THE ETA INVARIANT AND EQUIVARIANT INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS

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ABSTRACT. We prove a formula for the multiplicities of the index of an equivariant transversally elliptic operator on a G-manifold. The formula is a sum of integrals over blowups of the strata of the group action and also involves eta invariants of associated elliptic operators. Among the applications, we obtain an index formula for basic Dirac operators on Riemannian foliations, a problem that was open for many years.

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1. Introduction

Suppose that a compact Lie group G acts by isometries on a compact, connected Riemannian manifold M, and let $E = E^+ \oplus E^-$ be a graded, G-equivariant Hermitian vector bundle over M. We consider a first order G-equivariant differential operator $D = D^+ \colon \Gamma(M, E^+) \to \Gamma(M, E^-)$ that is transversally elliptic, and let D^- be the formal adjoint of D^+.

The group G acts on $\Gamma(M, E^\pm)$ by $(gs)(x) = g \cdot s(g^{-1}x)$, and the (possibly infinite-dimensional) subspaces $\ker(D^+)$ and $\ker(D^-)$ are G-invariant subspaces. Let $\rho \colon G \to U(V_\rho)$ be an irreducible unitary representation of G, and let $\chi_\rho = \operatorname{tr}(\rho)$ denote its character. Let $\Gamma(M, E^\pm)^\rho$ be the subspace of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma(M, E^\pm)$ that are unitarily equivalent to the representation ρ. It can be shown that the extended operators

$$\overline{D}_{\rho,s} \colon H^s\left(\Gamma\left(M, E^+\right)^\rho\right) \to H^{s-1}\left(\Gamma\left(M, E^-\right)^\rho\right)$$

are Fredholm and independent of s, so that each irreducible representation of G appears with finite multiplicity in $\ker D^\pm$ (see Corollary 3.3). Let $a_\rho^\pm \in \mathbb{Z}_{\geq 0}$ be the multiplicity of ρ in $\ker(D^\pm)$.

The study of index theory for such transversally elliptic operators was initiated by M. Atiyah in the early 1970s ([2]). The virtual representation-valued index of D is given by

$$\operatorname{ind}^G (D) := \sum_\rho \left(a_\rho^+ - a_\rho^- \right) [\rho],$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ. The index multiplicity is

$$\operatorname{ind}^\rho (D) := a_\rho^+ - a_\rho^- = \frac{1}{\dim V_\rho} \operatorname{ind} \left(D|_{\Gamma(M,E^+)^\rho \to \Gamma(M,E^-)^\rho} \right).$$

In particular, if ρ_0 is the trivial representation of G, then

$$\operatorname{ind}^{\rho_0} (D) = \operatorname{ind} \left(D|_{\Gamma(M,E^+)^G \to \Gamma(M,E^-)^G} \right),$$

where the superscript G implies restriction to G-invariant sections.

There is a clear relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\operatorname{ind}_g (D)$; the multiplicities determine the distributional index, and vice versa. Let $\{X_1, \ldots, X_r\}$ be an orthonormal basis of the Lie algebra of G. Let $\mathcal{L}_{X_j}$ denote the induced Lie derivative with respect to X_j on sections of E, and let $C = \sum_j \mathcal{L}_{X_j}^* \mathcal{L}_{X_j}$ be the Casimir operator on sections of E. The space $\Gamma(M, E^\pm)^\rho$ is a subspace of the λ_ρ-eigenspace of C. The virtual character $\operatorname{ind}_g (D)$ is given by (see [2])

$$\operatorname{ind}_g (D) := \text{``} \operatorname{tr} \left(g|_{\ker D^+} \right) - \operatorname{tr} \left(g|_{\ker D^-} \right) \text{''}$$

$$= \sum_\rho \operatorname{ind}^\rho (D) \chi_\rho (g).$$

Note that the sum above does not in general converge, since $\ker D^+$ and $\ker D^-$ are in general infinite-dimensional, but it does make sense as a distribution on G. That is, if dg is the normalized, biinvariant Haar measure on G, and if $\phi = \beta + \sum c_\rho \chi_\rho \in C^\infty (G)$, with β orthogonal to the subspace of class functions on G, then

$$\operatorname{ind}_* (D) (\phi) = \text{``} \int_G \phi (g) \; \overline{\operatorname{ind}_g (D)} \; dg \text{''}$$

$$= \sum_\rho \operatorname{ind}^\rho (D) \int \phi (g) \; \overline{\chi_\rho (g)} \; dg = \sum_\rho \operatorname{ind}^\rho (D) c_\rho,$$

an expression which converges because c_ρ is rapidly decreasing and $\operatorname{ind}^\rho (D)$ grows at most polynomially as ρ varies over the irreducible representations of G. From this calculation, we see that the multiplicities determine Atiyah's distributional index. Conversely, let $\alpha \colon G \to U (V_\alpha)$ be an irreducible unitary representation. Then

$$\operatorname{ind}_* (D) (\chi_\alpha) = \sum_\rho \operatorname{ind}^\rho (D) \int \chi_\alpha (g) \overline{\chi_\rho (g)} \, dg = \operatorname{ind}^\alpha D,$$

so that complete knowledge of the equivariant distributional index is equivalent to knowing all of the multiplicities $\operatorname{ind}^{\rho}(D)$. Because the operator $D|_{\Gamma(M,E^+)^{\rho} \to \Gamma(M,E^-)^{\rho}}$ is Fredholm, all of the indices $\operatorname{ind}^{G}(D)$, $\operatorname{ind}_{g}(D)$, and $\operatorname{ind}^{\rho}(D)$ depend only on the stable homotopy class of the principal transverse symbol of D.

Let us now consider the heat kernel expression for the index multiplicities. The usual McKean-Singer argument shows that, in particular, for every $t > 0$, the index $\operatorname{ind}^{\rho}(D)$ may be expressed as the following iterated integral:

$$\operatorname{ind}^{\rho}(D) = \int_{x \in M} \int_{g \in G} \operatorname{str} g \cdot K\left(t, g^{-1}x, x\right) \overline{\chi_{\rho}(g)} \, dg \, |dx|$$

(1.1)

$$= \int_{x \in M} \int_{g \in G} \left(\operatorname{tr} g \cdot K^+\left(t, g^{-1}x, x\right) - \operatorname{tr} g \cdot K^-\left(t, g^{-1}x, x\right)\right) \overline{\chi_{\rho}(g)} \, dg \, |dx|$$

where $K^{\pm}(t, \cdot, \cdot) \in \Gamma\left(M \times M, E^{\mp} \boxtimes (E^{\pm})^*\right)$ is the kernel for $e^{-t\left(D^{\mp}D^{\pm}+C-\lambda_{\rho}\right)}$ on $\Gamma(M, E^{\pm})$, letting $|dx|$ denote the Riemannian density over M.

A priori, the integral above is singular near sets of the form

$$\bigcup_{G_x \in [H]} x \times G_x \subset M \times G,$$

where the isotropy subgroup G_x is the subgroup of G that fixes $x \in M$, and $[H]$ is a conjugacy class of isotropy subgroups.

A large body of work over the last twenty years has yielded theorems that express $\operatorname{ind}_g(D)$ and $\int_M \left(\operatorname{tr} g \cdot K^+\left(t, g^{-1}x, x\right) - \operatorname{tr} g \cdot K^-\left(t, g^{-1}x, x\right)\right) |dx|$ in terms of topological and geometric quantities, as in the Atiyah-Segal-Singer index theorem for elliptic operators [5] or the Berline-Vergne Theorem for transversally elliptic operators [9],[10]. However, until now there has been very little known about the problem of expressing $\operatorname{ind}^{\rho}(D)$ in terms of topological or geometric quantities which are determined at the different strata of the G-manifold M. The special case when all of the isotropy groups are the same dimension was solved by M. Atiyah in [2], and this result was utilized by T. Kawasaki to prove the Orbifold Index Theorem (see [32]). Our analysis is new in that the integral over the group in (1.1) is performed first, before integration over the manifold, and thus the invariants in our index theorem are very different from those seen in other equivariant index formulas. Theorem 9.2 gives a formula for the Fourier coefficients of the virtual character instead of the value of the character at a particular $g \in G$.

Our main theorem (Theorem 9.2) expresses $\operatorname{ind}^{\rho}(D)$ as a sum of integrals over the different strata of the action of G on M, and it involves the eta invariant of associated equivariant elliptic operators on spheres normal to the strata. The result is

$$\operatorname{ind}^\rho(D) = \int_{G \backslash \widetilde{M_0}} A_0^\rho(x) \ \widetilde{|dx|} \ + \sum_j \beta\left(\Sigma_{\alpha_j}\right) \ ,$$

$$\beta\left(\Sigma_{\alpha_j}\right) = \frac{1}{2 \dim V_\rho} \sum_j \sum_{b \in B} \frac{1}{n_b \operatorname{rank} W^b}\left(-\eta\left(D_j^{S+,b}\right)\right.$$

$$\left.+ h\left(D_j^{S+,b}\right)\right) \int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A_{j,b}^\rho(x) \ \widetilde{|dx|} \ .$$

(The notation will be explained later; the integrands $A_0^\rho(x)$ and $A_{j,b}^\rho(x)$ are the familar Atiyah-Singer integrands corresponding to local heat kernel supertraces of induced elliptic operators over closed manifolds.) Even in the case when the operator D is elliptic, this result was not known previously. Further, the formula above gives a method for computing eta invariants of Dirac-type operators on quotients of spheres by compact group actions; these have been computed previously only in some special cases. We emphasize that every part of the formula is explicitly computable from local information provided by the operator and manifold. Even the eta invariant of the operator $D_j^{S+,b}$ on a sphere is calculated directly from the principal symbol of the operator D at one point of a singular stratum. The de Rham operator provides an important example illustrating the computability of the formula, yielding a new theorem expressing the equivariant Euler characteristic in terms of ordinary Euler characteristics of the strata of the group action (Theorem 10.1).

One of the primary motivations for obtaining an explicit formula for $\operatorname{ind}^\rho(D)$ was to use it to produce a basic index theorem for Riemannian foliations, thereby solving a problem that has been open since the 1980s (it is mentioned, for example, in [18]). In fact, the basic index theorem is a consequence of the invariant index theorem (Theorem 9.6), corresponding to the trivial representation ρ_0. This theorem is stated in Section 10.4. We note that a recent paper of Gorokhovsky and Lott addresses this transverse index question on Riemannian foliations. Using a different technique, they are able to prove a formula for the basic index of a basic Dirac operator that is distinct from our formula, in the case where all the infinitessimal holonomy groups of the foliation are connected tori and if Molino's commuting sheaf is abelian and has trivial holonomy (see [22]). Our result requires at most mild topological assumptions on the transverse structure of the strata of the Riemannian foliation and has a similar form to the formula above for $\operatorname{ind}^{\rho_0}(D)$. In particular, the analogue for the Gauss-Bonnet Theorem for Riemannian foliations (Theorem 10.7) is a corollary and requires no assumptions on the structure of the Riemannian foliation.

There are several new techniques in this paper that have not been explored previously. First, the fact that $\operatorname{ind}^\rho(D)$ is invariant under G-equivariant homotopies is used in a very specific way, and we keep track of the effects of these homotopies so that the formula for the index reflects data coming from the original operator and manifold. In Section 7 we describe a process of blowing up, cutting,

and reassembling the G-manifold into what is called the desingularization, which also involves modifying the operator and vector bundles near the singular strata as well. The result is a G-manifold that has less intricate structure and for which the heat kernels are easier to evaluate. The key idea is to relate the local asymptotics of the equivariant heat kernel of the original manifold to the desingularized manifold; at this stage the eta invariant appears through a direct calculation on the normal bundle to the singular stratum. We note that our desingularization process and the equivariant index theorem were stated and announced in [41] and [42]; recently Albin and Melrose have taken it a step further in tracking the effects of the desingularization on equivariant cohomology and equivariant K-theory ([1]).

Another new idea in this paper is the decomposition of equivariant vector bundles over G-manifolds with one orbit type. A crucial step in the proof required the construction of a subbundle of an equivariant bundle over a G-invariant part of a stratum that is the minimal G-bundle decomposition that consists of direct sums of isotypical components of the bundle. We call this decomposition the *fine decomposition* and define it in Section 4.4. A more detailed account of this method will appear in [24].

There are certain assumptions on the operator D that are required to produce the formula in the main theorem, mainly that it has product-like structure near the singular strata after a G-equivariant homotopy. We note that most of the major examples of transversally elliptic differential operators have the required properties. In fact, in [39], a large variety of examples of naturally defined transversal operators similar to Dirac operators are explored and shown under most conditions to provide all possible index classes of equivariant transversally elliptic operators. These operators almost always satisfy the required product condition at the singular strata (see Section 7.4). Further, we note that when the basic index problem for transverse Dirac-type operators on Riemannian foliations is converted to an invariant index problem, the resulting operator on the G-manifold again satisfies the required assumptions, under mild topological conditions on the bundle and principal transverse symbol of the operator.

The outline of the paper is as follows. We first review the stratification of G-manifolds in Section 2 and establish the elementary properties of transversally elliptic operators in Section 3. In Section 4, we discuss equivariant analysis on manifolds with one orbit type and decompose equivariant bundles over them in various ways, including the fine decomposition mentioned above. The relevant properties of the supertrace of the equivariant heat kernel are discussed in Section 5. In Section 6.2, we compute the local contribution of the supertrace of a general constant coefficient equivariant heat operator in the neighborhood of a singular point of an orthogonal group action on a sphere. It is here that the equivariant index is related to a boundary value problem, which explains the presence of eta invariants in the main theorem. In Section 7, we describe the iterative process of desingularizing the G-manifold near a minimal stratum

and producing a double of the manifold with one less stratum. We apply the heat kernel analysis, representation theory, and fine decomposition to produce a heat kernel splitting formula in Section 8. This process leads to a reduction theorem for the equivariant heat supertrace (Theorem 9.1). The main theorem follows and is stated in Section 9.2. Several examples are discussed in Section 10; these examples show that all of the terms in the formula above are nontrivial. The basic index theorem for Riemannian foliations is stated and discussed in Section 10.4.

We note that other researchers have investigated transverse index theory of group actions and foliations in noncommutative geometry and topology; these papers answer different questions about the analysis and topology of certain groupoids (see [16]). A survey of index theory, primarily in the noncommutative geometry setting, can be found in [33].

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2. Stratifications of G-manifolds

In the following, we will describe some standard results from the theory of Lie group actions (see [12], [30]). Such G-manifolds are stratified spaces, and the stratification can be described explicitly. In the following discussion, G is a compact Lie group acting on a smooth, connected, closed manifold M. We assume that the action is effective, meaning that no $g \in G$ fixes all of M. (Otherwise, replace G with $G \diagup \{g \in G : gx = x \text{ for all } x \in M\}$.) Choose a Riemannian metric for which G acts by isometries; average the pullbacks of any fixed Riemannian metric over the group of diffeomorphisms to obtain such a metric.

Given such an action and $x \in M$, the isotropy or stabilizer subgroup $G_x < G$ is defined to be $\{g \in G : gx = x\}$. The orbit $\mathcal{O}_x$ of a point x is defined to be $\{gx : g \in G\}$. Since $G_{xg} = gG_xg^{-1}$, the conjugacy class of the isotropy subgroup of a point is fixed along an orbit.

On any such G-manifold, the conjugacy class of the isotropy subgroups along an orbit is called the **orbit type**. On any such G-manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types. Given subgroups H and K of G, we say that $[H] \leq [K]$ if H is conjugate to a subgroup of K, and we say $[H] < [K]$ if $[H] \leq [K]$ and $[H] \neq [K]$. We may enumerate the conjugacy classes of isotropy subgroups as $[G_0], \ldots, [G_r]$ such that $[G_i] \leq [G_j]$ implies that $i \leq j$. It is well-known that the union of the principal orbits (those with type $[G_0]$) form an open dense subset M_0 of the manifold M, and the other

orbits are called **singular**. As a consequence, every isotropy subgroup H satisfies $[G_0] \leq [H]$. Let M_j denote the set of points of M of orbit type $[G_j]$ for each j; the set M_j is called the **stratum** corresponding to $[G_j]$. If $[G_j] \leq [G_k]$, it follows that the closure of M_j contains the closure of M_k. A stratum M_j is called a **minimal stratum** if there does not exist a stratum M_k such that $[G_j] < [G_k]$ (equivalently, such that $\overline{M_k} \subsetneq \overline{M_j}$). It is known that each stratum is a G-invariant submanifold of M, and in fact a minimal stratum is a closed (but not necessarily connected) submanifold. Also, for each j, the submanifold $M_{\geq j} := \bigcup\limits_{[G_k] \geq [G_j]} M_k$ is a closed, G-invariant submanifold.

Now, given a proper, G-invariant submanifold S of M and $\varepsilon > 0$, let $T_\varepsilon(S)$ denote the union of the images of the exponential map at s for $s \in S$ restricted to the open ball of radius ε in the normal bundle at S. It follows that $T_\varepsilon(S)$ is also G-invariant. If M_j is a stratum and ε is sufficiently small, then all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$. This implies that if $j < k$, $\overline{M_j} \cap \overline{M_k} \neq \varnothing$, and $M_k \subsetneq \overline{M_j}$, then $\overline{M_j}$ and $\overline{M_k}$ intersect at right angles, and their intersection consists of more singular strata (with isotropy groups containing conjugates of both G_k and G_j).

Fix $\varepsilon > 0$. We now decompose M as a disjoint union of sets $M_0^\varepsilon, \ldots, M_r^\varepsilon$. If there is only one isotropy type on M, then $r = 0$, and we let $M_0^\varepsilon = \Sigma_0^\varepsilon = M_0 = M$. Otherwise, for $j = r, r-1, \ldots, 0$, let $\varepsilon_j = 2^j \varepsilon$, and let

$$(2.1) \qquad\qquad \Sigma_j^\varepsilon = M_j \setminus \overline{\bigcup_{k>j} M_k^\varepsilon}$$

$$(2.2) \qquad\qquad M_j^\varepsilon = T_{\varepsilon_j}(M_j) \setminus \overline{\bigcup_{k>j} M_k^\varepsilon},$$

Thus,

$$T_\varepsilon\left(\Sigma_j^\varepsilon\right) \subset M_j^\varepsilon, \ \Sigma_j^\varepsilon \subset M_j.$$

The following facts about this decomposition are contained in [30, pp. 203ff]:

Lemma 2.1. *For sufficiently small $\varepsilon > 0$, we have, for every $i \in \{0, \ldots, r\}$:*

(1) $M = \coprod\limits_{i=0}^{r} M_i^\varepsilon$ *(disjoint union).*
(2) M_i^ε *is a union of G-orbits; Σ_i^ε is a union of G-orbits.*
(3) *The manifold M_i^ε is diffeomorphic to the interior of a compact G-manifold with corners; the orbit space M_i^ε / G is a smooth manifold that is isometric to the interior of a triangulable, compact manifold with corners. The same is true for each Σ_i^ε.*
(4) *If $[G_j]$ is the isotropy type of an orbit in M_i^ε, then $j \leq i$ and $[G_j] \leq [G_i]$.*
(5) *The distance between the submanifold M_j and M_i^ε for $j > i$ is at least ε.*

Remark 2.2. The lemma above remains true if at each stage $T_\varepsilon(M_j)$ is replaced by any sufficiently small open neighborhood of M_j that contains $T_\varepsilon(M_j)$, that is a union of G-orbits, and whose closure is a manifold with corners.

Let γ be a geodesic orthogonal to M_j through $w \in M_j$. This situation occurs exactly when this geodesic is orthogonal both to the fixed point set M^{G_j} of G_j and to the orbit $\mathcal{O}_w$ of G containing w. For any $h \in G_j$, right multiplication by h maps geodesics orthogonal to M^{G_j} through w to themselves and likewise maps geodesics orthogonal to $\mathcal{O}_w$ through w to themselves. Thus, the group G_j acts orthogonally on the normal space to $w \in M_j$ by the differential of the left action. Observe in addition that there are no fixed points for this action; that is, there is no element of the normal space that is fixed by every $h \in G_j$. Since G_j acts without fixed points, the codimension of M_j is at least two if G_j preserves orientation.

3. Properties of transversally elliptic operators

Let G be a compact Lie group, M be a compact, Riemannian G-manifold, and $E \to M$ a G-equivariant Hermitian vector bundle. For the following constructions, we refer to [13]. The unitary G-representation on $L^2(M, E)$ admits an orthogonal Hilbert sum decomposition with respect to equivalence classes $[\rho]$ of irreducible representations $\rho \colon G \to U(V_\rho)$:

$$(3.1) \qquad L^2(M, E) \cong \bigoplus_{[\rho] \in \widehat{G}} L^2(M, E)_\rho \otimes_{\mathbb{C}} V_\rho \,.$$

The component $L^2(M, E)_\rho$ of type $[\rho]$ is given by

$$L^2(M, E)_\rho = \operatorname{Hom}_G\left(V_\rho, L^2(M, E)\right) \cong L^2\left(M, E \otimes_{\mathbb{C}} V_\rho^*\right)^G$$

and (3.1) is determined up to unitary equivalence by the inclusions (evaluation maps)

$$(3.2) \qquad i_\rho : L^2(M, E)_\rho \otimes_{\mathbb{C}} V_\rho \to L^2(M, E)\,.$$

We define $L^2(M, E)^\rho \subset L^2(M, E)$ to be the image $i_\rho\left(L^2(M, E)_\rho \otimes_{\mathbb{C}} V_\rho\right)$ and $\Gamma(M, E)^\rho = \Gamma(M, E) \cap L^2(M, E)^\rho$.

The orthogonal projections

$$P_\rho \colon L^2(M, E) \to L^2(M, E)^\rho$$

are given by integration over G:

$$L^2(M, E) \overset{\otimes I_\rho}{\to} L^2(M, E) \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(V_\rho) \cong L^2\left(M, E \otimes_{\mathbb{C}} V_\rho^*\right) \otimes_{\mathbb{C}} V_\rho$$

$$\overset{\int_G \otimes I_\rho}{\to} L^2\left(M, E \otimes_{\mathbb{C}} V_\rho^*\right)^G \otimes_{\mathbb{C}} V_\rho \overset{i_\rho}{\to} L^2(M, E)^\rho$$

Lemma 3.1. *Let ρ be an irreducible unitary representation of a compact Lie group G on a complex vector space V_ρ. Let $\chi_\rho(\cdot) = \operatorname{tr}\rho(\cdot)$ be the character of ρ, and let $\alpha \colon G \to U(Q)$ be any other representation on a Hermitian vector space Q. Let dh denote the volume form induced by the biinvariant, normalized metric on G. Let $P_\rho^\alpha \colon Q \to Q^\rho$ denote the projection onto the the subspace of Q on which the restriction of α is of type ρ. Then the following equation of endomorphisms of Q holds:*

$$\int_G \overline{\chi_\rho}(h)\,\alpha(h)\ dh = \frac{1}{\dim V_\rho}P_\rho^\alpha,$$

Proof. If F is a class function on G, then for any element $g \in G$,

$$\begin{aligned}
\alpha(g)\int_G F(h)\,\alpha(h)\ dh &= \int_G F(h)\,\alpha(g)\,\alpha(h)\ dh \\
&= \int_G F(h)\,\alpha\left(ghg^{-1}\right)\ dh\ \alpha(g),\ \text{and letting}\ \widetilde{h} = ghg^{-1} \\
&= \int_G F\left(g^{-1}\widetilde{h}g\right)\alpha\left(\widetilde{h}\right)\ d\widetilde{h}\ \alpha(g)\ \text{since}\ dh\ \text{is biinvariant} \\
&= \int_G F\left(\widetilde{h}\right)\alpha\left(\widetilde{h}\right)\ d\widetilde{h}\ \alpha(g) = \int_G F(h)\,\alpha(h)\ dh\ \alpha(g),
\end{aligned}$$

so $\int_G F(h)\,\alpha(h)\ dh$ commutes with $\alpha(g)$ for all $g \in H$. One can see easily that this remains true if we restrict the endomorphisms to an irreducible component Q^β of type β. Schur's Lemma implies that on this component $\int_G F(h)\,\alpha(h)\ dh$ is a constant multiple of the identity I_β. We evaluate the constant by taking traces. In particular, on this component,

$$\int_G \overline{\chi_\rho}(h)\ \alpha(h)\ dh = \int_G \overline{\chi_\rho}(h)\ \beta(h)\ dh = cI_\beta\ \text{implies}$$

$$\int_G \overline{\chi_\rho}(h)\,\chi_\beta(h)\ dh = c\ \dim V_\rho\ ,\ \text{so}$$

$$c = \begin{cases} \frac{1}{\dim V_\rho} & \text{if}\ \beta = \rho \\ 0 & \text{otherwise}. \end{cases} \qquad \square$$

Given a group representation $\theta \colon G \to U(V_\theta)$ and an irreducible representation $\rho \colon G \to U(V_\tau)$ of the compact Lie group G, the *multiplicity space* of ρ in θ is

$$\operatorname{Hom}_G(V_\rho, V_\theta),$$

and the *multiplicity* of ρ in θ is

$$\dim \operatorname{Hom}_G(V_\rho, V_\theta).$$

Given a subgroup H of G, and a representation $\sigma \colon H \to U(W_\sigma)$, we may form the homogeneous vector bundle $G \times_H W_\sigma \to G/H$. The space

$$\operatorname{Ind}(\sigma) = L^2(G/H, G \times_H W_\sigma)$$

of sections is a representation space for G, called the *representation of G induced by σ*, or simply the *induced representation*. By the Frobenius reciprocity theorem, Ind is the right adjoint functor to Res, the restriction functor. That is, given a representation $\tau\colon G \to U(V_\tau)$, $V_{\mathrm{Res}(\tau)}$ is the representation space V_τ with H-representation $\mathrm{Res}(\tau) = \tau|_H$, and the multiplicity space of τ in Ind (σ) satisfies

$$\mathrm{Hom}_G\left(V_\tau, \mathrm{Ind}(\sigma)\right) \cong L^2\left(G/H, (G \times_H W_\sigma) \otimes_{\mathbb{C}} V_\tau^*\right)^G$$
$$\cong \mathrm{Hom}_H\left(V_{\mathrm{Res}(\tau)}, W_\sigma\right)$$

for every representation $\sigma\colon H \to U(W_\sigma)$. This follows from the fact that a section s of the homogeneous vector bundle $G \times_H W_\sigma$ is given by an H-equivariant function $f_s\colon G \to W_\sigma$, and f_s is a G-invariant section if and only if f_s is a constant in W_σ^H.

Let D be a G-equivariant, symmetric differential operator of order k on $\Gamma(M, E)$, which is transversally elliptic with respect to the G-action (see [2]). This means that the principal symbol $\sigma_k(D)(\xi)$ of D is invertible on all nonzero

$$\xi \in T_G^*(M)_x = \{\xi \in T_x^*(M) \mid \xi(X) = 0$$
$$\text{for every } X \text{ tangent to the orbit at } x.\}$$

Then D^2 is a G-equivariant, symmetric differential operator of order $2k$ on $\Gamma(M, E)$, which is transversally strongly elliptic.

The following proposition is contained to some extent in [2] and [13].

Proposition 3.2. *Let D be a G-equivariant, symmetric, transversally elliptic differential operator of order k on $\Gamma(M, E)$.*

(1) *The operators*

$$D_\rho = (D \otimes I_{\rho^*})^G = D \otimes I_{\rho^*}|_{\Gamma(M,E)_\rho}$$

(and their powers) are essentially self-adjoint on $L^2(M, E)_\rho$ and generate the self-adjoint operators $D_\rho^* = \mathcal{S}_\rho$.

(2) *The operator D (and its powers) is essentially self-adjoint on $L^2(M, E)$ and generates a G-equivariant, self-adjoint operator $\mathcal{R}$. With respect to the Hilbert sum decomposition (3.1), the operator $\mathcal{R}$ decomposes as a sum*

$$\mathcal{R} = \bigoplus_{[\rho] \in \widehat{G}} \mathcal{S}_\rho \otimes I_\rho \ .$$

(3) *Each component $\mathcal{S}_\rho$ has discrete spectrum without finite accumulation points and admits a complete system of smooth eigensections, and we have*

$$E_\lambda(\mathcal{R})_\rho \cong E_\lambda(\mathcal{S}_\rho) = E_\lambda(D_\rho) .$$

(4) *For each $[\rho] \in \widehat{G}$, the eigenspaces of $\mathcal{S}_\rho \otimes I_\rho$ are finite G-modules of the form*

$$E_\lambda(\mathcal{S}_\rho \otimes I_\rho) \cong E_\lambda(\mathcal{S}_\rho) \otimes_{\mathbb{C}} V_\rho \ .$$

Proof. D_ρ^2 is completed to a strongly elliptic operator $\widetilde{D}_\rho$ by adding the kth power of an appropriate Casimir operator, restricting to D_ρ^2 on $L^2\left(M, E\right)_\rho = L^2\left(M, E \otimes_{\mathbb{C}} V_\rho^*\right)^G$. It follows that D_ρ^2 is essentially self-adjoint. Furthermore, $\left(\widetilde{D}_\rho + I\right)^*$ is an isomorphism from the Sobolev space $H^{2k}\left(M, E\right)$ to $L^2\left(M, E\right)$. To see that the operators D_ρ are essentially self-adjoint and generate the self-adjoint operators $\mathcal{S}_\rho$ on $L^2\left(M, E\right)_\rho$, suppose that $D_\rho^* u = \pm\iota\, u$, with u nonzero in the domain $\text{Dom}\left(D_\rho^*\right)$. Then we have

$$\langle D_\rho s, u\rangle = \langle s, D_\rho^* u\rangle = \mp\iota\, \langle s, u\rangle$$

for all $s \in \Gamma\left(M, E\right)_\rho$. Therefore,

$$\left\langle D_\rho^2 s, u\right\rangle = \left\langle D_\rho s, D_\rho^* u\right\rangle = \mp\iota\, \left\langle D_\rho s, u\right\rangle = \mp\iota\, \left\langle s, D_\rho^* u\right\rangle = -\left\langle s, u\right\rangle.$$

That is,

$$\left\langle s, \left(\widetilde{D}_\rho + I\right)^* u\right\rangle = \left\langle s, \left(D_\rho^2 + I\right)^* u\right\rangle = \left\langle \left(D_\rho^2 + I\right) s, u\right\rangle = 0,$$

a contradiction.

