ON GRAPHS WITH FEW DISJOINT t-STARS

COLIN MCDIARMID

ABSTRACT. For fixed positive integers $t \geq 3$ and k, almost all graphs which have at most k disjoint minors isomorphic to a t-star contain k vertices such that deleting them leaves a graph with no such minor. This holds for both labelled and unlabelled graphs, and answers a question of Bernardi, Noy and Welsh.

1. INTRODUCTION AND STATEMENT OF RESULTS

For $t \geq 3$ let us call the star with t leaves joined to a centre vertex the *t-star* and denote it by S_t. Bernardi, Noy and Welsh [1] recently proposed the following problem on graphs with at most k disjoint minors S_t, where 'disjoint' means 'pairwise vertex disjoint'.

Denote the class of graphs with no minor H by $\mathrm{Ex}\, H$. Thus $\mathrm{Ex}\, S_3$ is the class of graphs with maximum degree at most 2; and for each $t \geq 3$, a graph is in $\mathrm{Ex}\, S_t$ if and only if each subtree has at most $t-1$ leaves.

Given a class $\mathcal{A}$ of graphs (closed under isomorphism), let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on the vertex set $V = \{1, \ldots, n\}$, and let $u_n(\mathcal{A})$ denote the number of unlabelled n-vertex graphs in $\mathcal{A}$. We say that $\mathcal{A}$ has *labelled growth constant* λ if $(|\mathcal{A}_n|/n!)^{1/n} \to \lambda$ as $n \to \infty$; and $\mathcal{A}$ has *unlabelled growth constant* γ if $u_n(\mathcal{A}_n)^{1/n} \to \gamma$ as $n \to \infty$. Also let $\mathrm{apex}^k \mathcal{A}$ denote the class of graphs G such that by deleting at most k vertices we may obtain a graph in $\mathcal{A}$.

Fix $t \geq 3$. We shall see that $\mathrm{Ex}\, S_t$ has both labelled and unlabelled growth constants equal to 1, and more generally that $\mathrm{apex}^k\, \mathrm{Ex}\, \mathrm{S_t}$ has both growth constants equal to 2^k for each $k \geq 0$. Clearly the class $\mathrm{Ex}\,(k+1)S_t$ of graphs which have at most k disjoint minors S_t satisfies

$$\mathrm{Ex}\,(k+1)S_t \supseteq \mathrm{apex}^k\, \mathrm{Ex}\, \mathrm{S_t}. \tag{1}$$

How much bigger is the class on the left than that on the right? Bernardi, Noy and Welsh [1] showed that $\mathrm{Ex}\ 2S_t$ has labelled growth constant 2, the same labelled growth constant as $\mathrm{apex}^1\, \mathrm{Ex}\, \mathrm{S_t}$. They asked whether it is true for all k that $\mathrm{Ex}\,(k+1)S_t$ has labelled growth constant 2^k, so that the two sides of the containment (1) are 'close', at least to the extent that they have the same labelled growth constant. We shall see that this is the case, and indeed much more is true. First let us state the basic result.

Theorem 1. *For fixed integers $t \geq 3$ and $k \geq 1$, the classes $\mathrm{Ex}\,(k+1)S_t$ and $\mathrm{apex}^k\, \mathrm{Ex}\, \mathrm{S_t}$ each have both labelled and unlabelled growth constants equal to 2^k.*

We shall obtain a more precise version of this result, by using a modified version of the approach in Kurauskas and McDiarmid [5]. Given $t \geq 3$ and $k \geq 1$ let $\mathcal{D}^{t,k}$ denote the 'difference class' $\mathrm{Ex}\,(k+1)S_t \ \setminus \ (\mathrm{apex^k\ Ex\,S_t})$. We are interested in how large the class $\mathcal{D}^{t,k}$ is relative to the classes $\mathrm{Ex}\,(k{+}1)S_t$ and $\mathrm{apex^k\ Ex\,S_t}$ from which it is defined. We find essentially the same behaviour for labelled and unlabelled graphs, but different behaviours for the cases $t = 3$ and $t \geq 4$. The first parts of the theorem below concern the relative size of $\mathcal{D}^{t,k}$ in the two cases, and the last part completes the story by describing the asymptotic size of $\mathrm{Ex}\,(k+1)S_t$.

Theorem 2. *Fix an integer $k \geq 1$.*
Then (a)

$$|(\mathrm{Ex}\,(k{+}1)S_3)_n| = (1{+}2^{-n+\Theta(n^{\frac{1}{2}})})\,|(\mathrm{apex^k Ex\,S_3})_n| \tag{2}$$

$$u_n(\mathrm{Ex}\,(k{+}1)S_3) = (1{+}2^{-n+\Theta(n^{\frac{1}{2}})})\,u_n(\mathrm{apex^k Ex\,S_3}); \tag{3}$$

(b) for each fixed integer $t \geq 4$

$$|(\mathrm{Ex}\,(k{+}1)S_t)_n| = (1{+}2^{-\Theta(n^{\frac{2t-5}{2t-4}})})\,|(\mathrm{apex^k Ex\,S_t})_n| \tag{4}$$

$$u_n(\mathrm{Ex}\,(k{+}1)S_t) = (1{+}2^{-\Theta(n^{\frac{2t-5}{2t-4}})})\,u_n(\mathrm{apex^k Ex\,S_t}); \tag{5}$$

and (c) for each fixed integer $t \geq 3$, both $|(\mathrm{Ex}\,(k{+}1)S_t)_n|/n!$ and $u_n(\mathrm{Ex}\,(k{+}1)S_t)$ are asymptotically $2^{kn+\Theta(n^{\frac{2t-5}{2t-4}})}$.

By this theorem, if R_n denotes a graph sampled uniformly at random from either the labelled or unlabelled n-vertex graphs in $\mathrm{Ex}\,(k+1)S_t$, then the probability that R_n contains k vertices such that deleting them leaves a graph in $\mathrm{Ex}\,S_t$ is $1 - 2^{-n+\Theta(n^{\frac{1}{2}})}$ when $t = 3$, and is $1 - 2^{-\Theta(n^{\frac{2t-5}{2t-4}})}$ when $t \geq 4$.

