WEIGHTED NORM INEQUALITIES FOR FOURIER TRANSFORMS OF RADIAL FUNCTIONS

D. GORBACHEV, E. LIFLYAND, AND S. TIKHONOV

ABSTRACT. Weighted $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ Fourier inequalities are studied. We prove Pitt–Boas type results on integrability with general weights of the Fourier transform of a radial function.

1. INTRODUCTION

Weighted norm inequalities for the Fourier transform provide a natural way to describe the balance between the relative sizes of a function and its Fourier transform at infinity. What is more, such inequalities with sharp constants imply the uncertainty principle relations ([1], [2]). The celebrated Pitt inequality illustrates this idea at the spectral level ([1]):

$$\int_{\mathbb{R}^n} \Phi(1/|y|)|\widehat{f}(y)|^2 dy \le C_\Phi \int_{\mathbb{R}^n} \Phi(|x|)|f(x)|^2 dx,$$

where Φ is an increasing function and $\widehat{f}$ is the Fourier transform of a function f from the Schwartz class $\mathcal{S}(\mathbb{R}^n)$,

$$\widehat{f}(y) = \mathcal{F}f(y) = \int_{\mathbb{R}^n} f(x)e^{ixy}dx. \tag{1}$$

In the (L^p, L^q) setting such inequalities have been studied extensively (see, for instance, [1]–[5], [9], [10], [11], [17], [22]). In this case Pitt's inequality is written as follows: for $1 < p \le q < \infty$, $0 \le \gamma < n/q$, $0 \le \beta < n/p'$ and $n \ge 1$

$$\left(\int_{\mathbb{R}^n} \left(|y|^{-\gamma}|\widehat{f}(y)|\right)^q dy\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} \left(|x|^{\beta}|f(x)|\right)^p dx\right)^{1/p} \tag{2}$$

with the index constraint

$$\beta - \gamma = n - n\left(\frac{1}{p} + \frac{1}{q}\right)$$

(primes denote the dual exponents, $1/p + 1/p' = 1$).

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The restrictions on γ and β can be written as

$$\max\left\{0, n\left(\frac{1}{p}+\frac{1}{q}-1\right)\right\} \le \gamma < \frac{n}{q}. \tag{3}$$

It is worth mentioning that inequality (2) contains classical (non-weighted) versions of the Plancherel theorem, that is, $\|\widehat{f}\|_2 \asymp \|f\|_2$, Hardy–Littlewood's theorem ($1 < p = q \le 2$, $\beta = 0$ or $p = q \ge 2$, $\gamma = 0$), and Hausdorff–Young's theorem ($q = p' \ge 2$, $\beta = \gamma = 0$).

For $n = 1$, inequality (2) can be found in [3], [15], [16], [20]; for $n \ge 1$ see [2], [3]. In [1], W. Beckner found a sharp constant in (2) for $p = q = 2$ and used this result to prove a logarithmic estimate for uncertainty.

In this paper we address the following two problems.

Problem 1**:** The range (3) is sharp if f is simply assumed to be in L^p_u, $u(x) = |x|^{p\beta}$. Is it possible to extend this range if additional regularity of f is assumed?

Problem 2**:** Under which additional assumption on f it is possible to reverse inequality (2) for $p = q$?

Let us first recall several known results in dimension 1. Some progress toward extending the range of γ in (3) was made in [4], [17], and [22], where the authors assumed that the function has vanishing moments up to certain order.

Another approach, which is related to both Problems 1 and 2, is due to Hardy, Littlewood, and, later, Boas. The well-known Hardy–Littlewood theorem (see [23, Ch.IV]) states that if $1 < p < \infty$ and f is an even non-increasing function which vanishes at infinity, then

$$C_1\left(\int_{\mathbb{R}} |\widehat{f}(x)|^p\,dx\right)^{1/p} \le \left(\int_{\mathbb{R}_+} |f(t)|^p\,t^{p-2}\,dt\right)^{1/p} \le C_2\left(\int_{\mathbb{R}} |\widehat{f}(x)|^p\,dx\right)^{1/p}. \tag{4}$$

Boas conjectured in [7] that the weighted version of (4) should also be true: under the same conditions on f and p,

$$|x|^{-\gamma}|\widehat{f}(x)| \in L^p(\mathbb{R}) \qquad \text{if and only if} \qquad t^{1+\gamma-2/p} f(t) \in L^p(\mathbb{R}_+), \tag{5}$$

provided $-1/p' = -1 + 1/p < \gamma < 1/p$.

Relation (5) was proved in [18]. Thus, assuming a function to be monotone allows one to extend the range of γ as well as to reverse inequality (2) for $p = q$.

In [12], Boas-type results were obtained for the cosine and sine Fourier transforms, separately. To describe it briefly, we denote

$$\widehat{f}_c(x) = \int_0^\infty f(t)\cos xt\,dt \qquad \text{and} \qquad \widehat{f}_s(x) = \int_0^\infty f(t)\sin xt\,dt.$$

We call a function *admissible* if it is locally of bounded variation on $(0,\infty)$ and vanishes at infinity. For any admissible non-negative function f satisfying

$$\int_t^{2t} |df(u)| \le C \int_{t/c}^{ct} u^{-1}|f(u)|\,du \tag{6}$$

for some $c > 1$, relation (5) holds for f and $\widehat{f}_c$ provided $-1/p' < \gamma < 1/p$, while for f and $\widehat{f}_s$ provided $-1/p' < \gamma < 1/p + 1$ (note the larger range).

In the higher-dimensional setting, the situation is expectedly more complex. For radial functions $f(x) = f_0(|x|)$, $x \in \mathbb{R}^n$, the Fourier transform is also radial, i.e., $\widehat{f}(x) = F_0(|x|)$. One can then apply the one-dimensional results. For example, in $\mathbb{R}^3$ the Fourier transform is given by

$$\widehat{f}(x) = 4\pi|x|^{-1} \int_0^\infty t f_0(t) \sin|x|t\,dt.$$

So, applying the result for the sine transform $\widehat{f}_s$ to the function $tf_0(t)$, we obtain

$$|x|^{-\gamma}\widehat{f}(x) \in L^p(\mathbb{R}^3) \quad \text{if and only if} \quad t^{3+\gamma-4/p} f_0(t) \in L^p(0,\infty), \tag{7}$$

provided $-2 + 3/p < \gamma < 3/p$. Note that it is enough to assume that f_0 itself satisfies (6), since this implies the same for $tf_0(t)$.