Next, the family $\{\mathcal{S}_\rho\}_{[\rho]\in\widehat{G}}$ of self-adjoint operators defines a G-equivariant, self-adjoint operator

$$\mathcal{R} = \bigoplus_{[\rho]\in\widehat{G}} \mathcal{S}_\rho \otimes I_\rho$$

on $L^2\left(M, E\right)$ whose domain is given by

$$\left\{ u = (u_\rho)_{[\rho]\in\widehat{G}} \,\middle|\, \begin{array}{c} u_\rho \in \text{Dom}\left(\mathcal{S}_\rho\right) \otimes_{\mathbb{C}} I_\rho, \\ \sum_{[\rho]} \left(\|u_\rho\|^2 + \|(\mathcal{S}_\rho \otimes I_\rho)\, u_\rho\|^2\right) < \infty \end{array} \right\}$$

and satisfies $P_\rho \text{Dom}\left(\mathcal{R}\right) = \text{Dom}\left(\mathcal{S}_\rho\right) \otimes I_\rho \subset \text{Dom}\left(\mathcal{R}\right)$. Thus

$$\mathcal{R}u = \sum_{[\rho]} \left(\mathcal{S}_\rho \otimes I_\rho\right) u_\rho\,, \ u \in \text{Dom}\left(\mathcal{R}\right),\ u_\rho = P_\rho u.$$

For $s \in \Gamma\left(M, E\right)$, $u \in \text{Dom}\left(\mathcal{R}\right)$, we have

$$\begin{aligned}
\langle s, \mathcal{R}u\rangle &= \sum_{[\rho]} \langle P_\rho s, \left(\mathcal{S}_\rho \otimes I_\rho\right) u_\rho\rangle \\
&= \sum_{[\rho]} \langle \left(D_\rho \otimes I_\rho\right) P_\rho s, u_\rho\rangle \\
&= \sum_{[\rho]} \langle P_\rho Ds, u_\rho\rangle = \sum_{[\rho]} \langle Ds, u_\rho\rangle = \langle Ds, u\rangle,
\end{aligned}$$

and thus $\text{Dom}\left(\mathcal{R}\right) \subseteq \text{Dom}\left(D^*\right)$. Since the operators $\mathcal{S}_\rho = D_\rho^{**} = D_\rho^*$ are the closures of the components D_ρ of D and $\mathcal{S}_\rho \otimes I_\rho = D_\rho^{**} \otimes I_\rho = \left(D_\rho \otimes I_\rho\right)^{**}$, it

follows that

$$D^{**} = \bigoplus_{[\rho]} (D_\rho \otimes I_\rho)^{**} = \bigoplus_{[\rho]} \mathcal{S}_\rho \otimes I_\rho = \mathcal{R}.$$

Since $\mathcal{R}$ is self-adjoint, it follows that D is essentially self-adjoint with closure $\mathcal{R}$. □

Corollary 3.3. *The operators D_ρ induce operators $\overline{D}_{\rho,s}$*

$$\overline{D}_{\rho,s} : H^s \left(\Gamma \left(M, E^+\right)^\rho\right) \to H^{s-1} \left(\Gamma \left(M, E^-\right)^\rho\right)$$

that are Fredholm and independent of s.

Corollary 3.4. *D admits a complete system of smooth eigensections. Since the spectrum is given by $Spec\,(D) = \bigcup_{[\rho]} Spec\,(D_\rho)$, it need not be discrete, and the eigenvalues may have infinite multiplicities.*

Note that the eigenvalue counting function of D_ρ^2 (of order $2k$) satisfies the asymptotic formula

$$N\,(\lambda) := \sum_{\lambda^\rho \leq \lambda} \dim E_{\lambda^\rho}\,(D_\rho)$$

(3.3)
$$\sim c\lambda^{m/2k},$$

where m is the dimension of M/G (see [13], applied to the strongly elliptic operator $D^2 + C - \lambda_\rho$).

4. The case of one isotropy type

In this section we consider the equivariant index problem and representation theory in the case where the group action has one isotropy type. The interested reader may consult the paper [24] to obtain more detailed exposition and more explicit and extensive results on G-bundles over manifolds with one isotropy type.

4.1. **Induced bundles over the orbit space.** The statements in this subsection are for the most part known (see, for example, [13]), but we will use them later in the paper. Let G be a compact Lie group, and let X be a (not necessarily closed) Riemannian G-manifold, and suppose that there is only one isotropy type. For all $x \in X$, let $\mathcal{O}_x$ denote the orbit Gx. Let $\overline{X}$ denote the manifold $G \setminus X$, and let $\pi \colon X \to \overline{X}$ denote the projection. The metric g on X induces a metric $\overline{g}$ on $\overline{X}$ defined uniquely by

$$\overline{g}_{\pi(x)} \left(\pi_* v, \pi_* w\right) = \left(\text{vol}\,(\mathcal{O}_x)\right)^{2/\dim\left(\overline{X}\right)} g\,(v, w)$$

for any $x \in X$ and $v, w \in N_x\,(\mathcal{O}_x) \subset T_x X$. The scale factor is chosen so that vol $X = $ vol $\overline{X}$. Let E be a Hermitian, G-equivariant vector bundle over X, and let $\rho \colon G \to U\,(V_\rho)$ be an irreducible unitary representation. Denote by G_x the isotropy group at $x \in X$. Define the vector spaces $\mathcal{E}_\rho\,(\mathcal{O}_x)$ and $\mathcal{E}^\rho\,(\mathcal{O}_x)$ by

$$\mathcal{E}_\rho\,(\mathcal{O}_x) := L^2\,(\mathcal{O}_x, E)_\rho = \text{Hom}_G\,(V_\rho, L^2\,(\mathcal{O}_x, E))$$

$$\cong \mathrm{Hom}_G \left(V_\rho, L^2 \left(\mathcal{O}_x, G \times_{G_x} E_x \right) \right)$$

$$\cong \mathrm{Hom}_{G_x} \left(V_\rho, E_x \right) \cong \left(E_x \otimes V_\rho^* \right)^{G_x},$$

$$\mathcal{E}^\rho \left(\mathcal{O}_x \right) := L^2 \left(\mathcal{O}_x, E \right)^\rho = i_\rho \left(\mathcal{E}_\rho \left(\pi \left(x \right) \right) \otimes_\mathbb{C} V_\rho \right).$$

Note that i_ρ above is not necessarily one to one. Both $\mathcal{E}_\rho \left(\mathcal{O}_x \right)$ and $\mathcal{E}^\rho \left(\mathcal{O}_x \right)$ are well-defined vector spaces whose dimensions are constant, and moreover $\mathcal{E}_\rho \left(\cdot \right)$ and $\mathcal{E}^\rho \left(\cdot \right)$ form Hermitian vector bundles over $\overline{X}$ (see [13, Lemma 1.2 ff]). In fact, the vector spaces $E_{\rho, x}$ defined for each $x \in X$ by

$$E_{\rho, x} = \mathrm{Hom}_{G_x} \left(V_\rho, E_x \right)$$

form a G-invariant subbundle E_ρ of $\mathrm{Hom} \left(V_\rho, E \right)$ over X. The Hermitian structure on $\mathcal{E}^\rho$ is defined uniquely by

$$\left\langle s_1, s_2 \right\rangle_{\pi(x)} = \left\langle s_1 \left(x \right), s_2 \left(x \right) \right\rangle_x$$

for every $x \in X$. With these choices, we have canonical isomorphisms

$$\Gamma \left(\overline{X}, \mathcal{E}_\rho \right) \cong \Gamma \left(X, E_\rho \right)^G \cong \Gamma \left(X, E \right)_\rho, \ \Gamma \left(\overline{X}, \mathcal{E}^\rho \right) \cong \Gamma \left(X, E \right)^\rho$$

(4.1) $$L^2 \left(\overline{X}, \mathcal{E}_\rho \right) \cong L^2 \left(X, E_\rho \right)^G \cong L^2 \left(X, E \right)_\rho, \ L^2 \left(\overline{X}, \mathcal{E}^\rho \right) \cong L^2 \left(X, E \right)^\rho.$$

If E has a grading, there are gradings induced on $\mathcal{E}_\rho$ and $\mathcal{E}^\rho$ in the obvious way.

4.2. **Equivariant operators on G-manifolds with one orbit type.** Given a transversally elliptic, G-equivariant operator D on sections of E over X, the operator

$$D^\rho = D|_{\Gamma(X,E)^\rho}$$

induces an operator, called $\mathcal{D}^\rho$, on sections of $\mathcal{E}^\rho$ over $\overline{X}$. Similarly, D induces an operator $\mathcal{D}_\rho$ on sections of $\mathcal{E}_\rho$ over $\overline{X}$. Furthermore, $\mathcal{D}^\rho$ and $\mathcal{D}_\rho$ are elliptic operators. If D is essentially self-adjoint on X, then $\mathcal{D}^\rho$ and $\mathcal{D}_\rho$ are also essentially self-adjoint on $L^2 \left(\overline{X}, \mathcal{E}^\rho \right)$ and $L^2 \left(\overline{X}, \mathcal{E}_\rho \right)$. The supertrace of the equivariant heat kernel corresponding to D restricted to $\Gamma \left(X, E \right)^\rho$ can be identified with the supertrace of the (ordinary) heat kernel associated to the elliptic operator $\mathcal{D}^\rho$ on $\Gamma \left(\overline{X}, \mathcal{E}^\rho \right)$ (similarly $\mathcal{D}_\rho$ on $\Gamma \left(\overline{X}, \mathcal{E}_\rho \right)$.)

Thus,

$$\mathrm{ind}^\rho \left(D \right) = \frac{1}{\dim V_\rho} \mathrm{ind} \left(\mathcal{D}^\rho \right)$$

$$= \mathrm{ind} \left(\mathcal{D}_\rho \right),$$

which is the Atiyah-Singer index of the elliptic operator $\mathcal{D}_\rho$ on the base manifold $\overline{X}$. This is also a special case of our main theorem (Theorem 9.2).

4.3. **The normalized isotypical decomposition.** We retain the notation of the previous subsections. For further detail regarding topics in this section, we refer the reader to [24]. Again, we emphasize that X has one isotropy type $[H]$, so that the orbits form a Riemannian foliation of X. Let $N = N(H)$; note that N-bundles over the fixed point set X^H induce G-bundles over X and vice versa.

Let $E \to X$ be a given G-equivariant vector bundle over X. For each $x \in X^H$, let $\Sigma_{E_x} = \{[\sigma \colon H \to U(W_\sigma)] \text{ irreducible} : \mathrm{Hom}_H(W_\sigma, E_x) \neq 0\}$ be the set of equivalence classes of irreducible representations of H present in E_x; by rigidity, Σ_{E_x} is locally constant in x. Let $\Sigma_E = \bigcup_{x \in X^H} \Sigma_{E_x}$. The isotypical (or primary) decomposition of the H-module E_x is of the form

$$E_x = \bigoplus_{[\sigma] \in \Sigma_{E_x}} E_x^{[\sigma]}$$

where

$$E_x^{[\sigma]} = i_\sigma \left(\mathrm{Hom}_H(W_\sigma, E_x) \otimes W_\sigma \right).$$

Note that the representations $E_x^{[\sigma]}$ are not necessarily irreducible. Further, these subspaces do not necessarily form N-equivariant bundles as x varies over $X^H \subset X$. In particular, if the normalizer N is disconnected it may be the case that an element $n \in N$ maps a primary component $E_x^{[\sigma]}$ to an inequivalent primary component of E_{nx}, and $nx \in X^H$. The representation of H on $nE_x^{[\sigma]} \subset E_{nx}$ is given by

$$hv = n \left(n^{-1}hn \right) \left(n^{-1}v \right), \ h \in H, v \in E_{nx}$$

so that the representation on $nE_x^{[\sigma]}$ is equivalent to a direct sum of representations of type $[\sigma_x^n \colon H \to U(W_{\sigma_x})]$ defined by

$$(4.2) \qquad\qquad \sigma_x^n(h) = \sigma_x \left(n^{-1}hn \right).$$

For this reason, we will instead use a more coarse decomposition of E_x. For each $n \in N$, we see that $E_x^{[\sigma^n]} = n^{-1} \left(E_{nx}^{[\sigma]} \right)$, an H-invariant subspace of E_x that is the primary $[\sigma^n]$-part of the representation of H on E_x. The representation of H on $E_x^{[\sigma^n]}$ is equivalent to the representation of H on $nE_x^{[\sigma]} \subset E_{nx}$. We say that the two representations σ_x and σ_x^n are **normalizer-conjugate**. It is often the case that σ_x^n is equivalent to σ_x; as explained in [24], the rigidity of irreducible representations implies that the set

$$\widetilde{N}_{[\sigma]} = \left\{ n \in N : [\sigma^n] = [\sigma] \right\},$$

is a subgroup of N such that $N \big/ \widetilde{N}_{[\sigma]}$ is finite. Let $n_1, \ldots, n_k \in N$ be elements such that $N \big/ \widetilde{N}_{[\sigma]} = \left\{ n_1 \widetilde{N}_{[\sigma]}, \ldots, n_k \widetilde{N}_{[\sigma]} \right\}$. Let

$$(E_x)^{\overline{\sigma}} = \bigoplus_{j=1}^{k} E_x^{\left[\sigma^{n_j} \right]} \subset E_x.$$

Here, $\overline{\sigma}$ is the set of equivalence classes of irreducible representations inside the set $\{\sigma^n : n \in N\}$, so that we see that the set of all $(E_x)^{\overline{\sigma}}$ in E_x is in fact indexed by the finite set of such $\overline{\sigma}$. We decompose for each $x \in X^H \subset X$,

$$E_x = \bigoplus_{\overline{\sigma}} (E_x)^{\overline{\sigma}} .$$

These subspaces $(E_x)^{\overline{\sigma}}$ are in fact invariant under the action of $n \in N$ and patch together to form well-defined N-equivariant subbundles of $E|_{X^H}$. Thus, G acts on these subspaces to produce well-defined subbundles of E over each orbit, and these subbundles patch together to form well-defined G-subbundles $E^1, \ldots, E^r$ over X. In summary, we may decompose the vector bundle E as

$$(4.3) \qquad\qquad E = \bigoplus_j E^j ,$$

where each G-bundle E^j is the direct sum of primary components E^{σ} corresponding to irreducible representations σ of the isotropy subgroup that are all normalizer-conjugate. Further, the representations present in E^j are not normalizer-conjugate to those present in E^k for $j \neq k$. We call this the **normalized isotypical decomposition**; see [24] for details. Let $\widetilde{N}$ be the connected component of N relative to H, so that $\pi_0(N \diagup H) \cong N \diagup \widetilde{N}$. If we denote by $\widetilde{N}_{[\sigma]}$ the subgroup (of finite index) of N fixing $[\sigma] \in \Sigma_E$, so that $\widetilde{N} \subseteq \widetilde{N}_{[\sigma]} \subseteq N$ and $E^{[\sigma]}$ is $\widetilde{N}_{[\sigma]}$-equivariant, we may rephrase the preceding construction by saying that the normalized isotypical components are obtained by 'inducing up' the isotypical components $E^{[\sigma]}$ from $\widetilde{N}_{[\sigma]}$ to N. That is, the normalized isotypical component over X^H induced from $[\sigma] \in \Sigma_E$ is the set

$$N \times_{\widetilde{N}_{[\sigma]}} E^{[\sigma]}$$

This gives an N-subbundle of E over X^H containing $E^{[\sigma]}$ that induces the normalized isotypical component over all of X.

4.4. **The refined isotypical decomposition.** With notation as above, let X^H be the fixed point set of H, and for $\alpha \in \pi_0(X^H)$, let X_α^H denote the corresponding connected component of X^H.

Definition 4.1. We denote $X_\alpha = G X_\alpha^H$, and X_α is called a **component of** X **relative to** G.

Remark 4.2. The space X_α is not necessarily connected, but it is the inverse image of a connected component of $G \diagdown X = N \diagdown X^H$ under the projection $X \to G \diagdown X$. Also, note that $X_\alpha = X_\beta$ if there exists $n \in N$ such that $n X_\alpha^H = X_\beta^H$. If X is a closed manifold, then there are a finite number of components of X relative to G.

We now introduce a decomposition of a G-bundle $E \to X$ over a G-space with single orbit type $[H]$ that is a priori finer than the normalized isotypical decomposition. Let E_α be the restriction $E|_{X_\alpha^H}$. As in the previous section, let $\widetilde{N}_{[\sigma]} = \{n \in N : [\sigma^n] \text{ is equivalent to } [\sigma]\}$. If the isotypical component $E_\alpha^{[\sigma]}$ is nontrivial, then it is invariant under the subgroup $\widetilde{N}_{\alpha,[\sigma]} \subseteq \widetilde{N}_{[\sigma]}$ that leaves in addition the connected component X_α^H invariant; again, this subgroup has finite index in N. The isotypical components transform under $n \in N$ as

$$n : E_\alpha^{[\sigma]} \xrightarrow{\cong} E_{\overline{n}(\alpha)}^{[\sigma^n]} \ ,$$

where $\overline{n}$ denotes the residue class class of $n \in N$ in $N \big/ \widetilde{N}_{\alpha,[\sigma]}$. Then a decomposition of E is obtained by 'inducing up' the isotypical components $E_\alpha^{[\sigma]}$ from $\widetilde{N}_{\alpha,[\sigma]}$ to N. That is,

$$E_{\alpha,[\sigma]}^N = N \times_{\widetilde{N}_{\alpha,[\sigma]}} E_\alpha^{[\sigma]}$$

is a bundle containing $E_\alpha^{[\sigma]}\big|_{X_\alpha^H}$. This is an N-bundle over $NX_\alpha^H \subseteq X^H$, and a similar bundle may be formed over each distinct NX_β^H, with $\beta \in \pi_0\left(X^H\right)$. Further, observe that since each bundle $E_{\alpha,[\sigma]}^N$ is an N-bundle over NX_α^H, it defines a unique G bundle $E_{\alpha,[\sigma]}^G$.

Definition 4.3. The G-bundle $E_{\alpha,[\sigma]}^G$ over the submanifold X_α is called a **fine component** or the **fine component of $E \to X$ associated to** $(\alpha, [\sigma])$.

If $G \backslash X$ is not connected, one must construct the fine components separately over each X_α. If E has finite rank, then E may be decomposed as a direct sum of distinct fine components over each X_α. In any case, $E_{\alpha,[\sigma]}^N$ is a finite direct sum of isotypical components over each X_α^H.

Definition 4.4. The direct sum decomposition of $E|_{X_\alpha}$ into subbundles E^b that are fine components $E_{\alpha,[\sigma]}^G$ for some $[\sigma]$, written

$$E|_{X_\alpha} = \bigoplus_b E^b \ ,$$

is called the **refined isotypical decomposition** (or **fine decomposition**) of $E|_{X_\alpha}$.

In the case where $G \backslash X$ is connected, the group $\pi_0\left(N \big/ H\right)$ acts transitively on the connected components $\pi_0\left(X^H\right)$, and thus $X_\alpha = X$. We comment that if $[\sigma, W_\sigma]$ is an irreducible H-representation present in E_x with $x \in X_\alpha^H$, then $E_x^{[\sigma]}$ is a subspace of a distinct E_x^b for some b. The subspace E_x^b also contains $E_x^{[\sigma^n]}$ for every n such that $nX_\alpha^H = X_\alpha^H$.

Remark 4.5. Observe that by construction, for $x \in X_\alpha^H$ the multiplicity and dimension of each $[\sigma]$ present in a specific E_x^b is independent of $[\sigma]$. Thus, $E_x^{[\sigma^n]}$ and $E_x^{[\sigma]}$ have the same multiplicity and dimension if $nX_\alpha^H = X_\alpha^H$.

Remark 4.6. The advantage of this decomposition over the isotypical decomposition is that each E^b is a G-bundle defined over all of X_α, and the isotypical decomposition may only be defined over X_α^H.

This new decomposition is a priori a finer decomposition than the normal isotypical decomposition into G-subbundles.

Definition 4.7. Now, let E be a G-equivariant vector bundle over X, and let E^b be a fine component as in Definition 4.3 corresponding to a specific component $X_\alpha = GX_\alpha^H$ of X relative to G. Suppose that another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E_y^b, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, and that W has a single component in the fine decomposition. Then we say that W is **adapted** to E^b.

Lemma 4.8. *In the definition above, if another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E_y^b, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, then it follows that W has a single component in the fine decomposition and hence is adapted to E^b. Thus, the last phrase in the corresponding sentence in the above definition is superfluous.*

Proof. Suppose that we choose an equivalence class $[\sigma]$ of H-representations present in $W_x, x \in X_\alpha^H$. Let $[\sigma']$ be any other equivalence class; then, by hypothesis, there exists $n \in N$ such that $nX_\alpha^H = X_\alpha^H$ and $[\sigma'] = [\sigma^n]$. Then, observe that $nW_x^{[\sigma]} = W_{nx}^{[\sigma^n]} = W_x^{[\sigma^n]}$, with the last equality coming from the rigidity of irreducible H-representations. Thus, W is contained in a single fine component, and so it must have a single component in the fine decomposition. $\square$

4.5. Canonical isotropy G-bundles.
In what follows, we show that there are naturally defined finite-dimensional vector bundles that are adapted to any fine components. Once and for all, we enumerate the irreducible representations $\left\{\left[\rho_j, V_{\rho_j}\right]\right\}_{j=1,2,\dots}$ of G. Let $[\sigma, W_\sigma]$ be any irreducible H-representation. Let $G \times_H W_\sigma$ be the corresponding homogeneous vector bundle over the homogeneous space G/H. Then the L^2-sections of this vector bundle decompose into irreducible G-representations. In particular, let $\left[\rho_{j_0}, V_{\rho_{j_0}}\right]$ be the equivalence class of irreducible representations that is present in $L^2\left(G/H, G \times_H W_\sigma\right)$ and that has the lowest index j_0. Then Frobenius reciprocity implies

$$0 \neq \operatorname{Hom}_G\left(V_{\rho_{j_0}}, L^2\left(G/H, G \times_H W_\sigma\right)\right) \cong \operatorname{Hom}_H\left(V_{\operatorname{Res}\left(\rho_{j_0}\right)}, W_\sigma\right),$$

so that the restriction of ρ_{j_0} to H contains the H-representation $[\sigma]$. Now, for a component X_α^H of X^H, with $X_\alpha = GX_\alpha^H$ its component in X relative to G, the trivial bundle

$$X_\alpha \times V_{\rho_{j_0}}$$

is a G-bundle (with diagonal action) that contains a nontrivial fine component $W_{\alpha,[\sigma]}$ containing $X_\alpha^H \times \left(V_{\rho_{j_0}}\right)^{[\sigma]}$.

Definition 4.9. We call $W_{\alpha,[\sigma]} \to X_\alpha$ the **canonical isotropy G-bundle associated to** $(\alpha, [\sigma]) \in \pi_0\left(X^H\right) \times \widehat{H}$. Observe that $W_{\alpha,[\sigma]}$ depends only on the enumeration of irreducible representations of G, the irreducible H-representation $[\sigma]$ and the component X_α^H. We also denote the following positive integers associated to $W_{\alpha,[\sigma]}$:

- $m_{\alpha,[\sigma]} = \dim \operatorname{Hom}_H\left(W_\sigma, W_{\alpha,[\sigma],x}\right) = \dim \operatorname{Hom}_H\left(W_\sigma, V_{\rho_{j_0}}\right)$ (the **associated multiplicity**), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$ (see Remark 4.5).
- $d_{\alpha,[\sigma]} = \dim W_\sigma$ (the **associated representation dimension**), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.
- $n_{\alpha,[\sigma]} = \frac{\operatorname{rank}\left(W_{\alpha,[\sigma]}\right)}{m_{\alpha,[\sigma]}d_{\alpha,[\sigma]}}$ (the **inequivalence number**), the number of inequivalent representations present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.

Remark 4.10. Observe that $W_{\alpha,[\sigma]} = W_{\alpha',[\sigma']}$ if $[\sigma'] = [\sigma^n]$ for some $n \in N$ such that $nX_\alpha^H = X_{\alpha'}^H$.

The lemma below follows immediately from Lemma 4.8.

Lemma 4.11. *Given any G-bundle $E \to X$ and any fine component E^b of E over some $X_\alpha = GX_\alpha^H$, there exists a canonical isotropy G-bundle $W_{\alpha,[\sigma]}$ adapted to $E^b \to X_\alpha$.*

4.6. **Decomposing sections in tensor products of equivariant bundles.** We now consider a method of decomposing sections of a tensor product of equivariant bundles. Suppose that $E_1 \to X$ is a (possibly infinite-dimensional) G-equivariant vector bundle over a manifold with one orbit type $[H]$. Let $X_\alpha = GX_\alpha^H \subseteq X$, and let $E_{1,X_\alpha} := E_1|_{X_\alpha} = \bigoplus_b E_1^b$ be the fine decomposition (Definition 4.4). Let $W^b = W_{\alpha,[\sigma]} \to X_\alpha$ be the canonical isotropy G-bundle adapted to E_1^b, as in Definition 4.9 and Lemma 4.11. Every $[\sigma] \in \widehat{H}$ that is present in W_x^b has associated multiplicity m_b , which is independent of $x \in X_\alpha^H$. Let $W_x^b = \bigoplus_j W_{j,x}^b$ be the isotypical decomposition, so that each j refers to a distinct irreducible H-representation type $[\sigma_{b,j}]$, each with dimension $m_b \cdot d_b$, where d_b is the dimension of that irreducible representation. Let $E_2 \to X$ be another G-equivariant vector bundle over X. Restricted to a single orbit $\mathcal{O}_x = Gx$ in X_α, let $E_{2,\mathcal{O}_x} = E_2|_{\mathcal{O}_x}$.

Then Frobenius reciprocity implies

$$\mathrm{Hom}_G\left(V_\rho, \Gamma\left(\mathcal{O}_x, E_{1,\mathcal{O}_x}^b \otimes E_{2,\mathcal{O}_x}\right)\right) \otimes \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, W_{j,x}^b\right)$$

$$\cong \mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, E_{1,x}^b \otimes E_{2,x}\right) \otimes \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, W_{j,x}^b\right)$$

$$\cong \bigoplus_j \mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, \mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right) \otimes W_{j,x}^b \otimes E_{2,x}\right),$$

Since $\mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right)$ is a trivial H-space,

$$\mathrm{Hom}_G\left(V_\rho, \Gamma\left(\mathcal{O}_x, E_{1,\mathcal{O}_x}^b \otimes E_{2,\mathcal{O}_x}\right)\right) \otimes \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, W_{j,x}^b\right)$$

$$\cong \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right) \otimes \mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, W_{j,x}^b \otimes E_{2,x}\right)$$

Then

$$\Gamma\left(\mathcal{O}_x, E_{1,\mathcal{O}_x}^b \otimes E_{2,\mathcal{O}_x}\right)^\rho \otimes \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, W_{j,x}^b\right)$$

$$\cong \mathrm{Hom}_G\left(V_\rho, \Gamma\left(\mathcal{O}_x, E_{1,\mathcal{O}_x}^b \otimes E_{2,\mathcal{O}_x}\right)\right) \otimes V_\rho \otimes \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, W_{j,x}^b\right)$$

$$\cong \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right) \otimes \mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, W_{j,x}^b \otimes E_{2,x}\right) \otimes V_\rho$$

$$\cong \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right) \otimes \Gamma\left(\mathcal{O}_x, \widetilde{W}_j^b \otimes E_2\right)^\rho,$$

where $\widetilde{W}_j^b \to \mathcal{O}_x$ is the bundle $G \times_H W_{j,x}^b \subseteq W^b\big|_{\mathcal{O}_x}$ with the identification $x = eH$. Observe that the isomorphism above is an isomorphism of G-modules, where G acts trivially on $\mathrm{Hom}_H\left(W_{j,x}^b, W_{j,x}^b\right)$ and on $\mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right)$. Let $L\colon E_1 \to E_1$ be a G-equivariant bundle map, and let $B\colon \Gamma\left(X, E_2\right) \to \Gamma\left(X, E_2\right)$ be a G-equivariant operator. Let $L^b\colon E_1^b \to E_1^b$ be the restriction. We note that L^b acts on $E_{1,x}^b \cong \bigoplus_j \mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right) \otimes W_{j,x}^b$ by $L'^{\,b} \otimes \mathbf{1}$, where $L'^{\,b}\colon$
$\mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right) \to \mathrm{Hom}_H\left(W_{j,x}^b, E_{1,x}^b\right)$ acts by post-composing with L^b.

The following Lemma is a consequence of the derivation above and the slice theorem.