Observe that for $t = 3$ the difference class $\mathcal{D}^{t,k}$ is exponentially smaller than $\mathrm{Ex}\,(k{+}1)S_t$ and $\mathrm{apex^k\ Ex\,S_t}$, but this is not the case for $t \geq 4$. This behaviour contrasts with that for labelled graphs and cycles [5]; and more generally for labelled graphs with few disjoint excluded minors, in the addable case [6]. Recall that a minor-closed class of graphs is *addable* if each excluded minor is 2-connected. For $t \geq 3$, the *fan* F_t is the graph obtained from a path with $t-1$ vertices by adding a new vertex and joining it to each vertex on the path. Observe that no fan has minor K_4, so $\mathrm{Ex}\,K_4$ contains all fans.

For an addable minor-closed class $\mathcal{A}$ of graphs with set $\mathcal{B}$ of excluded minors, by [6] we have two cases: (a) if $\mathcal{A}$ does not contain all fans then, for each k, $\mathrm{Ex}\,(k+1)\mathcal{B}$ consists of $\mathrm{apex^k}\mathcal{A}$ together with an exponentially smaller class; and (b) if $\mathcal{A}$ contains all fans then $\mathrm{Ex}\,(k+1)\mathcal{B}$ is exponentially larger than $\mathrm{apex^k}\mathcal{A}$ for all sufficiently large k.

We can improve the estimate in Theorem 2 for $|(\mathrm{Ex}\,(k{+}1)S_3)_n|$ to an asymptotic counting formula: for each $k \geq 0$

$$|(\mathrm{Ex}\,(k+1)S_3)_n| \sim c \cdot 2^{kn} n^{-\frac{1}{2}} e^{(2n)^{\frac{1}{2}}}\, n! \tag{6}$$

where the constant c is $(2^{k^2+k+2}\pi e)^{-\frac{1}{2}}(k!)^{-1}$. We may obtain results about random graphs, as in Theorems 1.4, 1.5 and 1.6 of [5]. For example, almost all graphs with no two disjoint copies of the 3-star S_3 have clique number and chromatic number equal to 4.

2. Proof plan

We shall prove the following 6 results.

$$\text{For each } t \geq 3, k \geq 0: \quad |(\mathrm{apex^k\ Ex\,S_t})_n|/n! \;=\; 2^{\mathrm{kn}+\Theta(\mathrm{n}^{\frac{2t-5}{2t-4}})}. \tag{7}$$

$$\text{For each } t \geq 3, k \geq 0: \quad u_n(\mathrm{apex^k\ Ex\,S_t}) \;=\; 2^{\mathrm{kn}+\Theta(\mathrm{n}^{\frac{2t-5}{2t-4}})}. \tag{8}$$

$$\text{For each } k \geq 1: \quad u_n(\mathcal{D}^{3,k}) \;\leq\; 2^{(k-1)n+O(n^{\frac{2t-5}{2t-4}})}. \tag{9}$$

$$\text{For each } k \geq 1: \quad |(\mathcal{D}^{3,k})_n|/n! \;\geq\; 2^{(k-1)n+\Omega(n^{\frac{2t-5}{2t-4}})}. \tag{10}$$

$$\text{For each } t \geq 4, k \geq 1: \quad u_n(\mathcal{D}^{t,k}) \;\leq\; 2^{kn+O(\ln n)}. \tag{11}$$

$$\text{For each } t \geq 4, k \geq 1: \quad |(\mathcal{D}^{t,k})_n|/n! \;=\; \Omega(2^{kn}). \tag{12}$$

Combined with the observation that always $|\mathcal{A}_n|/n! \;\leq\; u_n(\mathcal{A})$, these yield Theorem 2 and thus also Theorem 1. The asymptotic counting formula (6) for $\mathrm{Ex}\,(k+1)S_3$ follows from the asymptotic counting formula (22) below for $\mathrm{apex^k\,Ex\,S_3}$, together with the inequality (9) which shows that $\mathcal{D}^{3,k}$ is negligibly small.

3. Preliminary lemmas and proofs of (7), (8)

Given a graph G, let $D_3(G)$ be the set of vertices of degree at least 3, and let $d_3(G) = |D_3|$. We need upper bounds on $d_3(G)$.

Lemma 3. *Let G be a connected graph in $\mathrm{Ex}\,S_t$, where $t \geq 4$. Then $\Delta(G) \leq t-1$, and $d_3(G) \leq 10(t-3)$.*

Proof. Trivially $\Delta(G) \leq t-1$. Given a connected graph H, let $\text{star}(H)$ be the maximum t such that H has a minor S_t. We will show that

$$d_3(G) \leq 10\,(\text{star}(G)-2) \tag{13}$$

for each connected graph G with at least two edges (so $\text{star}(G) \geq 2$), by induction on the number of edges of G.

The result is clearly true if G has 2 edges. Let G have at least 3 edges and assume that the result holds for each connected graph with fewer edges. Suppose that in G two vertices of degree at least 4 are joined by an edge e. If e is not a bridge, then by the induction hypothesis,

$$d_3(G) = d_3(G-e) \leq 10(\text{star}(G-e)-2) \leq 10(\text{star}(G)-2).$$

If e is a bridge, and $G-e$ has components G_1 and G_2 then $\text{star}(G) \geq \text{star}(G_1) + \text{star}(G_2) - 2$, and both G_1 and G_2 have at least 3 edges. Thus by the induction hypothesis,

$$\begin{aligned} d_3(G) &= d_3(G_1) + d_3(G_2) \\ &\leq 10(\text{star}(G_1)-2) + 10(\text{star}(G_2)-2) \\ &= 10(\text{star}(G_1)+\text{star}(G_2)) - 40 \\ &\leq 10(\text{star}(G)+2) - 40 \;=\; 10(\text{star}(G)-2). \end{aligned}$$

Hence we may assume that no two vertices of degree at least 4 are adjacent. Pick a vertex v of maximum degree in G, remove v and its neighbours from G, and repeat as long as there is a vertex of degree at least 3 in the graph at that time. Suppose that we pick $v_1, \ldots, v_j$ with degrees $x_1, \ldots, x_j$ in the graph at the time. Since G is connected we may form a tree by adding $j-1$ paths between the x_i-stars centred on the v_i, and this tree will have at least $\sum_i x_i - 2(j-1)$ leaves. Thus $\sum_i (x_i - 2) \leq \text{star}(G) - 2$, and since each $x_i \geq 3$ we have $j \leq \text{star}(G) - 2$. Hence the number of vertices within distance 2 of a v_i is at most

$$\sum_i (1+3x_i) = 3\sum_i x_i + j \leq 3\,\text{star}(G) + 7j - 6 \leq 10(\text{star}(G)-2).$$