For $n \ne 3$, we can also apply (5) using fractional integrals. If f_0 is such that

$$\int_0^\infty t^{n-1}(1+t)^{(1-n)/2}|f_0(t)|\,dt < \infty, \tag{8}$$

one has the following Leray's formula (see, e.g., Lemma 25.1′ in [19]):

$$\widehat{f}(x) = 2\pi^{(n-1)/2} \int_0^\infty I(t) \cos|x|t\,dt, \tag{9}$$

where the fractional integral I is given by

$$I(t) = \frac{2}{\Gamma\left(\frac{n-1}{2}\right)} \int_t^\infty s f_0(s)(s^2 - t^2)^{(n-3)/2} ds.$$

Then, the one-dimensional Boas' relation (5) implies that if $f_0 \ge 0$ satisfies (8), then

$$|x|^{-\gamma}\widehat{f}(x) \in L^p(\mathbb{R}^n) \qquad \text{if and only if} \qquad t^{1+\gamma-(n+1)/p} I(t) \in L^p(0,\infty),$$

provided $-1 + n/p < \gamma < n/p$. However, the condition on I is difficult to verify and so it is desirable to obtain more direct Boas-type conditions. This is the main goal of the present paper.

Definition. *We call an admissible function f_0 general monotone, written GM, if for any $t > 0$*

$$\int_t^\infty |df_0(u)| \le C \int_{t/c}^\infty |f_0(u)|\,\frac{du}{u} \tag{10}$$

for some $c > 1$.

In the context of our results, we always deal with functions satisfying

$\int_1^\infty |f_0(u)|\,du/u < \infty$. It is clear that any such function being monotone, or satisfying (6), is general monotone. However, this class also contains functions with much more complex structure (see, e.g., [13]-[14]).

It is natural in our study that $f_0 \in GM$ satisfies a less restrictive condition than (8):

$$\int_0^1 t^{n-1}|f_0(t)|\,dt + \int_1^\infty t^{(n-1)/2}\,|df_0(t)| < \infty. \tag{11}$$

Let us present the main result of this paper with power weights.

Theorem 1. *Let $1 \le p < \infty$ and $n \ge 1$. Then, for any radial function $f(x) = f_0(|x|)$, $x \in \mathbb{R}^n$, such that $f_0 \ge 0$, $f_0 \in GM$, and satisfying (11),*

$$\Big\||x|^{-\gamma}\widehat{f}(x)\Big\|_{L^p(\mathbb{R}^n)} \asymp \Big\|t^\beta f_0(t)\Big\|_{L^p(0,\infty)} \tag{12}$$

if and only if

$$\beta = \gamma + n - \frac{n+1}{p} \qquad \textit{and} \qquad -\frac{n+1}{2} + \frac{n}{p} < \gamma < \frac{n}{p}.$$

We immediately have the following generalization of Hardy–Littlewood's theorem (4).

Corollary 1. *Let $1 < p < \infty$ and $n \ge 1$. Then, for any radial function $f(x) = f_0(|x|)$, $x \in \mathbb{R}^n$, such that $f_0 \ge 0$, $f_0 \in GM$, and satisfying (11),*

$$C_1\left(\int_{\mathbb{R}^n} |\widehat{f}(x)|^p\,dx\right)^{1/p} \le \left(\int_{\mathbb{R}^n} |f(t)|^p\,t^{n(p-2)}\,dt\right)^{1/p} \le C_2\left(\int_{\mathbb{R}^n} |\widehat{f}(x)|^p\,dx\right)^{1/p}.$$

if and only if

$$\frac{2n}{n+1} < p < \infty;$$

and

$$C_1\left(\int_{\mathbb{R}^n} |\widehat{f}(x)|^p\,|x|^{n(p-2)}\,dx\right)^{1/p} \le \left(\int_{\mathbb{R}^n} |f(t)|^p\,dt\right)^{1/p} \le C_2\left(\int_{\mathbb{R}^n} |\widehat{f}(x)|^p\,|x|^{n(p-2)}\,dx\right)^{1/p}$$

if and only if

$$1 < p < \frac{2n}{n-1}.$$

The paper is organized as follows. Section 2 provides some useful facts about the Fourier transform of a radial function. In Sections 3 and 4, we prove auxiliary upper and lower estimates for the Fourier transform; these estimates are used in the next sections to obtain (L^p, L^q) Fourier inequalities with general weights and partial cases for power weights.

Concerning Problem 1, we observe that the upper estimate of $\widehat{f}$ in Theorem 3 is Pitt's inequality, which holds in the case of general monotone functions only when $\dfrac{n}{q} - \dfrac{n+1}{2} < \gamma < \dfrac{n}{p}$. Since in any case

$$\frac{n}{q} - \frac{n+1}{2} < \max\left\{0, n\left(\frac{1}{p} + \frac{1}{q} - 1\right)\right\},$$

we extend the range of γ given by (3). Theorem 1 exhibits a solution of Problem 2. Note that for $n = 1$ and $n = 3$ Theorem 1 gives (5) and (7), correspondingly.

The notation "$\lesssim$" and "$\gtrsim$" means "$\le C$" and "$\ge C$", respectively (with C independent of essential quantities), while "$\asymp$" stands for "$\lesssim$" and "$\gtrsim$" to hold simultaneously.

2. The Fourier transform of radial functions

The facts we are going to make use of can be found in [6, 19, 21]. For $n \ge 1$, $x \in \mathbb{R}^n$, let $f(x) = f_0(|x|)$ be a radial function. Then

$$\int_{\mathbb{R}^n} f(x)\,dx = |S^{n-1}| \int_0^\infty f_0(t) t^{n-1}\,dt, \tag{13}$$

where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n\colon |x| = 1\}$.