Lemma 4.12. *Let $T_\varepsilon \to \mathcal{O}_x$ be a tubular neighborhood of $\mathcal{O}_x \subset X$ such that $T_\varepsilon \cong G \times_H D_\varepsilon$, where $x \in D_\varepsilon \subset X^H$ is a ball transverse to $\mathcal{O}_x$. Then the operator $L \otimes B \otimes \mathbf{1}$ corresponds to $L'^{\,b} \otimes (\mathbf{1} \otimes B)^\rho$ through the isomorphism*

$$\Gamma\left(T_\varepsilon, E_1^b \otimes E_2\right)^\rho \otimes \bigoplus_j \mathrm{Hom}_H\left(W_j^b\big|_{D_\varepsilon}, W_j^b\big|_{D_\varepsilon}\right)$$

$$\cong \bigoplus_j \operatorname{Hom}_H \left(W_j^b \big|_{D_\varepsilon}, E_1^b \big|_{D_\varepsilon} \right) \otimes \Gamma \left(T_\varepsilon, \widetilde{W}_j^b \otimes E_2 \right)^\rho,$$

with the isomorphism given by evaluation. If $\widehat{\otimes}$ denotes the graded tensor product, then

$$\Gamma \left(T_\varepsilon, E_1^b \widehat{\otimes} E_2 \right)^\rho \otimes \bigoplus_j \operatorname{Hom}_H \left(W_j^b \big|_{D_\varepsilon}, W_j^b \big|_{D_\varepsilon} \right)$$

$$\cong \bigoplus_j \operatorname{Hom}_H \left(W_j^b \big|_{D_\varepsilon}, E_1^b \big|_{D_\varepsilon} \right) \widehat{\otimes} \Gamma \left(T_\varepsilon, \widetilde{W}_j^b \otimes E_2 \right)^\rho.$$

In the above, $W^b = \bigoplus_j W_j^b$ is the isotypical decomposition.

5. Properties of the Equivariant Heat Kernel and Equivariant Index

5.1. The equivariant heat kernel and index.

We now review some properties of the equivariant index and equivariant heat kernel that are known to experts in the field but are not written in the literature in the form and generality required in this paper (see [2], [13], [14], [8]). With notation as in the introduction, let $E = E^+ \oplus E^-$ be a graded, G-equivariant vector bundle over M. We consider a first order G-equivariant differential operator $D^+ \colon \Gamma(M, E^+) \to \Gamma(M, E^-)$ which is transversally elliptic, and let D^- be the formal adjoint of D^+. The restriction $D^{\pm,\rho} = D^\pm|_{\Gamma(M,E)^\rho}$ behaves in a similar way to an elliptic operator. Let $C \colon \Gamma(M, E) \to \Gamma(M, E)$ be the Casimir operator described in the introduction, and let λ_ρ be the eigenvalue of C associated to the representation type $[\rho]$. The following argument can be seen in some form in [2]. Given a section $\alpha \in \Gamma(M, E^+)^\rho$, we have

$$D^- D^+ \alpha = \left(D^- D^+ + C - \lambda_\rho \right) \alpha.$$

Then $D^- D^+ + C - \lambda_\rho$ is self-adjoint and elliptic and has finite dimensional eigenspaces consisting of smooth sections. Thus, the eigenspaces of $D^- D^+$ restricted to $\Gamma(M, E^+)^\rho$ are finite dimensional and consist of smooth sections, and the index $\operatorname{ind}^\rho(D)$ is well-defined. Further, the $[\rho]$-part $K^{[\rho]}$ of the heat kernel of $e^{-tD^- D^+}$ is the same as the $[\rho]$-part of the heat kernel $K(t, \cdot, \cdot)$ of $e^{-t\left(D^- D^+ + C - \lambda_\rho\right)}$. Let

$$\beta_1 \leq \beta_2 \leq \ldots$$

be the eigenvalues of $D^- D^+ + C - \lambda_\rho$ repeated according to multiplicities, which correspond to the L^2 orthonormal set of eigensections $\{\alpha_1, \alpha_2, \ldots\}$. We may choose that basis so that each α_j belongs to a specific irreducible representation space of G. The kernel K^+ of $e^{-t\left(D^- D^+ + C - \lambda_\rho\right)}$ satisfies

$$K^+(t, x, y) = \sum_k e^{-t\beta_k} \alpha_k(x) \otimes (\alpha_k(y))^* \in \operatorname{Hom}\left(E_y^+, E_x^+ \right).$$

Observe that for all $g \in G$,

$$g \cdot K^+ \left(t, g^{-1}x, x \right) = \sum_k e^{-t\beta_k} g \cdot \alpha_k \left(g^{-1}x \right) \otimes \left(\alpha_k \left(x \right) \right)^*$$

$$= \sum_k e^{-t\beta_k} \sum_{[\eta]} \eta \left(g \right) \left(P_\eta \alpha_k \right) \left(x \right) \otimes \left(\alpha_k \left(x \right) \right)^*,$$

where the interior sum is over all equivalence classes of irreducible unitary representations η of G, and P_η is the projection onto the $[\eta]$-component. By the way we have chosen the basis $\{\alpha_1, \alpha_2, \ldots\}$, each $P_\eta \alpha_k$ is either α_k or 0, so in fact $\left(P_\eta \alpha_k \right) \left(x \right) \otimes \left(\alpha_k \left(x \right) \right)^* = \left(P_\eta \alpha_k \right) \left(x \right) \otimes \left(P_\eta \alpha_k \left(x \right) \right)^*$. Next, let ρ be a particular irreducible representation, and let χ_ρ be its character. By Lemma 3.1, if dg is the biinvariant, normalized Haar volume form,

$$\int_{g \in G} \eta \left(g \right) P_\eta \overline{\chi_\rho \left(g \right)} \, dg = \frac{1}{\dim V_\rho} P_\rho^\eta P_\eta,$$

and

$$\int_{x \in M} \int_{g \in G} \mathrm{tr} \left(\eta \left(g \right) \left(P_\eta \alpha_k \right) \left(x \right) \otimes \left(\alpha_k \left(x \right) \right)^* \right) \overline{\chi_\rho \left(g \right)} \, dg \, |dx|$$

$$= \int_{x \in M} \int_{g \in G} \mathrm{tr} \left(\eta \left(g \right) \left(P_\eta \alpha_k \right) \left(x \right) \otimes \left(P_\eta \alpha_k \left(x \right) \right)^* \right) \overline{\chi_\rho \left(g \right)} \, dg \, |dx|$$

$$= \frac{1}{\dim V_\rho} \varepsilon_k^\eta \delta_{\eta\rho} \int_{x \in M} \mathrm{tr} \left(\alpha_k \left(x \right) \otimes \alpha_k \left(x \right)^* \right) \, |dx| = \frac{1}{\dim V_\rho} \varepsilon_k^\eta \delta_{\eta\rho},$$

where ε_k^η is 1 if α_k is a section of type η and 0 otherwise, and where $\delta_{\eta\rho} = 1$ if η is equivalent to ρ and is zero otherwise. Using this information, we have that

$$\int_{x \in M} \int_{g \in G} \mathrm{tr} \left(g \cdot K^+ \left(t, g^{-1}x, x \right) \right) \overline{\chi_\rho \left(g \right)} \, dg \, |dx| = \frac{1}{\dim V_\rho} \sum_k e^{-t\beta_k} \varepsilon_k^\rho$$

$$= \frac{1}{\dim V_\rho} \sum_{k^{[\rho]}} e^{-t\beta_{k^{[\rho]}}},$$

where $k^{[\rho]}$ are the specific positive integers corresponding to eigensections $\alpha_{k^{[\rho]}}$ of type $[\rho]$. In summary,

$$\mathrm{tr} \left(e^{-tD^+D^-} \Big|_{\Gamma(M, E^+)^\rho} \right) = \sum_{k^{[\rho]}} e^{-t\beta_{k^{[\rho]}}}$$

$$= \dim V_\rho \int_{x \in M} \int_{g \in G} \mathrm{tr} \left(g \cdot K^+ \left(t, g^{-1}x, x \right) \right) \overline{\chi_\rho \left(g \right)} \, dg \, |dx|$$

Similar arguments apply to the operator D^+D^-. The usual McKean-Singer argument implies that the supertrace of the heat kernel K corresponding to

$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ restricted to $\Gamma\left(M, E^+ \oplus E^-\right)^\rho$ is the same as the index $\mathrm{ind}^\rho(D)$. By the argument above,

$$(5.1) \qquad \mathrm{ind}^\rho(D) = \int_{x \in M} \int_{g \in G} \mathrm{str}\ g \cdot K\left(t, g^{-1}x, x\right)\ \overline{\chi_\rho(g)}\ dg\ |dx|,$$

where K is the heat kernel of $e^{-t\left(D^2 + C - \lambda_\rho\right)}$. Since the heat kernel K changes smoothly with respect to G-equivariant deformations of the metric and of the operator D and the right hand side is an integer, we see that $\mathrm{ind}^\rho(D)$ is stable under such homotopies of the operator D^+ through G-equivariant transversally elliptic operators. This implies that the indices $\mathrm{ind}^G(D)$ and $\mathrm{ind}_g(D)$ mentioned in the introduction depend only on the G-equivariant homotopy class of the principal transverse symbol of D^+. (We knew this already because of the Fredholm properties discussed in Section 3.)

Note also that $e^{-t\left(D^2 + C - \lambda_\rho\right)}$ differs from $e^{-t\left(D^2 + C\right)}$ by a factor of $e^{-\lambda_\rho t}$, so we also have that

$$\mathrm{ind}^\rho(D) = \lim_{t \to 0} \int_{x \in M} \int_{g \in G} \mathrm{str}\ g \cdot K^0\left(t, g^{-1}x, x\right)\ \overline{\chi_\rho(g)}\ dg\ |dx|,$$

where K^0 is the heat kernel of $e^{-t\left(D^2 + C\right)}$.

5.2. The asymptotic expansion of the supertrace of the equivariant heat kernel.

We now use Lemma 2.1 to decompose $M = \bigcup_{i=0}^r M_i^\varepsilon$ into a disjoint union of pieces so that the integral (5.1) can be simplified. Also, observe that the kernel K and the metric on M are smooth functions of the operator D^+ and the metric on M. As we have mentioned, the index $\mathrm{ind}^\rho(D)$ is invariant under smooth, G-equivariant perturbations of D^+ and the metric. In particular, on each M_i^ε for $i > 0$ (ie not including M_0, the union of the principal orbits), in normal directions to the singular stratum M_i, we may flatten the metric, trivialize the bundles $E^\pm$, and make the operator D have constant coefficients with respect to the trivialized normal coordinates. Hence,

$$\mathrm{ind}^\rho(D) = \sum_i \int_{w \in M_i^\varepsilon} \int_{g \in G} \mathrm{str}\left(g \cdot K\left(t, g^{-1}w, w\right)\right)\ \overline{\chi_\rho(g)}\ dg\ |dw|$$

$$(5.2)$$

$$= \sum_i \int_{(x, z_x) \in M_i^\varepsilon} \int_{g \in G} \mathrm{str}\left(g \cdot K\left(t, g^{-1}\left(x, z_x\right), \left(x, z_x\right)\right)\right)\ \overline{\chi_\rho(g)}\ dg\ |dz_x|\ |dx|_i,$$

where we choose coordinates $w = (x, z_x)$ for the point $\exp_x^\perp(z_x)$, where $x \in M_i$, $z_x \in B_{\varepsilon_i} = B_{2^i\varepsilon}\left(N_x\left(M_i\right)\right)$, and $\exp_x^\perp$ is the normal exponential map. In the expression above, $|dz_x|$ is the Euclidean density on $N_x\left(M_i\right)$, and $|dx|_i$ is the Riemannian density on M_i. We note that the outer integral is not a true product integral if M_i is not the lowest stratum, because for the most part $x \in \Sigma_i^\varepsilon$

and $z_x \in B_{\varepsilon_i} = B_{2^i\varepsilon}(N_x(\Sigma_i^\varepsilon))$, but to include all of M_i^ε, some $x \in M_i \setminus \Sigma_i^\varepsilon$ and z_x away from the zero section may be needed; metrically, this is the integral over a cylinder of small radius whose ends are concave spherical (of larger radius) instead of flat. Recall that $K(t, g^{-1}(x, z_x), (x, z_x))$ is a map from $E_{(x,z_x)} = E_x$ to $E_{g^{-1}(x,z_x)} = E_{g^{-1}x}$.

One important idea is that the asymptotics of $K(t, g^{-1}w, w)$ as $t \to 0$ are completely determined by the operator's local expression along the minimal geodesic connecting $g^{-1}w$ and w, if $g^{-1}w$ and w are sufficiently close together. If the distance between $g^{-1}w$ and w is bounded away from zero, there is a constant $c > 0$ such that $K(t, g^{-1}w, w) = \mathcal{O}(e^{-c/t})$ as $t \to 0$. For these reasons, it is clear that the asymptotics of the supertrace $\mathrm{ind}^\rho(D)$ are locally determined over spaces of orbits in M, meaning that the contribution to the index is a sum of t^0-asymptotics of integrals of $\int_U \int_G \mathrm{str}(g \cdot K(t, g^{-1}w, w)) \overline{\chi_\rho(g)} \, dg \, |dw|$ over a finite collection of saturated sets $U \subset M$ that are unions of orbits that intersect a neighborhood of a point of M. In particular, the index $\mathrm{ind}^\rho(D)$ is the sum of the t^0 asymptotics of the integrals over M_i^ε. However, the integral of the t^0 asymptotic coefficient of $\int_G \mathrm{str}(g \cdot K(t, g^{-1}w, w)) \overline{\chi_\rho(g)} \, dg$ over all of M is not the index, and the integral of the t^0 asymptotic coefficient of $\int_M \mathrm{str}(g \cdot K(t, g^{-1}w, w)) \, \mathrm{dvol}_M$ over all of G is not the index. Thus, the integrals over M and G may not be separated when computing the local index contributions. In particular, the singular strata may not be ignored.

5.3. **Heat kernel equivariance property.** With notation as in the previous sections, the kernel K^+ of $e^{-t(D^-D^+ + C - \lambda_\rho)}$ satisfies

$$K^+(t, x, y) = \sum_k e^{-t\beta_k} \alpha_k(x) \otimes (\alpha_k(y))^* \in \mathrm{Hom}\left(E_y^+, E_x^+\right),$$

with $\{\alpha_j \in \Gamma(E^+)\}$ is an L^2-orthogonal basis of eigensections of $\Delta = D^-D^+ + C - \lambda_\rho$ corresponding to eigenvalues $\beta_j \geq 0$ (counted with multiplicity).

If $g\cdot$ denotes the action of $g \in G$ on sections of E, note that $(g \cdot u)(x) = g \cdot u(g^{-1}x)$. The G-equivariance implies

$$g \cdot (\Delta \alpha_j) = (\Delta(g \cdot \alpha_j)),$$

or

$$\beta_j(g \cdot \alpha_j) = \Delta(g \cdot \alpha_j),$$

so that $\{g \cdot \alpha_j\}$ is another L^2-orthonormal basis of eigensections, and

$$
\begin{aligned}
K^+(t, x, y) &= \sum e^{-t\beta_k} g \cdot \alpha_k(g^{-1}x) \otimes \left[g \cdot \alpha_k(g^{-1}y)\right]^* \\
&= \sum e^{-t\beta_k} g \cdot \alpha_k(g^{-1}x) \otimes \left[\alpha_k(g^{-1}y)\right]^* \circ g^{-1}\cdot \\
&= g \cdot K^+\left(t, g^{-1}x, g^{-1}y\right) \circ g^{-1}\cdot \quad \in \mathrm{Hom}(E_y, E_x).
\end{aligned}
$$

(5.3)

Note that $\mathrm{Hom}\,(E,E)$ is a G-bundle, and it carries the left G-action defined on $L_x \in \mathrm{Hom}\,(E_x,E_x) = E_x \otimes E_x^*$ by

$$g \bullet L_x = g \cdot L_x \circ g^{-1} \cdot \ \in \mathrm{Hom}\,(E_{gx},E_{gx})\,,$$

or

$$g \bullet (v \otimes w^*) = (g \cdot v) \otimes \left(w^* \circ g^{-1} \cdot\right).$$

With this action,

$$g \bullet \left(K^+\left(t,g^{-1}x,g^{-1}x\right)\right) = K^+\left(t,x,x\right).$$

Note $g \cdot K^+\left(t,g^{-1}x,x\right) \in \mathrm{Hom}\,(E_x,E_x)$. Observe that if we let $L_{g,x} = g \cdot K^+\left(t,g^{-1}x,x\right) \in \mathrm{Hom}\,(E_x,E_x)$, then

$$g' \bullet L_{g,g'^{-1}x} = \qquad g' \cdot \left(g \cdot K^+\left(t,g^{-1}\left(g'^{-1}x\right),g'^{-1}x\right)\right) \circ g'^{-1}$$

$$= \qquad g' \cdot g \cdot g'^{-1} K^+\left(t,g'g^{-1}\left(g'^{-1}x\right),g'g'^{-1}x\right) \circ g' \circ g'^{-1}$$

by (5.3), or

$$g' \bullet L_{g,g'^{-1}x} = \left(g'gg'^{-1}\right) \cdot K^+\left(t,\left(g'gg'^{-1}\right)^{-1}x,x\right) = L_{g'gg'^{-1},x}\,.$$

This implies that

$$K^+\left(t,x,x\right)^\rho = \dim V^\rho \int_G g \cdot K^+\left(t,g^{-1}x,x\right)\,\overline{\chi_\rho\,(g)}\,dg$$

$$= \dim V^\rho \int_G \left(g'gg'^{-1}\right) \cdot K^+\left(t,\left(g'gg'^{-1}\right)^{-1}x,x\right)\,\overline{\chi_\rho\,(g)}\,dg,$$

by changing variables from g to $g'gg'^{-1}$ for some fixed $g' \in G$. By the formula above, we obtain

$$K^+\left(t,x,x\right)^\rho = \left(\dim V^\rho\right) g' \cdot \int_G g \cdot K^+\left(t,g^{-1}\left(g'^{-1}x\right),g'^{-1}x\right)\,\overline{\chi_\rho\,(g)}\,dg \circ g'^{-1}$$

$$= g' \cdot K^+\left(t,g'^{-1}x,g'^{-1}x\right)^\rho \circ g'^{-1}.$$

Equivalently, as sections of $\mathrm{Hom}\,(E,E)$, for all $g \in G$,

$$K^+\left(t,gx,gx\right)^\rho = g \bullet K^+\left(t,x,x\right)^\rho = g \cdot K^+\left(t,x,x\right)^\rho \circ g^{-1}.$$

Furthermore, since the calculations above certainly apply to K^-,

$$\mathrm{tr}K^\pm\left(t,gx,gx\right)^\rho = \mathrm{tr}K^\pm\left(t,x,x\right)^\rho\,,$$

(5.4) $$\mathrm{str}K\left(t,gx,gx\right)^\rho = \mathrm{str}K\left(t,x,x\right)^\rho\,.$$

Thus, these traces descend to functions on the orbit space. In order to evaluate the integral of $K^\pm\left(t,x,x\right)^\rho$ over a neighborhood of a singular stratum Σ, it is sufficient to evaluate it over the normal bundle to Σ restricted to local sections of $\Sigma \to G\backslash\Sigma$.

Note that the calculations above may be applied to nonelliptic kernels. In particular, we may let D be a fiberwise equivariant elliptic operator; then the calculations above apply to it.

6. The Equivariant Index of an operator on a sphere

The purpose of this section is to do an explicit calculation of local heat trace asymptotics of a very specific type of operator; this analysis will apply to a much more general situation later in the paper. We will relate the local equivariant supertrace of a heat operator on Euclidean space to that of the natural pullback of the operator to the radial blowup of the origin. In what follows, we will make very explicit choices in order to do the calculation, but the final result will be independent of those choices.

6.1. **Polar coordinate form of constant coefficient operator on $\mathbb{R}^k$.** In this section, we consider the following situation. Let $Q\colon C^\infty\left(\mathbb{R}^k, \mathbb{C}^d\right) \to C^\infty\left(\mathbb{R}^k, \mathbb{C}^d\right)$ be an elliptic, first order, constant coefficient differential operator, where $k \geq 2$. Let

$$Q_1 = \sum_{j=1}^{k} A_j \partial_j,$$

where by assumption $A_j \in GL\left(d, \mathbb{C}\right)$, and for any $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$, $\det\left(\sum c_j A_j\right) = 0$ implies $c = 0$. Note that

$$Q_1^* = -\sum_{j=1}^{k} A_j^* \partial_j$$

We form the symmetric operator $Q\colon C^\infty\left(\mathbb{R}^k, \mathbb{C}^d \oplus \mathbb{C}^d\right) \to C^\infty\left(\mathbb{R}^k, \mathbb{C}^d \oplus \mathbb{C}^d\right)$

$$Q = \begin{pmatrix} 0 & Q_1^* \\ Q_1 & 0 \end{pmatrix} = \sum_{j=1}^{k} \begin{pmatrix} 0 & -A_j^* \\ A_j & 0 \end{pmatrix} \partial_j.$$

We write Q in polar coordinates $(r, \theta) \in (0, \infty) \times S^{k-1}$, with $x = r\theta = r\left(\theta_1, \ldots, \theta_k\right)$, as in Section 11

$$Q_1 = Z^+\left(\partial_r + \frac{1}{r}Q^{S+}\right), \text{ so that}$$

$$Q = \begin{pmatrix} 0 & \left(\partial_r + \frac{1}{r}Q^{S+}\right)^* Z^{+*} \\ Z^+\left(\partial_r + \frac{1}{r}Q^{S+}\right) & 0 \end{pmatrix}.$$

The purpose of this section is to prove the formula

$$\int_{B_\varepsilon(0)} \alpha^\beta = \int_{B_\varepsilon\left(\partial\frac{1}{2}DS^k\right)} \widetilde{\alpha}^\beta + \frac{-\eta\left(Q^{S+,\beta}\right) + h\left(Q^{S+,\beta}\right)}{2},$$

where $\alpha^\beta = \operatorname{str} K_Q\left(t, x, x\right)^\beta$ is the supertrace of the equivariant heat kernel of Q^2 restricted to sections of type β (β being an irreducible representation of a group that commutes with Q), and where $B_\varepsilon\left(0\right)$ is the ball of radius ε around the origin of $\mathbb{R}^k$, and where $\widetilde{\alpha}^\beta$ and $B_\varepsilon\left(\partial\frac{1}{2}DS^k\right)$ correspond to the same neighborhood after a radial blowup of the origin confined to a smaller ball.

In what follows, we will require another result. Assume $Z = -Z^*$, and $Q^S = Q^{S*}$ on the sphere, so that $Q^S Z + Z Q^S = (k-1) Z$. (We note that the second assumption is satisfied automatically; see Section 11.2.) Suppose that q is a smooth, positive, real-valued function on a subinterval I of $(0, \infty)$, and let $\widetilde{Q} \colon C^\infty \left(I \times S^{k-1}, \mathbb{C}^d \right) \to C^\infty \left(I \times S^{k-1}, \mathbb{C}^d \right)$ be defined by

$$\widetilde{Q} = Z \left(\partial_r + \frac{1}{q(r)} Q^S \right).$$

Let

$$ds^2 = [g(r)]^2 \, d\theta^2 + dr^2$$

be the metric on $I \times S^{k-1}$, where g is a smooth positive function. The formal adjoint of ∂_r with respect to this metric is then

$$\partial_r^* = -\partial_r - \frac{(k-1) g'(r)}{g(r)},$$

so that

$$\widetilde{Q}^* = \left(-\partial_r - \frac{(k-1) g'(r)}{g(r)} + \frac{1}{q(r)} Q^S \right) Z^*$$

$$= Z \left(\partial_r + \frac{(k-1) g'(r)}{g(r)} + \frac{1}{q(r)} Q^S - \frac{k-1}{q(r)} \right)$$

$$(6.1) \qquad = \widetilde{Q} + \left[\frac{(k-1) g'(r)}{g(r)} - \frac{k-1}{q(r)} \right] Z.$$

6.2. **An induced operator on a sphere.** We now consider a new operator induced from the operator in Section 6.1 as a symmetric operator on $\mathbb{C}^d \oplus \mathbb{C}^d$-valued functions on $S^k \subset \mathbb{R}^{k+1}$; in particular, the metric and operator will be chosen to agree exactly with the original operator near the south pole. Suppose that $H < O(k)$ acts on $\mathbb{R}^{k+1}$ in the first k coordinates and isometrically on the fibers $\mathbb{C}^d$ and commutes with Q defined in (11.2). Thus H represents on each copy of $\mathbb{C}^d$ as a unitary representation. Further, every element of H commutes with Z and with Q^s. Using these operators, we will define a new operator on a rank $2d$ bundle $E = E^+ \oplus E^-$ over $S^k \subset \mathbb{R}^{k+1}$ to obtain an H-equivariant elliptic operator on $\Gamma \left(S^k, E^+ \oplus E^- \right)$. In spherical coordinates $(\phi, \theta) \in [0, \pi] \times S^{k-1}$, with ϕ corresponding to the (spherical) distance from the south pole $x_{k+1} = -1$, the operator we consider is defined on the lower hemisphere $S^{k,s}$ to be

$$\mathbf{Q} = \begin{pmatrix} 0 & \left(\partial_\phi + \frac{1}{f(\phi)} Q^{S+} \right)^* Z^- \\ Z^+ \left(\partial_\phi + \frac{1}{f(\phi)} Q^{S+} \right) & 0 \end{pmatrix}$$

mapping $C^\infty \left(S^{k,s}, \mathbb{C}^d \oplus \mathbb{C}^d \right)$ to itself, where $f(\phi)$ is a function that is smooth and positive on $(0, \pi)$ and such that $f(\phi) = \phi$ for ϕ near zero and $f(\phi) = 1$ near $\phi = \frac{\pi}{2}$, so that the operator exactly agrees with the Euclidean operator Q

near the south pole and is a cylindrical operator near the equator. We choose the manifold metric $ds^2 = [g(\phi)]^2 d\theta^2 + d\phi^2$ near the south pole and equator to agree with the standard Euclidean metric and cylindrical metric, so that $g(\phi) = \phi$ for ϕ near zero and $g(\phi) = 1$ for ϕ near $\frac{\pi}{2}$. By Equation (6.1),

$$\left(\partial_\phi + \frac{1}{f(\phi)} Q^{S+}\right)^* = -\partial_\phi - \frac{(k-1)g'(\phi)}{g(\phi)} + \frac{1}{f(\phi)} Q^{S+},$$

Again, we must choose the bundle metric so that Z is a unitary transformation, so in fact Z^+ and $Z^- = (Z^+)^*$ are unitary as maps from $\mathbb{C}^d$ to itself. We now use the approach used in the index theory for manifolds with boundary to double the operator (see, for example, [11]). On the upper hemisphere, we use the same coordinates as on the reflection in the lower hemisphere so that the orientation is reversed, and we identify the opposite parts of the bundle $\mathbb{C}^d \oplus \mathbb{C}^d$ via the map Z over the cylinder. The operator Q has the same description in the upper hemisphere as in the lower hemisphere. That is, let (θ_s, ϕ_s) be the coordinates in the lower hemisphere, and let (θ_n, ϕ_n) be coordinates in the upper hemisphere. Near the equator, the transition function is

$$\theta_n = \theta_s; \ \phi_n = \pi - \phi_s.$$

The bundles $\mathbb{C}^d \oplus \mathbb{C}^d$ over the upper and lower hemispheres are identified as follows. A section $(L_+(\theta_s, \phi_s), L_-(\theta_s, \phi_s))$ in the lower hemisphere is identified with the section $(U_+(\theta_n, \phi_n), U_-(\theta_n, \phi_n))$ in the upper hemisphere near the equator if

$$Z^+(\theta_s)^{-1} L_-(\theta_s, \phi_s) = U_+(\theta_s, \pi - \phi_s) \text{ and}$$
$$Z^-(\theta_s)^{-1} L_+(\theta_s, \phi_s) = U_-(\theta_s, \pi - \phi_s), \text{ or}$$
$$\begin{pmatrix} 0 & Z^+(\theta_s)^{-1} \\ Z^-(\theta_s)^{-1} & 0 \end{pmatrix} \begin{pmatrix} L_+(\theta_s, \phi_s) \\ L_-(\theta_s, \phi_s) \end{pmatrix} = \begin{pmatrix} U_+(\theta_s, \pi - \phi_s) \\ U_-(\theta_s, \pi - \phi_s) \end{pmatrix}$$

The bundle E^+ is defined to be $\mathbb{C}^d \oplus \{0\}$ over the lower hemisphere and $\{0\} \oplus \mathbb{C}^d$ (with identifications as above), and the bundle E^- is defined similarly. The operator Q can be extended to be well-defined on entire sphere, as follows. Define it as before on the two hemispheres, and so that near the equator

$$\mathbf{Q} \begin{pmatrix} L_+(\theta_s, \phi_s) \\ L_-(\theta_s, \phi_s) \end{pmatrix} = \begin{pmatrix} 0 & \left(-\partial_{\phi_s} + Q^{S+}\right) Z^- \\ Z^+ \left(\partial_{\phi_s} + Q^{S+}\right) & 0 \end{pmatrix} \begin{pmatrix} L_+(\theta_s, \phi_s) \\ L_-(\theta_s, \phi_s) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \left(-\partial_{\phi_n} + Q^{S+}\right) Z^- \\ Z^+ \left(\partial_{\phi_n} + Q^{S+}\right) & 0 \end{pmatrix} \begin{pmatrix} U_+(\theta_s, \phi_n) \\ U_-(\theta_s, \phi_n) \end{pmatrix}$$

To check compatibility, we see that

$$
\begin{pmatrix} 0 & \left(-\partial_{\phi_n} + Q^{S+}\right) Z^- \\ Z^+ \left(\partial_{\phi_n} + Q^{S+}\right) & 0 \end{pmatrix} \begin{pmatrix} U_+ \left(\theta_s, \phi_n\right) \\ U_- \left(\theta_s, \phi_n\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & \left(\partial_{\phi_s} + Q^{S+}\right) Z^- \\ Z^+ \left(-\partial_{\phi_s} + Q^{S+}\right) & 0 \end{pmatrix} \begin{pmatrix} 0 & \left(Z^+\right)^{-1} \\ \left(Z^-\right)^{-1} & 0 \end{pmatrix} \begin{pmatrix} L_+ \left(\theta_s, \phi_s\right) \\ L_- \left(\theta_s, \phi_s\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} \left(-\partial_{\phi_s} + Q^{S+}\right) L_+ \left(\theta_s, \phi_s\right) \\ Z^+ \left(-\partial_{\phi_s} + Q^{S+}\right) \left(Z^+\right)^{-1} L_- \left(\theta_s, \phi_s\right) \end{pmatrix}
$$

We continue to simplify to obtain

$$
\begin{pmatrix} 0 & \left(-\partial_{\phi_n} + Q^{S+}\right) Z^- \\ Z^+ \left(\partial_{\phi_n} + Q^{S+}\right) & 0 \end{pmatrix} \begin{pmatrix} U_+ \left(\theta_s, \phi_n\right) \\ U_- \left(\theta_s, \phi_n\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & \left(Z^+\right)^{-1} \\ \left(Z^-\right)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \left(-\partial_{\phi_s} + Q^{S+}\right) Z^- \\ Z^+ \left(\partial_{\phi_s} + Q^{S+}\right) & 0 \end{pmatrix} \begin{pmatrix} L_+ \left(\theta_s, \phi_s\right) \\ L_- \left(\theta_s, \phi_s\right) \end{pmatrix}.
$$

Thus, the newly defined elliptic differential operator Q on $\Gamma\left(S^k, E^+ \oplus E^-\right)$ is well-defined. This operator is self adjoint, and remains H-equivariant by construction. In fact Q has trivial kernel and thus zero index, and all possible equivariant indices are likewise zero. Note that the bundle E is no longer trivial.

