A DOUBLING MEASURE ON $\mathbb{R}^d$ CAN CHARGE A RECTIFIABLE CURVE

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ABSTRACT. For $d \geq 2$, we construct a doubling measure ν on $\mathbb{R}^d$ and a rectifiable curve Γ such that $\nu(\Gamma) > 0$.

1. INTRODUCTION

A Borel measure ν on $\mathbb{R}^d$ is said to be *doubling* if there is a constant $C_\nu < \infty$ such that for any $x \in \mathbb{R}^d$ and $0 < r < \infty$ we have

$$\nu(B(x, 2r)) \leq C_\nu(B(x, r)) \tag{1.1}$$

where $B(x, r)$ is the ball $\{y : |y - x| < r\}$. A *rectifiable curve* is a continuous map $\gamma : [0, 1] \to \mathbb{R}^d$ with

$$\text{length}(\gamma) := \sup_{0 \leq t_0 \leq \cdots \leq t_n \leq 1} \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| < \infty.$$

By reparametrization, one may assume that γ is Lipschitz with constant equal to length(γ). We will also make use of the following simple (and well-known) criterion: a compact set Γ is the image of a rectifiable curve if and only if it is connected and $\mathcal{H}^1(\Gamma) < \infty$. Indeed, one may choose γ so that length$(\gamma) \leq C\mathcal{H}^1(\Gamma)$; see, for example, [1, 2]. Here and below, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

The purpose of this note is to prove

Theorem 1.1. *Let $d \geq 2$. There exists a doubling measure ν on $\mathbb{R}^d$ and a rectifiable curve Γ such that $\nu(\Gamma) > 0$.*

We note that doubling measures cannot charge even slightly more regular curves; indeed the authors' initial belief was that a rectifiable curve could not carry any weight. As discussed in [4, §I.8.6] doubling measures give zero weight to any smooth hyper-surface. The argument, based on Lebesgue's density theorem (for ν), adapts without difficulty to show that for any connected set Γ,

$$\nu\Big(\{x \in \Gamma : \liminf_{r \to 0} r^{-1}\mathcal{H}^1(B(x, r) \cap \Gamma) < \infty\}\Big) = 0.$$

Therefore if Γ is a rectifiable curve, then $\nu|_\Gamma$ must be singular to $\mathcal{H}^1|_\Gamma$. Similarly, no doubling measure can charge an Ahlfors regular curve.

We will prove Theorem 1.1 by explicitly constructing a measure and a rectifiable curve.

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2. Proof

2.1. **The Measure.** Our measure ν will be the d-fold product of a doubling measure μ on $\mathbb{R}$. The latter is constructed by a simple iterative procedure that we will now describe. It may be viewed as a variant of the classic Riesz product construction and a 'lift the middle' idea of Kahane (cf. [3]). A very general form of this construction appears in [6].

Let $h\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic function

$$h(x) = \begin{cases} 2 & : x \in [\frac{1}{3}, \frac{2}{3}) + \mathbb{Z} \\ -1 & : \text{otherwise.} \end{cases}$$

Then given $\delta \in (0, \frac{1}{3}]$, we define μ as the weak-$*$ limit of

$$d\mu_n := \prod_{j=0}^{n-1} \big[1 + (1-3\delta)h(3^j x)\big]\ dx$$

When $\delta = 1/3$, μ is Lebesgue measure.

By viewing points $x \in \mathbb{R}$ in terms of their ternary (i.e., base 3) expansion, we may interpret μ as the result of a sequence of independent trials. More precisely, let $\mathcal{T}_n$ denote the collection of triadic intervals of size 3^{-n}, that is,

$$\mathcal{T}_n = \big\{ \big[i3^{-n}, (i+1)3^{-n}\big) : i \in \mathbb{Z} \big\}. \tag{2.1}$$

Then the measure of a triadic interval $I = [i3^{-n}, (i+1)3^{-n})$ is related to that of its parent $\hat{I}$, the unique interval in $\mathcal{T}_{n-1}$ containing I, by

$$\mu(I) = \begin{cases} (1-2\delta)\mu(\hat{I}) & : i \equiv 1 \mod 3 \\ \delta\mu(\hat{I}) & : \text{otherwise.} \end{cases} \tag{2.2}$$

Coupled with the fact that $\mu([i, i+1)) = 1$ for $i \in \mathbb{Z}$, condition (2.2) uniquely determines μ. In particular, we note that if j, $n \geq 0$, and $0 \leq i < 3^n$ are integers, then

$$\mu\big([j+i3^{-n}, j+(i+1)3^{-n})\big) = \delta^{n-k(i)}(1-2\delta)^{k(i)} \tag{2.3}$$

where $k(i)$ is the number of times the digit 1 appears in the ternary expansion of i.