The Fourier transform (1) of the radial function f is also radial and is given via the Hankel–Fourier transform [21] as

$$\widehat{f}(y) = F_0(|y|) = |S^{n-1}| \int_0^\infty f_0(t) j_\alpha(|y|t) t^{n-1}\,dt. \tag{14}$$

Here $j_\alpha(z)$ is the normed Bessel function

$$j_\alpha(z) = \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) = \prod_{k=1}^\infty \left(1 - \frac{z^2}{\rho_{\alpha,k}^2}\right), \tag{15}$$

where $J_\alpha(z)$ is the classical Bessel function of first kind and order α, and $0 < \rho_{\alpha,1} < \rho_{\alpha,2} < \ldots$ are the positive zeros of $J_\alpha(z)$. We denote

$$\alpha := \frac{n}{2} - 1 \ge -\frac{1}{2}.$$

Let us give several useful properties of the function $j_\alpha(z)$, $\alpha \ge -1/2$, which follow from the known properties of $J_\alpha(z)$ (see, e.g., [6, Ch.VII]): $j_{-1/2}(z) = \cos z$, $j_{1/2}(z) = \frac{\sin z}{z}$;

$$|j_\alpha(z)| \le j_\alpha(0) = 1, \quad z \ge 0; \tag{16}$$

$$\frac{d}{dz}\left(z^{2\alpha+2} j_{\alpha+1}(z)\right) = (2\alpha+2) z^{2\alpha+1} j_\alpha(z); \tag{17}$$

(18)
$$j_\alpha(z) = \frac{2^\alpha\Gamma(\alpha+1)(2/\pi)^{1/2}}{z^{\alpha+1/2}} \cos\left(z - \frac{\pi(\alpha+1/2)}{2}\right) + O(z^{-\alpha-3/2}), \quad z \to \infty;$$

(19)
$$|j_\alpha(z)| \le \frac{M_\alpha}{z^{\alpha+1/2}}, \quad z > 0;$$

(20)
$$\rho_{\alpha,k} = \pi k + O(1/k), \quad k \to \infty;$$

the zeros of the Bessel function are separated:

(21)
$$0 < \rho_{\alpha,1} < \rho_{\alpha+1,1} < \rho_{\alpha,2} < \rho_{\alpha+1,2} < \rho_{\alpha,3} < \dots.$$

It follows from (17) and (21) that the function $z^{2\alpha+2}j_{\alpha+1}(z)$ increases when $z \in [0, \rho_{\alpha,1}]$ and decreases when $z \in [\rho_{\alpha,1}, \rho_{\alpha+1,1}]$. The function $j_{\alpha+1}(z)$ decreases on the interval $[0, \rho_{\alpha+1,1}]$. This yields the estimate

(22)
$$z^{2\alpha+2}j_{\alpha+1}^2(z) \ge m_b > 0, \quad 1/b \le z \le b, \quad 1 < b = b_\alpha < \rho_{\alpha+1,1}.$$

In what follows we understand integral (14) as improper:

(23)
$$F_0(s) = |S^{n-1}| \lim_{\substack{a\to 0\\ A\to\infty}} \int_a^A f_0(t) j_\alpha(st) t^{n-1}\, dt, \quad s = |y| > 0.$$

Note that for admissible f_0, (16) implies

$$\left| \int_a^A f_0(t) j_\alpha(st) t^{n-1}\, dt \right| \le \int_a^A |f_0(t)| t^{n-1}\, dt < \infty.$$

Further, for a radial function $f(x) = f_0(|x|)$, by properties (16) and (19), the integral in (14) converges uniformly for $s > 0$ in improper sense to the continuous function $F_0(s)$, provided (8) holds (see [19]). In Lemma 1 below, we prove this fact for $F_0(s)$ via a pointwise estimate of F_0. Note that for $n \ge 2$ condition (11), as well as condition (8), is less restrictive than $f \in L^1(\mathbb{R}^n)$.

3. Estimates from above for the Fourier transforms

Let $f(x) = f_0(|x|)$ with f_0 admissible and satisfying (11), that is, $\int_0^1 t^{n-1}|f_0(t)|\, dt + \int_1^\infty t^{(n-1)/2}\, |df_0(t)| < \infty$. We observe that (11) implies for $t > 1$

$$t^{(n-1)/2}|f_0(t)| \le t^{(n-1)/2} \int_t^\infty |df_0(s)| \le \int_t^\infty s^{(n-1)/2} |df_0(s)|.$$

Therefore

(24)
$$t^{(n-1)/2} f_0(t) \to 0 \qquad \text{as} \qquad t \to \infty.$$

Lemma 1. *Given f_0 as above, for $s > 0$ the Fourier transform $F_0(s)$ is continuous, and satisfies*

$$|F_0(s)| \lesssim \int_0^{1/s} t^{n-1}|f_0(t)|\,dt + s^{-(n+1)/2}\int_{1/s}^{\infty} t^{(n-1)/2}\,|df_0(t)|.$$

Proof. Let for $s > 0$

$$I = \int_0^{\infty} f_0(t) j_\alpha(st) t^{n-1}\,dt = \frac{F_0(s)}{|S^{n-1}|}. \tag{25}$$

Let $\rho > 1$ be a zero of the Bessel function $J_{\alpha+1}(\cdot)$. Then, by (16),
(26)

$$I \le \int_0^{1/s} |f_0(t)|\,t^{n-1}\,dt + \int_{1/s}^{\rho/s} |f_0(t)|\,t^{n-1}\,dt + \Big|\int_{\rho/s}^{\infty} f_0(t) j_\alpha(st) t^{n-1}\,dt\Big| = I_1 + I_2 + I_3.$$

Estimating I_2 we obtain

$$\begin{aligned} I_2 &\lesssim \int_{1/s}^{\rho/s} t^{n-1}\Big(\int_t^{1/s} |df_0(u)| + \int_{1/s}^{\infty} |df_0(u)|\Big)\,dt \\ &\lesssim \int_{1/s}^{\rho/s} u^n |df_0(u)| + s^{-n}\int_{1/s}^{\infty} |df_0(u)| \lesssim s^{-(n+1)/2}\int_{1/s}^{\infty} t^{(n-1)/2}\,|df_0(t)|. \qquad (27) \end{aligned}$$

It follows from (17) that

$$\frac{d}{dt}\left(t^n j_{\alpha+1}(st)\right) = n t^{n-1} j_\alpha(st). \tag{28}$$

Integrating by parts, we obtain

$$I_3 = \frac{1}{n}\int_{\rho/s}^{\infty} f_0(t)\,d(t^n j_{\alpha+1}(st)) = \frac{1}{n}\,f_0(t) t^n j_{\alpha+1}(st)\Big|_{\rho/s}^{\infty} - \frac{1}{n}\int_{\rho/s}^{\infty} t^n j_{\alpha+1}(st)\,df_0(t).$$