6.3. **Local contribution of the heat supertrace near the south pole.**

6.3.1. *Local elements of the kernel.* We now wish to compute the local supertrace of the equivariant heat kernel associated to Q, in particular the part near the south pole. Let $\beta \colon H \to GL\left(V_\beta\right)$ denote an irreducible representation of H. We know that the supertrace of $e^{-t\mathbf{Q}^2}$ restricted to sections of type β is the index of $Q^\beta = \mathbf{Q}|_{\Gamma\left(S^k, E^+ \oplus E^-\right)^\beta}$, which is zero; however, the local supertrace need not be trivial. Let $u \in \Gamma\left(S^k, E^+\right)^\beta$. Since H commutes with Q^S, $\Gamma\left(S^k, E^+\right)^\beta$ is a direct sum of eigenspaces of Q^S. Near the south pole we write

$$
u\left(\theta_s, \phi_s\right) = \sum_\lambda u_\lambda\left(\phi_s\right) f_\lambda\left(\theta_s\right),
$$

where $Q^S f_\lambda = Q^{S+} f_\lambda = \lambda f_\lambda$.

If $u \in \Gamma_{\mathrm{loc}}\left(S^k, E^+\right)^\beta$ is a local smooth solution to the equation $Q^\beta u = 0$ near the south pole, then

$$
\partial_{\phi_s} u_\lambda + \frac{\lambda}{\phi_s} u_\lambda = 0,
$$

or

$$
u_\lambda\left(\phi_s\right) = c_\lambda \phi_s^{-\lambda}
$$

for some $c_\lambda \in \mathbb{C}$. Note that for $k > 2$, $\lambda < \frac{k}{2} - 1$ if and only if $\phi_s^{-\lambda} f_\lambda\left(\theta_s\right)$ provides an element of the Sobolev space H^1 that locally solves the differential equation

(for $k = 2$, the condition is $\lambda \leq 0$). Because $\ker Q$ is (globally) trivial, it is not possible to continue this solution to a global H^1 solution of $Qu = 0$ on the whole sphere.

On the other hand, if $v \in \Gamma_{\text{loc}} \left(S^k, E^- \right)^\beta$ is a local smooth solution to $Q^\beta u = 0$, then

$$\mathbf{Q}^\beta v = \left(\partial_{\phi_s} + \frac{1}{\phi_s} Q^{S+} \right)^* Z^- v$$

$$= \left(-\partial_{\phi_s} - \frac{k-1}{\phi_s} + \frac{1}{\phi_s} Q^{S+} \right) Z^- v$$

$$= Z^- \left(\partial_{\phi_s} + \frac{1}{\phi_s} Q^{S-} \right) v = 0$$

so that $w = Z^- v \in \Gamma_{\text{loc}} \left(S^k, E^+ \right)^\beta$ satisfies

$$\left(-\partial_{\phi_s} - \frac{k-1}{\phi_s} + \frac{1}{\phi_s} Q^{S+} \right) w = 0.$$

We write

$$w \left(\theta_s, \phi_s \right) = \sum_\lambda w_\lambda \left(\phi_s \right) f_\lambda \left(\theta_s \right),$$

where as before $Q^S f_\lambda = Q^{S+} f_\lambda = \lambda f_\lambda$.

Then $Q^\beta \left(Z^- \right)^{-1} w = 0$ near the south pole implies

$$\partial_{\phi_s} w_\lambda + \frac{k - 1 - \lambda}{\phi_s} w_\lambda = 0,$$

or

$$w_\lambda \left(\phi_s \right) = e_\lambda \phi_s^{-k+1+\lambda}$$

for some $e_\lambda \in \mathbb{C}$. Observe that for $k > 2$, $\lambda > \frac{k}{2}$ if and only if $\phi_s^{-k+1+\lambda} \left(Z^- \right)^{-1} f_\lambda \left(\theta_s \right)$ provides an element of H^1 that locally solves $Qv = 0$ (the condition is $\lambda \geq 1$ for $k = 2$). Because $\ker Q$ is (globally) trivial, it is not possible to continue this solution (or a constant solution) to a global H^1 solution of $Q(Z^-)^{-1} w = 0$ on the whole sphere.

We remark further that if $k > 2$, the sections of E^+ that satisfy $Q^\beta u = 0$ on the sphere minus a small neighborhood of the south pole that do extend to the north pole are exactly those of the form $\sum_{\lambda > \frac{k}{2}} c_\lambda \phi_s^{-\lambda} f_\lambda \left(\theta_s \right)$ near the south pole (for $k = 2$ the sum is over all $\lambda \geq 1$). If $k > 2$, the sections of E^- that satisfy $Q^\beta u = 0$ on the sphere minus a small neighborhood of the south pole that do extend to the north pole are exactly those of the form $\sum_{\lambda < \frac{k}{2} - 1} c_\lambda \phi_s^{-\lambda} f_\lambda \left(\theta_s \right)$ near the south pole (for $k = 2$ the sum is over all $\lambda \leq 0$).

6.3.2. *A related boundary value problem.* In this section, we will replace the index problem for Q^β on the sphere with a related Atiyah-Patodi-Singer boundary problem, and we will explicitly determine the difference between these indices in terms of local heat kernel supertraces. Choose $\varepsilon > 0$ such that $f(y) = y$ on the 5ε-ball centered at the south pole. Consider all the data similar as before, but we modify the operator near the south pole $x_{k+1} = -1$. We replace ϕ_s by $r(\phi_s) = \exp\left(C_\varepsilon + \int_0^{\phi_s} \frac{1}{\psi(y)} dy\right)$ for small ϕ_s, where $\psi \colon \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\psi(y) = y$ in a neighborhood of $y = 3\varepsilon$ and $\psi(y) = \varepsilon$ for $y \le \varepsilon$, and where C_ε is a constant chosen so that $C_\varepsilon + \int_0^{3\varepsilon} \frac{1}{\psi(y)} dy = \log(3\varepsilon)$. Suppose that $u(\phi_s, \theta_s) = \sum_{\lambda < \frac{k}{2}-1} c_\lambda \phi_s^{-\lambda} f_\lambda(\theta_s) \in \Gamma\left(S^k, E^+\right)^\beta$ is a local H^1 solution to $Q^\beta u = 0$ near the south pole. Then

$$\widetilde{u}(\phi_s, \theta) := u(r(\phi_s), \theta) = \sum_{\lambda < \frac{k}{2}-1} c_\lambda r^{-\lambda} f_\lambda(\theta)$$

$$= \sum_{\lambda < \frac{k}{2}-1} c_\lambda \exp\left(-\lambda\left(C_\varepsilon + \int_0^{\phi_s} \frac{1}{\psi(y)} dy\right)\right) f_\lambda(\theta)$$

are local solutions to the equation

$$\widetilde{Q}\widetilde{u} = Z^+\left(\partial_{\phi_s} + \frac{1}{\psi(\phi_s)} Q^{S+}\right)\widetilde{u} = 0.$$

(In the above sums, for $k = 2$ the condition $\lambda < \frac{k}{2} - 1$ is replaced by $\lambda \le 0$.) For $\phi_s < \varepsilon$, the equation reads

$$\left(\partial_{\phi_s} + \frac{1}{\varepsilon} Q^{S+}\right)\widetilde{u} = 0,$$

and for $\phi_s > 3\varepsilon$, the equation reads

$$\left(\partial_{\phi_s} + \frac{1}{f(\phi_s)} Q^{S+}\right) u = 0.$$

Note that $\widetilde{u}$ is H-equivariant, because the transformation above affects the ϕ_s variable alone. Further, observe that the map $u \to \widetilde{u}$ is continuous with respect to the C^∞ norm and is injective, and the H^1 condition on u translates to the boundary condition $(\phi_s = 0)$

$$P_{\ge \frac{k}{2}-1}(\widetilde{u}|_\partial) = 0$$
$$\text{(or } P_{>0}(\widetilde{u}|_\partial) = 0 \text{ if } k = 2\text{)},$$

where $P_{\ge \frac{k}{2}-1}$ is the spectral projection onto the span of the eigenspaces of $\left(Q^{S+}\right)^\beta$ corresponding to eigenvalues at least $\frac{k}{2}-1$. Note that no such $\widetilde{u}$ may be continued to a global element of the kernel of the modified Q^β, by the reasoning used in the previous section. Thus the boundary value problem for $\widetilde{u}$ with spectral boundary

condition as above yields no kernel, as does the original problem for Q^β on the sphere.

Next, we modify the metric in a very specific way. We will choose the metric so that

(6.2) $$ds^2 = [\tilde{g}(\phi_s)]^2 \, d\theta^2 + d\phi_s^2,$$

where $\tilde{g}(\phi_s) = \phi_s$ in a neighborhood of $\phi_s = 3\varepsilon$ (so that it agrees with the Euclidean metric), and such that $\tilde{g}(\phi_s)$ is constant if $\phi_s < \varepsilon$ (the cylindrical metric). Then, if $v \in \Gamma\left(S^k, E^-\right)^\beta$ is a local solution to the equation $\left(\mathbf{Q}^\beta\right)^* v = 0$, then $w = \left(Z^-\right)^{-1} v$ satisfies

$$\widetilde{w}(\phi_s, \theta) := w\left(r(\phi_s), \theta\right) = \sum_{\lambda > \frac{k}{2}} c_\lambda r^{-\lambda} f_\lambda(\theta)$$

$$= \sum_{\lambda > \frac{k}{2}} c_\lambda g(\phi_s)^{-(k-1)} \exp\left(\lambda\left(C_\varepsilon + \int_0^{\phi_s} \frac{1}{\psi(y)} dy\right)\right) f_\lambda(\theta)$$

are local solutions to the equation

$$\left(-\partial_{\phi_s} - \frac{(k-1)\tilde{g}'(\phi_s)}{\tilde{g}(\phi_s)} + \frac{1}{\psi(\phi_s)} Q^{S+}\right) \widetilde{w} = 0.$$

(In the sums above, if $k = 2$ then the condition $\lambda > \frac{k}{2}$ is replaced by $\lambda \geq 1$.) For $\phi_s < \varepsilon$, the equation reads

$$\left(-\partial_{\phi_s} + \frac{1}{\varepsilon} Q^{S+}\right) \widetilde{w} = 0,$$

and for $\phi_s > 3\varepsilon$, the equation reads

$$\left(-\partial_{\phi_s} - \frac{(k-1)\tilde{g}'(\phi_s)}{\tilde{g}(\phi_s)} + \frac{1}{f(\phi_s)} Q^{S+}\right) \widetilde{w} = 0.$$

This time $\widetilde{w}$ satisfies the boundary condition ($\phi_s = 0$)

$$P_{\leq \frac{k}{2}}\left(\widetilde{w}|_\partial\right) = 0$$
$$\left(\text{or } P_{<1}\left(\widetilde{w}|_\partial\right) = 0 \text{ if } k = 2\right),$$

where $P_{\leq \frac{k}{2}}$ is the spectral projection onto the span of the eigenspaces of $\left(Q^{S+}\right)^\beta$ corresponding to eigenvalues at most $\frac{k}{2}$. Note that no such $\widetilde{w}$ may be continued to a global element of the kernel of the modified Q^β, by the reasoning used in the previous section. Thus the index problem for the original Q^β (for u and w) is the same as the boundary value "index problem" for $\widetilde{u}$ and $\widetilde{w}$ described as above.

Observe that the operator and metric now fit the conditions of the Atiyah-Patodi-Singer theorem ([3]) for manifolds with boundary, but the boundary conditions are not the same. More precisely, the related situation is that of the equivariant version of the Atiyah-Patodi-Singer for elliptic operators in

[17, Theorem 1.2]. The equivariant theorem actually gives a formula for the character $\mathrm{ind}_g(\mathbf{Q})$, which by ellipticity is a smooth function on H. We are interested in $\mathrm{ind}^\beta(\mathbf{Q})$, which is easily computable from $\mathrm{ind}_g(\mathbf{Q})$ by determining the β-component of the character. All of the parts of the equivariant index formula (the integral of the heat supertrace, the eta invariant, and the dimension of the kernel) are smooth functions on H and can be computed similarly. The resulting equivariant eta invariant of any H-equivariant operator L is defined to be

$$(6.3) \qquad \eta\left(L^\beta\right) = \sum_{\lambda^\beta \neq 0} \mathrm{sign}\left(\lambda^\beta\right)\left|\lambda^\beta\right|^{-s}\Bigg|_{s=0},$$

where $\beta\colon H \to U\left(V_\beta\right)$ is an irreducible representation and where the sum is over all nonzero eigenvalues (including multiplicities) of L restricted to the space of sections of type β. (Recall that the eigenvalues grow as in Formula (3.3).)

The Atiyah-Patodi-Singer boundary conditions on the punctured sphere would be

$$P_{\geq 0}\widetilde{u} = 0;\ \ P_{<0}\widetilde{w} = 0.$$

If these conditions are used instead of the conditions induced from the closed manifold problem, note that again no such $\widetilde{u}$ could be continued to the north pole, because this would require that $\lambda < 0$ and $\lambda > \frac{k}{2}$ (or $\lambda \geq 1$ for $k = 2$) simultaneously. However, the possibility exists that some choices of $\widetilde{w}$ could be continued to sections of $\ker Q^{\beta-}$ over the north pole, requiring only that $0 \leq \lambda < \frac{k}{2} - 1$ (or $0 \leq \lambda \leq 0$ for $k = 2$). By Proposition 11.12, the only part of the spectrum of Q^{S+} in this range is the zero eigenvalue, and thus the equivariant Atiyah-Patodi-Singer index of the modified Q^β is

$$(6.4) \qquad -h\left(Q^{S+,\beta}\right) := -\dim\ker Q^{S+,\beta} = -\dim \mathbb{C}^{d,\beta} = -\dim E^{+,\beta}.$$

By the equivariant version of the Atiyah-Patodi-Singer Theorem, we have that

$$-h\left(Q^{S+,\beta}\right) = \int_{\frac{1}{2}DS^k} \widetilde{\alpha}^\beta - \frac{1}{2}\left(\eta\left(Q^{S+,\beta}\right) + h\left(Q^{S+,\beta}\right)\right)$$

$$= \int_{S^k - B_\varepsilon(SP)}^\beta \widetilde{\alpha}^\beta + \int_{B_\varepsilon\left(\partial\frac{1}{2}DS^k\right)} \widetilde{\alpha}^\beta - \frac{1}{2}\left(\eta\left(Q^{S+,\beta}\right) + h\left(Q^{S+,\beta}\right)\right),\ \text{ or}$$

$$0 = \int_{S^k - B_\varepsilon(SP)} \widetilde{\alpha}^\beta + \int_{B_\varepsilon\left(\partial\frac{1}{2}DS^k\right)} \widetilde{\alpha}^\beta + \frac{1}{2}\left(-\eta\left(Q^{S+,\beta}\right) + h\left(Q^{S+,\beta}\right)\right),$$

where $\widetilde{\alpha}^\beta$ is the the supertrace of the heat kernel of Q^β over the double DS^k of S^k (blown up at the south pole), where $\frac{1}{2}DS^k$ means the half of the double, where $B_\varepsilon(SP)$ means an ε-neighborhood of the south pole, and where $\eta\left(Q^{S+,\beta}\right)$ is the eta invariant of $Q^{S+,\beta}$. The set $B_\varepsilon\left(\partial\frac{1}{2}DS^k\right)$ means the ε-collar around the boundary $\partial\frac{1}{2}DS^k$, so that $\frac{1}{2}DS^k = \left(S^k - B_\varepsilon(SP)\right) \cup B_\varepsilon\left(\partial\frac{1}{2}DS^k\right)$. Thus, if α_S^β

denotes the local heat supertrace of the operator on the sphere,

$$0 = \text{ind}\left(\mathbf{Q}^{\beta}\right) = \int_{S^k} \alpha_S^{\beta}$$

$$= \int_{S^k - B_\varepsilon(SP)} \alpha_S^{\beta} + \int_{B_\varepsilon(SP)} \alpha_S^{\beta}$$

$$= \int_{S^k - B_\varepsilon(SP)} \widetilde{\alpha}^{\beta} + \int_{B_\varepsilon\left(\partial \frac{1}{2} DS^k\right)} \widetilde{\alpha}^{\beta} + \frac{1}{2}\left(-\eta\left(Q^{S+,\beta}\right) + h\left(Q^{S+,\beta}\right)\right).$$

If α^{β} denotes the Euclidean heat supertrace corresponding to the original operator Q^{β} on $\mathbb{R}^k$,

$$\int_{S^k - B_\varepsilon(SP)} \alpha_S^{\beta} = \int_{S^k - B_\varepsilon(SP)} \widetilde{\alpha}^{\beta} + \mathcal{O}\left(e^{-c/t}\right), \text{ and}$$

$$\int_{B_\varepsilon(0)} \alpha^{\beta} = \int_{B_\varepsilon(SP)} \alpha_S^{\beta} + \mathcal{O}\left(e^{-c/t}\right),$$

for some $c > 0$, since the local data is the same. Then

$$(6.5) \quad \int_{B_\varepsilon(0)} \alpha^{\beta} = \int_{B_\varepsilon\left(\partial \frac{1}{2} DS^k\right)} \widetilde{\alpha}^{\beta} + \frac{1}{2}\left(-\eta\left(Q^{S+,\beta}\right) + h\left(Q^{S+,\beta}\right)\right) + \mathcal{O}\left(e^{-c/t}\right),$$

the contribution of the equivariant heat supertrace of the constant coefficient operator Q^{β} in an ε-ball around the origin of $\mathbb{R}^k$. As explained in Remark 11.13, $\eta\left(Q^{S+,\beta}\right)$ and $h\left(Q^{S+,\beta}\right)$ are invariant under stable H-equivariant homotopies of the original operator Q. We note that the formula above is scale invariant and independent of all the various choices we made through the calculation. It expresses the index contribution of the Euclidean neighborhood in terms of the index contribution from the same neighborhood of the blown-up operator, where a small ball around the center of the Euclidean neighborhood is replaced by a collar. Because of the homotopy and scale invariance, this formula remains valid if the operator is blown up in any way so that it is constant on the small collared neighborhood and so that the spherical part $Q^{S+,\beta}$ comes from the polar coordinate expression. Similarly, the metric may be chosen in any way as long as it is cylindrical on the collar.

6.4. **The case $k = 1$.** We repeat the analysis for the case $k = 1$. We consider a operator $Q \colon C^{\infty}\left(\mathbb{R}, \mathbb{C}^d \oplus \mathbb{C}^d\right) \to C^{\infty}\left(\mathbb{R}, \mathbb{C}^d \oplus \mathbb{C}^d\right)$ of the form

$$Q = A\partial_x$$

$$= \begin{pmatrix} 0 & Q_1^* \\ Q_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -A_1^* \\ A_1 & 0 \end{pmatrix} \partial_x = \begin{cases} A\partial_r & \text{if } x > 0 \\ -A\partial_r & \text{if } x < 0, \end{cases}$$

where by assumption $A_1 \in GL(d, \mathbb{C})$ and $r = |x|$. Suppose that H acts nontrivially on $\mathbb{R}$ by multiplication by ± 1, that H acts unitarily on each copy of $\mathbb{C}^d$, and that Q is H-equivariant. Let H_0 be the (normal) subgroup of H that fixes $\mathbb{R}$; then H_0 contains the identity component of H, and $H \diagup H_0 \cong \mathbb{Z}_2$. Let $\gamma \in H$ be chosen so that γH_0 generates $H \diagup H_0$. Then we have that

$$hA = Ah \text{ for all } h \in H_0, \text{ and}$$
$$\gamma A = -A\gamma,$$

or if we write

$$h = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}, \ \gamma = \begin{pmatrix} \gamma_+ & 0 \\ 0 & \gamma_- \end{pmatrix},$$

we have

$$h_- = A_1 h_+ A_1^{-1}$$
(6.6)
$$\gamma_- = -A_1 \gamma_+ A_1^{-1}.$$

Further, if $i\lambda \in i\mathbb{R} \setminus \{0\}$ is an eigenvalue of A corresponding to eigenspace

$$E_{i\lambda} = \left\{ \begin{pmatrix} v_+ \\ v_- \end{pmatrix} : A \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = i\lambda \begin{pmatrix} v_+ \\ v_- \end{pmatrix} \right\},$$

then

$$v_- = \frac{-i}{\lambda} A_1 v_+,$$

and

$$E_{-i\lambda} = \left\{ \begin{pmatrix} v_+ \\ -v_- \end{pmatrix} : \begin{pmatrix} v_+ \\ v_- \end{pmatrix} \in E_{i\lambda} \right\}.$$

By (6.6), if $W_{+,|\lambda|}^\beta$ is the subspace of $E_{i\lambda} \cap \mathbb{C}^{d+} = E_{-i\lambda} \cap \mathbb{C}^{d+}$ consisting of the vectors of H-representation type $[\beta, V_\beta]$, then

$$W_{-,|\lambda|}^{\overline{\beta}} := \frac{-i}{\lambda} A_1 W_{+,|\lambda|}^\beta \cong W_{+,|\lambda|}^\beta$$

is the subspace of $E_{i\lambda} \cap \mathbb{C}^{d-} = E_{-i\lambda} \cap \mathbb{C}^{d-}$ consisting of vectors of H-representation type $\overline{\beta}$, where $\overline{\beta}$ is the "conjugate" of β, defined up to equivalence as

$$\overline{\beta}(h) = \beta(h) \text{ for all } h \in H_0, \text{ and}$$
$$\overline{\beta}(\gamma) = -\beta(\gamma).$$

Observe that $W_{+,|\lambda|}^\beta$ is the β part of the λ^2 eigenspace of $A_1^* A_1$ consisting of vectors of type β, and $W_{-,|\lambda|}^\beta$ is the β part of the λ^2 eigenspace of $A_1 A_1^*$ consisting of vectors of type β. Let $P_{\pm,|\lambda|}^\beta \colon \mathbb{C}^{d\pm} \to W_{\pm,|\lambda|}^\beta$ denote the orthogonal projections.

We now calculate the local contribution $\int_{x=-\varepsilon}^{\varepsilon} \alpha^\beta$ of the equivariant heat supertrace corresponding to

$$Q^2 = \begin{pmatrix} -A_1^* A_1 & 0 \\ 0 & -A_1 A_1^* \end{pmatrix} \partial_x^2.$$

We have that

$$Q^2 P^\beta_{\pm,|\lambda|} = -\lambda^2 \partial_x^2 P^\beta_{\pm,|\lambda|} = -\lambda^2 P^\beta_{\pm,|\lambda|} \partial_x^2$$

Since the fundamental solution of $\frac{\partial}{\partial t} - \lambda^2 \frac{\partial^2}{\partial x^2}$ on $\mathbb{R}$ is $\frac{1}{|\lambda|\sqrt{4\pi t}} \exp\left(-(x-y)^2 / (4\lambda^2 t)\right)$, if $K(t, x, y)$ is the kernel of the operator e^{-tQ^2} on $\mathbb{R}$, for $0 < \delta < \varepsilon$, we have

$$\frac{1}{\dim V_\beta} \int_{x=-\delta}^\delta \alpha^\beta = \int_{x=-\delta}^\delta \int_{h \in H_0} \left(\text{tr } h \cdot K^+(t, x, x) - \text{tr } h \cdot K^-(t, x, x)\right) \overline{\chi_\beta(h)} \, dh$$

$$+ \int_{h \in H_0} \left(\text{tr } \gamma h \cdot K^+(t, \gamma^{-1}x, x) - \text{tr } \gamma h \cdot K^-(t, \gamma^{-1}x, x)\right) \overline{\chi_\beta(h)} \, dh \, dx$$

$$= \int_{h \in H_0} \sum_{|\lambda|,\tau} \frac{2\delta}{|\lambda|\sqrt{4\pi t}} \left(\text{tr } \left(h \cdot P^\tau_{+,|\lambda|}\right) - \text{tr } \left(h \cdot P^\tau_{-,|\lambda|}\right)\right) \overline{\chi_\beta(h)} \, dh$$

$$+ \int_{x=-\delta}^\delta \int_{h \in H_0} \sum_{|\lambda|,\tau} \frac{1}{|\lambda|\sqrt{4\pi t}} \exp\left(\frac{-|2x|^2}{4\lambda^2 t}\right) \left(\text{tr } \left(\gamma h \cdot \sum_{|\lambda|,\tau} P^\tau_{+,|\lambda|}\right)\right.$$

$$\left. - \text{tr } \left(\gamma h \cdot \sum_{|\lambda|,\tau} P^\tau_{-,|\lambda|}\right)\right) \overline{\chi_\beta(h)} \, dh \, dx.$$

Next, since $\int_{-\infty}^\infty \frac{1}{|\lambda|\sqrt{4\pi t}} \exp\left(\frac{-|2x|^2}{4\lambda^2 t}\right) dx = \frac{1}{2}$,

$$\frac{1}{\dim V_\beta} \int_{x=-\delta}^\delta \alpha^\beta = \sum_{|\lambda|,\tau} \frac{2\delta}{|\lambda|\sqrt{4\pi t}} \int_{h \in H_0} \left(\text{tr } \left(h \cdot P^\tau_{+,|\lambda|}\right) - \text{tr } \left(h \cdot P^\tau_{-,|\lambda|}\right)\right) \overline{\chi_\beta(h)} \, dh$$

$$+ \frac{1}{2} \sum_{|\lambda|,\tau} \int_{h \in H_0} \left(\text{tr } \left(\gamma h \cdot P^\tau_{+,|\lambda|}\right) - \text{tr } \left(\gamma h \cdot P^\tau_{-,|\lambda|}\right)\right) \overline{\chi_\beta(h)} \, dh + O\left(e^{-c/t}\right).$$

By (3.1) we conclude that the contribution of the equivariant heat supertrace on $(-\varepsilon, \varepsilon) \subset \mathbb{R}^1$ is, for any δ such that $0 < \delta < \varepsilon$,

$$\int_{-\varepsilon}^\varepsilon \alpha^\beta = \int_{[-\varepsilon,\varepsilon] \setminus [-\delta,\delta]} \alpha^\beta + \frac{1}{2}\left(\dim \left(\mathbb{C}^{d+}\right)^\beta - \dim \left(\mathbb{C}^{d+}\right)^{\overline{\beta}}\right) + O\left(e^{-c/t}\right).$$

In order to make our formulas consistent throughout the paper, we define for the (equivariant) zero operator $0 : \mathbb{C}^{d+} \to \mathbb{C}^{d-}$

(6.7) $$h\left(0^\beta\right) := \dim \left(\mathbb{C}^{d+}\right)^\beta; \quad \eta\left(0^\beta\right) := \dim \left(\mathbb{C}^{d+}\right)^{\overline{\beta}},$$

so that

(6.8) $$\int_{-\varepsilon}^\varepsilon \alpha^\beta = \int_{[-\varepsilon,\varepsilon] \setminus [-\delta,\delta]} \alpha^\beta + \frac{1}{2}\left(h\left(0^\beta\right) - \eta\left(0^\beta\right)\right) + \mathcal{O}\left(e^{-c/t}\right).$$

7. Desingularizing along a singular stratum

7.1. **Topological Desingularization.** Assume that G is a compact Lie group that acts on a Riemannian manifold M by isometries. We will construct a new G-manifold N that has a single stratum (of type $[G_0]$) and that is a branched cover of M, branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the desingularization of M and is denoted $\widetilde{M}$. We also refer to [1] for their recent related explanation of this desingularization (which they call *resolution*).

