d, "p_2"] \\ B \arrow[r, hook] & Y \end{tikzcd}$$

Proposition 10. *Suppose that $f : V \to A$ is a map with mapping cone X, and $p : (X, A) \to (Y, B)$ is a relative homeomorphism. Suppose further that $p_0 : A \to B$ is a k-equivalence $(k > 0)$.*

Then each of the maps $(X, A) \to (Y, B)$ and $X \to Y$ is a k-equivalence.

Proof: In Diagram (G) below, q is the identity map and thus the diagram is commutative. The map of double mapping cylinders of this $\mathbf{Top}^2$-cotriad is precisely our map p.

(G)
$$\begin{tikzcd} * \arrow[d, "q'"'] & V \arrow[l] \arrow[r, "f"] \arrow[d, "q"] & A \arrow[d, "p_0"'] \\ * & V \arrow[l] \arrow[r, "p_0 \circ f"'] & B \end{tikzcd}$$

The $\mathbf{Top}^2$-morphism $q \to q'$ is a weak equivalence of homotopy fibres. Since the homotopy fibres of p_0 are $(k-1)$-connected, $q \to p_0$ is a $(k-1)$-equivalence of homotopy fibres. Thus by Theorem 7, the $\mathbf{Top}^2$-morphism $p_0 \to p$ is a $(k-1)$-equivalence of homotopy fibres. From Proposition 3(b), it follows that $(X, A) \to (Y, B)$ is a k-equivalence, and then from Proposition 3(c), $X \to Y$ is a k-equivalence. ■

Lemma 11. *Suppose that G is a subspace of a space F such that the inclusion $G \subset F$ is an m-equivalence ($m > 0$). Let W_n be a bouquet of n-dimensional spheres, and let C_n be the subset $F \times * \cup G \times W_n$ of $F \times W_n$.*

Then the restriction of the projection map $F \times W_n \to W_n$, is an $(m+n)$-equivalence of pairs $r_n : (C_n, F \times *) \to (W_n, *)$.

Proof: We proceed by induction on n. The case $n = 0$ is obviously true. Now let us assume the statement to be true for all n such that $0 \leq n \leq t-1$, where $t \geq 1$, and show that it is also true for $n = t$.

We apply Corollary 8 to Diagram (H) below. V is the corresponding bouquet of the cones of the spheres of W_{t-1}. The map q is the restriction of the projection map $F \times W_{t-1} \to W_{t-1}$, and h is the restriction of the projection map $F \times W_{t-1} \to F$.

$$\text{(H)} \qquad \begin{tikzcd} F \times * \cup G \times V \arrow[d, "q"'] & F \times * \cup G \times W_{t-1} \arrow[l, hook'] \arrow[r, "h"] \arrow[d, "\alpha"'] & F \arrow[d] \\ V & W_{t-1} \arrow[l, hook'] \arrow[r] & * \end{tikzcd}$$

The diagram is commutative and the push-out of the **Top**^{2}-cotriad is a map of the form $r_t : C_t \to W_t$. Note that there is a one to one correspondence between the spheres in W_{t-1} and those in W_t.

In accordance with Theorem 7, we must choose a set $T \subset W_{t-1}$. If $t = 1$, then we choose $T = W_{t-1}$, otherwise we choose $T = \{*\}$. The subsets F_i^x required in Theorem 7 are chosen to be the complete inverse images (of the relevant point with respect to the relevant vertical arrow in Diagram (H)). For the horizontal arrows pointing to the left, every induced map between fibres is a homeomorphism. For the horizontal arrows pointing to the right, the induced map between fibres over $*$ is a homeomorphism, and otherwise (only relevant in the case $t = 1$) it is an m-equivalence. So, the conditions of Theorem 7 can be seen to be fulfilled for $k = m + t - 2$, using the induction assumption. Thus the push-out of the **Top**^{2}-cotriad is a $(m + t - 1)$-equivalence. This completes the induction and hence the proof of the lemma. ■

Proposition 12. *Let $p_0 : A \to B$ be an m-equivalence ($m > 1$). Let (X, A) be a relative CW-complex having cells of dimension n only, and let $p : (X, A) \to (Y, B)$ be a relative homeomorphism.*

Then $p : (X, A) \to (Y, B)$ is an $(m + n - 1)$-equivalence.

Proof: We first prove the result assuming that $p_0 : A \to B$ is a fibration, and thereafter we deduce the general case.

So let us assume that $p_0 : A \to B$ is a fibration. Then the fibres of p_0 are $(m-1)$-connected. Let $g : W \to A$ be the attaching map for the cells of X, where W is a bouquet of $(n-1)$-spheres. The cone on W is denoted by V. Let $F = p_0^{-1}(*)$ be the fibre of p_0 over the base point of B. In Diagram (I), q and q' are relative homeomorphisms which collapse the subspace F. The map g' has restrictions $g'|W = g$ and $g'|F$ is a homeomorphism onto the subspace F of A. Thus Diagram (I) is commutative.

(I)
$$
\begin{tikzcd}
F \vee V \arrow[d, "q'"'] & F \vee W \arrow[l, hook'] \arrow[r, "g'"] \arrow[d, "q"'] & A \arrow[d, "p_0"'] \\
V & W \arrow[l, hook'] \arrow[r, "g \circ p_0"'] & B
\end{tikzcd}
$$

By Lemma 11, $q : (W \vee F, F) \to (W, *)$ is an $(m+n-2)$-equivalence (we choose the space G required in Lemma 11 to be the one-point set $\{*\}$). Furthermore, we have weak equivalences $q' : (F \vee V, F) \to (V, *)$, and $p_0 : (A, F) \to (B, *)$. By Corollary 8 it follows that $p : (X, A) \to (Y, B)$ is an $(m+n-1)$-equivalence, and the special case is proved.