Then (19) and (24) yield

$$|f_0(t) t^n j_{\alpha+1}(st)| \lesssim |f_0(t)| t^n (st)^{-(n+1)/2} \lesssim |f_0(t)| t^{(n-1)/2} \to 0 \quad \text{as} \quad t \to \infty,$$

and hence

$$I_3 \lesssim \int_{\rho/s}^{\infty} t^n (st)^{-(n+1)/2}\,|df_0(t)| \lesssim s^{-(n+1)/2}\int_{1/s}^{\infty} t^{(n-1)/2}\,|df_0(t)|. \tag{29}$$

Combining (27) and (29), we finish the proof of the lemma. □

We will also use similar estimates of the Fourier transform in terms of the following functions:

$$\Phi^*(t) = \int_t^{2t} |df_0(u)|, \quad \Phi(t) = \int_t^{\infty} |df_0(u)|, \quad \Psi(t) = \int_t^{\infty} s^{(n-1)/2}|df_0(s)|.$$

These functions are continuous for $t > 0$, and $\Phi^*(t) \le \Phi(t)$.

Corollary 2. *The estimate holds for $s > 0$*

$$\begin{aligned}|F_0(s)| &\lesssim \int_0^{1/s} t^{n-1}\Phi^*(t)\,dt + s^{-(n+1)/2}\int_{1/s}^{\infty} t^{(n-3)/2}\Phi^*(t)\,dt\\ &\lesssim \int_0^{1/s} t^{n-1}\Phi(t)\,dt + s^{-(n+1)/2}\int_{1/s}^{\infty} t^{(n-3)/2}\Phi(t)\,dt.\end{aligned}$$

Proof. Similar to (27), we first get

$$\int_0^{1/s} t^{n-1}|f_0(t)|\,dt \lesssim \int_0^{1/s} t^n\,|df_0(t)| + s^{-(n+1)/2}\int_{1/s}^{\infty} t^{(n-1)/2}\,|df_0(t)|. \tag{30}$$

Then the required estimates follows from Lemma 1 and inequalities

$$\ln 2\int_0^B |\psi(u)|\,du \le \int_0^B t^{-1}\int_t^{2t} |\psi(u)|\,du\,dt, \tag{31}$$

$$\ln 2\int_{2A}^{\infty} |\psi(u)|\,du \le \int_A^{\infty} t^{-1}\int_t^{2t} |\psi(u)|\,du\,dt, \tag{32}$$

valid for any integrable ψ. □

Corollary 3. *The estimate holds for $s > 0$*

$$|F_0(s)| \lesssim \int_0^{1/s} t^{(n-1)/2}\Psi(t)\,dt. \tag{33}$$

Proof. Indeed, by Lemma 1 and (30),

$$|F_0(s)| \lesssim \int_0^{1/s} t^n\,|df_0(t)| + s^{-(n+1)/2}\int_{1/s}^{\infty} t^{(n-1)/2}\,|df_0(t)| = I_1 + I_2.$$

We have

$$\begin{aligned}I_2 = s^{-(n+1)/2}\Psi(1/s) &\asymp \Psi(1/s)\int_{1/(2s)}^{1/s} t^{(n-1)/2}\,dt\\ &\le \int_{1/(2s)}^{1/s} t^{(n-1)/2}\Psi(t)\,dt \le \int_0^{1/s} t^{(n-1)/2}\Psi(t)\,dt.\end{aligned}$$

Using (31), we get

$$\begin{aligned}I_1 \lesssim \int_0^{1/s} t^{n-1}\left(\int_t^{2t} |df_0(s)|\right)dt &\asymp \int_0^{1/s} t^{(n-1)/2}\left(\int_t^{2t} s^{(n-1)/2}\,|df_0(s)|\right)dt\\ &\le \int_0^{1/s} t^{(n-1)/2}\Psi(t)\,dt.\end{aligned}$$

The obtained bounds for I_1 and I_2 give (33). □

Note that in this section we did not assume the positivity of f_0 so far. This will come into play in the next section.

4. Estimates from below for the Fourier transforms

Let us consider a radial function $f(x) = f_0(|x|)$ such that f_0 is admissible and $f_0(t) \geq 0$ when $t > 0$. We assume that f_0 satisfies condition (11). Then, by Lemma 1, the integral in (23) converges uniformly on any compact set away from zero and $F_0(s)$ is continuous for $s > 0$. Suppose also that

$$\int_0^1 |F_0(s)| s^{(n-1)/2}\, ds < \infty. \tag{34}$$

In particular, this implies that $\widehat{f}$ is integrable in a neighborhood of zero. We will need the following

Lemma 2. *For $u > 0$ and $1 < b < \rho_{\alpha+1,1}$, the inequality holds*

$$u^{(1-n)/2} \int_0^{2/u} s^{(n-1)/2} |F_0(s)|\, ds \gtrsim \int_{u/b}^{bu} \frac{f_0(t)}{t}\, dt.$$

Proof. We denote by $B^n = \{x \in \mathbb{R}^n \colon |x| \leq 1\}$ the unit ball, $|B^n| = |S^{n-1}|/n$ is the volume of this ball.

Let us consider the following well-known compactly supported function

$$k(y) = |B^n|^{-1} (\chi * \chi)(y),$$

where χ is the indicator function of the unit ball B^n. For $n = 1$, it is the Fejér kernel $(1 - |y|/2)_+$.