A sequence of modifications is used to construct N and $\widetilde{M} \subset N$. Let M_j be a minimal stratum. Let $T_\varepsilon(M_j)$ denote a tubular neighborhood of radius ε around M_j, with ε chosen sufficiently small so that all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$. Let

$$N^1 = (M \setminus T_\varepsilon(M_j)) \cup_{\partial T_\varepsilon(M_j)} (M \setminus T_\varepsilon(M_j))$$

be the manifold constructed by gluing two copies of $(M \setminus T_\varepsilon(M_j))$ smoothly along the boundary (see Section 7.3 for the codimension one case). Since the $T_\varepsilon(M_j)$ is saturated (a union of G-orbits), the G-action lifts to N^1. Note that the strata of the G-action on N^1 correspond to strata in $M \setminus T_\varepsilon(M_j)$. If $M_k \cap (M \setminus T_\varepsilon(M_j))$ is nontrivial, then the stratum corresponding to isotropy type $[G_k]$ on N^1 is

$$N_k^1 = (M_k \cap (M \setminus T_\varepsilon(M_j))) \cup_{(M_k \cap \partial T_\varepsilon(M_j))} (M_k \cap (M \setminus T_\varepsilon(M_j))).$$

Thus, N^1 is a G-manifold with one fewer stratum than M, and $M \setminus M_j$ is diffeomorphic to one copy of $(M \setminus T_\varepsilon(M_j))$, denoted $\widetilde{M}^1$ in N^1. In fact, N^1 is a branched double cover of M, branched over M_j. If N^1 has one orbit type, then we set $N = N^1$ and $\widetilde{M} = \widetilde{M}^1$. If N^1 has more than one orbit type, we repeat the process with the G-manifold N^1 to produce a new G-manifold N^2 with two fewer orbit types than M and that is a 4-fold branched cover of M. Again, $\widetilde{M}^2$ is a fundamental domain of $\widetilde{M}^1 \setminus \{$a minimal stratum$\}$, which is a fundamental domain of M with two strata removed. We continue until $N = N^r$ is a G-manifold with all orbits of type $[G_0]$ and is a 2^r-fold branched cover of M, branched over $M \setminus M_0$. We set $\widetilde{M} = \widetilde{M}^r$, which is a fundamental domain of M_0 in N.

Further, one may independently desingularize $M_{\geq j}$, since this submanifold is itself a closed G-manifold. If $M_{\geq j}$ has more than one connected component, we may desingularize all components simultaneously. The isotropy type of all points of $\widetilde{M_{\geq j}}$ is $[G_j]$, and $\widetilde{M_{\geq j}}/G$ is a smooth (open) manifold.

7.2. **Modification of the metric and differential operator.** We now more precisely describe the desingularization. If M is equipped with a G-equivariant, transversally elliptic differential operator on sections of an equivariant vector bundle over M, then this data may be pulled back to the desingularization $\widetilde{M}$. Given the bundle and operator over N^j, simply form the invertible double of the

operator on N^{j+1}, which is the double of the manifold with boundary $N^j \setminus T_\varepsilon(\Sigma)$, where Σ is a minimal stratum on N^j.

Specifically, we modify the metric equivariantly so that there exists $\varepsilon > 0$ such that the tubular neighborhood $B_{2\varepsilon}\Sigma$ of Σ in N^j is isometric to a ball of radius 2ε in the normal bundle $N\Sigma$. In polar coordinates, this metric is $ds^2 = dr^2 + d\sigma^2 + r^2 d\theta_\sigma^2$, with $r \in (0, 2\varepsilon)$, $d\sigma^2$ is the metric on Σ, and $d\theta_\sigma^2$ is the metric on $S(N_\sigma\Sigma)$, the unit sphere in $N_\sigma\Sigma$; note that $d\theta_\sigma^2$ is isometric to the Euclidean metric on the unit sphere. We simply choose the horizontal metric on $B_{2\varepsilon}\Sigma$ to be the pullback of the metric on the base Σ, the fiber metric to be Euclidean, and we require that horizontal and vertical vectors be orthogonal. We do not assume that the horizontal distribution is integrable.

Next, we replace r^2 with $f(r) = [\tilde{g}(r)]^2$ in the expression for the metric, where $\tilde{g}$ is defined as in (6.2), so that the metric is cylindrical for small r.

In our description of the modification of the differential operator, we will need the notation for the (external) product of differential operators. Suppose that $F \hookrightarrow X \xrightarrow{\pi} B$ is a fiber bundle that is locally a metric product. Given an operator $A_{1,x} : \Gamma(\pi^{-1}(x), E_1) \to \Gamma(\pi^{-1}(x), F_1)$ that is locally given as a differential operator $A_1 : \Gamma(F, E_1) \to \Gamma(F, F_1)$ and $A_2 : \Gamma(B, E_2) \to \Gamma(B, F_2)$ on Hermitian bundles, we define the product

$$A_{1,x} * A_2 : \Gamma(X, (E_1 \boxtimes E_2) \oplus (F_1 \boxtimes F_2)) \to \Gamma(X, (F_1 \boxtimes E_2) \oplus (E_1 \boxtimes F_2))$$

as the unique linear operator that satisfies locally

$$A_{1,x} * A_2 = \begin{pmatrix} A_1 \boxtimes \mathbf{1} & -\mathbf{1} \boxtimes A_2^* \\ \mathbf{1} \boxtimes A_2 & A_1^* \boxtimes \mathbf{1} \end{pmatrix}$$

on sections of

$$\begin{pmatrix} E_1 \boxtimes E_2 \\ F_1 \boxtimes F_2 \end{pmatrix}$$

of the form $\begin{pmatrix} u_1 \boxtimes u_2 \\ v_1 \boxtimes v_2 \end{pmatrix}$, where $u_1 \in \Gamma(F, E_1)$, $u_2 \in \Gamma(B, E_2)$, $v_1 \in \Gamma(F, F_1)$, $v_2 \in \Gamma(B, E_2)$. This coincides with the product in various versions of K-theory (see, for example, [2], [35, pp. 384ff]), which is used to define the Thom Isomorphism in vector bundles.

Let $D = D^+ : \Gamma(N^j, E^+) \to \Gamma(N^j, E^-)$ be the given first order, transversally elliptic, G-equivariant differential operator. Let Σ be a minimal stratum of N^j. Here we assume that Σ has codimension at least two. We modify the metrics and bundles equivariantly so that there exists $\varepsilon > 0$ such that the tubular neighborhood $B_\varepsilon(\Sigma)$ of Σ in M is isometric to a ball of radius ε in the normal bundle $N\Sigma$, and so that the G-equivariant bundle E over $B_\varepsilon(\Sigma)$ is a pullback of the bundle $E|_\Sigma \to \Sigma$. We assume that near Σ, after a G-equivariant homotopy D^+ can be written on $B_\varepsilon(\Sigma)$ locally as the product

$$D^+ = (D_N * D_\Sigma)^+,$$

where D_Σ is a transversally elliptic, G-equivariant, first order operator on the stratum Σ, and D_N is a G-equivariant, first order operator on $B_\varepsilon(\Sigma)$ that is elliptic on the fibers. If r is the distance from Σ, we write D_N in polar coordinates as

$$D_N = Z\left(\nabla^E_{\partial_r} + \frac{1}{r}D^S\right)$$

where $Z = -i\sigma(D_N)(\partial_r)$ is a local bundle isomorphism and the map D^S is a purely first order operator that differentiates in the unit normal bundle directions tangent to $S_x\Sigma$.

We modify the operator D_N on each Euclidean fiber of $N\Sigma \xrightarrow{\pi} \Sigma$ exactly as in Section 6.3.2; the result is a G-manifold $\widetilde{M}^j$ with boundary $\partial\widetilde{M}^j$, a G-vector bundle $\widetilde{E}^j$, and the induced operator $\widetilde{D}^j$, all of which locally agree with the original counterparts outside $B_\varepsilon(\Sigma)$. We may double $\widetilde{M}^j$ along the boundary $\partial\widetilde{M}^j$ and reverse the chirality of $\widetilde{E}^j$ as described in Section 6.2 for the case of a hemisphere and in general in [11, Ch. 9]. Doubling produces a closed G-manifold N^j, a G-vector bundle E^j, and a first-order transversally elliptic differential operator D^j. This process may be iterated until all orbits of the resulting G-manifold are principal. The case where some strata have codimension 1 is addressed in the following section.

7.3. **Codimension one case.** We now give the definitions of the previous section for the case when there is a minimal stratum Σ of codimension 1. Note that this means that the isotropy subgroup H corresponding to Σ contains a principal isotropy subgroup of index two. Again we modify the metrics and bundles equivariantly so that the tubular neighborhood $B_\varepsilon\Sigma$ of Σ in M is isometric to a ball of radius ε in the normal bundle $N\Sigma$ and so that the G-equivariant bundle E over $B_\varepsilon(\Sigma)$ is a pullback of the bundle $E|_\Sigma \to \Sigma$. Near Σ, D^+ can be written on $B_\varepsilon(\Sigma)$ locally as the product

(7.1) $$D^+ = (D_N * D_\Sigma)^+,$$

where D_Σ is a transversally elliptic, G-equivariant, first order operator on the stratum Σ, and D_N is a G-equivariant, first order operator on $B_\varepsilon(\Sigma)$ that is elliptic on the fibers. If r is the distance from Σ, we write D_N away from Σ as

$$D_N = Z\left(\nabla^E_{\partial_r} + \frac{1}{r}D^S\right) = Z\nabla^E_{\partial_r}$$

where $Z = -i\sigma(D_N)(\partial_r)$ is a local bundle isomorphism and the map $D^S = 0$.

In this case, there is no reason to modify the metric inside $B_\varepsilon(\Sigma)$. The equivariant "desingularization" of M along Σ is the manifold with boundary $\widetilde{M} = M \setminus B_\delta(\Sigma)$ for some $0 < \delta < \varepsilon$; the singular stratum is replaced by the boundary $\partial\widetilde{M} = S_\delta(\Sigma)$, which is a two-fold cover of Σ and whose normal bundle

is necessarily oriented (via ∂_r). The double M' is identical to the double of $\widetilde{M}$ along its boundary, and as in the previous section M' contains one less stratum.

7.4. Discussion of operator product assumption.
We now explain specific situations that guarantee that, after a G-equivariant homotopy, D^+ may be written locally as a product of operators as in (7.1) over the tubular neighborhhood $B_\varepsilon(\Sigma)$ over a singular stratum Σ. This demonstrates that this assumption is not overly restrictive. In [39], a large variety of examples of naturally defined transversal operators similar to Dirac operators are explored and shown under suitable conditions to provide all possible index classes of equivariant transversally elliptic operators. The principal symbols of these operators always have the form described below, and they almost always satisfy one of the spin^c conditions. We also emphasize that one might think that this assumption places conditions on the curvature of the normal bundle $N\Sigma$; however, this is not the case for the following reason. The condition is on the G-homotopy class of the principal transverse symbol of D. The curvature of the bundle only effects the zeroth order part of the symbol. For example, if $Y \to X$ is any fiber bundle over a spin^c manifold X with fiber F, then a Dirac-type operator D on Y has the form $D = \partial_X * D_F + Z$, where D_F is a family of fiberwise Dirac-type operators, ∂_X is the spin^c Dirac operator on X, and Z is a bundle endomorphism.

First, we show that if D^+ is G-homotopic to a transversal Dirac operator at points of Σ, and if either Σ is spin^c or its normal bundle $N\Sigma \to \Sigma$ is (fiberwise) spin^c, then it has the desired form. Moreover, we also remark that certain operators, like those resembling transversal de Rham operators, always satisfy this splitting condition with no assumptions on Σ.

Let $N\mathcal{F}$ be normal bundle of the foliation$\mathcal{F}$ by G-orbits in Σ, and let $N\Sigma$ be the normal bundle of Σ in M. Suppose that (after a G-homotopy) the principal transverse symbol of D^+ (evaluated at $\xi \in N_x^*\mathcal{F} \oplus N_x^*\Sigma$) at points $x \in \Sigma$ takes the form of a constant multiple of Clifford multiplication. That is, we assume there is an action c of $\mathbb{Cl}(N\mathcal{F} \oplus N\Sigma)$ on E and a Clifford connection ∇ on E such that the local expression for D is given by the composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*M) \xrightarrow{\text{proj}} \Gamma(E \otimes (N^*\mathcal{F} \oplus N^*\Sigma)) \xrightarrow{\cong} \Gamma(E \otimes (N\mathcal{F} \oplus N\Sigma)) \xrightarrow{c} \Gamma(E).$$

The principal transverse symbol $\sigma(D^+)$ at $\xi_x \in T_x^*\Sigma$ is

$$\sigma(D^+)(\xi_x) = \sum_{j=1}^{q} ic(\xi_x) : E_x^+ \to E_x^-$$

Suppose $N\Sigma$ is spin^c; then there exists a vector bundle $S = S^+ \oplus S^- \to \Sigma$ that is an irreducible representation of $\mathbb{Cl}(N\Sigma)$ over each point of Σ, and we let $E^\Sigma = \text{End}_{\mathbb{Cl}(N\Sigma)}(E)$ and have

$$E \cong S \widehat{\otimes} E^\Sigma$$

as a graded tensor product, such that the action of $\mathbb{Cl}\left(N\mathcal{F}\oplus N\Sigma\right)\cong \mathbb{Cl}\left(N\Sigma\right)\widehat{\otimes}\mathbb{Cl}\left(N\mathcal{F}\right)$ (as a graded tensor product) on E^+ decomposes as

$$\begin{pmatrix} c\left(x\right)\otimes\mathbf{1} & -\mathbf{1}\otimes c\left(y\right)^* \\ \mathbf{1}\otimes c\left(y\right) & c\left(x\right)^*\otimes\mathbf{1} \end{pmatrix} : \begin{pmatrix} S^+\otimes E^{\Sigma+} \\ S^-\otimes E^{\Sigma-} \end{pmatrix} \to \begin{pmatrix} S^-\otimes E^{\Sigma+} \\ S^+\otimes E^{\Sigma-} \end{pmatrix}$$

(see [6], [35]). If we let the operator ∂^N denote the spin^c transversal Dirac operator on sections of $\pi^*S\to N\Sigma$, and let D_Σ be the transversal Dirac operator defined by the action of $\mathbb{Cl}\left(N\mathcal{F}\right)$ on E^Σ, then we have

$$D^+ = \left(\partial^N * D_\Sigma\right)^+$$

up to zero$^{\text{th}}$ order terms (coming from curvature of the fiber).

The same argument works if instead we have that the bundle $N\mathcal{F}\to\Sigma$ is spin^c in addition to the assumption that D restricts to a transversal Dirac operator on the stratum Σ. In this case there exists a spin^c Dirac operator ∂^Σ on sections of a complex spinor bundle over Σ is transversally elliptic to the orbit foliation $\mathcal{F}$, and we have a formula of the form

$$D^+ = \left(D_N * \partial^\Sigma\right)^+,$$

again up to zeroth order terms.

Even if $N\Sigma\to\Sigma$ and $N\mathcal{F}\to\Sigma$ are not spin^c, many other first order operators have splittings as in Equation (7.1). For example, if D^+ is a transversal de Rham operator from even to odd forms, then D^+ is the product of de Rham operators in the $N\Sigma$ and $N\mathcal{F}$ directions.

8. Heat Kernel Splitting Formula

8.1. **Tensor Products of Representations.** If H is a compact Lie group and $V = V_1\otimes V_2$ is a tensor product of H-modules, and if $\left\{\left[\sigma_j,W_{\sigma_j}\right]:j\geq 0\right\}$ are the irreducible unitary representations of H, counted only up to equivalence, then we define the integers γ_{ab}^j (Clebsch-Gordan coefficients) as

$$\gamma_{ab}^j = \text{multiplicity of } \sigma_j \text{ in } \sigma_a\otimes\sigma_b$$
$$= \text{rank Hom}_H\left(W_{\sigma_j},W_{\sigma_a}\otimes W_{\sigma_b}\right),$$

and

$$W_{\sigma_a}\otimes W_{\sigma_b}\cong\bigoplus_j\gamma_{ab}^jW_{\sigma_j}.$$

Quite often these integers are 1 or 0 (see [38], [34]). For example, if σ_0 denotes the trivial representation, then

$$\gamma_{ab}^0 = \begin{cases} 1 & \text{if } \sigma_b = \sigma_a^* \\ 0 & \text{otherwise} \end{cases}$$

It is important to realize that for fixed j, as above there may be an infinite number of γ_{ab}^j that are nonzero. On the other hand, if two of the numbers a,b,j are fixed, then only a finite number of the γ_{ab}^j are nonzero as the third index varies over all possibilities.

8.2. **Decomposition near an orbit.** With notation as before, let Σ be a minimal stratum of the G-action on the vector bundle $E \to M$, corresponding to an isotropy type $[H]$. We modify the metrics and bundles equivariantly so that there exists $\varepsilon > 0$ such that the tubular neighborhood $B_\varepsilon \Sigma$ of Σ in M is isometric to a ball of radius ε in the normal bundle $N\Sigma$ (endowed with the natural product-like metric), and so that the G-equivariant bundle E over $B_\varepsilon \Sigma$ is a pullback of the bundle $E|_\Sigma \to \Sigma$.

Choose $\Sigma_\alpha = G\Sigma_\alpha^H \subseteq \Sigma$ to be a component of Σ relative to G (Definition 4.1). Note that Σ is partitioned into a finite number of such components, which is the number of components of $G\backslash\Sigma$. We assume that

$$E|_{\Sigma_\alpha} = E_N \widehat{\otimes} E_\Sigma,$$

the graded tensor product of two G-vector bundles over Σ_α as in Section 7. We extend $B_\varepsilon \Sigma$ to a sphere bundle $S\Sigma \xrightarrow{\pi} \Sigma$ by modifying the metric near the boundary and doubling. We wish to compute the local supertrace of a heat operator corresponding to a G-equivariant, transversally elliptic operator, restricted to sections of type ρ, on $S\Sigma|_{\Sigma_\alpha}$. Let $\mathcal{O}_x$ be an orbit in $\Sigma_\alpha \subseteq \Sigma$, with $x \in \Sigma^H$, and let $S_{\mathcal{O}_x} \to \mathcal{O}_x$, respectively $B_{\mathcal{O}_x} \to \mathcal{O}_x$, be the restriction of the Euclidean sphere bundle $S\Sigma \xrightarrow{\pi} \Sigma$, respectively $B_\varepsilon \Sigma \to \Sigma$. By the constructions in Section 7, the group action, tensor product structure, and the equivariant operator naturally extend from $B_\varepsilon \Sigma$ to $S\Sigma$, and the local supertraces of the heat operators on $B_\varepsilon \Sigma$ and $S\Sigma$ agree in a neighborhood of Σ, up to $\mathcal{O}\left(e^{-c/t}\right)$ for some $c > 0$. The space of sections $\Gamma\left(S_{\mathcal{O}_x}, E\right)$ satisfies

$$\Gamma\left(S_{\mathcal{O}_x}, E|_{\mathcal{O}_x}\right) \cong \Gamma\left(\mathcal{O}_x, F_N \widehat{\otimes} E_\Sigma|_{\mathcal{O}_x}\right),$$

$$\Gamma\left(S\Sigma|_{\Sigma_\alpha}, E|_{\Sigma_\alpha}\right) \cong \Gamma\left(\Sigma_\alpha, F_N \widehat{\otimes} E_\Sigma\right),$$

where $F_N \to \Sigma$ is the infinite dimensional equivariant vector bundle whose fiber is

$$F_{N,z} = \Gamma\left(S\Sigma_z, E_N|_{S\Sigma_z}\right), \ z \in \Sigma_\alpha.$$

Observe that $F_{N,gx}$ and $E_{\Sigma,gx}$ are G_{gx}-modules, where $G_{gx} = gG_xg^{-1} \cong H$. If F_N^b is a fine component of F_N associated to $(\alpha, [\sigma])$ as in Definition 4.3, let

- $W^b = W_{\alpha,[\sigma]}$ be the canonical isotropy G-bundle adapted to F_N^b (Definition 4.9, Lemma 4.8),
- m_b be the associated multiplicity of each irreducible isotropy representation present in W^b,
- d_b be the associated representation dimension of each irreducible isotropy representation present in W^b (and thus F_N^b), and
- n_b be the inequivalence number, the number of inequivalent irreducible representation types present in W^b (and thus F_N^b).
- Note rank $\left(W^b\right) = m_b d_b n_b$.

We remark that at this time there are an infinite number of such components F_N^b, but soon we will restrict to a finite number.

With notation as above, let $[\rho, V_\rho]$ be an irreducible unitary G-representation. We will restrict a transversally elliptic, G-equivariant differential operator to the space $\Gamma\left(\Sigma, F_N\widehat{\otimes}E_\Sigma\right)^\rho$ of sections of type ρ. By Frobenius reciprocity,

$$\mathrm{Hom}_G\left(V_\rho, \Gamma\left(\mathcal{O}_x, F_N\widehat{\otimes}E_\Sigma\right)\right) \cong \mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, \left(F_N\widehat{\otimes}E_\Sigma\right)_x\right)$$

$$(8.1) \qquad\qquad \cong \bigoplus_b \mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, \left(F_N^b\widehat{\otimes}E_\Sigma\right)_x\right).$$

If $\{\sigma_a\}$ is the finite set of H-representation types present in $E_{\Sigma,x}$ and $\{\sigma_j\}$ is the finite set of H-representation types present in $V_{\mathrm{Res}(\rho)}$, then by the remarks in the previous subsection, the Clebsch-Gordan coefficients γ_{ap}^j are nonzero only for a finite number of irreducible representation types $[\sigma_p]$. Therefore, only a finite number of b satisfy

$$\mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, \left(F_N^b\widehat{\otimes}E_\Sigma\right)_x\right) \neq 0.$$

Moreover, if we consider all such b over every $x \in \Sigma_\alpha \subseteq \Sigma$, only a finite number of b would yield a nonzero $\mathrm{Hom}_H\left(V_{\mathrm{Res}(\rho)}, \left(F_N^b\widehat{\otimes}E_\Sigma\right)_x\right)$. Let B be this finite set of indices, and thus we have

$$\Gamma\left(\Sigma_\alpha, F_N\widehat{\otimes}E_\Sigma\right)^\rho = \bigoplus_{b\in B}\Gamma\left(\Sigma_\alpha, F_N^b\widehat{\otimes}E_\Sigma\right)^\rho.$$

In the equation above, we may of course replace the smooth sections with the L^2 sections.

Let $K\left(t, y_x, y_x\right)^\rho$, $y \in S\Sigma_x$ denote the heat kernel of a G-equivariant transversally elliptic first-order symmetric operator D^*D of the form $D = D_N * D_\Sigma$ on $\Gamma\left(\Sigma_\alpha, F_N\widehat{\otimes}E_\Sigma\right)$; we further restrict this kernel to $\Gamma\left(B_{\mathcal{O}_x}, E\right)^\rho$ or $\Gamma\left(S_{\mathcal{O}_x}, E\right)^\rho$. We assume that D_N is a family of first order elliptic operators on fibers $F_{N,x}$, and we also assume D_N is G-equivariant, with corresponding local equivariant fiberwise heat kernel K_N. We likewise assume that D_Σ is a first order transversally elliptic operator on sections of E_Σ over Σ with equivariant heat kernel K_Σ.

Consider a small ε-tubular neighborhood $T_\varepsilon \subset \Sigma_\alpha \subseteq \Sigma$, $T_\varepsilon \cong G \times_H D_\varepsilon$ around a fixed orbit $\mathcal{O}_x$ in Σ, where $x \in D_\varepsilon \subset \Sigma^H$ is a ball transverse to $\mathcal{O}_x$. The space of sections $L^2\left(T_\varepsilon, F_N^b\widehat{\otimes}E_\Sigma\right)^\rho$ decomposes using Lemma 4.12 as

$$L^2\left(T_\varepsilon, F_N^b\widehat{\otimes}E_\Sigma\right)^\rho \otimes \bigoplus_{j=1}^{n_b}\mathrm{Hom}_H\left(\left.W_j^b\right|_{D_\varepsilon}, \left.W_j^b\right|_{D_\varepsilon}\right)$$

$$\cong \bigoplus_{j=1}^{n_b}\mathrm{Hom}_H\left(\left.W_j^b\right|_{D_\varepsilon}, \left.F_N^b\right|_{D_\varepsilon}\right)\widehat{\otimes}L^2\left(T_\varepsilon, \widetilde{W}_j^b \otimes E_\Sigma\right)^\rho$$

$$(8.2)$$

$$\cong \bigoplus_{j=1}^{n_b}\mathrm{Hom}_H\left(\left.W_j^b\right|_{D_\varepsilon}, W_{\sigma_j}\right) \otimes \mathrm{Hom}_H\left(W_{\sigma_j}, \left.F_N^b\right|_{D_\varepsilon}\right)\widehat{\otimes}L^2\left(T_\varepsilon, \widetilde{W}_j^b \otimes E_\Sigma\right)^\rho$$

with the isomorphism given by evaluation, and then $D \otimes \mathbf{1}$ acts by $\mathbf{1} \otimes D_N \widehat{\otimes} (\mathbf{1} \otimes D_\Sigma)^\rho$. Here, corresponding to the normalizer-conjugate representations $[\sigma_j, W_{\sigma_j}]$ is the isotypical decomposition $W^b|_{D_\varepsilon} = \bigoplus_j W_j^b$ of the canonical isotropy G-bundle W^b, which we may define over the small transversal neighborhood D_ε, and $\widetilde{W}_j^b \to T_\varepsilon$ is the bundle $G \times_H W_j^b \to G \times_H D_\varepsilon \cong T_\varepsilon$. Note that $W^b|_{T_\varepsilon} = \bigoplus_j \widetilde{W}_j^b$, and the multiplicity of the irreducible H-representation actually present in each $\widetilde{W}_{j,p}^b$ is m_b, with $p \in \Sigma^H \cap T_\varepsilon$. Each bundle $\mathrm{Hom}_H \left(W_j^b|_{D_\varepsilon}, W_j^b|_{D_\varepsilon} \right) \cong \mathbb{C}^{m_b^2} \times D_\varepsilon$ is trivial.

Given an operator P, let $E_t(P) = \exp(-tP^*P)$ denote the associated heat operator (defined through context). We wish to calculate the local supertrace of $E_t(D) = \exp(-tD^*D)$ on $L^2 \left(T_\varepsilon, F_N^b \widehat{\otimes} E_\Sigma \right)^\rho$, which is the same as $\frac{1}{m_b^2 n_b}$ times the local supertrace of $E_t(D) \otimes \mathbf{1} = E_t(D \otimes \mathbf{1})$ on $L^2 \left(T_\varepsilon, F_N^b \widehat{\otimes} E_\Sigma \right)^\rho \otimes \bigoplus_{j=1}^{n_b} \mathrm{Hom}_H \left(W_j^b|_{D_\varepsilon}, W_j^b|_{D_\varepsilon} \right)$. According to the operator product assumption and decomposition (8.2), we then have the pointwise supertrace formula

$$\mathrm{str} \left(E_t(D)^{b,\rho} \text{ on } T_\varepsilon \right)_y = \frac{1}{m_b^2 n_b} \sum_{j=1}^{n_b} \mathrm{tr} \, (\mathbf{1}^{\sigma_j})_x \, \mathrm{str} \left(E_t \left(D_N^{b,j} \right) \right)_x \mathrm{str} \left(E_t \left(\mathbf{1}^{b,j} \otimes D_\Sigma \right)^\rho \right)_y,$$
$$x \in D_\varepsilon, y \in [(x,p)] \in T_\varepsilon,$$

since supertraces are multiplicative on graded tensor products and where $\mathbf{1}^{\sigma_j}$ is the identity bundle map on $\mathrm{Hom}_H \left(W_j^b|_{D_\varepsilon}, W_{\sigma_j} \right) \to D_\varepsilon$ and $\mathbf{1}^{b,j}$ is the identity bundle map on $\widetilde{W}_j^b \to T_\varepsilon$. Here $D_N^{b,j}$ means the restriction of D_N to $\mathrm{Hom}_H \left(W_{\sigma_j}, F_N^b \right)$, and so the supertrace $\mathrm{str} \left(E_t \left(D_N^{b,j} \right) \right)_x$ is the same as $\frac{1}{d_b} \mathrm{str} \left(E_t \left(D_N^{\sigma_j} \right) \right)_x$, where $D_N^{\sigma_j}$ is the restriction of D_N to

$$F_{N,x}^{\sigma_j} = \Gamma \left(S\Sigma_x, E_N|_{S\Sigma_x} \right)^{\sigma_j}, x \in D_\varepsilon.$$

Then

$$\mathrm{str} \left(E_t(D)^{b,\rho} \text{ on } T_\varepsilon \right)_y = \frac{1}{m_b n_b d_b} \sum_{j=1}^{n_b} \mathrm{str} \left(E_t \left(D_N^{\sigma_j} \right) \right)_x \mathrm{str} \left(E_t \left(\mathbf{1}^{b,j} \otimes D_\Sigma \right)^\rho \right)_y,$$
$$x \in D_\varepsilon, y \in [(x,p)] \in T_\varepsilon.$$

By (6.5) and (6.8), the integral of the pointwise supertrace $\mathrm{str} \left(E_t \left(D_N^{\sigma_j} \right) \right)_p (z_p, z_p)$ over a small normal ball $z_p \in B_\varepsilon^N$ containing p in $S_p\Sigma$ is the same as

$$\int_{z_p \in B_\varepsilon^N} \mathrm{str} \left(E_t \left(D_N^{\sigma_j} \right) \right)_p (z_p, z_p) \sim \int_{z_p \in \widetilde{B_\varepsilon^N}} \mathrm{str} \left(E_t \left(\widetilde{D_N^{\sigma_j}} \right) \right)_p (z_p, z_p)$$
$$+ \frac{1}{2} \left(-\eta \left(D^{S+,b,j} \right) + h \left(D^{S+,b,j} \right) \right),$$

with the tilde corresponding to the equivariant desingularization of the origin as in Section 7.