Hence if (13) failed then some vertex would still have degree at least 3 in the last graph G and so the process should not have stopped: so (13) must hold. This completes the proof of the induction step, and thus of the lemma. □

We will obtain finer results later, but note that the last lemma already shows that $\text{Ex}\, S_t$ has unlabelled growth constant at most 1 (and hence also labelled growth constant at most 1, and thus both constants equal to 1). For when we construct an unlabelled graph G in $\text{Ex}\, S_t$ on $[n]$ where $n > 10t$, we may assume that $D_3(G) \subseteq [10t]$; and there are $n^{O(1)}$ ways to choose a graph on $[10t]$ and the edges between $[10t]$ and $\{10t+1, \ldots, n\}$, and $(1+o(1))^n$ ways to choose a graph with maximum degree at most 2 on $\{10t+1, \ldots, n\}$.

Lemma 4. *For all positive integers $t \geq 4$, k and d there is a constant $c = c(t,k,d)$ such that each connected graph $G \in \mathrm{Ex}\,(k+1)S_t$ with $\Delta(G) \leq d$ satisfies $d_3(G) \leq c$.*

Proof. We fix $t \geq 4$ and d, and use induction on k. By Lemma 3, the result is true when $k = 0$. Let $j \geq 1$ and suppose that the result holds when $k = j-1$. Let c be such that each connected graph $G \in \mathrm{Ex}\, jS_t$ with $\Delta(G) \leq d$ satisfies $d_3(G) \leq c$. Let $c' = 20td(c+1)$. Let G be a connected graph in $\mathrm{Ex}\,(j+1)S_t$ with $\Delta(G) \leq d$. By induction it will suffice for us to prove that $d_3(G) \leq c'$ to complete the proof of the lemma.

Suppose for a contradiction that $d_3(G) > c'$. Let H be a connected subgraph of G with $d_3(H) > 10(t-2)$ and $|V(H) \cap D_3(G)| \leq 20t$. We may pick such a graph as follows. Suppress vertices of degree 2 in G to form the connected multigraph M. Pick a vertex $v_0 \in D_3(M)$ and add 3 neighbours: this forms the inital set W. Repeatedly pick a vertex $w \in W$ with a neighbour in $V(M) - W$, choosing a vertex of degree less than 3 in the induced subgraph $M[W]$ if possible, and add 1 or 2 such neighbours of w to W until w has degree at least 3 in $M[W]$. Continue as long as $d_3(M[W]) \leq 10(t-2)$. When we finish, $d_3(M[W]) > 10(t-2)$, and $|W \cap D_3(G)| \leq 20(t-2)+4 \leq 20t$. So the induced subgraph H of G corresponding to the induced subgraph $M[W]$ is as required.

By Lemma 3, there is a tree T in H with t leaves. Let N denote the set of neighbours in G of vertices in T. Then each vertex in $D_3(G)$ contributes at most $d-2$ vertices to N unless it is a leaf when it may contribute one more, and similarly each vertex of degree 2 contibutes 0 unless it is a leaf when it may contribute one more. Hence

$$|N| \leq |V(T) \cap D_3(G)|(d-2) + t \leq 20t(d-2) + t < 20t(d-1).$$

Consider $G' = G - V(T)$. Then

$$d_3(G') \geq d_3(G) - |V(T) \cap D_3(G)| - |N \cap D_3(G)| > d_3(G) - 20td$$

and $\kappa(G') \leq |N| < 20dt$. Hence some component $\tilde{G}$ of G' has

$$d_3(\tilde{G}) \geq \frac{d_3(G')}{\kappa(G')} > \frac{d_3(G)}{20dt} - 1 \geq c.$$

But now $\tilde{G}$ must have at least j disjoint minors S_t, so $G \notin \mathrm{Ex}\,(j+1)S_t$; and this contradiction completes the proof. □

Lemma 5. *Let $t \geq 4$ and $k \geq 1$.*

(a) Given a graph $G \in \mathrm{Ex}\,(k+1)S_t$, let $U(G)$ denote the set of vertices in G with degree at least $k(t+1)+t$: then $|U(G)| \leq k$.

(b) There is a positive integer $\alpha = \alpha(t,k)$ such that each graph G in $\mathrm{Ex}\,(k+1)S_t$ contains a 'blocking' set Q of at most α vertices such that $G - Q$ is in $\mathrm{Ex}\,(S_t)$.

Proof. (a) If at least $k+1$ vertices each had degree at least $k(t+1)+t$ then we could find $k+1$ disjoint t-stars centred on these vertices by greedily picking disjoint t-stars one after another, since at each stage before the last at most $k(t+1)$ vertices would have been used.

(b) Let $G' = G - U(G)$. Then $G' \in \mathrm{Ex}\,(k+1)S_t$ and $\Delta(G') \leq k(t+1)+t-1$. G' can have at most k components G_i containing a minor S_t. By Lemma 4 we may take Q as $U(G)$ together with the sets of vertices of degree at least 3 in these components G_i. □

Lemma 5 is a step on our way to prove the main result Theorem 2, which extends Theorem 1, but note that it already yields Theorem 1. We may see this as in the discussion following Lemma 3 on the unlabelled growth constant of $\mathrm{Ex}\,S_t$.

Since the graph $(k+1)S_t$ is planar, we could alternatively have proved part (b) of Lemma 5 above by using the extension by Robertson and Seymour [8] of the classical Erdős - Pósa theorem [3] on disjoint cycles (see for example Corollary 12.4.10 in [2]); and we could then have used Lemma 3 to prove Lemma 4.

Let us record a basic combinatorial fact as a lemma, as we will use it several times.