The kernel k is radial $k(y) = k_0(|y|)$ and possesses the following properties:

$$0 \leq k_0(s) \leq k_0(0) = 1, \quad 0 \leq s \leq 2; \qquad k_0(s) = 0, \quad s \geq 2; \tag{35}$$

and the Fourier transform of k is

$$\widehat{k}(x) = K_0(|x|) = |B^n|^{-1} (\widehat{\chi}(x))^2 \geq 0.$$

By (28), for $t = |x|$

$$\widehat{\chi}(x) = |S^{n-1}| \int_0^1 j_\alpha(ts) s^{n-1}\, ds = \frac{|S^{n-1}|}{n}\, j_{\alpha+1}(t) = |B^n| j_{\alpha+1}(t). \tag{36}$$

Therefore,

$$K_0(t) = |S^{n-1}| \int_0^2 k_0(s) j_\alpha(ts) s^{n-1}\, ds = |B^n| j_{\alpha+1}^2(t). \tag{37}$$

Let ε be small enough. Denoting

$$J_\varepsilon := \int_{\varepsilon/u}^{2/u} F_0(s) k_0(us) s^{n-1}\, ds = u^{-n} \int_\varepsilon^2 F_0(s/u) k_0(s) s^{n-1}\, ds.$$

We have, by (34) and (35),

$$|J_\varepsilon| \leq \int_0^{2/u} |F_0(s)| s^{n-1}\, ds \lesssim u^{(1-n)/2} \int_0^{2/u} s^{(n-1)/2} |F_0(s)|\, ds. \tag{38}$$

The uniform convergence of integral (23) implies

$$J_\varepsilon = u^{-n}\int_\varepsilon^2 \left(|S^{n-1}|\int_0^\infty f_0(t)j_\alpha(st/u)t^{n-1}\,dt\right)k_0(s)s^{n-1}\,ds$$
$$= u^{-n}\int_0^\infty f_0(t)\left(|S^{n-1}|\int_\varepsilon^2 k_0(s)j_\alpha(st/u)s^{n-1}\,ds\right)t^{n-1}\,dt.$$

Using (37), we get

$$|S^{n-1}|\int_\varepsilon^2 k_0(s)j_\alpha(st/u)s^{n-1}\,ds = K_0(t/u) - \lambda_\varepsilon(t),$$

where

$$\lambda_\varepsilon(t) = |S^{n-1}|\int_0^\varepsilon k_0(s)j_\alpha(st/u)s^{n-1}\,ds.$$

Taking into account (22) and (37), we have $(t/u)^n K_0(t/u) \gtrsim 1$ for $u/b \le t \le bu$. Therefore,

$$J_\varepsilon \gtrsim \int_{u/b}^{bu}\frac{f_0(t)}{t}\,dt - J'_\varepsilon, \qquad J'_\varepsilon = u^{-n}\int_0^\infty f_0(t)\lambda_\varepsilon(t)t^{n-1}\,dt. \tag{39}$$

We are going to prove that $J'_\varepsilon \to 0$ as $\varepsilon \to 0$. Take $A > 1$. It follows from (35) and (16) that

$$|\lambda_\varepsilon(t)| \le |S^{n-1}|\int_0^\varepsilon s^{n-1}\,ds \lesssim \varepsilon^n, \tag{40}$$

and hence

$$\left|u^{-n}\int_0^A f_0(t)\lambda_\varepsilon(t)t^{n-1}\,dt\right| \lesssim \varepsilon^n\int_0^A |f_0(t)|t^{n-1}\,dt. \tag{41}$$

Let $t \ge A$. Define

$$\Lambda_\varepsilon(t) = \int_0^t \lambda_\varepsilon(v)v^{n-1}\,dv = |S^{n-1}|\int_0^\varepsilon k_0(s)s^{n-1}\left(\int_0^t j_\alpha(sv/u)v^{n-1}\,dv\right)ds.$$

Making use of (28), we obtain

$$\Lambda_\varepsilon(t) = \frac{|S^{n-1}|t^n}{n}\int_0^\varepsilon k_0(s)j_{\alpha+1}(st/u)s^{n-1}\,ds. \tag{42}$$

For $n = 1$,

$$|\Lambda_\varepsilon(t)| = \left|2t\int_0^\varepsilon (1-s/2)\,\frac{\sin(st/u)}{st/u}\,ds\right| = \left|2u\int_0^{\varepsilon t/u}\frac{\sin s}{s}\,ds - \frac{u^2(1-\cos(\varepsilon t/u))}{t}\right|.$$

It is well-known that $\left|\int_0^v \frac{\sin s}{s}\,ds\right| \le \int_0^\pi \frac{\sin s}{s}\,ds$ for $v > 0$, and $|\Lambda_\varepsilon(t)| \lesssim 1 \lesssim t^{(n-1)/2}$.

Let now $n \geq 2$. We have

$$\Lambda_\varepsilon(t) = \frac{|S^{n-1}|t^n}{n} \left(\int_0^{\varepsilon/t} + \int_{\varepsilon/t}^{\varepsilon} \right) k_0(s) j_{\alpha+1}(st/u) s^{n-1}\, ds.$$

As above

$$\left| \frac{|S^{n-1}|t^n}{n} \int_0^{\varepsilon/t} k_0(s) j_{\alpha+1}(st/u) s^{n-1}\, ds \right| \lesssim t^n \int_0^{\varepsilon/t} s^{n-1}\, ds \lesssim \varepsilon^n \lesssim 1 \lesssim t^{(n-1)/2}.$$

Applying (19), we get

$$\left| \frac{|S^{n-1}|t^n}{n} \int_{\varepsilon/t}^{\varepsilon} k_0(s) j_{\alpha+1}(st/u) s^{n-1}\, ds \right| \lesssim t^n \int_{\varepsilon/t}^{\varepsilon} |j_{\alpha+1}(st/u)| s^{n-1}\, ds$$
$$\lesssim t^n (t/u)^{-(n+1)/2} \int_{\varepsilon/t}^{\varepsilon} s^{(n-1)/2-1}\, ds \lesssim t^{(n-1)/2} \varepsilon^{(n-1)/2} \lesssim t^{(n-1)/2}.$$

Therefore, $|\Lambda_\varepsilon(t)| \lesssim t^{(n-1)/2}$ for $t \geq A$ and $n \geq 1$.