We now discuss a general situation. Let u be an eigenvector of a G-equivariant fiberwise operator L on a fiber E_x of an equivariant bundle over a G-manifold X with single orbit type G/H. Assume that $x \in X^H$, and $n \in N(H)$. Then nu is an eigensection of L over E_{nx} with the same eigenvalue. Moreover, if u is an element of an H-irreducible subspace of type σ, then nu is an element of an irreducible subspace of type σ^n. This implies that the Schwarz kernel K_L of L at x is mapped to the kernel K_L of L at nx by conjugation by n, and thus the kernel $K_{L,x}^\sigma$ restricted to sections of type σ at x is conjugate to the kernel $K_{L,nx}^{\sigma^n}$ at nx (see also equations (5.3), (5.4)). Moreover, the kernels K_L, K_L^σ, $K_L^{\sigma^n}$ vary smoothly as a function of $x \in X^H$, and in fact any fiberwise spectral invariant of L must vary continuously with $x \in X^H$.

Next, consider the special case where $L \colon \Gamma(X, E) \to \Gamma(X, E)$ has integer eigenvalues (as in the case of the operators D^{S+} – see Corollary 11.12 and Remark 11.13). Then the eigenvalues of L, L^σ, and L^{σ^n} must be locally constant on X^H. This implies that if $n \in N(H)$ maps a connected component $X_\alpha^H \subset X^H$ to itself, then all spectral invariants of L^σ and L^{σ^n} are identical and constant, when restricted to X_α^H. Now, let U be an open subset of X_α^H, and let E^b be a fine component of E. If $[\sigma], [\sigma'] \in \widehat{H}$ are different representation types present in a specific $E^b|_U$, then a consequence of the argument above is that all spectral invariants of $L^\sigma|_U$ and $L^{\sigma'}|_U$ are identical and constant.

As a result, in the displayed formula above, $\frac{1}{2}\left(-\eta\left(D^{S+,b,j}\right) + h\left(D^{S+,b,j}\right)\right)$ is independent of j and is a constant on each connected component Σ_α^H of Σ^H (and thus on $\Sigma_\alpha = G\Sigma_\alpha^H \subset \Sigma$). In particular,

$$\frac{1}{2}\left(-\eta\left(D^{S+,b,j}\right) + h\left(D^{S+,b,j}\right)\right) = \frac{1}{2n_b}\left(-\eta\left(D^{S+,b}\right) + h\left(D^{S+,b}\right)\right),$$

where $D^{S+,b}$ refers to the restriction of D^{S+} to the sections in the fine component F_N^b. We now integrate over $p \in T_\varepsilon$ to get

$$\int_{B_\varepsilon^N(T_\varepsilon)} \mathrm{str}\left(E_t(D)^{b,\rho}\right)(z_p, z_p)$$

$$= \frac{1}{m_b n_b d_b} \sum_{j=1}^{n_b} \int_{T_\varepsilon} \int_{z_p \in B_\varepsilon^N} \mathrm{str}\left(E_t\left(D_N^{\sigma_j}\right)\right)_p (z_p, z_p)\, \mathrm{str}\left(E_t\left(\mathbf{1}^{b,j} \otimes D_\Sigma\right)^\rho\right)(p, p)$$

$$\sim \frac{1}{m_b n_b d_b} \sum_{j=1}^{n_b} \int_{T_\varepsilon} \int_{z_p \in \widetilde{B_\varepsilon^N}} \mathrm{str}\left(E_t\left(\widetilde{D_N^{\sigma_j}}\right)\right)_p (z_p, z_p)\, \mathrm{str}\left(E_t\left(\mathbf{1}^{b,j} \otimes D_\Sigma\right)^\rho\right)(p, p)$$

$$+ \frac{1}{m_b n_b d_b} \int_{T_\varepsilon} \frac{1}{2n_b}\left(-\eta\left(D^{S+,b}\right) + h\left(D^{S+,b}\right)\right) \sum_{j=1}^{n_b} \mathrm{str}\left(E_t\left(\mathbf{1}^{b,j} \otimes D_\Sigma\right)^\rho\right)(p, p)$$

$$= \int_{\widetilde{B_\varepsilon^N (T_\varepsilon)}} \mathrm{str} \left(E_t \left(\widetilde{D} \right)^{b,\rho} \right) (z_p, z_p)$$

$$+ \frac{1}{2m_b n_b^2 d_b} \left(-\eta \left(D^{S+,b} \right) + h \left(D^{S+,b} \right) \right) \int_{T_\varepsilon} \mathrm{str} \left(E_t \left(\mathbf{1}^b \otimes D_\Sigma \right)^\rho \right) (p, p) ,$$

where $\mathbf{1}^b \otimes D_\Sigma$ is the differential operator on $W^b \otimes E_\Sigma$ over the G-manifold Σ_α.

Thus, in calculating the small t asymptotics of $\int_{B_\varepsilon(U)} \mathrm{str} K (t, z_p, z_p)^\rho = \int \mathrm{str} (E_t (D)^\rho) (z_p, z_p)$ with $E_t (D)^\rho = \exp (-tD^*D)^\rho$, it suffices to calculate the right hand side of the formula above over the region B_ε and sum over fine components $b \in B$, using the heat kernel coming from the blown up manifold. Thus, by integrating over any open saturated subset U of $\Sigma_\alpha \subseteq \Sigma$, we conclude that as $t \to 0$,

$$\int_{B_\varepsilon(U)} \mathrm{str} K (t, z_p, z_p)^\rho \sim \int_{\widetilde{B_\varepsilon(U)}} \mathrm{str} K (t, z_p, z_p)^\rho$$

$$(8.3) \qquad + \sum_b \frac{1}{2n_b \mathrm{rank} (W^b)} \left(-\eta \left(D^{S+,b} \right) + h \left(D^{S+,b} \right) \right) \int_{p \in U} \mathrm{str} K_\Sigma^b (t, p, p)^\rho ,$$

with $\mathrm{str} K_\Sigma^b (t, p, p)^\rho = \mathrm{str} \left(E_t \left(\mathbf{1}^b \otimes D_\Sigma \right)^\rho \right) (p, p)$ is the local heat supertrace corresponding to the operator $\mathbf{1}^b \otimes D_\Sigma$ on $\Gamma \left(U, W^b \otimes E_\Sigma \right)^\rho$.

9. The Equivariant Index Theorem

9.1. **Reduction formula for the heat supertrace.** As before, let E be a graded Hermitian vector bundle over a closed Riemannian manifold M, such that a compact Lie group G acts on (M, E) by isometries. Let $D = D^+ \colon \Gamma (M, E^+) \to \Gamma (M, E^-)$ be a first order, transversally elliptic, G-equivariant differential operator. Let Σ be a minimal stratum of the G-manifold M. We modify the metrics and bundles equivariantly so that there exists $\varepsilon > 0$ such that the tubular neighborhood $B_\varepsilon \Sigma$ of Σ in M is isometric to a ball of radius ε in the normal bundle $N\Sigma$, and so that the G-equivariant bundle E over $B_\varepsilon (\Sigma)$ is a pullback of the bundle $E|_\Sigma \to \Sigma$. We assume that D^+ can be written on each saturated open set U contained in $B_\varepsilon (\Sigma)$ as the product

$$D^+ = (D_N * D_\Sigma)^+ ,$$

where D_Σ is a transversally elliptic, G-equivariant, first order operator on the stratum Σ, and D_N is a G-equivariant, family of elliptic first order operators on the fibers of $B_\varepsilon (\Sigma)$. In fact, we only require that D^+ is of this form after an equivariant homotopy. Further, we may assume that D_N is a first order operator with coefficients frozen at Σ so its restriction to a normal exponential neighborhood $B_\varepsilon \Sigma_x$, $x \in \Sigma$, is a constant coefficient operator. If r is the distance from Σ, we can write D_N in polar coordinates as

$$D_N = Z \left(\nabla_{\partial_r}^E + \frac{1}{r} D^S \right)$$

where $Z = -i\sigma(D_N)(\partial_r)$ is a local bundle isomorphism and the map D^S is a purely first order operator that differentiates in the unit normal bundle directions tangent to $S_x\Sigma$. Note that we allow this to make sense if Σ has codimension one by setting $D^S = 0$ and ∂_r to be the inward-pointing vector.

Let $\rho\colon G \to U(V_\rho)$ be an irreducible representation. Let U be a saturated (ie G-invariant) open subset of M such that $G \backslash U$ is connected. From Equation (5.2), we wish to compute the quantity

$$I_U = \int_{x \in \Sigma \cap U} \int_{z_x \in B_\varepsilon(\Sigma)_x \cap U} \mathrm{str}\left(K\left(t, (x, z_x), (x, z_x)\right)^\rho\right) \; |dz_x| \; |dx|,$$

where we choose coordinates $w = (x, z_x)$ for the point $\exp_x^\perp(z_x)$, where $x \in \Sigma$, $z_x \in B_\varepsilon = B_\varepsilon(N_x(\Sigma), 0)$, and $\exp_x^\perp$ is the normal exponential map. In the expression above, $|dz_x|$ is the Euclidean density on $N_x(\Sigma)$, and $|dx|$ is the Riemannian density on Σ. Recall that $K\left(t, (x, z_x), (x, z_x)\right)^\rho$ is a map from $E_{(x,z_x)}^{\mathrm{Res}(\rho)} = E_x^{\mathrm{Res}(\rho)}$ to itself.

First, observe that the small t asymptotics of this integral over $B_\varepsilon(\Sigma) \cap U$ may be computed using a heat kernel for another differential on a neighborhood of a different manifold, as long as all the local data (differential operator, vector bundles, metrics) are the same. In particular we may use the manifold $S\Sigma$ as in the previous section. Using Equation (8.3),

$$I_{i,U} \sim \int_{\widetilde{B_\varepsilon(\Sigma) \cap U}} \mathrm{str}\left(\widetilde{K}(t, \widetilde{x}, \widetilde{x})^\rho\right) |d\widetilde{x}|$$
$$- \frac{1}{2} \sum_{b \in B} \frac{1}{n_b \mathrm{rank}\, W^b} \left(\eta\left(D^{S+,b}\right) - h\left(D^{S+,b}\right)\right) \int_{x \in \Sigma \cap U} \mathrm{str}\left(K_\Sigma^b(t, x, x)^\rho\right) \; |dx| \;,$$

where $\sim$ means equal up to $\mathcal{O}(t^k)$ for large k, where $\widetilde{B_\varepsilon(\Sigma) \cap U}$ is the fundamental domain in the desingularization of $B_\varepsilon(\Sigma) \cap U$ and where $\widetilde{K}$ is the suitably modified heat kernel. We note that the equation above remains true if Σ is of codimension 1, because in this case $D^{S+,b} = 0^b$, and we use the definitions (6.7).

Let $\widetilde{M}$ be the equivariant desingularization of M along Σ, and let $\widetilde{E^\pm}$ be the equivariant bundles over $\widetilde{M}$ induced from $E^\pm$ with transversally elliptic operators $\widetilde{D^\pm}$ (see Section 7.1). Let $\widetilde{K}^\pm(t, \cdot, \cdot)^\beta$ denote the equivariant heat kernels corresponding to the extensions of $\widetilde{D^\mp}\widetilde{D^\pm} + C - \lambda_\beta$ to the double of $\widetilde{M}$ restricted to sections of type β (with reversed orientations of the induced bundles $\widetilde{E^\pm}$ in the usual way). Then the heat kernel supertrace formula becomes

$$\int_{x \in U} \mathrm{str}\left(K(t, x, x)^\rho\right) \; |dx| \sim \int_{\widetilde{U}} \mathrm{str}\left(\widetilde{K}(t, \widetilde{x}, \widetilde{x})^\rho\right) |d\widetilde{x}|$$
$$+ \frac{1}{2} \sum_{b \in B} \frac{1}{n_b \mathrm{rank}\, W^b} \left(-\eta\left(D^{S+,b}\right) + h\left(D^{S+,b}\right)\right) \int_{G \backslash (U \cap \Sigma)} \mathrm{str}\left(\overline{K_\Sigma^b}(t, \overline{x}, \overline{x})^\rho\right) \; |d\overline{x}| \;.$$

The heat kernel $\overline{K_\Sigma^b}$ is that corresponding to the operator $(\mathbf{1} \otimes D_\Sigma)^\rho$ induced by the operator $\mathbf{1} \otimes D_\Sigma$ on $\Gamma\left(\Sigma, W^b \otimes E_\Sigma\right)$ on the quotient $G \backslash (U \cap \Sigma)$ as described in Section 4.1.

We assemble the results above into the following theorem.

Theorem 9.1. *(Reduction Formula for the Equivariant Supertrace) Let E be a graded Hermitian vector bundle over a closed Riemannian manifold M, such that a compact Lie group G acts on (M, E) by isometries. Let Σ denote a minimal stratum. Let $D^+ : \Gamma(M, E^+) \to \Gamma(M, E^-)$ be a first order, transversally elliptic, G-equivariant differential operator. We modify the metrics and bundles equivariantly so that there exists $\varepsilon > 0$ such that the tubular neighborhood $B_\varepsilon \Sigma$ of Σ in M is isometric to a ball of radius ε in the normal bundle $N\Sigma$, and so that the G-equivariant bundle E over $B_\varepsilon(\Sigma)$ is a pullback of the bundle $E|_\Sigma \to \Sigma$. We assume that near Σ, D is G-homotopic to the product*

$$D^+ = (D_N * D_\Sigma)^+,$$

where D_Σ is a global transversally elliptic, G-equivariant, first order operator on the stratum Σ, and D_N is a G-equivariant, first order operator on $B_\varepsilon \Sigma$ that is elliptic and has constant coefficients on the fibers. If r is the distance from Σ, we write D_N in polar coordinates as

$$D_N = Z\left(\nabla_{\partial_r}^E + \frac{1}{r} D^S\right)$$

where Z is a local bundle isomorphism and the map D^S is a purely first order operator that differentiates in the unit normal bundle directions tangent to $S_x \Sigma$. Let U be a saturated (ie G-invariant) open subset of M such that $G \backslash U$ is connected. Then as $t \to 0$,

$$\int_{x \in U} \mathrm{str}\, (K(t, x, x)^\rho)\ |dx| \sim \int_{\widetilde{U}} \mathrm{str}\left(\widetilde{K}(t, \widetilde{x}, \widetilde{x})^\rho\right) |d\widetilde{x}|$$
$$+ \frac{1}{2} \sum_{b \in B} \frac{1}{n_b \mathrm{rank}\, W^b} \left(-\eta\left(D^{S+,b}\right) + h\left(D^{S+,b}\right)\right) \int_{G \backslash (U \cap \Sigma)} \mathrm{str}\left(\overline{K_\Sigma^b}(t, \overline{x}, \overline{x})^\rho\right)\ |d\overline{x}|_i\ .$$

Here, the superscript $W^b \to \Sigma_\alpha$ is a canonical isotropy G-bundle over the component Σ_α relative to G that contains U, K_Σ^b refers to the equivariant heat kernel corresponding to a twist of D_Σ by $W^b \to \Sigma_\alpha$. Since K_Σ^b is an equivariant heat kernel on the component Σ_α, it induces a heat kernel $\overline{K_\Sigma^b} = \left(K_\Sigma^b\right)^\rho$ on the quotient $G \backslash \Sigma_\alpha$. Also, $\widetilde{U}$ is the equivariant desingularization of U along the stratum Σ, and the kernel $\widetilde{K}$ is the corresponding heat kernel of the double.

We refer the reader to Definitions 4.1 and 4.9. Note that only a finite number ($b \in B$) of W^b result in heat kernels $\overline{K_\Sigma^b}(t, \overline{x}, \overline{x})^\rho$ that are not identically zero, by the discussion following (8.1).

9.2. **The main theorem.** To evaluate $\mathrm{ind}^{\rho}(D)$ as in Equation (5.2), we apply Theorem 9.1 repeatedly, starting with a minimal stratum and then applying to each double of the equivariant desingularization. After all the strata are blown up and doubled, all of the resulting manifolds have a single stratum, and we may apply the results of Section 4.1. We obtain the following result. In what follows, if U denotes an open subset of a stratum of the action of G on M, U' denotes the equivariant desingularization of U, and $\widetilde{U}$ denotes the fundamental domain of U inside U', as in Section 7. We also refer the reader to Definitions 4.1 and 4.9.

Theorem 9.2. *(Equivariant Index Theorem) Let M_0 be the principal stratum of the action of a compact Lie group G on the closed Riemannian M, and let $\Sigma_{\alpha_1}, \ldots, \Sigma_{\alpha_r}$ denote all the components of all singular strata relative to G. Let $E \to M$ be a Hermitian vector bundle on which G acts by isometries. Let $D : \Gamma(M, E^+) \to \Gamma(M, E^-)$ be a first order, transversally elliptic, G-equivariant differential operator. We assume that near each Σ_{α_j}, D is G-homotopic to the product $D_N * D^{\alpha_j}$, where D_N is a G-equivariant, first order differential operator on $B_{\varepsilon}\Sigma$ that is elliptic and has constant coefficients on the fibers and D^{α_j} is a global transversally elliptic, G-equivariant, first order operator on the Σ_{α_j}. In polar coordinates*

$$D_N = Z_j \left(\nabla^E_{\partial_r} + \frac{1}{r} D^S_j \right) \; ,$$

where r is the distance from Σ_{α_j}, where Z_j is a local bundle isometry (dependent on the spherical parameter), the map D^S_j is a family of purely first order operators that differentiates in directions tangent to the unit normal bundle of Σ_j. Then the equivariant index $\mathrm{ind}^{\rho}(D)$ satisfies

$$\mathrm{ind}^{\rho}(D) = \int_{G \backslash \widetilde{M_0}} A^{\rho}_0(x) \; \widetilde{|dx|} \; + \sum_j \beta\left(\Sigma_{\alpha_j}\right) \; ,$$

$$\beta\left(\Sigma_{\alpha_j}\right) = \frac{1}{2 \dim V_{\rho}} \sum_j \sum_{b \in B} \frac{1}{n_b \mathrm{rank}\, W^b} \left(-\eta\left(D^{S+,b}_j\right) \right.$$

$$\left. + h\left(D^{S+,b}_j\right) \right) \int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A^{\rho}_{j,b}(x) \; \widetilde{|dx|} \; ,$$

where

(1) *$A^{\rho}_0(x)$ is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from D' (blown-up and doubled from D) on the quotient $M'_0 \big/ G$, where the bundle E is replaced by the bundle $\mathcal{E}_{\rho}$ (see Section 4.1).*

(2) *Similarly, $A^{\rho}_{i,b}$ is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $(\mathbf{1} \otimes D^{\alpha_j})'$ (blown-up and doubled from $\mathbf{1} \otimes D^{\alpha_j}$, the twist of D^{α_j} by the canonical isotropy bundle $W^b \to$*

Σ_{α_j}) on the quotient Σ'_{α_j} / G, where the bundle is replaced by the space of sections of type ρ over each orbit.

(3) $\eta\left(D_j^{S+,b}\right)$ is the eta invariant of the operator D_j^{S+} induced on any unit normal sphere $S_x \Sigma_{\alpha_j}$, restricted to sections of isotropy representation types in W_x^b, defined in (6.3) and in (6.7) for the codimension one case when $D_j^{S+,b} = 0_j^{+,b}$. This is constant on Σ_{α_j}.

(4) $h\left(D_j^{S+,b}\right)$ is the dimension of the kernel of $D_j^{S+,b}$, restricted to sections of isotropy representation types in W_x^b, again constant on on Σ_{α_j}.

(5) n_b is the number of different inequivalent G_x-representation types present in each W_x^b, $x \in \Sigma_{\alpha_j}$.

Remark 9.3. Note that only a finite number ($b \in B$) of W^b result in integrands $A_{j,b}^\rho(x)$ that are not identically zero, by the discussion following (8.1).

Remark 9.4. If the stratum Σ_{α_j} is a single orbit, then we have in the Theorem above that

$$\frac{1}{2 \dim V_\rho} \sum_j \sum_{b \in B} \frac{1}{n_b \operatorname{rank} W^b} \left(-\eta\left(D_j^{S+,b}\right) + h\left(D_j^{S+,b}\right)\right) \int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A_{j,b}^\rho(x) \ \widetilde{|dx|}$$

$$= \frac{1}{2 \dim V_\rho} \sum_\sigma \frac{n_\sigma^\rho}{\dim(W_\sigma)} \left(-\eta\left(D^{S+,\sigma}\right) + h\left(D^{S+,\sigma}\right)\right),$$

where the restriction of ρ to H is $\operatorname{Res}([\rho, V_\rho]) = \bigoplus_\sigma n_\sigma^\rho [\sigma, W_\sigma]$.

Remark 9.5. Note that the quantities $\eta\left(D_j^{S+,b}\right)$ and $h\left(D_j^{S+,b}\right)$ are invariant under equivariant stable homotopies of the operator D, as explained in Remark 11.13.

Theorem 9.6. *(Invariant Index Theorem) With notation as in the last theorem, if ρ_0 is the trivial representation, then*

$$\operatorname{ind}^{\rho_0}(D) = \int_{G \backslash \widetilde{M_0}} A_0^{\rho_0}(x) \ \widetilde{|dx|} \ + \sum_j \beta\left(\Sigma_{\alpha_j}\right) \ ,$$

$$\beta\left(\Sigma_{\alpha_j}\right) = \frac{1}{2} \sum_j \sum_{b \in B} \frac{1}{n_b \operatorname{rank} W^b} \left(-\eta\left(D_j^{S+,b}\right)\right.$$

$$\left. + h\left(D_j^{S+,b}\right)\right) \int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A_{j,b}^{\rho_0}(x) \ \widetilde{|dx|} \ ,$$

where $b \in B$ only if W^b corresponds to irreducible isotropy representations whose duals are present in E^{α_j}, the bundle on which D_{α_j} acts.

Proof. If $[\sigma, W_\sigma], [\tau, W_\tau] \in \widehat{H}$, we have $(W_\sigma \otimes W_\tau)^H = 0$ unless $\tau \simeq \sigma^*$, and in this case $(W_\sigma \otimes W_\tau)^H \cong \mathbb{C}$. $\qquad \square$

10. Examples and Applications

10.1. **Equivariant indices of elliptic operators.** Clearly, Theorem 9.2 and Theorem 9.6 apply when the G-equivariant differential operator $D\colon \Gamma\left(M, E^+\right) \to \Gamma\left(M, E^-\right)$ is elliptic. In this case, the kernel and cokernel are finite-dimensional representations of G, and thus we have the fact that the Atiyah-Singer index satisfies

$$\operatorname{ind}\left(D\right) = \sum_{[\rho]} \operatorname{ind}^\rho\left(D\right) \dim\left(V_\rho\right),$$

and the equivariant indices satisfy

$$\operatorname{ind}^G\left(D\right) = \sum_{[\rho]} \operatorname{ind}^\rho\left(D\right)\left[\rho\right] \in R\left(G\right)$$

$$\operatorname{ind}_g\left(D\right) = \sum_{[\rho]} \operatorname{ind}^\rho\left(D\right) \chi_\rho\left(g\right),$$

with all sums, integers, and functions above being finite. Even in this elliptic case, the result of the main theorem was not known previously.

For example, suppose that D is a general Dirac operator over sections of a Clifford bundle E that is equivariant with respect to the action of a Lie group G of isometries of M. If we apply Theorem 9.2 to this operator, all of the operators in the theorem are in fact Dirac operators. That is, the operators D_N and D_Σ are Dirac operators formed by restricting Clifford multiplication and the connections to the normal and tangent bundles of the stratum Σ_i. The spherical operators $D_i^{S^+}$ are also Dirac operators induced on the unit normal bundles of the Σ_i. Further, in calculating the Atiyah-Singer integrands on the quotients, note that the operators on the orbit spaces are Dirac operators twisted by a bundle.

10.2. **The de Rham operator.** It is well known that if M is a Riemannian manifold and $f\colon M \to M$ is an isometry that is homotopic to the identity, then the Euler characteristic of M is the sum of the Euler characteristics of the fixed point sets of f. We generalize this result as follows. We consider the de Rham operator

$$d + d^*\colon \Omega^{\mathrm{even}}\left(M\right) \to \Omega^{\mathrm{odd}}\left(M\right)$$

on a G-manifold, and the invariant index of this operator is the equivariant Euler characteristic $\chi^G\left(M\right)$, the Euler characteristic of the elliptic complex consisting of invariant forms. If G is connected and the Euler characteristic is expressed in terms of its ρ-components, only the invariant part $\chi^G\left(M\right) = \chi^{\rho_0}\left(M\right)$ appears. This is a consequence of the homotopy invariance of de Rham cohomology. Thus $\chi^G\left(M\right) = \chi\left(M\right)$ for connected Lie groups G. In general the Euler characteristic is a sum of components

$$\chi\left(M\right) = \sum_{[\rho]} \chi^\rho\left(M\right),$$

where $\chi^\rho(M)$ is the alternating sum of the dimensions of the $[\rho]$-parts of the cohomology groups (or spaces of harmonic forms). Since the connected component G_0 of the identity in G acts trivially on the harmonic forms, the only nontrivial components $\chi^\rho(M)$ correspond to representations induced from unitary representations of the finite group G/G_0.

In any case, at each stratum Σ_{α_j} of codimension at least two in a G-manifold, we may write the de Rham operator (up to lower order perturbations) as

$$d + d^* = D_N * D_\Sigma,$$

where D_N and D_Σ are both de Rham operators on the respective spaces of forms. Further, the spherical operator D^S is simply

$$D^S = -c\left(\partial_r\right)\left(d + d^*\right)^S, \ \ c\left(\partial_r\right) = dr \wedge - dr\lrcorner$$

where $\left(d + d^*\right)^S$ is a vector-valued de Rham operator on the sphere. We will need to compute the kernel of this operator, which is related to the harmonic forms on the sphere; it is spanned by $\{1, dr, \omega, dr \wedge \omega\}$, where ω is the sphere volume form. The eigenforms of this operator are integers, and if $\alpha + dr \wedge \beta$ is an eigenform of D^S corresponding to eigenvalue λ, then

$$\lambda\left(\alpha + dr \wedge \beta\right) = -c\left(\partial_r\right)\left(d + d^*\right)^S\left(\alpha + dr \wedge \beta\right)$$
$$= -dr \wedge \left(d + d^*\right)^S \alpha - \left(d + d^*\right)^S \beta,$$

so that

$$\left(d + d^*\right)^S \alpha = -\lambda\beta; \left(d + d^*\right)^S \beta = -\lambda\alpha, \ \text{or}$$
$$\left(d + d^*\right)^S \left(\alpha \pm \beta\right) = \mp\lambda\left(\alpha \pm \beta\right).$$

Also

$$D^S\left(\alpha - dr \wedge \beta\right) = -c\left(\partial_r\right)\left(d + d^*\right)^S\left(\alpha - dr \wedge \beta\right)$$
$$= \lambda dr \wedge \beta - \lambda\alpha = -\lambda\left(\alpha - dr \wedge \beta\right), \ \text{and}$$
$$D^S\left(\beta \pm dr \wedge \alpha\right) = \pm\lambda\left(\beta \pm dr \wedge \alpha\right).$$

Thus the spectra of both D^S and $\left(d + d^*\right)^S$ are symmetric. Let $[G_j]$ be the isotropy type corresponding to the stratum Σ_{α_j}. If we restrict to forms of G_j-representation type σ, the forms α, β would necessarily be of that type, but the symmetry of the spectra would remain. Thus for the de Rham operator,

$$\eta\left(D^{S+,\sigma}\right) = 0$$

at each stratum. The dimension $h\left(D^{S+,\sigma}\right)$ of the kernel is not always the same; the + component of the forms contains at most the span of $\{1, \omega\}$ if the codimension of the stratum is odd and at most the span of $\{1, dr \wedge \omega\}$ if the codimension is even. Observe that 1, ω, and $dr \wedge \omega$ are invariant under every orientation-preserving isometry, but ω and $dr \wedge \omega$ change sign under orientation reversing isometries. Thus, the only representations of G_j that appear are the induced

one-dimensional representations of G_j on the transverse volume form to Σ_{α_j}. If some elements of G_j reverse orientation of the normal bundle, then let ξ_{G_j} denote the relevant one-dimensional representation of G_j as ± 1. Then

$$(10.1) \quad h\left(D_j^{S+,\sigma}\right) = \begin{cases} 2 & \text{if } \sigma = \mathbf{1} \text{ and } G_j \text{ preserves orientation} \\ 1 & \text{if } \sigma = \mathbf{1} \text{ and } G_j \text{ does not preserve orientation} \\ 1 & \text{if } \sigma = \xi_{G_j} \text{ and } G_j \text{ does not preserve orientation} \\ 0 & \text{otherwise} \end{cases}$$

The orientation line bundle $\mathcal{L}_{N_j} \to \Sigma_j$ of the normal bundle to Σ_j is a pointwise representation space for the representation ξ_{G_j}, and it is the canonical isotropy G-bundle W^b corresponding to $\left(\alpha_j, [\xi_{G_j}]\right)$. We may also take it to be a representation bundle for the trivial G_j-representation $\mathbf{1}$ (although the trivial line bundle is the canonical one). The main theorem takes the form

$$\text{ind}^\rho\left(d + d^*\right) = \frac{1}{\dim V_\rho} \int_{G \backslash \widetilde{M_0}} A_0^\rho(x) \ \widetilde{|dx|} \ + \sum_j \beta\left(\Sigma_{\alpha_j}\right) \ ,$$

$$\beta\left(\Sigma_{\alpha_j}\right) = \frac{1}{2 \dim V_\rho} \sum_j \left(h\left(D_j^{S+,\xi_{G_j}}\right) + h\left(D_j^{S+,\mathbf{1}}\right)\right) \int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A_j^\rho\left(x, \mathcal{L}_{N_j}\right) \ \widetilde{|dx|}$$

$$= \frac{1}{\dim V_\rho} \sum_j \int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A_j^\rho\left(x, \mathcal{L}_{N_j}\right) \ \widetilde{|dx|}.$$

In this case it is useful to rewrite $\int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A_j^\rho\left(x, \mathcal{L}_{N_j}\right) \ \widetilde{|dx|}$ as $\int_{\widetilde{\Sigma_{\alpha_j}}} K_j^\rho\left(x, \mathcal{L}_{N_j}\right) \ \widetilde{|dx|}$ before taking it to the quotient. We see that $K_j^\rho\left(x, \mathcal{L}_{N_j}\right)$ is the Gauss-Bonnet integrand on the orbit space of the desingularized stratum $\widetilde{\Sigma_{\alpha_j}}$, restricted to $\mathcal{L}_{N_j}$-twisted forms of type $[\rho]$. The result is the relative Euler characteristic

$$\int_{\widetilde{\Sigma_{\alpha_j}}} K_j^\rho\left(x, \mathcal{L}_{N_j}\right) \ \widetilde{|dx|} = \chi^\rho\left(\overline{\Sigma_{\alpha_j}}, \text{lower strata}, \mathcal{L}_{N_j}\right) ,$$

Here, the relative Euler characteristic is defined for X a closed subset of a manifold Y as $\chi\left(Y, X, \mathcal{V}\right) = \chi\left(Y, \mathcal{V}\right) - \chi\left(X, \mathcal{V}\right)$, which is also the alternating sum of the dimensions of the relative de Rham cohomology groups with coefficients in a complex vector bundle $\mathcal{V} \to Y$. The superscript ρ denotes the restriction to the subgroups of these cohomology groups of G-representation type $[\rho]$. Note that if Y is a G/G_j-bundle, $\chi\left(Y, \mathcal{V}\right)^\rho$ is the alternating sum of the dimensions of subspaces of harmonic forms with coefficients in the G-bundle $\mathcal{V} \to Y$ with representation type $[\rho]$. Further, since Σ_{α_j} is a fiber bundle over $G \backslash \Sigma_{\alpha_j}$ with fiber G-diffeomorphic to G/G_j, we have

$$\int_{\widetilde{\Sigma_{\alpha_j}}} K_j^\rho\left(x, \mathcal{L}_{N_j}\right) \ \widetilde{|dx|} = \chi^\rho\left(G/G_j, \mathcal{L}_{N_j}\right) \chi\left(G \backslash \overline{\Sigma_{\alpha_j}}, G \backslash \text{lower strata}\right) ,$$

by the formula for the Euler characteristic on fiber bundles. A similar formula holds for the principal stratum, with no orientation bundle. Putting these facts together with Theorem 9.2 and $\mathrm{ind}^{\rho}(d + d^*) = \frac{1}{\dim V_{\rho}} \chi^{\rho}(M)$, we have the following result.

