On Strategy-proofness and Symmetric Single-peakedness*

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Abstract: We characterize the class of strategy-proof social choice functions on the domain of symmetric single-peaked preferences. This class is strictly larger than the set of generalized median voter schemes (the class of strategy-proof and tops-only social choice functions on the domain of single-peaked preferences characterized by Moulin (1980)) since, under the domain of symmetric single-peaked preferences, generalized median voter schemes can be disturbed by discontinuity points and remain strategy-proof on the smaller domain. Our result identifies the specific nature of these discontinuities which allow to design non-onto social choice functions to deal with feasibility constraints.

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Keywords: Strategy-proofness, Single-peaked Preferences, Median Voter, Feasibility Constraints.

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1 Introduction

The aim of this paper is to identify the class of strategy-proof social choice functions on the domain of symmetric single-peaked preferences. The characterization of this class allows us to design social choice functions that are strategy-proof in cases in which there are feasibility constraints; *i.e.*, when the set of possible alternatives is not convex. Although we think that the restriction to the domain of symmetric single-peaked preferences is interesting in its own right, the ability of designing strategy-proof social choice functions under feasibility constraints is certainly relevant in many applications.

Consider a society with n agents who have to collectively choose one alternative from a given set of social alternatives. Assume that this set is endowed with a natural strict order because alternatives have a common characteristic that makes the comparison between pairs of alternatives meaningful and objective. For instance, the set of alternatives may consist of physical locations (a public facility on a road or street), properties of a political project in terms of its left-right characteristics, the expenditure level on a public good, indexes reflecting the quality of a product, feasible temperatures in a room, and so on.[1] In all these cases, and in many others, this linear order structure permits to identify the set of alternatives with a subset of the real line. Agents have (potentially different) preferences on the set of alternatives. Black (1948) is the first to suggest that, given the linear order on the set of alternatives, agents' preferences ought to be single-peaked. The preference of an agent is *single-peaked* if there exists an alternative (called the top) which is strictly preferred to any other alternative and on each side of the top the preference is strictly monotonic, increasing on its left and decreasing on its right.[2]

Society would like to select an alternative according to agents' preferences. But since they constitute private information, agents have to be asked about them. A social choice function on a domain of preferences requires each agent to report a preference and associates an alternative with the reported preference profile. Hence, a social choice function on a Cartesian product domain induces an (ordinal) direct revelation game where each agent's set of strategies is his set of possible preferences. A social choice

[1] There is an extensive literature studying collective choice problems where the set of social alternatives is a linearly ordered set. See Moulin (1980), for instance. This class of problems also plays a fundamental role in Sprumont (1995) and Barberà (2001), two excellent surveys on strategy-proofness.

[2] The set of single-peaked preferences is extremely large and rich; for instance, for each alternative there are many single-peaked preferences that have as top this alternative. Moreover, no *a priori* restriction is imposed on how pairs of alternatives lying in different sides of the top are ordered. Ballester and Haeringer (2007) identify two properties that are both necessary and sufficient to characterize the domain of single-peaked preference profiles.

function is *strategy-proof* if no agent has never incentives to strategically misrepresent his preference; in other words, truth-telling is a (weakly) dominant strategy in the direct revelation game induced by the social choice function.

Moulin (1980) characterizes the class of strategy-proof and tops-only social choice functions on the domain of single-peaked preferences as the set of generalized median voter schemes.[3] A generalized median voter scheme is, in general, a non-anonymous extension of the median voter. It can be interpreted as a particular way of distributing the power to influence the social outcome among all coalitions of agents. In addition, Moulin (1980) also identifies the two nested subclasses of strategy-proof, tops-only and anonymous social choice functions, and strategy-proof, tops-only, anonymous and efficient social choice functions.[4] All the functions in these characterizations have convex range meaning that the set of implementable alternatives is convex. This implies that if some alternatives were banned or infeasible, either the social choice function would have to request from the agents more information than just their top, or there would be a single-peaked preference profile for which truthtelling is strictly dominated by some agent.