Lemma 6. *For fixed integers $k \geq 1$ and $j \geq 0$, the number of ways of choosing k integers each at least j to sum to n is $\Theta(n^{k-1})$.*

Proof. Consider k fixed, and let $S(n,j)$ be the set of k-tuples $\mathbf{x}$ of integers at least j which sum to n. There is a familiar natural bijection between $S(n,0)$ and the $(k-1)$-subsets of $[n+k-1]$. Further, the map $\mathbf{x} \rightarrow \mathbf{x} + j\mathbf{1}$ gives a bijection between $S(n-kj,0)$ and $S(n,j)$. Thus $|S(n,j)| = \binom{n-(j-1)k-1}{k-1}$. □

Clearly the connected graphs in $\mathrm{Ex}\,S_3$ are the paths and cycles. Thus for $n \geq 3$ vertices the number of such graphs is 2 in the unlabelled case, and is $\frac{1}{2}n! + \frac{1}{2}(n-1)! \sim \frac{1}{2}n!$ in the labelled case. The next lemma is an approximate version of this result for $\mathrm{Ex}\,S_t$ with any fixed $t \geq 3$.

Lemma 7. *Let $t \geq 3$ and let $\mathcal{C}$ be the class of connected graphs in* $\mathrm{Ex}\,S_t$. *Then both $|\mathcal{C}_n|/n!$ and $u_n(\mathcal{C})$ are $\Theta(n^{2t-6})$.*

Proof. For the lower bound consider a tree T with $t-1$ leaves and $t-3$ internal vertices each of degree 3, which thus has $2t-4$ vertices and $2t-5$ edges. By Lemma 6, the number of graphs on $\{1,\ldots,n\}$ homeomorphic to T is $n!\,\Theta(n^{2t-6})$, and each of these graphs is in $\mathrm{Ex}\,S_t$.

The upper bound needs more work. Given a multigraph G we let $s(G)$ be the pair $(\tilde{G}, \mathbf{r})$ defined as follows: $\tilde{G}$ is the multigraph obtained from G by suppressing vertices of degree 2, and $\mathbf{r}$ is the vector indexed by the edges e of $\tilde{G}$ where r_e is the number of degree-2 vertices suppressed when forming e. Conversely, given a pair

$(H, \mathbf{r})$ where H is a multigraph and $\mathbf{r}$ is a vector of non-negative integers indexed by the edges e of H, we let $G(H, \mathbf{r})$ be the multigraph G such that $s(G) = (H, \mathbf{r})$. (We are considering graphs as unlabelled here.)

Fix $t \geq 3$. Let G be a connected graph in $\mathrm{Ex}\, S_t$ and let $s(G) = (H, \mathbf{r})$. By Lemma 3, $\Delta(H) \leq t-1$ and H has at most $10t$ vertices of degree at least 3, and so H has at most $10t^2$ leaves. Thus there is a finite number of possible multigraphs H that can appear as the first co-ordinate of $s(G)$ where G is a connected graph in $\mathrm{Ex}\, S_t$. Given a vector $\mathbf{r}$ let $\mathbf{r} \wedge 2$ denote the vector with co-ordinates the minimum $r_e \wedge 2$ of r_e and 2. Then for any multigraph H and corresponding vector $\mathbf{r}$

$$G(H, \mathbf{r}) \in \mathrm{Ex}\, S_t \iff G(H, \mathbf{r} \wedge 2) \in \mathrm{Ex}\, S_t. \tag{14}$$

Let $\mathcal{H}$ be the set of all pairs $s(G) = (H, \mathbf{r})$ for (simple) graphs $G \in \mathrm{Ex}\,(S_t)$. Let $\mathcal{H}_0$ be the set of all pairs $(H, \mathbf{r})$ in $\mathcal{H}$ where each $r_e \in \{0, 1, 2\}$. Given $(H, \mathbf{r})$ in $\mathcal{H}_0$ let

$$\mathcal{G}(H, \mathbf{r}) = \{G(H, \mathbf{s}) : \mathbf{s} \wedge 2 = \mathbf{r}\}.$$

Then by (14) $\mathrm{Ex}\,(S_t)$ is partitioned into the finite collection of sets $\mathcal{G}(H, \mathbf{r})$ for $(H, \mathbf{r})$ in $\mathcal{H}_0$. Given a vector $\mathbf{r}$, let $f(\mathbf{r})$ be the number of co-ordinates equal to 2. For $(H, \mathbf{r})$ in $\mathcal{H}_0$, the value of $f(\mathbf{r})$ essentially determines the growth of the number of graphs $\mathcal{G}(H, \mathbf{r})$: we claim that

$$|\mathcal{G}(H, \mathbf{r})_n| = \Theta(n^{f(\mathbf{r})-1})\; n! \tag{15}$$

and

$$u_n(\mathcal{G}(H, \mathbf{r}) = \Theta(n^{f(\mathbf{r})-1}). \tag{16}$$

Once we have established these claims, since $\mathcal{H}_0$ is finite it will suffice for us to show that the maximum value of $f(\mathbf{r})$ over the pairs $(H, \mathbf{r})$ in $\mathcal{H}_0$ is $2t - 5$.

To prove (15), list the vertices of H in a fixed order; and similarly list the edges of H in a fixed order, starting with the edges with r-value 1 (say there are n_1 of them), followed by the $f(\mathbf{r})$ edges with r-value 2, then the remaining edges (with r-value 0). We may construct graphs in $\mathcal{G}(H, \mathbf{r})$ as follows. List the vertices $1, \ldots, n$ in any order. Assign the vertices (labels) in this order to the vertices of H then the 'midpoints' of the n_1 edges with r-value 1. This leaves the list of the remaining $m = n - |V(H)| - n_1$ vertices to be be divided into an ordered list of $f(\mathbf{r})$ sublists each of length at least 2, which will then be assigned to the edges with r-value 2. Thus by Lemma 6 there are $\Theta(n^{f(\mathbf{r})-1})\; n!$ constructions, and (15) follows since H has a finite number of automorphisms. The claim (16) may be proved in a similar way.

It remains for us to show that the maximum value of $f(\mathbf{r})$ over the pairs $(H, \mathbf{r})$ in $\mathcal{H}_0$ is $2t - 5$. Suppose that $(H, \mathbf{r})$ achieves this maximum, and amongst such pairs H has fewest edges. Let F be the set of edges e in H with $r_e = 2$. Then each edge in F is a bridge of H: for if $e = \{u, v\} \in F$ were not a bridge then we could introduce two new vertices u' and v', and replace e by the two new edges $\{u, u'\}$ and $\{v, v'\}$ both with corresponding r-value 2, which would contradict the choice of $(H,\mathbf{r})$ to maximise $f(\mathbf{r})$. But now by the minimality of the number of edges in H it follows that H is a tree. Further each vertex degree in this tree must be 1 or 3, since no vertices can have degree 2 and any vertex of degree > 3 could be split to contradict the maximality of $f(\mathbf{r})$. Hence the tree must be as described in the initial lower bound part of the proof, and we are done. □

The last lemma gave estimations of the numbers of connected graphs in $\mathrm{Ex}\, S_t$: the next lemma will let us use these results to estimate the number of graphs which are not necessarily connected. If a class of graphs is such that G is in $\mathcal{A}$ if and only if each component is in $\mathcal{A}$ then we call $\mathcal{A}$ *decomposable*.