Integrating by parts yields

$$\int_A^\infty f_0(t)\lambda_\varepsilon(t) t^{n-1}\, dt = \int_A^\infty f_0(t)\, d\Lambda_\varepsilon(t) = f_0(t)\Lambda_\varepsilon(t)\big|_A^\infty - \int_A^\infty \Lambda_\varepsilon(t)\, df_0(t).$$

It follows from (24) and $|\Lambda_\varepsilon(t)| \lesssim t^{(n-1)/2}$ that $f_0(t)\Lambda_\varepsilon(t) \to 0$ as $t \to \infty$. Since (40) and (42) imply $|\Lambda_\varepsilon(A)| \lesssim \varepsilon^n A^n$,

$$\left| \int_A^\infty f_0(t)\lambda_\varepsilon(t) t^{n-1}\, dt \right| \leq \varepsilon^n |f_0(A)| A^n + \int_A^\infty t^{(n-1)/2}\, |df_0(t)|. \tag{43}$$

Combining (41) and (43), we get

$$|J'_\varepsilon| \lesssim \varepsilon^n \left(\int_0^A |f_0(t)| t^{n-1}\, dt + |f_0(A)| A^n \right) + \int_A^\infty t^{(n-1)/2}\, |df_0(t)|.$$

Letting first $\varepsilon \to 0$ and then $A \to \infty$, we obtain the claimed $J'_\varepsilon \to 0$. Using this, (38), and (39), we arrive at the assertion of the lemma. □

5. L^p–L^q Fourier inequalities with general weights

For any weights $\mathbf{u}, \mathbf{v} \colon \mathbb{R}^n \to \mathbb{R}_+$, consider their radial parts

$$U(t) = \int_{S^{n-1}} \mathbf{u}(t\xi)\, d\xi, \qquad V(t) = \int_{S^{n-1}} \mathbf{v}(t\xi)\, d\xi.$$

We set $x' = x/|x|$, $\overline{\mathbf{v}}(x) = \mathbf{v}(x'/|x|)$.

Theorem 2. *Let $1 \leq p, q < \infty$ and $n \in \mathbb{N}$. Let f be radial on $\mathbb{R}^n$ such that f_0 is a non-negative general monotone function on $\mathbb{R}_+$ satisfying* (11).

(A) *If* $p \le q$*, and* V*,* U *satisfy*

$$\sup_{r>0}\left(\int_r^\infty t^{-n-1}V(ct)\,dt\right)^{1/q}\left(\int_0^r \Big[U(t)t^{(n-1)(1-p)}\Big]^{1/(1-p)}\,dt\right)^{1/p'}<\infty; \tag{44}$$

$$\sup_{r>0}\left(\int_0^r t^{(1-\frac{p}{2})(n-1)}U(t)\,dt\right)^{1/p}\left(\int_r^\infty \Big[U(t)t^{(1-\frac{p}{2})(n-1)+p}\Big]^{1/(1-p)}\,dt\right)^{1/p'}<\infty, \tag{45}$$

then

$$\|\widehat{f}\|_{L^q_{\mathbf{v}}}\equiv\left(\int_{\mathbb{R}^n}|\widehat{f}(x)|^q\mathbf{v}(x)\,dx\right)^{1/q}\lesssim\left(\int_{\mathbb{R}^n}|f(x)|^p\mathbf{u}(x)\,dx\right)^{1/p}\equiv\|f\|_{L^p_{\mathbf{u}}}.$$

(B) *If* $q \le p$*, and* U*,* V *satisfy*

$$\sup_{r>0}\left(\int_0^r t^{n-1}U(2bct)\,dt\right)^{1/p}\left(\int_r^\infty \Big[V(t)t^{(n+1)(q-1)}\Big]^{1/(1-q)}\,dt\right)^{1/q'}<\infty; \tag{46}$$

$$\sup_{r>0}\left(\int_0^r t^{(n+1)(\frac{q}{2}-1)}V(t)\,dt\right)^{1/q}\left(\int_r^\infty \Big[V(t)t^{(n+1)(\frac{q}{2}-1)+q}\Big]^{1/(1-q)}\,dt\right)^{1/q'}<\infty, \tag{47}$$

then

$$\|f\|_{L^p_{\mathbf{u}}}\lesssim\|\widehat{f}\|_{L^q_{\mathbf{v}}}.$$

Proof. We will use the (p,q) version of Hardy's inequalities ([8]) with general weights $u,v\ge 0$: for $1\le\alpha\le\beta<\infty$,

$$\left[\int_0^\infty u(t)\left(\int_0^t \psi(s)\,ds\right)^\beta dt\right]^{1/\beta}\le C\left[\int_0^\infty v(t)\psi(t)^\alpha\,dt\right]^{1/\alpha} \tag{48}$$

holds for every $\psi\ge 0$ if and only if

$$\sup_{r>0}\left(\int_r^\infty u(t)\,dt\right)^{1/\beta}\left(\int_0^r v(t)^{1-\alpha'}\,dt\right)^{1/\alpha'}<\infty,$$

and

$$\left[\int_0^\infty u(t)\left(\int_t^\infty \psi(s)\,ds\right)^\beta dt\right]^{1/\beta}\le C\left[\int_0^\infty v(t)\psi(t)^\alpha\,dt\right]^{1/\alpha} \tag{49}$$

if and only if

$$\sup_{r>0}\left(\int_0^r u(t)\,dt\right)^{1/\beta}\left(\int_r^\infty v(t)^{1-\alpha'}\,dt\right)^{1/\alpha'}<\infty.$$

Here we consider the usual modification of the integral $\Big[\int v(t)^\theta\,dt\Big]^{1/\theta}$ when $\theta=\infty$.

Remark 1. In particular, (48) holds with $u(t) = t^{\varepsilon-1}$ and $v(t) = t^{\delta-1}$ if and only if $\varepsilon < 0$ and $\delta = \varepsilon\alpha/\beta + \alpha$.

To prove (A), we use the pointwise Fourier transform inequality (33) from Corollary 3. First, we have to check the accuracy of (11). By Hölder's inequality,

$$\int_0^1 t^{n-1}|f_0(t)|\,dt \le \left(\int_0^1 t^{n-1}|f_0(t)|^p U(t)dt\right)^{1/p}\left(\int_0^1 \left[U(t)t^{(n-1)(1-p)}\right]^{1/(1-p)}dt\right)^{1/p'},$$

and the right-hand side is finite since $f \in L^p_{\mathbf{u}}$ and the last integral in (44) is finite. Further, it follows from the definition of GM and simple calculations that

$$\int_1^\infty t^{\frac{n-1}{2}}|df_0(t)| \lesssim \int_{1/c}^\infty t^{\frac{n-3}{2}}|f_0(t)|\,dt.$$

Therefore,

$$\int_{1/c}^\infty t^{\frac{n-3}{2}}|f_0(t)|\,dt \le \left(\int_{1/c}^\infty t^{n-1}|f_0(t)|^p U(t)\,dt\right)^{1/p}$$
$$\left(\int_{1/c}^\infty t^{-\frac{n-1}{p}\frac{p}{p-1}}t^{\frac{n-3}{2}\frac{p}{p-1}}U(t)^{1/(1-p)}\,dt\right)^{1/p'}.$$

The last integral is finite due to (45).