We claim that μ is a doubling measure on $\mathbb{R}$. First let I and J be adjacent triadic intervals of equal size. By (2.3) we have that $\mu(I)/\mu(J) \leq \frac{1-2\delta}{\delta}$. Several applications of this shows that $\mu(I)/\mu(J) \leq C(\delta)$ for any pair I and J of adjacent intervals of equal size. Thus μ is doubling.

Let ν be the product measure $\mu \times \cdots \times \mu$ on $\mathbb{R}^d$. This is a doubling measure: $\nu(I_1 \times ... \times I_d) \leq C(\delta)^d\, \nu(J_1 \times ... \times J_d)$ for any d pairs of identical or adjacent intervals I_l, J_l that obey $|I_l| = |J_l|$. Indeed, this holds even without the requirement that $|I_l| = |I_{l'}|$ for $l \neq l'$.

2.2. **The Basic Building Blocks.**

Definition 2.1. Given integer parameters $0 \leq k \leq n$, we define $K(n,k) \subset [0,1)$ via

$$K(n,k) = \cup\big\{I \in \mathcal{T}_n : I \subseteq [0,1) \text{ and } \mu(I) \geq \delta^k(1-2\delta)^{n-k}\big\}. \tag{2.4}$$

Equivalently, if $\delta < \frac{1}{3}$, $K(n,k)$ is the set of those $x \in [0,1)$ whose ternary expansion contains at most k zeros or twos amongst the first n digits.

Lemma 2.2. *For $2\delta n \leq k \leq \frac{2}{3}n$ and $K = K(n,k)$ defined as in (2.4), we have*

$$1 - \mu(K) \leq \exp\big\{-2n\big(\tfrac{k}{n} - 2\delta\big)^2\big\} \tag{2.5}$$

and

$$|K| \leq 3^{-n} e^{k[1+\log(\delta^{-1})]} \tag{2.6}$$

Proof. Both inequalities rest on standard estimates for tail probabilities for the binomial distribution. These are proved by the usual large deviation technique of Cramér (cf. [5, Theorem 1.3.13]):

$$\begin{aligned}\sum_{m\geq an}^{n} \tbinom{n}{m} p^m (1-p)^{n-m} &\leq \inf_{t\geq 0} \sum_{m=0}^{n} \tbinom{n}{m} p^m (1-p)^{n-m} e^{(m-an)t} \\ &= \inf_{t\geq 0} \big[e^{-at}(1-p+p\,e^t)\big]^n\end{aligned}$$

This infimum can be determined exactly and for $0 < p \leq a < 1$ we obtain

$$\sum_{m\geq an}^{n} \tbinom{n}{m} p^m (1-p)^{n-m} \leq e^{-nH(a,p)}$$

where

$$H(a,p) = a\log\big(\tfrac{a}{p}\big) + (1-a)\log\big(\tfrac{1-a}{1-p}\big).$$

For (2.5) we set $a = k/n$ and $p = 2\delta$ and make use of the fact that

$$H(a, p) \geq 2(a - p)^2.$$

Indeed, H and $\partial_a H$ vanish at $a = p$, while $\partial_a^2 H = a^{-1}(1-a)^{-1} \geq 4$.

To obtain (2.6), we set $p = \frac{1}{3}$ and $a = \frac{n-k}{n}$. We simplified the answer by using

$$H(a, p) \geq \log\big(\tfrac{1}{p}\big) - (1-a)\Big[\log\big(\tfrac{1-p}{p}\big) + 1 + \log\big(\tfrac{1}{1-a}\big)\Big],$$

which amounts simply to $a\log(a) + 1 - a = -\int_a^1 \log(t)\,dt \geq 0$. □

Remark 2.3. Choosing $\delta < 2/9$ and $k = 3\delta n$ and sending $n \to \infty$, we see by Lemma 2.2 that μ gives all its weight to a set of Hausdorff dimension $O(\delta\log(\delta^{-1}))$. The precise dimension of μ is not important to us; however, we will exploit the fact that it can be made as small as we wish by sending $\delta \downarrow 0$. Indeed, the product measure ν cannot charge a set of Hausdorff dimension one (not to mention a rectifiable curve) unless μ gives positive weight to a set of dimension d^{-1} or smaller.