Lemma 8. *Let $\mathcal{A}$ be a decomposable class of graphs, let $\mathcal{C}$ be the class of connected graphs in $\mathcal{A}$, let $c > -1$ and let $\gamma > 0$. If $|\mathcal{C}_n|/n! = \Theta(n^c)\gamma^n$ then $|\mathcal{A}_n|/n! = e^{\Theta(n^{\frac{c+1}{c+2}})}\gamma^n$; and if $\gamma \geq 1$ and $u_n(\mathcal{C}) = \Theta(n^c)\gamma^n$ then $u_n(\mathcal{A}) = e^{\Theta(n^{\frac{c+1}{c+2}})}\gamma^n$.*

Proof. This is really four results, two upper bounds and two lower bounds. Note that $[x^n](1-x)^{-b} = \Theta(n^{b-1})$. Let $b = c+1 > 0$, let $a > 0$ and let $D(x) = a(1-x)^{-b}$.

(a) *Upper bounds.* Consider first the labelled case. If a is sufficiently large then $|\mathcal{C}_n|/(\gamma^n n!) \leq [x^n]D(x)$ for each n, and so $|\mathcal{A}_n|/(\gamma^n n!) \leq [x^n]e^{D(x)}$ for each n. Thus it suffices to show that

$$[x^n]e^{D(x)} = [x^n]\ e^{a(1-x)^{-b}} = e^{O(n^{\frac{b}{b+1}})}.$$

This follows from results in [10], but we shall use a cruder method which will work also for the unlabelled case.

Let $r = r(n) = 1 - n^{-1/(1+b)}$. Note that $1 - x \geq e^{-2x}$ for $x > 0$ sufficiently small, so $r^n \geq e^{-2n^{b/(1+b)}}$ for n sufficiently large. Also $D(r) = an^{-b/(1+b)}$, so

$$e^{D(r)}/r^n \leq \frac{e^{an^{b/(1+b)}}}{e^{-2n^{b/(1+b)}}} = e^{(a+2)n^{b/(1+b)}}.$$

Hence by a standard saddle point bound, see for example (19) in Proposition IV.1 of [4], we have that for n sufficiently large $[x^n]e^{D(x)} \leq e^{(a+2)n^{b/(1+b)}} = e^{O(n^{b/(1+b)})}$.

Now consider the unlabelled case. As above, if a is sufficiently large then $u_n(\mathcal{C})/\gamma^n \leq [x^n]D(x)$ for each n. Let $S(x) = \sum_{k=1}^{n} D(x^k)/k$, and $F(x) = e^{S(x)}$. Since $\gamma \geq 1$ and each coefficient in the power series for $F(x)$ is non-negative, we have $u_n(\mathcal{A})/\gamma^n \leq [x^n]F(x)$ for each n. (We need not consider $D(x^k)/k$ for $k > n$.)

Let $\alpha = 1 - 1/e \approx 0.63$. We will use the inequality that, for all $k \geq 1$ and $0 \leq x \leq 1/k$,

$$(1-x)^k \leq 1 - \alpha k x. \tag{17}$$

To prove this, let $f(x) = (1-x)^k$ and $g(x) = 1 - \alpha k x$: then $f(0) = 1 = g(0)$; $f(1/k) = (1-1/k)^k \leq 1/e = g(1/k)$; and f is convex and g is linear on $(0, 1/k)$.

Let $x = x(n) = n^{-1/(1+b)}$ and let $r = r(n) = 1 - x$ as above. In the sums below, k runs from 1 to $k_0 = \lfloor 1/x \rfloor$. By (17)

$$\sum_{k \le k_0} \frac{1}{k} D(r^k) = a \sum_{k \le k_0} \frac{1}{k} (1 - (1-x)^k)^{-b}$$
$$\le a \sum_{k \le k_0} \frac{1}{k} (\alpha k x)^{-b} \;=\; a \alpha^{-b} x^{-b} \sum_{k \le k_0} k^{-(b+1)}$$
$$= O(x^{-b}) \;=\; O(n^{b/(1+b)}).$$

Further, for $k \ge k_0$,

$$D(x^k) \le D(x^{k_0}) \le a(\alpha k_0 x)^{-b} \le a\alpha^{-b}(1-x)^{-b} \le 2a\alpha^{-b}$$

for n sufficiently large; and then

$$\sum_{k_0 < k \le n} D(r^k)/k \le 2a\alpha^{-b} \sum_{k_0 < k \le n} 1/k = O(\ln n).$$

Putting these bounds together gives $S(r) = O(n^{b/(1+b)})$. Hence by a saddle point bound as above,

$$u_n(\mathcal{A})/\gamma^n \le [x^n]F(x) \le \frac{F(r)}{r^n} = e^{O(n^{b/(1+b)})}$$

as required.

(b) *Lower bounds.* We can handle the labelled and unlabelled cases together. If $a > 0$ is sufficiently small and d is sufficiently large, then, for each n, $|\mathcal{C}_n|/(\gamma^n n!) \ge [x^n]x^d D(x)$ and so $|\mathcal{A}_n|/(\gamma^n n!) \ge [x^n]e^{x^d D(x)}$; and similarly $u_n(\mathcal{C})/\gamma^n \ge [x^n]x^d D(x)$ and so $u_n(\mathcal{A})/\gamma^n \ge [x^n]e^{x^d D(x)}$. Let $\tau > 0$ and let $t = \lfloor \tau n^{\frac{b}{b+1}} \rfloor$. Then

$$[x^n]e^{x^d D(x)} \ge [x^n]\frac{a^t x^{dt}(1-x)^{-bt}}{t!} = [x^{n-dt}]\frac{a^t(1-x)^{-bt}}{t!}.$$