We now turn to the general case. For the mapping path fibration factorization of p_0 below, the inclusion is a homotopy equivalence and f is a fibration.

$$
A \hookrightarrow E \xrightarrow{f} B
$$

This factorization induces relative homeomorphisms between relative CW-complexes,

$$
(X, A) \xrightarrow{h} (Z, E) \xrightarrow{f} (Y, B),
$$

and the composition of the two maps coincides with p. By Proposition 10, the map $h : (X, A) \to (Z, E)$ is a weak equivalence. From the special case of Proposition 12 that we have already proved, it follows that $f : (Z, E) \to (Y, B)$ is an $(n+m-1)$-equivalence. Thus $f \circ h : (X, A) \to (Y, B)$ is an $(m+n-1)$-equivalence. ■

Theorem 13. *Let $p_0 : A \to B$ be an m-equivalence, $m > 0$, and (X, A) a relative CW-complex having only cells of dimension n and higher.*

Then the relative homeomorphism $p : (X, A) \to (Y, B)$ is an $(m+n-1)$-equivalence.

Proof: By direct limit considerations, it suffices to prove that for every $r \geq n$, the induced map $p_r : (X_r, A) \to (Y_r, B)$ on relative r-skeleta is an $(n+m-1)$-equivalence. We do it by induction on r. For $r = n$ the statement follows by Proposition 12.

Let t be an integer such that our statement is true for every r such that $n \leq r \leq t-1$. Then, in particular, $p_{t-1} : (X_{t-1}, A) \to (Y_{t-1}, B)$ is an $(n+m-1)$-equivalence. Note that by repeated application of Proposition 10, it follows that $X_{t-1} \to Y_{t-1}$ is an m-equivalence, for every r. Thus by Proposition 12, $(X_t, X_{t-1}) \to (Y_t, Y_{t-1})$ is an $(t+m-1)$-equivalence, $t > n$. By the five-lemma applied to the ladder of which a portion is shown in Diagram (J), it follows that $(X_t, A) \to (Y_t, B)$ is an $(n+m-1)$-equivalence.

$$\text{(J)} \quad \begin{array}{ccccccc} \pi_k(X_{t-1}, A) & \longrightarrow & \pi_k(X_t, A) & \longrightarrow & \pi_k(X_t, X_{t-1}) & \longrightarrow & \pi_{k-1}(X_{t-1}, A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_k(Y_{t-1}, B) & \longrightarrow & \pi_k(Y_t, B) & \longrightarrow & \pi_k(Y_t, Y_{t-1}) & \longrightarrow & \pi_{k-1}(Y_{t-1}, B) \end{array}$$

This completes the induction and hence the proof of the theorem. ■

Acknowledgment. This article forms part of a doctoral thesis [**15**], prepared under supervision of Prof. K. A. Hardie at the University of Cape Town. The author is grateful to Professor Hardie for helpful advice with the preparation of this paper.

References

1. Baues, H. J., "*Algebraic Homotopy*," Cambridge University Press, 1989.
2. Brown, R., "*Topology: A geometric account of general Topology, homotopy types and the fundamental groupoid*," Ellis Horwood Limited, Chichester, West Sussex, England, 1988.
3. Brown, R. and Loday, J.-L., Homotopical excision and Hurewicz theorems for n-cubes of spaces, *Proc. London Math. Soc.* **45(3)** (1987), 176–192.
4. Blakers, A. L. and Massey, W. S., The homotopy groups of a triad II, *Ann. of Math.* **55** (1952), 192–201.
5. Blakers, A. L. and Massey, W. S., The homotopy groups of a triad III, *Ann. of Math.* **58** (1953), 409–417.

6. tom Dieck, T., Kamps, K. H. and Puppe, V., "*Homotopietheorie*," Lecture Notes in Mathematics **157**, Springer-Verlag, Berlin, 1970.
7. Dold, A. and Thom, R., Quasifaserungen und unendliche Symmetrische Produkte, *Ann. of Math.* **67** (1958), 239–281.
8. Gray, B., "*Homotopy Theory: An Introduction to Algebraic Topology*," Academic Press, New York, 1975.
9. Hardie, K. A., Quasifibration and Adjunction., *Pacific J. Math.* **35** (1970), 389–397.
10. May, J. P., Weak Equivalence and Quasifibration, in "*Groups of self-equivalences and related topics*," Piccinini, R. (Ed.), Lecture Notes in Mathematics **1425**, Springer-Verlag, Berlin, 1990.
11. Moore, J. C., Some Applications of Homology Theory to Homotopy Theory, *Ann. of Math.* **58** (1953), 325–350.
12. Namioka, I., Maps of Pairs in Homotopy Theory, *Proc. London Math. Soc.* **12(3)** (1962), 725–738.
13. Puppe, V., A remark on homotopy fibrations, *Manuskripta Math.* **12** (1974), 113–130.
14. Switzer, R. M., "*Algebraic Topology - Homology and Homotopy*," Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen - Band 212, Springer-Verlag, Berlin and New York, 1975.
15. Witbooi, P. J., A contribution to the foundations of the theory of quasifibration, Doctoral dissertation, University of Cape Town, Cape Town, South Africa, 1995.

1991 *Mathematics subject classifications*: 55Q30, 55R65

Department of Mathematics and Applied Mathematics
University of the Western Cape
Private Bag X17
Bellville 7535
SOUTH AFRICA

e-mail: pwitbooi@math.uwc.ac.za

Rebut el 26 d'Abril de 1995