By definition, $K(n, k)$ is a union of intervals from $\mathcal{T}_n$. Correspondingly, the d-fold Cartesian product $K(n, k)^d$ can be viewed as a union of triadic cubes $Q \subseteq \mathbb{R}^d$ (with side-length 3^{-n}). We denote this collection of cubes by $\mathcal{K}^d(n, k)$. By (2.6),

$$\#\mathcal{K}^d(n, k) \leq e^{kd[1+\log(\delta^{-1})]} \tag{2.7}$$

Similarly, we write $\mathcal{G}(n, k)$ for the gaps in $K(n, k)$, that is, the bounded connected components of $\mathbb{R} \setminus K(n, k)$. As each gap has a right end-point, (2.6) gives

$$\#\mathcal{G}(n, k) \leq e^{k[1+\log(\delta^{-1})]}. \tag{2.8}$$

Note also that $|\cup\,\mathcal{G}(n, k)| \leq 1$, as $K(n, k) \subseteq [0, 1)$.

We now define a curve $\Gamma(n, k) \subset \mathbb{R}^d$ which visits each cube $Q \in \mathcal{K}^d(n, k)$. Actually, we merely construct a connected family of line segments $\Gamma(n, k)$ that do this, and bound its total length. As noted in the introduction, all segments in $\Gamma(n, k)$ can be traversed by a single curve of comparable total length.

The family $\Gamma(n, k)$ is the union of skeletons of rectangular boxes, where we define the *skeleton* of a box is

$$\mathrm{Sk}(I_1 \times \cdots \times I_d) = \bigcup_{j=1}^{d} \partial I_1 \times \cdots \times \partial I_{j-1} \times I_j \times \partial I_{j+1} \times \cdots \times \partial I_d.$$

Thus $\mathrm{Sk}(Q)$ is the union of the edges — as opposed to vertices, faces, 3-faces, etc. — of the box Q. With this notation,

$$\Gamma(n, k) = \bigcup_{Q \in \mathcal{K}^d(n,k)} \mathrm{Sk}(Q) \quad \cup \bigcup_{I_1,\ldots,I_d \in \mathcal{G}(n,k)} \mathrm{Sk}(I_1 \times \cdots \times I_d).$$

Note that $\Gamma(n, k)$ is connected. We now estimate the total length of this set.

Lemma 2.4 (The length of the $\Gamma(n,k)$)**.** *Assuming* $2\delta n \le k \le \frac{2}{3}n$*,*

$$\mathcal{H}^1(\Gamma(n,k)) \le d2^d e^{dk[1+\log(\delta^{-1})]}. \tag{2.9}$$

Proof. By (2.7) and (2.8),

$$\begin{aligned}\mathcal{H}^1(\Gamma(n,k)) &\le \sum_{Q\in\mathcal{K}^d(n,k)} \mathcal{H}^1\big(\mathrm{Sk}(Q)\big) + \sum_{I_1,\dots,I_d\in\mathcal{G}(n,k)} \mathcal{H}^1\big(\mathrm{Sk}(I_1\times\cdots\times I_d)\big)\\ &= d\,2^{d-1}3^{-n}\big[\#\,\mathcal{K}^d(n,k)\big] + d\,2^{d-1}\big[\#\,\mathcal{G}(n,k)\big]^{d-1}\sum_{I\in\mathcal{G}(n,k)}|I|\\ &\le d\,2^{d-1}3^{-n}e^{kd[1+\log(\delta^{-1})]} + d\,2^{d-1}e^{k(d-1)[1+\log(\delta^{-1})]}\end{aligned}$$

which easily yields (2.9). □

2.3. **The Curve.** Using $\Gamma(n,k)$ as a building-block, we now explain the iterative construction of the full curve Γ. It depends upon a collection of parameters $\{n_j,k_j\}_{j=1}^{\infty}$. The guiding principle is to replace each cube in $\mathcal{K}^d(n_j,k_j)$ by rescaled/translated copies of $\mathcal{K}^d(n_{j+1},k_{j+1})$ and $\Gamma(n_{j+1},k_{j+1})$.

To this end, we define a version $\Gamma_Q(n,k)$ of $\Gamma(n,k)$ adapted to any cube Q:

$$\Gamma_Q(n,k) = A_Q\big(\Gamma(n,k)\big)$$

where A_Q is the affine transformation that maps $[0,1)^d$ to Q. Similarly, we inductively define

$$\mathcal{K}_0 = \big\{[0,1)^d\big\} \quad\text{and}\quad \mathcal{K}_l = \bigcup_{Q\in\mathcal{K}_{l-1}} \big\{A_Q(Q') : Q'\in\mathcal{K}^d(n_l,k_l)\big\} \quad \text{for } l\ge 1.$$

Thus $\mathcal{K}_l$ is the collection of cubes remaining after the l^{th} iteration in the construction of Γ. Subsequent iterations will not modify Γ outside their union,

$$K_l = \cup\{Q : Q\in\mathcal{K}_l\}.$$

We note that the cubes in $\mathcal{K}_l$ have disjoint interiors, and that by (2.7),

$$\begin{aligned}\#\mathcal{K}_l &\le [\#\mathcal{K}_{l-1}]\exp\big\{k_l d[1+\log(\delta^{-1})]\big\}\\ &\le \exp\big\{(k_1+\cdots+k_l)d[1+\log(\delta^{-1})]\big\}\end{aligned} \tag{2.10}$$