Since $dt = o(n)$ it suffices to show that

$$[x^n]\frac{a^t(1-x)^{-bt}}{t!} = e^{\Omega(n^{\frac{b}{b+1}})}. \tag{18}$$

But (assuming $bt \ge 1$)

$$[x^n]\frac{a^t(1-x)^{-bt}}{t!} = \frac{a^t}{t!}\binom{-bt}{n} = \frac{a^t}{t!}\frac{(n+bt-1)_{bt-1}}{\Gamma(bt)}$$
$$\ge \frac{a^t n^{bt-1}}{t^t (bt)^{bt-1}} \ge \frac{1}{n}\left(\frac{an^b}{b^b t^{b+1}}\right)^t.$$

Now let us assume that $\tau > 0$ is sufficiently small that $\frac{a}{b^b \tau^{b+1}} \geq e$. Then, using the definition of t, by the above inequality

$$[x^n]\frac{a^t(1-x)^{-bt}}{t!} \geq \frac{1}{n}\left(\frac{a}{b^b\tau^{b+1}}\right)^t \geq \frac{1}{n}e^t$$

and (18) follows. □

Lemmas 7 and 8 yield immediately:

Lemma 9. *For each $t \geq 3$, both $|(\mathrm{Ex}\ S_t)_n|/n!$ and $u_n(\mathrm{Ex}\ S_t)$ are $e^{\Theta(n^{\frac{2t-5}{2t-4}})}$.*

When $t = 3$ we can easily be more precise: known results give that $|(\mathrm{Ex}\ S_3)_n|/n!$ is $e^{(2n)^{\frac{1}{2}}+O(\ln n)}$ and $u_n(\mathrm{Ex}\, S_3)$ is $e^{2\pi(n/3)^{\frac{1}{2}}+O(\ln n)}$. We may see this as follows. For the labelled case, the generating function is

$$(1-x)^{-\frac{1}{2}}e^{-\frac{1}{2}-\frac{x^2}{4}}e^{\frac{\frac{1}{2}}{1-x}}$$

and so

$$|(\mathrm{Ex}\ S_3)_n| \sim (4\pi e n)^{-\frac{1}{2}}e^{(2n)^{\frac{1}{2}}}n! \tag{19}$$

by Theorem 2 of Wright [9]. For the unlabelled case, the number p_n of partitions of n, and the number $\tilde{p}_n$ where we insist that each part is at least 3, are both $e^{\pi(2n/3)^{\frac{1}{2}}+O(\ln n)}$. But $u_n(\mathrm{Ex}\, S_3) = \sum_{j=3}^n \tilde{p}_j \cdot p_{n-j}$ and the result follows.

Lemma 10. *Let $\mathcal{A}$ be any class of graphs with bounded maximum degree. Then for each $k \geq 1$*

$$|(\mathrm{apex}^{\mathrm{k}}\mathcal{A})_{\mathrm{n}}| \sim \left(\mathrm{k}!2^{\binom{\mathrm{k}+1}{2}}\right)^{-1} 2^{\mathrm{kn}}(\mathrm{n})_{\mathrm{k}}\ |\mathcal{A}_{\mathrm{n-k}}|. \tag{20}$$

Proof. We may prove this as in the proof of (3) in [7]. Let $V = \{1, \ldots, n\}$. We may construct the graphs in $\mathrm{apex}^{\mathrm{k}}\mathcal{A}$ on $V = \{1, \ldots, n\}$ by picking a set $S \subseteq V$ of k vertices ($\binom{n}{k}$ choices), picking a graph in $\mathcal{A}$ on $V \setminus S$ ($|\mathcal{A}_{n-k}|$ choices), and adding any set of edges incident to the vertices in S ($2^{kn-\binom{k+1}{2}}$ choices). Thus the number of constructions is as on the right hand side in (20). But for large n, almost all the constructions give each vertex in S degree at least $n/3$, and then the graph constructed is unique; and the lemma follows. □

The last lemma shows in particular that, for each $t \geq 3$ and $k \geq 1$

$$|(\mathrm{apex}^{\mathrm{k}}\mathrm{Ex}\,\mathrm{S_t})_{\mathrm{n}}| \sim \left(\mathrm{k}!2^{\binom{\mathrm{k}+1}{2}}\right)^{-1} 2^{\mathrm{kn}}(\mathrm{n})_{\mathrm{k}}\ |(\mathrm{Ex}\,\mathrm{S_t})_{\mathrm{n-k}}|.$$

Hence by Lemma 9, for each $t \geq 3$ and $k \geq 0$,

$$|(\mathrm{apex}^{\mathrm{k}}\mathrm{Ex}\,\mathrm{S_t})_{\mathrm{n}}| = 2^{\mathrm{kn}+\Theta(\mathrm{n}^{\frac{2t-5}{2t-4}})} \tag{21}$$

and we have proved (7). A more precise result when $t = 3$ follows from Lemma 10 and (19): we have

$$|(\mathrm{apex^k Ex\, S_3})_n| \sim \mathrm{c} \cdot 2^{\mathrm{kn}} \mathrm{n}^{-\frac{1}{2}} \mathrm{e}^{(2\mathrm{n})^{\frac{1}{2}}}\, \mathrm{n}! \tag{22}$$

where the constant c is $(2^{k^2+k+2}\pi e)^{-\frac{1}{2}}(k!)^{-1}$.

We need a result for unlabelled graphs corresponding to (21). Lemma 9 and the following lemma give (8).

Lemma 11. *Let $t \geq 3$ and $k \geq 1$. Then*

$$u_n(\mathrm{apex^k Ex\, S_t}) = 2^{\mathrm{kn}+\Theta(\mathrm{n}^{\frac{2t-5}{2t-4}})}. \tag{23}$$

Proof. Let $\mathcal{C}$ be the set of connected graphs in $\mathrm{Ex}\, S_t$. We say that a graph is 2^k-*coloured* if we assign a vector in $\{0,1\}^k$ to each vertex. Let $\tilde{\mathcal{A}}$ be the set of 2^k-coloured graphs in $\mathrm{Ex}\, S_t$, and let $\tilde{\mathcal{C}}$ the set of connected graphs in $\tilde{\mathcal{A}}$. Since the graphs in $\mathcal{C}$ have a bounded number of automorphisms,

$$u_n(\tilde{\mathcal{C}}) = \Theta(2^{kn} u_n(\mathcal{C})) = \Theta(2^{kn} n^{2t-6})$$

by Lemma 7. Hence by Lemma 8

$$u_n(\tilde{\mathcal{A}}) = 2^{kn} e^{\Theta(n^{\frac{2t-5}{2t-4}})}.$$

But $u_n(\mathrm{apex^k Ex\, S_t}) = \Theta(\mathrm{u_n}(\tilde{\mathcal{A}}))$, and the proof is complete. □

We need one last preliminary lemma.