Theorem 10.1. *Let M be a compact G-manifold, with G a compact Lie group and principal isotropy subgroup H_{pr}. Let M_0 denote the principal stratum, and let $\Sigma_{\alpha_1}, \ldots, \Sigma_{\alpha_r}$ denote all the components of all singular strata relative to G. We use the notations for $\chi^{\rho}(Y, X)$ and $\chi^{\rho}(Y)$ as in the discussion above. Then*

$$\chi^{\rho}(M) = \chi^{\rho}(G/H_{\mathrm{pr}}) \chi(G\backslash M, G\backslash singular\ strata)$$
$$+ \sum_j \chi^{\rho}\left(G/G_j, \mathcal{L}_{N_j}\right) \chi\left(G\backslash\overline{\Sigma_{\alpha_j}}, G\backslash lower\ strata\right),$$

where $\mathcal{L}_{N_j}$ is the orientation line bundle of normal bundle of the stratum component Σ_{α_j}.

Example 10.2. Let $M = S^n$, let $G = O(n)$ acting on latitude spheres (principal orbits, diffeomorphic to S^{n-1}). Then there are two strata, with the singular strata being the two poles (each with Euler characteristic 1). Without using the theorem, since the only harmonic forms are the constants and multiples of the volume form, we expect that

$$\chi^{\rho}(S^n) = \begin{cases} (-1)^n & \text{if } \rho = \xi \\ 1 & \text{if } \rho = \rho_0 = \mathbf{1} \end{cases},$$

where ξ is the induced one dimensional representation of $O(n)$ on the volume forms. We have that

$$\chi^{\rho}(G/H_{\mathrm{pr}}) = \chi^{\rho}(S^{n-1}) = \begin{cases} (-1)^{n-1} & \text{if } \rho = \xi \\ 1 & \text{if } \rho = \rho_0 = \mathbf{1} \end{cases},$$

and

$$\chi(G\backslash M, G\backslash singular\ strata) = \chi([-1, 1], \{-1, 1\})$$
$$= -1.$$

At each pole, the isotropy subgroup is the full $O(n)$, which may reverse the orientation of the normal space. Then

$$\chi^{\rho}\left(G/G_j, \mathcal{L}_{N_j}\right) = \chi^{\rho}(\mathrm{pt}) = \begin{cases} 1 & \text{if } \rho = \rho_0 = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Also, for each pole Σ_{α_j},

$$\chi\left(G\backslash\overline{\Sigma_{\alpha_j}}, G\backslash lower\ strata\right) = \chi(\mathrm{pt}) = 1.$$

By Theorem 10.1, we have

$$\chi^{\rho}(S^n) = \begin{cases} (-1)^n & \text{if } \rho = \xi \\ -1 & \text{if } \rho = \rho_0 = \mathbf{1} \end{cases}$$

$$+ \begin{cases} 0 & \text{if } \rho = \xi \\ 2 \cdot 1 & \text{if } \rho = \rho_0 \end{cases}$$

$$= \begin{cases} (-1)^n & \text{if } \rho = \xi \\ 1 & \text{if } \rho = \rho_0 \, , \end{cases}$$

which agrees with the predicted result.

Example 10.3. If instead the group $\mathbb{Z}_2$ acts on S^n by the antipodal map, note that

$$\chi^\rho(S^n) = \begin{cases} 1 - 1 = 0 & \text{if } \rho = \rho_0 \text{ and } n \text{ is odd} \\ 1 & \text{if } \rho = \rho_0 \text{ and } n \text{ is even} \\ 1 & \text{if } \rho = \xi \text{ and and } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

since the antipodal map is orientation preserving in odd dimensions and orientation reversing in even dimensions. On the other hand, by Theorem 10.1, since there are no singular strata, we have

$$\chi^\rho(S^n) = \chi^\rho(G/H_{\mathrm{pr}}) \chi(G \backslash M, G \backslash \text{singular strata})$$

$$= \begin{cases} 1 \cdot \chi(\mathbb{R}P^n) & \text{if } \rho = \rho_0 \text{ or } \xi \text{ and } n \text{ is odd} \\ 1 \cdot \chi(\mathbb{R}P^n) & \text{if } \rho = \rho_0 \text{ or } \xi \text{ and } n \text{ is even} \\ 0 \cdot \chi(\mathbb{R}P^n) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & \text{if } \rho = \rho_0 \text{ or } \xi \text{ and } n \text{ is odd} \\ 1 & \text{if } \rho = \rho_0 \text{ or } \xi \text{ and } n \text{ is even} \\ 0 & \text{otherwise} \, , \end{cases}$$

which agrees with the direct computation.

Example 10.4. Consider the action of $\mathbb{Z}_4$ on the torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$, where the action is generated by a $\frac{\pi}{2}$ rotation. Explicitly, $k \in \mathbb{Z}_4$ acts on $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ by

$$\phi(k) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Endow T^2 with the standard flat metric. The harmonic forms have basis $\{1, dy_1, dy_2, dy_1 \wedge dy_2\}$. Let ρ_j be the irreducible character defined by $k \in \mathbb{Z}_4 \mapsto e^{ikj\pi/2}$. Then the de Rham operator $(d + d^*)^{\rho_0}$ on $\mathbb{Z}_4$-invariant forms has kernel $\{c_0 + c_1 dy_1 \wedge dy_2 : c_0, c_1 \in \mathbb{C}\}$. One also sees that $\ker(d + d^*)^{\rho_1} = \text{span} \{idy_1 + dy_2\}$, $\ker(d + d^*)^{\rho_2} = \{0\}$, and $\ker(d + d^*)^{\rho_3} = \text{span} \{-idy_1 + dy_2\}$. Then

$$\chi^{\rho_0}(T^2) = 2, \chi^{\rho_1}(T^2) = \chi^{\rho_3}(T^2) = -1, \chi^{\rho_2}(T^2) = 0.$$

This illustrates the point that it is not possible to use the Atiyah-Singer integrand on the quotient of the principal stratum to compute even the invariant index alone. Indeed, the Atiyah-Singer integrand would be a constant times the Gauss curvature, which is identically zero. In these cases, the three singular points

$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, a_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ certainly contribute to the index. The quotient $T^2 / \mathbb{Z}_4$ is an orbifold homeomorphic to a sphere.

We now compute the Euler characteristics $\chi^\rho (T^2)$ using Theorem 10.1. First, $G / H_{\mathrm{pr}} = G$ is four points on which G acts transitively, and thus $\chi^\rho (G / H_{\mathrm{pr}}) = 1$ for each possible choice of ρ. We have $\chi (\mathbb{Z}_4 \backslash T^2, \{3 \text{ points}\}) = \chi (S^2, \{3 \text{ points}\}) = -1$. Each of the singular points a_1, a_2, a_3 is one of the strata components Σ_{α_j}, and thus $\chi (G \backslash \overline{\Sigma_{\alpha_j}}, G \backslash \text{lower strata}) = 1$ in each of these cases. We have the isotropy subgroup at a_j is

$$G_j = G_{a_j} = \begin{cases} \mathbb{Z}_4 & \text{if } j = 1 \text{ or } 3 \\ \mathbb{Z}_2 & \text{if } j = 2 \end{cases}$$

so that

$$\chi^\rho (G / G_j, \mathcal{L}_{N_j}) = \chi^\rho (G / G_j) = \begin{cases} 1 & \text{if } j = 1 \text{ or } 3 \text{ and } \rho = \rho_0 \\ 1 & \text{if } j = 2 \text{ and } \rho = \rho_0 \text{ or } \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

Then Theorem 10.1 implies

$$\chi^\rho (M) = 1 \cdot (-1) + \begin{cases} 3 & \text{if } \rho = \rho_0 \\ 1 & \text{if } \rho = \rho_2 \\ 0 & \text{if } \rho = \rho_1 \text{ or } \rho_3 \end{cases}$$

$$= \begin{cases} 2 & \text{if } \rho = \rho_0 \\ 0 & \text{if } \rho = \rho_2 \\ -1 & \text{if } \rho = \rho_1 \text{ or } \rho_3, \end{cases}$$

which agrees with the previous direct calculation.

10.3. **Transverse Signature Operator.** In this section we give an example of a transverse signature operator that arises from an S^1 action on a 5-manifold. This is essentially a modification of an example from [2, pp. 84ff], and it illustrates the fact that the eta invariant term does not always vanish. Let Z^4 be a closed, oriented, 4-dimensional Riemannian manifold on which $\mathbb{Z}_p$ (p prime > 2) acts by isometries with isolated fixed points x_i, $i = 1, \dots, N$. Let $M = S^1 \times_{\mathbb{Z}_p} Z^4$, where $\mathbb{Z}_p$ acts on S^1 by rotation by multiples of $\frac{2\pi}{p}$. Then S^1 acts on M, and $S^1 \backslash M \cong \mathbb{Z}_p \backslash Z^4$. Observe that this group action can be viewed as a Riemannian foliation by circles.

Next, let D^+ denote the signature operator $d + d^*$ from self-dual to anti-self-dual forms on Z^4; this induces a transversally elliptic operator (also denoted by D^+). Then the S^1-invariant index of D^+ satisfies

$$\mathrm{ind}^{\rho_0} (D^+) = \mathrm{Sign} (S^1 \backslash M) = \mathrm{Sign} (\mathbb{Z}_p \backslash Z^4).$$

By the Invariant Index Theorem (Theorem 9.6), Remark 9.4, and the fact that the Atiyah-Singer integrand is the Hirzebruch L-polynomial $\frac{1}{3}p_1$,

$$\operatorname{ind}^{\rho_0}(D) = \frac{1}{3}\int_{\widetilde{M}/S^1} p_1 + \frac{1}{2}\sum_{j=1}^{N}\left(-\eta\left(D_j^{S+,\rho_0}\right) + h\left(D_j^{S+,\rho_0}\right)\right),$$

where each D_j^{S+,ρ_0} is two copies of the boundary signature operator

$$B = (-1)^p (*d - d*)$$

on $2l$-forms $(l = 0, 1)$ on the lens space $S^3/\mathbb{Z}_p$. We have $h\left(D_j^{S+,\rho_0}\right) = 2h(B) = 2$ (corresponding to constants), and in [4] the eta invariant is explicitly calculated to be

$$\eta\left(D_j^{S+,\rho_0}\right) = 2\eta(B) = -\frac{2}{p}\sum_{k=1}^{p-1}\cot\left(\frac{km_j\pi}{p}\right)\cot\left(\frac{kn_j\pi}{p}\right),$$

where the action of the generator ζ of $\mathbb{Z}_p$ on S^3 is

$$\zeta \cdot (z_1, z_2) = \left(e^{\frac{2m_j\pi i}{p}}z_1, e^{\frac{2n_j\pi i}{p}}z_2\right),$$

with $(m_j, p) = (n_j, p) = 1$. Thus,

$$(10.2)\quad \operatorname{Sign}\left(S^1\backslash M\right) = \frac{1}{3}\int_{\mathbb{Z}_p\backslash\widetilde{Z^4}} p_1 + \frac{1}{p}\sum_{j=1}^{N}\sum_{k=1}^{p-1}\cot\left(\frac{km_j\pi}{p}\right)\cot\left(\frac{kn_j\pi}{p}\right) + N$$

Note that in [2, pp. 84ff] it is shown that

$$\operatorname{Sign}\left(S^1\backslash M\right) = \frac{1}{3}\int_{\mathbb{Z}_p\backslash Z^4} p_1 + \frac{1}{p}\sum_{j=1}^{N}\sum_{k=1}^{p-1}\cot\left(\frac{km_j\pi}{p}\right)\cot\left(\frac{kn_j\pi}{p}\right),$$

which demonstrates that

$$\frac{1}{3}\int_{\mathbb{Z}_p\backslash Z^4} p_1 - \frac{1}{3}\int_{\mathbb{Z}_p\backslash\widetilde{Z^4}} p_1 = N,$$

illustrating the difference in total curvature between the desingularization $\widetilde{M}$ and the original M.

10.4. **Basic index theorem for Riemannian foliations.** The content of this section will be proved, generalized, and expanded in detail in [15]. Let M be an n-dimensional, closed, connected manifold, and let $\mathcal{F}$ be a codimension q foliation on M. Let Q denote the quotient bundle $TM/T\mathcal{F}$ over M. Such a foliation is called a *Riemannian foliation* if it is endowed with a metric on Q (called the transverse metric) that is *holonomy-invariant*; that is, the Lie derivative of that transverse metric with respect to every leafwise tangent vector is zero. The metric on Q can always be extended to a Riemannian metric on M; the extended metric restricted to the normal bundle $N\mathcal{F} = (T\mathcal{F})^\perp$ agrees with the transverse metric

via the isomorphism $Q \cong N\mathcal{F}$. We refer the reader to [36], [37], and [43] for introductions to the geometric and analytic properties of Riemannian foliations.

Let $\widehat{M}$ be the transverse orthonormal frame bundle of $(M, \mathcal{F})$, and let p be the natural projection $p \colon \widehat{M} \to M$. The Bott connection is a natural connection on Q that induces a connection on $\widehat{M}$ (see [36, pp. 80ff]). The manifold $\widehat{M}$ is a principal $O(q)$-bundle over M. Given $\hat{x} \in \widehat{M}$, let $\hat{x}g$ denote the well-defined right action of $g \in G = O(q)$ applied to $\hat{x}$. Associated to $\mathcal{F}$ is the lifted foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$; the distribution $T\widehat{\mathcal{F}}$ is the horizontal lift of $T\mathcal{F}$. By the results of Molino (see [36, pp. 105-108, p. 147ff]), the lifted foliation is transversally parallelizable (meaning that there exists a global basis of the normal bundle consisting of vector fields whose flows preserve $\widehat{\mathcal{F}}$), and the closures of the leaves are fibers of a fiber bundle $\widehat{\pi} \colon \widehat{M} \to \widehat{W}$. The manifold $\widehat{W}$ is smooth and is called the basic manifold. Let $\overline{\widehat{\mathcal{F}}}$ denote the foliation of $\widehat{M}$ by leaf closures of $\widehat{\mathcal{F}}$, which is shown by Molino to be a fiber bundle. The leaf closure space of $(M, \mathcal{F})$ is denoted $W = M \big/ \overline{\mathcal{F}} = \widehat{W} \big/ G$.

$$
\begin{array}{ccccc}
p^*E & & & & \mathcal{E} \\
& \searrow & & & \downarrow \\
SO(q) & \hookrightarrow & \left(\widehat{M}, \widehat{\mathcal{F}}\right) & \xrightarrow{\widehat{\pi}} & \widehat{W} \\
& & \downarrow p & \circlearrowright \downarrow & \\
E & \to & (M, \mathcal{F}) & \xrightarrow{\pi} & W
\end{array}
$$

Endow $\left(\widehat{M}, \widehat{\mathcal{F}}\right)$ with the transverse metric $g^Q \oplus g^{O(q)}$, where g^Q is the pullback of metric on Q, and $g^{O(q)}$ is the standard, normalized, biinvariant metric on the fibers. We require that vertical vectors are orthogonal to horizontal vectors. This transverse metric gives each of $\left(\widehat{M}, \widehat{\mathcal{F}}\right)$ and $\left(\widehat{M}, \overline{\widehat{\mathcal{F}}}\right)$ the structure of a Riemannian foliation. The transverse metric on $\left(\widehat{M}, \overline{\widehat{\mathcal{F}}}\right)$ induces a well–defined Riemannian metric on $\widehat{W}$. The action of $G = O(q)$ on $\widehat{M}$ induces an isometric action on $\widehat{W}$.

For each leaf closure $\overline{\widehat{L}} \in \overline{\widehat{\mathcal{F}}}$ and $\hat{x} \in \overline{\widehat{L}}$, the restricted map $p \colon \overline{\widehat{L}} \to \overline{L}$ is a principal bundle with fiber isomorphic to a subgroup $H_{\hat{x}} \subset O(q)$, which is the isotropy subgroup at the point $\widehat{\pi}(\hat{x}) \in \widehat{W}$. The conjugacy class of this group is an invariant of the leaf closure $\overline{L}$, and the strata of the group action on $\widehat{W}$ correspond to the strata of the leaf closures of $(M, \mathcal{F})$.

A *basic form* over a foliation is a global differential form that is locally the pullback of a form on the leaf space; more precisely, $\alpha \in \Omega^*(M)$ is basic if for any vector tangent to the foliation, the interior product with both α and $d\alpha$ is zero. A *basic vector field* is a vector field V whose flow preserves the foliation. In a Riemannian foliation, near any point it is possible to choose a local orthonormal frame of Q represented by basic vector fields.

A vector bundle $E \to (M, \mathcal{F})$ that is *foliated* may be endowed with a basic connection ∇^E (one for which the associated curvature forms are basic – see [25]). An example of such a bundle is the normal bundle Q. Given such a foliated bundle, a section $s \in \Gamma(E)$ is called a *basic section* if for every $X \in T\mathcal{F}$, $\nabla^E_X s = 0$. Let $\Gamma_b(E)$ denote the space of basic sections of E. Note that the basic sections of Q correspond to basic normal vector fields.

An example of another foliated bundle over a component of a stratum M_j is the bundle defined as follows. Let $E \to M$ be any foliated vector bundle. Let $\Sigma_{\alpha_j} = \widehat{\pi}(p^{-1}(M_j))$ be the corresponding stratum on the basic manifold $\widehat{W}$, and let $W^\tau \to \Sigma_{\alpha_j}$ be a canonical isotropy bundle (Definition 4.9). Consider the bundle $\widehat{\pi}^* W^\tau \otimes p^* E \to p^{-1}(M_j)$, which is foliated and basic for the lifted foliation restricted to $p^{-1}(M_j)$. This defines a new foliated bundle $E^\tau \to M_j$ by letting E^τ_x be the space of $O(q)$-invariant sections of $\widehat{\pi}^* W^\tau \otimes p^* E$ restricted to $p^{-1}(x)$. We call this bundle **the W^τ-twist of** $E \to M_j$.

Suppose that E is a foliated $\mathrm{Cl}(Q)$ module with basic $\mathrm{Cl}(Q)$ connection ∇^E over a Riemannian foliation $(M, \mathcal{F})$. Then it can be shown that Clifford multiplication by basic vector fields preserves $\Gamma_b(E)$, and we have the operator

$$D^E_b : \Gamma_b(E^+) \to \Gamma_b(E^-)$$

defined for any local orthonormal frame $\{e_1, \ldots, e_q\}$ for Q by

$$D^E_b = \sum_{j=1}^q c(e_j) \nabla^E_{e_j} \bigg|_{\Gamma_b(E)}.$$

Then D^E_b can be shown to be well-defined and is called the basic Dirac operator corresponding to the foliated $\mathrm{Cl}(Q)$ module E (see [21]). We note that this operator is not symmetric unless a zero[th] order term involving the mean curvature is added; see [26], [28], [29], [21], [39], [23], [15] for more information regarding essential self-adjointness of the modified operator and its spectrum. In the formulas below, any lower order terms that preserve the basic sections may be added without changing the index.

Definition 10.5. The *analytic basic index* of D^E_b is

$$\mathrm{ind}_b\left(D^E_b\right) = \dim \ker D^E_b - \dim \ker \left(D^E_b\right)^*.$$

As we will show in [15], these dimensions are finite, and it is possible to identify $\mathrm{ind}_b\left(D^E_b\right)$ with the invariant index of a first order, $O(q)$-equivariant differential operator $\widehat{D}$ over a vector bundle over the basic manifold $\widehat{W}$. By applying the invariant index theorem (Theorem 9.6), we obtain the following formula for the index. In what follows, if U denotes an open subset of a stratum of $(M, \mathcal{F})$, U' denotes the desingularization of U very similar to that in Section 7, and $\widetilde{U}$ denotes the fundamental domain of U inside U'.

Theorem 10.6. *(Basic Index Theorem for Riemannian foliations, in* [15]*) Let M_0 be the principal stratum of the Riemannian foliation $(M, \mathcal{F})$, and let $M_1, \ldots, M_r$ denote all the components of all singular strata, corresponding to $O(q)$-isotropy types $[G_1], \ldots, [G_r]$ on the basic manifold. With notation as in the discussion above, we have*

$$\mathrm{ind}_b\left(D_b^E\right) = \int_{\widetilde{M_0/\mathcal{F}}} A_{0,b}(x) \; \widetilde{|dx|} + \sum_j \beta(M_j)$$

$$\beta(M_j) = \frac{1}{2} \sum_{j,\tau} \frac{1}{n_\tau \mathrm{rank}\, W^\tau} \left(-\eta\left(D_j^{S+,\tau}\right) + h\left(D_j^{S+,\tau}\right)\right) \int_{\widetilde{M_j/\mathcal{F}}} A_{j,b}^\tau(x) \; \widetilde{|dx|},$$

where the sum is over all components of singular strata and over all canonical isotropy bundles W^τ, only a finite number of which yield nonzero $A_{j,b}^\tau$, and where

(1) *$A_{0,b}(x)$ is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\widetilde{D_b^E}$ (a desingularization of D_b^E) on the quotient $\widetilde{M_0/\mathcal{F}}$, where the bundle E is replaced by the space of basic sections of over each leaf closure;*

(2) *$\eta\left(D_j^{S+,b}\right)$ and $h\left(D_j^{S+,b}\right)$ are defined in a similar way as in Theorem 9.2, using a decomposition $D_b^E = D_N * D_{M_j}$ at each singular stratum;*

(3) *$A_{j,b}^\tau(x)$ is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\left(\mathbf{1} \otimes D_{M_j}\right)'$ (blown-up and doubled from $\mathbf{1} \otimes D_{M_j}$, the twist of D_{M_j} by the canonical isotropy bundle W^τ) on the quotient $\widetilde{M_j/\mathcal{F}}$, where the bundle is replaced by the space of basic sections over each leaf closure; and*

(4) *n_τ is the number of different inequivalent G_j-representation types present in a typical fiber of W^τ.*

An example of this result is the generalization of the Gauss-Bonnet Theorem to the basic Euler characteristic. The basic forms $\Omega(M, \mathcal{F})$ are preserved by the exterior derivative, and the resulting cohomology is called basic cohomology $H^*(M, \mathcal{F})$. It is known that the basic cohomology groups are finite-dimensional in the Riemannian foliation case. See [19], [29], [27], [28], [20] for facts about basic cohomology and Riemannian foliations. The basic Euler characteristic is defined to be

$$\chi(M, \mathcal{F}) = \sum (-1)^j \dim H^j(M, \mathcal{F}).$$

We have two independent proofs of the following Basic Gauss-Bonnet Theorem; one proof uses the result in [7], and the other proof is a direct consequence of the basic index theorem stated above (proved in [15]).

In the theorem that follows, we express the basic Euler characteristic in terms of the ordinary Euler characteristic, which in turn can be expressed in terms of an integral of curvature. We extend the Euler characteristic notation $\chi(Y)$ for Y

any open (noncompact without boundary) or closed (compact without boundary) manifold to mean

$$\chi\left(Y\right) = \begin{array}{ll} \chi\left(Y\right) & \text{if } Y \text{ is closed} \\ \chi\left(1\text{-point compactification of } Y\right) - 1 & \text{if } Y \text{ is open} \end{array}$$

Also, if $\mathcal{L}$ is a foliated line bundle over a Riemannian foliation $(X, \mathcal{F})$, we define the basic Euler characteristic $\chi\left(X, \mathcal{F}, \mathcal{L}\right)$ as before, using the basic cohomology groups with coefficients in the line bundle $\mathcal{L}$.

Theorem 10.7. *(Basic Gauss-Bonnet Theorem, announced in* [40]*, proved in* [15]*) Let* $(M, \mathcal{F})$ *be a Riemannian foliation. Let* $M_0, \ldots, M_r$ *be the strata of the Riemannian foliation* $(M, \mathcal{F})$*, and let* $\mathcal{O}_{M_j/\overline{\mathcal{F}}}$ *denote the orientation line bundle of the normal bundle to* $\overline{\mathcal{F}}$ *in* M_j*. Let* L_j *denote a representative leaf closure in* M_j*. With notation as above, the basic Euler characteristic satisfies*

$$\chi\left(M, \mathcal{F}\right) = \sum_j \chi\left(M_j/\overline{\mathcal{F}}\right) \chi\left(L_j, \mathcal{F}, \mathcal{O}_{M_j/\overline{\mathcal{F}}}\right).$$

Remark 10.8. In [22, Corollary 1], they show that in special cases the only term that appears is one corresponding to a most singular stratum.

10.5. **The orbifold index theorem.** A smooth orbifold or V-manifold is locally diffeomorphic to the orbit space of discrete group action on a Euclidean ball. Globally, such a space may always be rendered as the orbit space of a $O\left(n\right)$ action on a smooth manifold, such that all of the isotropy groups are finite and the principal isotropy groups are trivial. The orbifold index theorem was proved in [31], [32]; but we may now use Theorem 9.6 to express the index of an orbifold elliptic operator in a slightly different way, because the orbifold index corresponds to the invariant index of the induced transversally elliptic operator lifted to the $O\left(n\right)$ action. The integrals over the quotients are integrals over the orbifold strata. According to Theorem 9.6, the index of an orbifold elliptic operator $D\colon \Gamma\left(M, E^+\right) \to \Gamma\left(M, E^-\right)$ over a Riemannian orbifold M is

$$\operatorname{ind}\left(D\right) = \int_{\widetilde{M_0}} A_0\left(x\right) \widetilde{|dx|} + \sum_j \beta\left(\Sigma_j\right),$$

$$\beta\left(\Sigma_j\right) = \frac{1}{2} \sum_j \sum_{b \in B} \frac{1}{n_b \operatorname{rank} W^b} \left(-\eta\left(D_j^{S+,b}\right) + h\left(D_j^{S+,b}\right)\right) \int_{\widetilde{\Sigma_j}} A_{j,b}\left(x\right) \widetilde{|dx|},$$

where $\widetilde{M_0}$ is the principal stratum of the orbifold (manifold points) after equivariantly desingularizing along all singular strata and taking the quotient, and each $\widetilde{\Sigma_i}$ is the orbit space of the fundamental domain of a component of the singular stratum Σ_j of the orbifold after desingularizing Σ_j along all singular strata properly contained in the closure of Σ_j. Also, $A_0\left(x\right)$ is the Atiyah-Singer integrand for the operator D, and $A_{j,b}\left(x\right)$ is the corresponding Atiyah-Singer integrand on the stratum Σ_j, where the restricted operator is modified by twisting

by the canonical isotropy bundle W^b of the isotropy subgroup of $O(n)$. The operators $D_j^{S+,b}$ on the sphere fibers are defined as in the rest of the paper and in Theorem 9.6.