Now, (32) and $f_0 \in GM$ yield

$$\begin{aligned}\Psi(t) = \int_t^\infty s^{(n-1)/2}|df_0(s)| &\lesssim \int_t^\infty y^{(n-1)/2-1}\int_y^{2y}|df_0(s)|dy\\ &\lesssim \int_t^\infty y^{(n-1)/2-1}\int_{y/c}^\infty |f_0(s)|\frac{ds}{s}dy\\ &\lesssim \int_{t/c}^\infty s^{(n-1)/2-1}|f_0(s)|ds.\end{aligned}$$

Using this and Corollary 3, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\mathbf{v}(x'/|x|)|\widehat{f}(x)|^q dx\right)^{1/q} &\lesssim \left(\int_0^\infty V(1/t)|F_0(t)|^q t^{n-1}\,dt\right)^{1/q}\\ &\lesssim \left(\int_0^\infty V(ct)\left(\int_0^t u^{(n-1)/2}\left(\int_u^\infty s^{(n-1)/2-1}|f_0(s)|\,ds\right)du\right)^q t^{-n-1}dt\right)^{1/q}.\end{aligned}$$

Applying now (48) to the first two integrals on the right, with $\beta = q$, $\alpha = p$, $u(t) = t^{-n-1}V(ct)$, and $v(t) = U(t)t^{n-1-p(n-1)}$, we obtain

$$\left(\int_{\mathbb{R}^n}\mathbf{v}(x'/|x|)|\widehat{f}(x)|^q dx\right)^{1/q} \lesssim \left(\int_0^\infty\left(\int_u^\infty s^{(n-1)/2-1}|f_0(s)|\,ds\right)^p U(t)t^{n-1-p(n-1)/2}dt\right)^{1/p},$$

provided (44) holds.

Making then use of (49), with $\alpha = \beta = p$, $u(t) = U(t)t^{n-1-p(n-1)/2}$, and $v(t) = U(t)t^{n-1-p(n-1)/2+p}$, we get

$$\left(\int_{\mathbb{R}^n} \mathbf{v}(x'/|x|)|\widehat{f}(x)|^q dx\right)^{1/q} \lesssim \left(\int_0^\infty U(t)t^{n-1}|f_0(t)|^p dt\right)^{1/p} \lesssim \|f\|_{L^p_{\mathbf{u}}},$$

provided (45) holds.

To prove (B), Lemma 2 is needed. For this we have to check (34). Hölder's inequality and simple substitution yield

$$\int_0^r |F_0(t)|t^{\frac{n-1}{2}} dt \lesssim \|\widehat{f}\|_{L^q_{\mathbf{v}}} \left(\int_{1/r}^\infty \left[V(t)t^{(n+1)(\frac{q}{2}-1)+q}\right]^{1/(1-q)} dt\right)^{1/q'}.$$

The finiteness of the last integral is ensured by (47).

We then note that for any $f_0 \in GM$ there holds

$$|f_0(x)| \le \int_x^\infty |df_0(t)| \lesssim \int_{x/c}^\infty |f_0(t)| \frac{dt}{t}. \tag{50}$$

Secondly, by (32) and Lemma 2, we have

$$\begin{aligned} |f_0(x)| &\le \int_x^\infty |df_0(t)| \lesssim \int_{x/bc}^\infty t^{-1}\Big(\int_{t/b}^{bt} \frac{f_0(s)}{s} ds\Big)\, dt \\ &\lesssim \int_{x/bc}^\infty t^{(1-n)/2-1}\Big(\int_0^{2/t} z^{(n-1)/2}|F_0(z)|dz\Big)\, dt \\ &\lesssim \int_0^{2bc/x} t^{(n-1)/2-1}\Big(\int_0^t z^{(n-1)/2}|F_0(z)|dz\Big)\, dt. \end{aligned} \tag{51}$$

Applying now (51), we obtain

$$\begin{aligned} &\|f\|_{L^p_{\mathbf{u}}} \lesssim \left(\int_0^\infty U(t)t^{n-1}|f_0(t)|^p dt\right)^{1/p} \\ &\lesssim \left(\int_0^\infty U(s)s^{n-1}\left(\int_s^\infty t^{-(n-1)/2-1}\left(\int_{2bct}^\infty z^{-(n-1)/2-2}|F_0(1/z)|\,dz\right)\, dt\right)^p ds\right)^{1/p}. \end{aligned}$$

As above, we then use Hardy's inequality (49) twice to obtain

$$\|f\|_{L^p_{\mathbf{u}}} \lesssim \left(\int_{\mathbb{R}^n} \mathbf{v}(x'/|x|)|\widehat{f}(x)|^q dx\right)^{1/q} = \|\widehat{f}\|_{L^q_{\mathbf{v}}}.$$

Under appropriate choice of the weights, the necessary and sufficient condition reduces to (46) and (47), respectively. The proof is complete. □

6. Applications. Pitt-Boas type results with power weights

For $\mathbf{v}(x) = |x|^{-q\gamma}$ and $\mathbf{u}(x) = |x|^{\gamma p - np/q + np - n}$, Theorem 2 implies the following result.

Theorem 3. *Let $1 \le p, q < \infty$ and $n \in \mathbb{N}$. Let f be radial on $\mathbb{R}^n$ such that f_0 is a general monotone function on $\mathbb{R}_+$.*

(A) *If $p \le q$ and*

$$\frac{n}{q} - \frac{n+1}{2} < \gamma < \frac{n}{q}, \tag{52}$$

then

$$t^{n+\gamma-n/q-1/p} f_0(t) \in L^p(0,\infty) \qquad \textit{implies} \qquad |x|^{-\gamma}\widehat{f}(x) \in L^q(\mathbb{R}^n);$$

(B) *Let a non-negative function f_0 satisfy* (11)*. If $q \le p$ and*

$$\frac{n}{q} - \frac{n+1}{2} < \gamma, \tag{53}$$

then

$$|x|^{-\gamma}\widehat{f}(x) \in L^q(\mathbb{R}^n) \qquad \textit{implies} \qquad t^{n+\gamma-n/q-1/p} f_0(t) \in L^p(0,\infty).$$

Note that the "if" part of Theorem 1 follows from Theorem 3 by taking $p = q$.