Lemma 12. *Fix positive integers a and b, and let $\mathcal{A}$ be the class of all connected graphs G with $\Delta(G) \leq a$ and $d_3(G) \leq b$. Then $u_n(\mathcal{A}) = \Theta(n^{(a-1)b})$.*

Proof. Consider the upper bound first. Let $G \in \mathcal{A}$, with $1 \leq b' \leq b$ vertices of degree at least 3. (The case $d_3(G) = 0$ is trivial.) As in the proof of lemma 7, let the multigraph $\tilde{G}$ be formed from G by suppressing all vertices of degree 2. Then each vertex of $\tilde{G}$ has degree 1 or at least 3, each edge of $\tilde{G}$ is incident to a vertex of degree at least 3, and at least $b' - 1$ are incident to two such vertices (since G is connected). Hence $\tilde{G}$ has at most $ab' - (b' - 1) \leq (a-1)b + 1$ edges. Hence by Lemma 6, the number of unlabelled simple graphs homeomorphic to $\tilde{G}$ is $O(n^{(a-1)b})$. But only a finite number of multigraphs can arise as $\tilde{G}$, and the upper bound follows.

Finally note that any tree consisting of b vertices of degree a together with leaves (of which there must be $(a-2)b+2$) has $(a-1)b+1$ edges, and the lower bound follows. □

4. Completing the proofs

We now prove (9), (10), (11) and (12). We first consider the upper bounds (9) and (11).

Fix $t \geq 3$ and $k \geq 1$. By Lemma 5 (b), there is a positive integer α such that each graph G in $\mathrm{Ex}\,(k+1)S_t$ contains a set Q of at most α vertices with $G \setminus Q$ is in $\mathrm{Ex}\,(S_t)$. We may assume that $\alpha \geq k$. Let $U(G)$ be the set of vertices in G with degree at least $k(t+1)+t$. By Lemma 5 (a), for each graph $G \in \mathrm{Ex}\,(k+1)S_t$ we have $|U(G)| \leq k$. For each $s = 0, 1, \ldots, k$ let $\mathcal{G}(s)$ denote the set of graphs $G \in \mathrm{Ex}\,(k+1)S_t$ such that $U(G) = \{1, \ldots, s\}$.

Now let $n > 2\alpha$, let $0 \leq s \leq k$ and let $S = \{1, \ldots, s\}$. From the above, for each graph $G \in \mathcal{G}(s)_n$ there is a set R of α vertices in $V \setminus S$ such that $G - (S \cup R) \in \mathrm{Ex}\,S_t$. We may assume that R is $\{s+1, \ldots, s+|R|\}$. Each vertex in $V \setminus S$ has degree less than $k(t+1)+t$ in G. Let $\mathcal{B}(s)$ be the class of graphs $G \in \mathcal{G}(s)$ such that $G - S \notin \mathrm{Ex}\,S_t$. We want to upper bound $u_n(\mathcal{B}(s))$.

Suppose first that $s \leq k-1$. We bound $u_n(\mathcal{G}(s))$. Consider the number of choices for (i) the size r of the set R and the graph on R ($O(1)$ choices), (ii) the graph on $V \setminus (S \cup R)$ (at most $u_n(\mathrm{Ex}\,S_t)$ choices), (iii) the neighbours in $V \setminus S$ of the vertices in R ($n^{O(1)}$ choices), and finally (iv) the neighbours of the vertices in S (at most $2^{(k-1)n}$ choices). Hence

$$u_n(\mathcal{B}(s)) \leq u_n(\mathcal{G}(s)) \leq n^{O(1)} \cdot u_n(\mathrm{Ex}\ S_t) \cdot 2^{(k-1)n}.$$

and using (8) we obtain

$$u_n(\mathcal{B}(s)) \leq 2^{(k-1)n + O(n^{\frac{2t-5}{2t-4}})}. \tag{24}$$

We now consider when $s = k$: we handle the cases $t = 3$ and $t \geq 4$ separately.

(a) *The case $s = k$ and $t = 3$.* Here $\mathcal{B}(s)$ is empty! For suppose that $G \in \mathcal{B}(s)$. Then we may pick $k+1$ disjoint 3-stars much as in Lemma 5 by starting with a 3-star with vertex set contained in $V \setminus S$, and greedily picking disjoint 3-stars centred on the vertices in S one after another. We can do this since when we have picked $i < k$ vertices from S, the next vertex in S is adjacent to at least $4k+3-4(i+1)-(k-i-1) = 3k-3i \geq 3$ vertices outside S, and so we can pick another 3-star. But then $G \notin \mathrm{Ex}\,(k+1)S_3$, a contradiction. This result together with (24) completes the proof of (9).

(b) *The case $s = k$ and $t \geq 4$.* Let $G \in \mathcal{B}(s)$; let $G_1, \ldots, G_{k'}$ be the $1 \leq k' \leq k$ components of $G[V \setminus S]$ which have a minor S_t; and let $W = \cup_i V(G_i)$. We upper bound $u_n(\mathcal{B}(s))$ by upper bounding the numbers of graphs in the sets

$$\mathcal{B}(s,j) = \{G \in \mathcal{B}(s) : |W| = j\}.$$

For $G \in \mathcal{B}(s,j)$, we may assume that $W = \{s+1, \ldots, s+j\}$. Some vertex in S must be adjacent to at most $(k-1)(t+1)+t-1 \leq k(t+1)$ vertices in $V \setminus (S \cup W)$, since otherwise $G \notin \mathrm{Ex}\,(k+1)S_t$ by Lemma 5. Hence the number of choices for edges between S and $V \setminus (S \cup W)$ is at most

$$k \sum_{i=0}^{k(t+1)} \binom{n}{i} \cdot (2^{n-k-j})^{k-1} = O(n^{k(t+1)}) \cdot 2^{(n-j)(k-1)}.$$