We define

$$\Gamma = \mathrm{Sk}\big([0,1)^d\big) \;\cup\; \bigcup_{l=1}^{\infty}\;\bigcup_{Q\in\mathcal{K}_{l-1}} \Gamma_Q(n_l,k_l) \;\cup\; \bigcap_{l=1}^{\infty} K_l. \tag{2.11}$$

Note that Γ is connected. The proof of Theorem 1.1 now reduces to the following two propositions, which show that $\mathcal{H}^1(\Gamma)<\infty$ and $\nu(\Gamma)>0$ for a certain explicit choice of parameters.

Proposition 2.5 (The length of Γ)**.** *Let $\delta > 0$ and $n_1 \in \mathbb{Z}$ be parameters so that*

$$18d\,[\delta + \delta\log(\delta^{-1})] \le \log(3) \tag{2.12}$$

and $k_1 := 3\delta n_1 \ge 1$ is an integer. If Γ is the curve defined above with parameters $n_l = ln_1$ and $k_l = lk_1$, then

$$\mathcal{H}^1(\Gamma) \le 3d\,2^d e^{3dn_1[\delta+\delta\log(\delta^{-1})]} \tag{2.13}$$

Proof. By (2.7) we have

$$\mathcal{H}^1\Big(\bigcap_{l=1}^{\infty} K_l\Big) \le \prod_{l=1}^{\infty}\big(3^{-n_l}e^{dk_l(1+\log(\delta^{-1}))}\big) = 0.$$

Hence by (2.9) and (2.10),

$$\begin{aligned}\mathcal{H}^1(\Gamma) &\le d\,2^{d-1} + \sum_{l=1}^{\infty}\sum_{Q\in\mathcal{K}_{l-1}} d\,2^d 3^{-(n_1+\cdots+n_{l-1})}\exp\big\{dk_l[1+\log(\delta^{-1})]\big\}\\ &\le d\,2^d\bigg[1+\sum_{l=1}^{\infty}3^{-(n_1+\cdots+n_{l-1})}\exp\big\{(k_1+\cdots+k_l)d\,[1+\log(\delta^{-1})]\big\}\bigg].\end{aligned}$$

Inserting the values of our parameters and performing a few elementary manipulations, we find

$$\mathcal{H}^1(\Gamma_0) \le d\,2^{d-1}e^{3dn_1[\delta+\delta\log(\delta^{-1})]}\bigg[2+\sum_{l=2}^{\infty}\exp\big\{-\tfrac{1}{4}l(l-1)\log(3)n_1\big\}\bigg]$$

which yields (2.13) with a few more manipulations. □

Proposition 2.6 (The measure of Γ)**.** *Let δ and $\{n_l, k_l\}_{l=1}^{\infty}$ be as in Proposition 2.5. Then*

$$\nu(\Gamma) \ge \exp\bigg\{-\frac{d\,e^{-2\delta^2 n_1}}{(1-e^{-2\delta^2 n_1})^2}\bigg\}. \tag{2.14}$$

Proof. By the dominated convergence theorem,

$$\nu(\Gamma) \ge \lim_{l\to\infty}\nu\big(\overline{\cup\{Q : Q\in\mathcal{K}_l\}}\big) \ge \lim_{l\to\infty}\nu\big(\cup\{Q : Q\in\mathcal{K}_l\}\big).$$

(In fact, since doubling measures cannot charge straight lines, equality actually holds above, but we will not need this.) Since the cubes in $\mathcal{K}_l$ have disjoint interiors, (2.5) and induction give us

$$\nu\big(\bigcup\{Q : Q\in\mathcal{K}_l\}\big) \ge \big[1-e^{-2\delta^2 n_l}\big]^d\sum_{Q\in\mathcal{K}_{l-1}}\nu(Q) \ge \prod_{j=1}^{l}\big[1-e^{-2\delta^2 n_j}\big]^d.$$

Inserting the values of our parameters and performing a few elementary manipulations, we conclude that

$$\nu(\Gamma) \geq \exp\left\{ d \sum_{j=1}^{\infty} \log(1 - Z^j) \right\} \geq \exp\left\{ -d \sum_{j,k=1}^{\infty} Z^{jk} \right\} \geq \exp\left\{ -\tfrac{dZ}{(1-Z)^2} \right\}$$

where $Z := \exp\{-2\delta^2 n_1\}$. That proves (2.14). □

In closing, we note that the curve Γ can be made to capture an arbitrarily large proportion of the ν-mass of the unit cube; one merely chooses the parameter n_1 large (with δ fixed).

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