Let us now discuss the sharpness of conditions on γ. We rewrite part (A) of Theorem 3 in the following way.

Theorem 3′. *Let $1 \le p \le q < \infty$ and $n \in \mathbb{N}$. Let f be radial on $\mathbb{R}^n$ such that f_0 is a general monotone function on $\mathbb{R}_+$. Then*

$$\Big\| |x|^{-\gamma}\widehat{f}(x)\Big\|_{L^q(\mathbb{R}^n)} \lesssim \Big\| t^{\beta} f_0(t)\Big\|_{L^p(0,\infty)} \tag{54}$$

if and only if

$$\beta = \gamma + n - \frac{n}{q} - \frac{1}{p} \qquad \textit{and} \qquad \frac{n}{q} - \frac{n+1}{2} < \gamma < \frac{n}{q}. \tag{55}$$

To prove Theorem 3′, we can restrict ourselves to the "only if" direction. This also captures the "only if" part in Theorem 1 when $p = q$.

Proof. Consider $f(x) = \chi(x)$, then $f_0(t) = \chi_{[0,1]}(t) \in GM$. Then we have

$$\|t^{n+\gamma-n/q-1/p} f_0(t)\|_{L^p(0,\infty)} = \left(\int_0^1 t^{pn+p\gamma-pn/q-1}\,dt\right)^{1/p}.$$

This integral converges if $pn + p\gamma - pn/q > 0$, or equivalently $\gamma > \frac{n}{q} - n$.

Let us figure out when $|y|^{-\gamma}\widehat{\chi}(y) \in L^q(\mathbb{R}^n)$. By (36), the Fourier transform of f is $\widehat{\chi}(y) = |B^n| j_{\alpha+1}(|y|) = F_0(s)$. Therefore, we obtain

$$\begin{aligned} \Big\| |y|^{-\gamma}\widehat{\chi}(y)\Big\|_{L^q(\mathbb{R}^n)} &\asymp \left(\int_0^\infty \big(s^{-\gamma}|F_0(s)|\big)^q s^{n-1}\,ds\right)^{1/q} \\ &\asymp \left(\int_0^\infty s^{n-q\gamma-1}|j_{\alpha+1}(s)|^q\,ds\right)^{1/q}. \end{aligned} \tag{56}$$

There holds $j_{\alpha+1}(s) \asymp 1$ in a neighborhood of zero, hence the integral in (56) converges if $n - q\gamma > 0$, that is, when $\gamma < \frac{n}{q}$. The upper bound is established.

There holds for s large, $j_{\alpha+1}(s) \lesssim s^{-(n+1)/2}$, therefore the integral in (56) converges if $\dfrac{n}{q} - \dfrac{n+1}{2} < \gamma$. We will now show that if this condition does not hold, then the integral in (56) diverges. It follows from (20) that for an integer number k_0 large enough

$$\rho_{\alpha+1,k} \asymp k, \quad \rho_{\alpha+1,k+1} - \rho_{\alpha+1,k} \asymp 1, \quad k \ge k_0,$$

and there is a small $\varepsilon > 0$, independent of k, such that

$$|j_{\alpha+1}(s)| \gtrsim s^{-(n+1)/2}, \qquad s \in [\rho_{\alpha+1,k} + \varepsilon, \rho_{\alpha+1,k+1} - \varepsilon], \quad k \ge k_0.$$

Therefore,

$$\begin{aligned} \int_0^\infty s^{n-q\gamma-1}|j_{\alpha+1}(s)|^q\,ds &\gtrsim \sum_{k=k_0}^\infty \int_{\rho_{\alpha+1,k}+\varepsilon}^{\rho_{\alpha+1,k+1}-\varepsilon} s^{n-q\gamma-1}s^{-q(n+1)/2}\,ds \\ &\gtrsim \sum_{k=k_0}^\infty (\rho_{\alpha+1,k+1}-\varepsilon)^{n-q\gamma-1-q(n+1)/2} \\ &\gtrsim \sum_{k=k_0}^\infty k^{n-q\gamma-1-q(n+1)/2}. \end{aligned}$$

The last series diverges provided $\gamma \le \dfrac{n}{q} - \dfrac{n+1}{2}$.

Let us verify that β and γ should be related by $\beta = \gamma + n - n/q - 1/p$. Let $u > 0$ and $g(x) = f_0(|x|/u) = \chi(x/u)$. Then for $t = |y|$ with $0 < t < 1/u$

$$\widehat{g}(y) = G_0(|y|) = u^n F_0(u|y|) = |B^n| u^n j_{\alpha+1}(ut) \asymp u^n.$$

We then have

$$\|t^\beta g_0(t)\|_{L^p(0,\infty)} \asymp \left(\int_0^u t^{\beta p+1}\,\frac{dt}{t}\right)^{1/p} \asymp u^{\beta+1/p},$$

and

$$\big\||x|^{-\gamma}\widehat{g}(x)\big\|_{L^q(\mathbb{R}^n)} \gtrsim \left(\int_0^u t^{-\gamma q+n}|G_0(t)|^q \,\frac{dt}{t}\right)^{1/q}$$
$$\gtrsim u^n \left(\int_0^u t^{-\gamma q+n}\,\frac{dt}{t}\right)^{1/q} \asymp u^{\gamma+n-n/q}.$$

These yield $u^{\beta+1/p} \gtrsim u^{\gamma+n-n/q}$ for any $u>0$, that is, $\beta=\gamma+n-\dfrac{n}{q}-\dfrac{1}{p}$. □

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D. Gorbachev
Tula State University
Department of Mechanics and Mathematics
300600 Tula, Russia
E-mail address: dvgmail@mail.ru

E. Liflyand
Department of Mathematics
Bar-Ilan University
52900 Ramat-Gan, Israel
E-mail address: liflyand@math.biu.ac.il

S. Tikhonov
ICREA and Centre de Recerca Matemàtica
Apartat 50 08193 (Bellaterra) Barcelona
E-mail address: stikhonov@crm.cat