Thus the number of choices for edges between S and $V \setminus S$ is at most

$$2^{(n-j)(k-1)+O(\ln n)} \cdot 2^{jk} = 2^{(k-1)n+j+O(\ln n)}.$$

Now let us bound the number of choices for $G[V \setminus S]$. By Lemma 6, there are $O(n^{k-1})$ ways to choose the sizes of the sets $V(G_i)$ to sum to j. Hence the number of choices for the graph induced on W is $n^{O(1)}$ by Lemmas 4 and 12. Let $W' = V \setminus (S \cup W)$, so $|W'| \leq n-j$. By Lemma 9, the number of choices for the graph on W' is at most

$$u_{n-j}(\mathrm{Ex}\ S_t) = e^{\Theta((n-j)^{\frac{2t-5}{2t-4}})}.$$

Hence the number of choices for $G[V \setminus S]$ is at most

$$n^{O(1)} \cdot e^{\Theta((n-j)^{\frac{2t-5}{2t-4}})} = n^{O(1)} \cdot \ e^{\Theta((n-j)^{\frac{2t-5}{2t-4}})}$$

Putting these estimates together,

$$\begin{aligned} u_n(\mathcal{B}(s,j)) &\leq 2^{(k-1)n+j+O(\ln n)}\ e^{\Theta((n-j)^{\frac{2t-5}{2t-4}})} \\ &= 2^{kn+O(\ln n)}\ e^{\Theta((n-j)^{\frac{2t-5}{2t-4}})} 2^{-(n-j)}. \end{aligned}$$

Hence

$$u_n(\mathcal{B}(s)) = \sum_{j=1}^{n-k} u_n(\mathcal{B}(s,j)) \leq 2^{kn+O(\ln n)} \cdot \sum_{j=1}^{n-k} e^{\Theta((n-j)^{\frac{2t-5}{2t-4}})} 2^{-(n-j)}.$$

But the last sum here is at most $\sum_{i \geq 0} e^{\Theta(i^{\frac{2t-5}{2t-4}})} 2^{-i}$, which is finite. Hence

$$u_n(\mathcal{B}(s)) \leq 2^{kn+O(\ln n)},$$

and this result together with (24) yields (11).

It remains to establish the lower bounds (10) and (12).

For (10) consider the following constructions of graphs in $\mathcal{D}^{3,k}$. Pick a subset S of V of size $k-1$; on $V \setminus S$ put a graph consisting of the disjoint union of K_5 and a graph in $\mathrm{Ex}\,S_3$; and add edges between S and $V \setminus S$ such that each vertex in S has degree at least $4k+3$. The number of (labelled) graphs constructed is

$$2^{(k-1)n+O(\log n)} |(\mathrm{Ex}\,S_3)_{n-k-4}| = 2^{(k-1)n+\Theta(n^{\frac{2t-5}{2t-4}})}$$

as required for (10).

For the case $t \geq 4$, let H_{t+2} be the tree of order $t+2$ with t leaves obtained by starting with adjacent vertices u and v, and joining 2 new vertices to u and

$t-2$ new vertices to v. For $j \geq t+3$ let H_j be the tree of order j formed from H_{t+2} by subdividing the edge uv $j-t-2$ times. Observe that H_j has a minor S_t and no subgraph obtained by deleting a vertex has this property. The number of graphs on $V = \{1, \ldots, n\}$ isomorphic to H_n is $\Theta(n!)$.

We may form graphs G in $(\mathcal{D}^{t,k})_n$ as follows. Choose a subset S of k vertices of V, put a copy of H_{n-k} on $V \setminus S$, and add edges between S and $V \setminus S$ in any way such that each vertex in S has degree at least $k+t$. (To see that G is not in $\mathrm{apex}^{\mathrm{k}}\mathrm{S}_{\mathrm{t}}$ consider $B \subseteq V$ with $|B| = k$. If $v \in S \setminus B$ then v has degree at least t in $G-B$. Thus if $G - B \in \mathrm{Ex}\, S_t$ then B must be S, but $G-S$ has an S_t minor.) The number of such graphs is $\Omega(n!\ 2^{kn})$, as required for (12).

5. Concluding remarks

As we noted in the first section, we know results about labelled graphs with few disjoint cycles [5]; and more generally about labelled graphs with few disjoint excluded minors, in the addable case (when the minors are all 2-connected) [6]. In this paper we have learned about labelled and unlabelled graphs with few disjoint S_t minors, where S_t is the t-leaf star and $t \geq 3$. Of course S_t is not 2-connected. Let us consider a more complicated such graph.

What if we let B denote the bowtie graph obtained from two triangles by identifying a vertex from the first triangle and a vertex from the second one? It is not hard to see that $\mathrm{Ex}\, B$ has labelled growth constant e. Also, the fan F_5 is not in $\mathrm{Ex}\, B$. Does $\mathrm{Ex}\, 2B$ have labelled growth constant $2e$? How large is $\mathrm{Ex}\, 2B \setminus \mathrm{apex}\, \mathrm{Ex}\, \mathrm{B}$? What about $\mathrm{Ex}\,(k+1)B$? What about unlabelled growth constants?

Acknowledgement

Thanks to Olivier Bernardi and Valentas Kurauskas for several helpful comments. Much of the work reported here was done while the author was visiting the CRM in Barcelona in 2009, on the research program in Probabilistic Techniques in Computer Science.

References

[1] O. Bernardi, M. Noy and D. Welsh, Growth constants of minor-closed classes of graphs. To appear.
[2] R. Diestel, *Graph Theory*, Springer, 2005.
[3] P. Erdős and L. Pósa, On independent circuits in a graph, *Canad. J. Math.* **17** (1965), 347–352.
[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics,* Cambridge University Press, 2009.
[5] V. Kurauskas and C. McDiarmid, On graphs with few disjoint cycles, manuscript, 2009.
[6] V. Kurauskas and C. McDiarmid, Random graphs containing few disjoint excluded minors, in preparation, 2010.
[7] C. McDiarmid, Random graphs on surfaces, *J. Combinatorial Theory B* **98** (2008), 778–797.
[8] N. Robertson and P.D. Seymour, Graph minors V: Excluding a planar graph, *J. Combinatorial Th. B* **41** (1986), 92–114.

[9] E.M. Wright, The coefficients of a certain power series, *J. London Math. Soc.* **7** (1932), 256–262.
[10] E.M. Wright, On the coefficients of power series having exponential singularities, *J. London Math. Soc.* **24** (1949), 304–309.

Colin McDiarmid
Department of Statistics
Oxford University