On the other hand, the orbifold index theorem in [32] states that

$$\text{ind}(D) = \int_{M_0} A_0(x)\ |dx| + \sum_i \frac{1}{m_i} \int_{M_i} A_i(x)\ |dx|,$$

where $A_0(x)$ is the Atiyah-Singer integrand for the operator D, and $A_i(x)$ is the corresponding Atiyah-Singer integrand on the set M_i. The set M_i is a connected component of the blown-up singular set ΣM of the orbifold: if a neighborhood of x is isometric to U_x / G_x, with $G_x < O(n)$ being the isotropy,

$$\Sigma M = \{(x,(h_x)) : x \in M,\ (h_x)\text{ is a nontrivial conjugacy class in } G_x\}.$$

The centralizer $Z_{G_x}(h)$ of $h \in G_x$ is not effective on U_x; the order of the trivially acting subgroup of $Z_{G_x}(h)$ the integer $m_{(x,(h_x))}$. This integer is constant on the connected components M_i of ΣM, so that one may write $m_i = m_{(x,(h_x))}$ for any $(x,(h_x)) \in M_i$. Clearly, the Kawasaki result is a different formula than that resulting from our theorem, mainly because in our case the singular strata are blown up. We note that in [31], a very similar formula to (10.2) above is provided for the orbifold signature theorem.

11. APPENDIX: GENERALIZED SPHERICAL HARMONICS

Collected here are the necessary facts concerning eigenvalues of spherical operators arising from writing constant coefficient, first order differential operators in polar coordinates.

11.1. Polar coordinate form of constant coefficient operator on $\mathbb{R}^k$.

We consider the following situation. Let $Q_1 \colon C^\infty(\mathbb{R}^k, \mathbb{C}^d) \to C^\infty(\mathbb{R}^k, \mathbb{C}^d)$ be an elliptic, first order, constant coefficient differential operator on $\mathbb{R}^k$ with $k \geq 2$. Let

$$Q_1 = \sum_{j=1}^k A_j \partial_j,$$

where by assumption $A_j \in GL(d, \mathbb{C})$, and for any $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$, $\det(\sum c_j A_j) = 0$ implies $c = 0$. Note that

$$Q_1^* = -\sum_{j=1}^k A_j^* \partial_j$$

We form the symmetric operator $Q \colon C^\infty(\mathbb{R}^k, \mathbb{C}^d \oplus \mathbb{C}^d) \to C^\infty(\mathbb{R}^k, \mathbb{C}^d \oplus \mathbb{C}^d)$

$$Q = \begin{pmatrix} 0 & Q_1^* \\ Q_1 & 0 \end{pmatrix} = \sum_{j=1}^k \begin{pmatrix} 0 & -A_j^* \\ A_j & 0 \end{pmatrix} \partial_j$$

We write Q in polar coordinates $(r, \theta) \in (0, \infty) \times S^{k-1}$, with $x = r\theta = r(\theta_1, \ldots, \theta_k)$, as

$$Q_1 = Z^+ \left(\partial_r + \frac{1}{r} Q^{S+} \right), \text{ so that}$$

$$Q = \begin{pmatrix} 0 & \left(\partial_r + \frac{1}{r} Q^{S+} \right)^* Z^{+*} \\ Z^+ \left(\partial_r + \frac{1}{r} Q^{S+} \right) & 0 \end{pmatrix}$$

$$= Z \left(\partial_r + \frac{1}{r} Q^S \right),$$

where

$$Z^+_{(r,\theta)} = \sum_{j=1}^k \theta_j A_j$$

and

$$Q^{S+}_{(r,\theta)} = r \sum_{j=1}^k \left(Z^+ \right)^{-1} A_j \partial_j - \sum_{j=1}^k x_j \partial_j$$

$$= r \sum_{j=1}^k \left(Z^+ \right)^{-1} A_j \left(\theta_j \partial_r + \frac{1}{r} \frac{\partial}{\partial \theta_j} \right) - \sum_{j=1}^k r\theta_j \left(\theta_j \partial_r + \frac{1}{r} \frac{\partial}{\partial \theta_j} \right)$$

$$= r \left(Z^+ \right)^{-1} \left(\sum_{j=1}^k A_j \theta_j \right) \partial_r + \left(Z^+ \right)^{-1} \sum_{j=1}^k A_j \frac{\partial}{\partial \theta_j} - r\partial_r - \sum_{j=1}^k \theta_j \frac{\partial}{\partial \theta_j}$$

$$(11.1) \qquad Q^{S+}_{(r,\theta)} = \left(Z^+ \right)^{-1} \sum_{j=1}^k A_j \frac{\partial}{\partial \theta_j} - \sum_{j=1}^k \theta_j \frac{\partial}{\partial \theta_j} = \left(Z^+ \right)^{-1} \sum_{j=1}^k A_j \frac{\partial}{\partial \theta_j}.$$

Note that $\frac{\partial}{\partial \theta_j}$ and Q^{S+} may be regarded as an operator on the sphere, and the matrix Z is independent of the radial parameter r. Thus the adjoint of Z is also independent of r. Since Q^{S+} does not differentiate in the r direction, the formal adjoint of Q^{S+} restricted to the sphere is the same as the formal adjoint of Q^{S+} as an operator on $\mathbb{R}^k$, restricted to functions that are locally constant in the radial directions. Note that the formula above implies that if u is a constant vector in $\mathbb{C}^d$, then $Q^{S+}u = 0$. Because the formal adjoint of ∂_r is $-\partial_r - \frac{k-1}{r}$ on $\mathbb{R}^k$ and by the properties indicated above, we have

$$\left(\partial_r + \frac{1}{r} Q^{S+} \right)^* Z^{+*} = \left(-\partial_r - \frac{k-1}{r} + Q^{S+*} \circ \frac{1}{r} \right) Z^{+*}$$

$$= -Z^{+*} \left(\partial_r + \frac{k-1}{r} - \left(Z^{+*} \right)^{-1} Q^{S+*} \left(Z^{+*} \right) \circ \frac{1}{r} \right).$$

Thus, we have that

$$Z = \begin{pmatrix} 0 & Z^- \\ Z^+ & 0 \end{pmatrix}, \ Q^S = \begin{pmatrix} Q^{S+} & 0 \\ 0 & Q^{S-} \end{pmatrix},$$

with

$$Z^- = -Z^{+*}, \ Q^{S-} = (k-1)I - r\left(Z^-\right)^{-1} Q^{S+*}\left(Z^-\right) \circ \frac{1}{r}.$$

Note that by the choice of polar coordinates, Q^{S-} is an operator on the sphere and does not differentiate in the radial direction, so that

$$Q^{S-}\left(r\left(Z^-\right)^{-1}\right) = \left[(k-1)I - r\left(Z^-\right)^{-1} Q^{S+*}\left(Z^-\right) \circ \frac{1}{r}\right]\left(r\left(Z^-\right)^{-1}\right), \text{ or}$$

$$rQ^{S-}\left(Z^-\right)^{-1} = r(k-1)\left(Z^-\right)^{-1} - r\left(Z^-\right)^{-1} Q^{S+*}.$$

Multiplying on the left by $\frac{1}{r}Z^-$ and solving for Q^{S+*}, we obtain

$$Q^{S+*} = (k-1)I - Z^- Q^{S-}\left(Z^-\right)^{-1}, \text{ or}$$

$$Q^{S-} = (k-1)I - \left(Z^-\right)^{-1} Q^{S+*}\left(Z^-\right).$$

In summary, our operator $Q \colon C^\infty\left(\mathbb{R}^k, \mathbb{C}^d \oplus \mathbb{C}^d\right) \to C^\infty\left(\mathbb{R}^k, \mathbb{C}^d \oplus \mathbb{C}^d\right)$ satisfies

(11.2)

$$\begin{aligned} Q &= \sum_{j=1}^k \begin{pmatrix} 0 & -A_j^* \\ A_j & 0 \end{pmatrix} \partial_j \\ &= Z\left(\partial_r + \frac{1}{r}Q^S\right) = \begin{pmatrix} 0 & Z^- \\ Z^+ & 0 \end{pmatrix}\left(\partial_r + \frac{1}{r}\begin{pmatrix} Q^{S+} & 0 \\ 0 & Q^{S-} \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & -Z^{+*} \\ Z^+ & 0 \end{pmatrix}\left(\partial_r + \frac{1}{r}\begin{pmatrix} Q^{S+} & 0 \\ 0 & (k-1)I - (Z^{+*})^{-1}Q^{S+*}(Z^{+*}) \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & Z^- \\ -Z^{-*} & 0 \end{pmatrix}\left(\partial_r + \frac{1}{r}\begin{pmatrix} (k-1)I - (Z^{-*})^{-1}Q^{S-*}(Z^{-*}) & 0 \\ 0 & Q^{S-} \end{pmatrix}\right), \end{aligned}$$

and

(11.3) $$Q^{S*} = (k-1)I - ZQ^S Z^{-1}$$

In the case of a Dirac type operator, Q^S restricts to a Dirac-type operator on the sphere that does not inherit the grading.

Note also that

(11.4) $$Z^+_{(r,\theta)} = \frac{1}{r}\sum_{j=1}^k x_j A_j, \quad \text{and} \quad Z^{+*}_{(r,\theta)} = \frac{1}{r}\sum_{j=1}^k x_j A_j^*$$

If we assume that Z^+ is an isometry, then

$$(11.5) \qquad \left(Z^+_{(r,\theta)}\right)^{-1} = \frac{1}{r} \sum_{j=1}^{k} x_j A_j^*.$$

Then

$$Q^{S+}_{(r,\theta)} = r \sum_{j=1}^{k} \left(Z^+\right)^{-1} A_j \partial_j - \sum_{j=1}^{k} x_j \partial_j = \sum_{j,l=1}^{k} x_l A_l^* A_j \partial_j - \sum_{j=1}^{k} x_j \partial_j,$$

so that Q^{S+} fixes the vector space of polynomials in x that are homogeneous of degree m, for $m \geq 0$. A similar fact is true for $\left(Q^{S+}\right)^*$. In the following section, it will be shown that in fact $\left(Q^{S+}\right)^* = Q^{S+}$ on appropriate subspaces of sections. For future reference, note that Q^{S+} is a continuous function of the matrices A_j.

11.2. **Eigenvalues of Q^{S+}.** Let

$$\Delta = Q_1^* Q_1 = \sum -A_i^* A_j \partial_i \partial_j$$
$$= \left(-\partial_r - \frac{k-1}{r} + \frac{1}{r}\left(Q^{S+}\right)^*\right)\left(Z^+\right)^* Z^+ \left(\partial_r + \frac{1}{r}Q^{S+}\right), \text{ and}$$
$$\Delta = \left(-\partial_r - \frac{k-1}{r} + \frac{1}{r}\left(Q^{S+}\right)^*\right)\left(\partial_r + \frac{1}{r}Q^{S+}\right)$$

if we let Z^+ be an isometry. Let

$$B(x) = \sum -x_i x_j A_i^* A_j,$$

which is, up to a sign, the principal symbol of Δ. Let $\mathcal{P}_m$ denote the vector space of polynomials in x that are homogeneous of degree m with coefficients that are d-dimensional complex vectors. We define the following Hermitian inner product $\langle \, \cdot \, , \, \cdot \, \rangle$ on $\mathcal{P}_m$. Given any $P, Q \in \mathcal{P}_m$, we write

$$P(x) = \sum P_\alpha x^\alpha, \ Q(x) = \sum Q_\alpha x^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multiindex and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}$. We define

$$\langle P, Q \rangle = \sum \alpha! P^\alpha \cdot \overline{Q^\alpha},$$

where $\alpha! = \alpha_1! \ldots \alpha_k!$. Observe that

$$\langle P, Q \rangle = P(\partial) \cdot \overline{Q(x)},$$

where $P(\partial)$ means $\sum P^\alpha \partial_\alpha$, and where $P(\partial) \cdot \overline{Q(x)}$ means $\sum \alpha! P^\alpha \cdot \overline{Q^\beta} \partial_\alpha x^\beta$. Observe that $\langle \, \cdot \, , \, \cdot \, \rangle$ is a positive definite Hermitian inner product on $\mathcal{P}_m$.

Proposition 11.1. *For $m \geq 2$, the linear map $\Delta \colon \mathcal{P}_m \to \mathcal{P}_{m-2}$ is onto. Also, letting*

$$\mathcal{H}_m = \ker \Delta \cap \mathcal{P}_m = \ker Q_1 \cap \mathcal{P}_m,$$

the space $\mathcal{P}_m$ is the orthogonal direct sum

$$\mathcal{P}_m = \bigoplus_{j=0}^{\lfloor m/2 \rfloor} \left(B\left(x\right) \right)^{2j} \mathcal{H}_{m-2j}.$$

Proof. Observe that for every $Q \in \mathcal{P}_{m-2}$ and every $P \in \mathcal{P}_m$,

$$\begin{aligned}
\langle Q, \Delta P \rangle &= Q\left(\partial\right) \cdot \overline{B\left(\partial\right) P\left(x\right)} \\
&= \sum_{\alpha,\beta,i,j} \alpha! Q_\alpha \cdot \overline{(-A_i^* A_j P_\beta)} \partial_\alpha \left(\partial_i \partial_j x^\beta \right) \\
&= \sum_{\alpha,\beta,i,j} \alpha! \left(-A_j^* A_i Q_\alpha \right) \cdot \overline{P_\beta} \partial_\alpha \left(\partial_j \partial_i x^\beta \right) \\
&= \left(BQ \right)\left(\partial\right) \cdot \overline{P\left(x\right)} = \langle BQ, P \rangle .
\end{aligned}$$

If $Q \in \mathcal{P}_{m-2}$ is orthogonal to $\Delta\left(\mathcal{P}_m\right)$, then the computation above implies that every $P \in \mathcal{P}_m$ is orthogonal to $BQ \in \mathcal{P}_m$, so $BQ = 0$ and thus $Q = 0$. (Here we use the fact that Q_1 and thus Δ is elliptic.) Thus $\Delta \colon \mathcal{P}_m \to \mathcal{P}_{m-2}$ is onto. Also, $\Delta P = 0$ if and only if $\langle BQ, P \rangle = 0$ for every $Q \in \mathcal{P}_{m-2}$, so $\mathcal{P}_m = \mathcal{H}_m \oplus B\left(x\right) \mathcal{P}_{m-2}$. The result follows. $\qquad\square$

Corollary 11.2. *Let d_m denote the dimension of $\mathcal{P}_m$, which is $d \begin{pmatrix} m+k-1 \\ m \end{pmatrix}$. The dimension h_m of the space $\mathcal{H}_m$ is*

$$h_m = \begin{cases} d_m - d_{m-2} & \text{if } m \geq 2 \\ d_m & \text{if } m = 0, 1 \end{cases}$$

Next, let

$$Q = \begin{pmatrix} 0 & Q_1^* \\ Q_1 & 0 \end{pmatrix} = \sum_{j=1}^{k} \begin{pmatrix} 0 & -A_j^* \\ A_j & 0 \end{pmatrix} \partial_j$$

as above, so that

$$Q^2 = \begin{pmatrix} \Delta & 0 \\ 0 & \left(Z^+\right) \widetilde{\Delta} \left(Z^+\right)^{-1} \end{pmatrix},$$

with

$$\widetilde{\Delta} = \left(Z^+\right)^{-1} Q_1 Q_1^* Z^+.$$

Writing Q in polar form

$$Q = Z \left(\partial_r + \frac{1}{r} Q^S \right),$$

note that a polynomial p_m of degree m is in the kernel $\mathcal{H}_m = \ker \Delta$ implies

$$\left(-\partial_r - \frac{k-1}{r} + \frac{1}{r} \left(Q^{S+} \right)^* \right) \left(\partial_r + \frac{1}{r} Q^{S+} \right) p_m = 0, \text{ or}$$

$$\left(-\frac{m+k-2}{r} + \frac{1}{r} \left(Q^{S+} \right)^* \right) \left(\frac{m}{r} + \frac{1}{r} Q^{S+} \right) p_m = 0, \text{ so that}$$

(11.6) $$\left(-(m+k-2)+\left(Q^{S+}\right)^*\right)\left(m+Q^{S+}\right)p_m=0.$$

Since the vector space $\mathcal{H}_m$ of polynomials that are homogeneous of degree m in $\ker\Delta$ is finite-dimensional and is fixed by Q^{S+} and $\left(Q^{S+}\right)^*$, the following linear algebra fact is relevant (thanks to George Gilbert and Igor Prokhorenkov):

Lemma 11.3. *Suppose that L is an $r\times r$ complex matrix that satisfies*

$$\left(L-aI\right)\left(L^*-bI\right)=0,$$

where L^* is the adjoint (which is the conjugate transpose), $a,b\in\mathbb{R}$, and I is the identity matrix. Then L is Hermitian, and each eigenvalue is a or b.

Proof. If $a=b$, then $L=aI$, since if M is an $r\times r$ matrix with $MM^*=0$, then $M=0$. If $a\neq b$, then by taking adjoints we also have that $\left(L-bI\right)\left(L^*-aI\right)=0$. Subtracting the two equations, we obtain

$$\left(a-b\right)L+\left(b-a\right)L^*=0,$$

so that $L=L^*$ and is thus Hermitian, and since its minimal polynomial is a factor of $(x-a)(x-b)$, the result follows. $\square$

Proposition 11.4. *For any $m\in\mathbb{Z}_{\geq0}$ the restriction $\mathcal{H}_m^S$ of $\mathcal{H}_m=\mathcal{P}_m\cap\ker\Delta$ to the unit sphere $S^{k-1}\subset\mathbb{R}^k$ is the L^2-orthogonal direct sum of the eigenspaces of Q^{S+} corresponding to eigenvalues $-m$ and $m+k-2$.*

Proof. Lemma 11.3 and Formula 11.6. $\square$

Observe that $d_0=\dim\mathcal{P}_0=\dim\mathcal{H}_0=h_0=d$, and $d_1=\dim\mathcal{P}_1=\dim\mathcal{H}_1=h_1=dk$. Letting E_λ denote the eigenspace of Q^{S+} corresponding to eigenvalue λ, by formula (11.1) we have

$$d=\dim E_0+\dim E_{k-2}\geq d+\dim E_{k-2}\text{, so}$$
$$\dim E_0=d;\ \dim E_{k-2}=0.$$

Note that we do have similar facts for $\ker\widetilde{\Delta}$, because also maps $\mathcal{P}_m$ to $\mathcal{P}_{m-2}$. If $p_m\in\mathcal{P}_m\cap\ker\widetilde{\Delta}$, we have

$$\widetilde{\Delta}p_m=\left(\partial_r+\frac{1}{r}Q^{S+}\right)\left(-\partial_r-\frac{k-1}{r}+\frac{1}{r}\left(Q^{S+}\right)^*\right)p_m$$
$$=\left(\frac{m-1}{r}+\frac{1}{r}Q^{S+}\right)\left(-\frac{m+k-1}{r}+\frac{1}{r}\left(Q^{S+}\right)^*\right)p_m=0,$$

which implies

$$\left(m-1+Q^{S+}\right)\left(-(m+k-1)+\left(Q^{S+}\right)^*\right)p_m=0.$$

Then, Lemma 11.3 implies the following.

Proposition 11.5. *For any $m\in\mathbb{Z}_{\geq0}$ the restriction $\widetilde{\mathcal{H}_m^S}$ of $\mathcal{P}_m\cap\ker\widetilde{\Delta}$ to the unit sphere $S^{k-1}\subset\mathcal{R}^k$ is the L^2-orthogonal direct sum of the eigenspaces of Q^{S+} corresponding to eigenvalues $-m+1$ and $m+k-1$.*

Proposition 11.6. *Let $\mathcal{P}_m^S$ denote the restriction of $\mathcal{P}_m$ to the unit sphere $S^{k-1} \subset \mathcal{R}^k$, and let $\mathcal{H}_m^S$ be defined similarly. Let E_λ denote the eigenspace of Q^{S+} corresponding to eigenvalue λ. Then*

$$\mathcal{P}_m^S = \mathcal{H}_m^S + \mathcal{H}_{m-2}^S + \ldots + \mathcal{H}_{m-2\lfloor m/2 \rfloor}^S$$
$$= E_{-m} + \ldots + E_0 + E_{k-1} + \ldots + E_{m+k-2}.$$

That is, $\mathcal{P}_m^S$ has a basis consisting of elements of $\ker \Delta$, restricted to S^{k-1}. Similarly, letting $\widetilde{\mathcal{H}_m^S}$ denote the restriction of $\mathcal{P}_m \cap \ker \widetilde{\Delta}$ to S^{k-1}, we have that

$$\mathcal{P}_m^S = \widetilde{\mathcal{H}_m^S} + \widetilde{\mathcal{H}_{m-2}^S} + \ldots + \widetilde{\mathcal{H}_{m-2\lfloor m/2 \rfloor}^S}$$
$$= E_{-m+1} + \ldots + E_0 + E_{k-1} + \ldots + E_{m+k-1}.$$

In particular, we have that for all $m \geq 0$,

$$\dim E_{-m} = \dim E_{m+k-1}$$

Proof. Clearly $\mathcal{H}_{m-2j}^S \subset \mathcal{P}_m^S$ for $0 \leq j \leq \lfloor m/2 \rfloor$, since $|x|^{2j} \mathcal{H}_{m-2j} \subset \mathcal{P}_m$. Also, the subspaces $\mathcal{H}_m^S, \mathcal{H}_{m-2}^S, \ldots, \mathcal{H}_{m-2\lfloor m/2 \rfloor}^S$ are linearly independent. Otherwise, a nontrivial polynomial in $\ker \Delta$ would be identically zero on S^{k-1}. By Proposition 11.4, we could write this vector-valued polynomial as a sum of eigenvectors of Q^{S+} and thus each eigenvector would have to be zero. Since eigenvalues of Q^{S+} corresponding to distinct $m \in \mathbb{Z}_{\geq 0}$ are distinct, each such eigenvector is written as a vector of polynomials of pure degree that is identically zero on the sphere. Because any such polynomial is the zero polynomial, this implies that there is no nontrivial polynomial in $\ker \Delta$ that is identically zero. Further, By counting dimensions, the result follows. The second part of the corollary is proved in a similar way. $\qquad \square$

The denseness of polynomials in $L^2\left(S^{k-1}\right)$ and the formulas above imply the following result.

Proposition 11.7. *The spectrum of Q^{S+} is discrete, and there is a basis of $L^2\left(S^{k-1}, \mathbb{C}^d\right)$ consisting of eigenvectors of Q^{S+}. The set of eigenvalues of Q^{S+} is a subset of $Q = \{\ldots, -5, -4, -3, -2, -1, 0, k-1, k, \ldots\}$. Let μ_j denote the multiplicity of the eigenvalue $j \in Q$. Then the nonnegative integers μ_j satisfy the equations*

$$\mu_0 = d \;, \;\; \mu_{-j} + \mu_{k-2+j} = h_j \;, \;\; and$$

$$\mu_{-j} = \mu_{j+k-1} = \sum_{m=0}^{j} (-1)^{j+m} h_m = d_j - d_{j-1}$$

$$= d\binom{j+k-2}{j} \; for \; all \; j \geq 0.$$

Remark 11.8. Observe that this gives a universal formula for the eigenvalues of any such Q^{S+} that depends only on the rank d and the dimension k.

11.3. **The equivariant case.** Suppose that we have the setup as in Section 11.1 and 11.2, but that in addition we are given a subgroup H of $O(k)$ that acts on $\mathbb{R}^k$ in the obvious way and also acts unitarily on each summand of $\mathbb{C}^d \oplus \mathbb{C}^d$, such that the operator Q_1 (and thus Q) commutes with the H-action. Then, given an irreducible representation $\alpha \colon H \to GL(V)$, we consider the restriction of each of the operators in the sections above to the space of vector-valued functions of type α. Then the following generalizations of Propositions 11.4, 11.5, 11.6, and 11.7 are immediate.

Let the superscript α denote the restriction to the space of vector-valued functions of type α.

Let d_m^α denote the dimension of $(\mathcal{P}_m)^\alpha = \mathcal{P}_m^\alpha$. As before, the dimension h_m^α of the space $\mathcal{H}_m^\alpha = \mathcal{P}_m \cap (\ker \Delta)^\alpha$ is then

$$h_m^\alpha = \begin{cases} d_m^\alpha - d_{m-2}^\alpha & \text{if } m \geq 2 \\ d_m^\alpha & \text{if } m = 1, 2 \end{cases}$$

Proposition 11.9. *For any $m \in \mathbb{Z}_{\geq 0}$ the restriction $\mathcal{H}_m^{S,\alpha}$ of $\mathcal{H}_m^\alpha = \mathcal{P}_m \cap (\ker \Delta)^\alpha$ to the unit sphere $S^{k-1} \subset \mathcal{R}^k$ is the L^2-orthogonal direct sum of the eigenspaces of $Q^{S+,\alpha}$ corresponding to eigenvalues $-m$ and $m + k - 2$.*

Proposition 11.10. *For any $m \in \mathbb{Z}_{\geq 0}$ the restriction $\widetilde{\mathcal{H}_m^{S,\alpha}}$ of $\mathcal{P}_m \cap \left(\ker \widetilde{\Delta}\right)^\alpha$ to the unit sphere $S^{k-1} \subset \mathcal{R}^k$ is the L^2-orthogonal direct sum of the eigenspaces of $Q^{S+,\alpha}$ corresponding to eigenvalues $-m + 1$ and $m + k - 1$.*

Proposition 11.11. *Let $\mathcal{P}_m^{S,\alpha}$ denote the restriction of $\mathcal{P}_m^\alpha$ to the unit sphere $S^{k-1} \subset \mathcal{R}^k$, and let $\mathcal{H}_m^{S,\alpha}$ be defined similarly. Let E_λ^α denote the eigenspace of $Q^{S+,\alpha}$ corresponding to eigenvalue λ. Then*

$$\mathcal{P}_m^{S,\alpha} = \mathcal{H}_m^{S,\alpha} + \mathcal{H}_{m-2}^{S,\alpha} + \ldots + \mathcal{H}_{m-2\lfloor m/2 \rfloor}^{S,\alpha}$$
$$= E_{-m}^\alpha + \ldots + E_0^\alpha + E_{k-1}^\alpha + \ldots + E_{m+k-2}^\alpha.$$

That is, $\mathcal{P}_m^{S,\alpha}$ has a basis consisting of elements of $(\ker \Delta)^\alpha$, restricted to S^{k-1}. Similarly, letting $\widetilde{\mathcal{H}_m^{S,\alpha}}$ denote the restriction of $\mathcal{P}_m \cap \left(\ker \widetilde{\Delta}\right)^\alpha$ to S^{k-1}, we have that

$$\mathcal{P}_m^{S,\alpha} = \widetilde{\mathcal{H}_m^{S,\alpha}} + \widetilde{\mathcal{H}_{m-2}^{S,\alpha}} + \ldots + \widetilde{\mathcal{H}_{m-2\lfloor m/2 \rfloor}^{S,\alpha}}$$
$$= E_{-m+1}^\alpha + \ldots + E_0^\alpha + E_{k-1}^\alpha + \ldots + E_{m+k-1}^\alpha.$$

In particular, we have that for all $m \geq 0$,

$$\dim E_{-m}^\alpha = \dim E_{m+k-1}^\alpha$$

Proposition 11.12. *The spectrum of $Q^{S+,\alpha}$ is discrete, and there is a basis of $L^2\left(S^{k-1}, \mathbb{C}^d\right)^\alpha$ consisting of eigenvectors of $Q^{S+,\alpha}$. The set of eigenvalues of $Q^{S+,\alpha}$ is a subset of $Q = \{\ldots, -5, -4, -3, -2, -1, 0, k-1, k, \ldots\}$. Let μ_j^α*

denote the (possibly zero) multiplicity of the eigenvalue $j \in Q$. Then the nonnegative integers μ_j^α satisfy the equations

$$\mu_0^\alpha = \dim \mathbb{C}^{d,\alpha} \ , \ \ \mu_{-j}^\alpha + \mu_{k-2+j}^\alpha = h_j^\alpha \ , \ \ and$$

$$\mu_{-j}^\alpha = \mu_{j+k-1}^\alpha = \sum_{m=0}^{j} (-1)^{j+m} \, h_m^\alpha = d_j^\alpha - d_{j-1}^\alpha \ for \ all \ j \geq 0.$$

Remark 11.13. Observe that this gives a universal formula for the eigenvalues of any such $Q^{S+,\alpha}$ that depends only on the irreducible representation α and the actions of H on $\mathbb{R}^k$ and on $\mathbb{C}^d$. Furthermore, since $Q^{S+,\alpha}$ varies continuously with the entries of the matrices A_j, we note that any continuous function of the set of eigenvalues (such as the eta invariant or the dimension of an eigenspace) remains constant as the matrices A_j vary continuously (and equivariantly). Furthermore, any spectral function of the form

$$\sum_{\lambda \in \sigma(Q^{S+,\alpha})} f(\lambda) \, ,$$

such as the trace of the heat kernel or the zeta or eta functions, is invariant under equivariant stable homotopies of the original operator Q.

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