In many applications however, the domain of preferences can be restricted even further because the linear order structure of the set of alternatives conveys to agents' preferences more than just an ordinal content. Often, an agent's preference on the set of alternatives is responsive also to the notion of distance, embedding to the preference its corresponding property of symmetry. A single-peaked preference is *symmetric* if the following additional condition holds: an alternative is strictly preferred to another one if and only if the former is strictly closer to the top. If an indifference class contains two alternatives then both are located in opposite sides of the top and are at the same distance of the top.

To restrict further the domain of a social choice function is equivalent to shrink the set of agents' strategies in its induced direct revelation game. Thus, strategies that were dominant remain dominant while strategies that were not dominant in the larger domain may become dominant after the domain reduction. Therefore, two important facts hold. First, any strategy-proof social choice function on a domain remains strategy-proof on all of its subdomains. Second, a manipulable social choice function on a domain

[3]A social choice function is *tops-only* if the chosen alternative only depends on the profile of tops. Tops-only social choice functions are especially simple in terms of the amount of information they require about individual preferences.

[4]A social choice function is *anonymous* if it is independent of the identities of the agents; it is *efficient* if it always selects a Pareto optimal alternative.

may become strategy-proof in a smaller subdomain.[5] Hence, we ask whether the set of strategy-proof and tops-only social choice functions on the domain of single-peaked preferences, identified by Moulin (1980) as the class of generalized median voter schemes, becomes larger when the domain of preferences where we want the social choice functions to operate is the subdomain of symmetric single-peaked preferences. We answer this question affirmatively by completely identifying the larger class of functions that emerge after restricting further the domain.

The new class of social choice functions can be described as generalized median voter schemes disturbed by discontinuity jumps. A social choice function f in the class coincides with a generalized median voter scheme except that at some (countable number of) discontinuity jumps (for instance, an interval (a, b) with middle point d), instead of taking the value prescribed by the generalized median voter scheme, f takes the constant value a at $[a, d)$, either the value a or b at d, and the constant value b at $(d, b]$. Our description of the class makes precise that the choice of either a or b at any of those profiles where the generalized median voter scheme would choose d must be monotonic in order to preserve strategy-proofness of the social choice function.

We want to stress the importance for applications of admitting discontinuous social choice functions that are non-onto because they have a disconnected range. Non-onto social choice functions are indispensable for the design of social choice functions that require that some subsets of alternatives are never chosen due to feasibility constraints. For instance when the range of the function has to be finite, or not all locations for a public facility are possible, or the set of indexes reflecting the quality of a product must be disconnected, or the thermostat controlling for the temperature in a room can not take all values, and so on. In all these cases, and in many others, discontinuities can not be regarded as pathological features of social choice functions but rather as indispensable requirements to deal with constraints on the set of feasible alternatives to be chosen.[6]

There is a large literature studying strategy-proofness on domains related to single-peakedness. Our result and its proof are closely related to the following papers. Theorem

[5]Observe that this is just a possibility. For instance, for the case where the set of social alternatives is the family of all subsets of a given set of candidates Barberà, Sonnenschein, and Zhou (1991) show that voting by committees is the class of strategy-proof and onto social choice functions on both, the domain of separable preferences as well as on the subdomain of additive preferences, although the set of additive preferences is strictly smaller than the set of separable preferences. No new strategy-proof social choice function appears after the domain reduction in this case.

[6]Barberà, Massó, and Neme (1997 and 2005) and Barberà, Massó, and Serizawa (1998) identify subclasses of strategy-proof social choice functions that are able to deal with constrained sets of alternatives in different environments.

1 partly retains the structure of Moulin (1980)'s characterization of strategy-proof and tops-only social choice functions under the single-peaked domain of preferences. Our result in Theorem 1 says that social choice functions that are strategy-proof on the symmetric single-peaked domain but they were manipulable in the larger single-peaked domain consists of generalized median voter schemes that are perturbed by specific discontinuities. Our result is also related to Theorem 3 in Barberà and Jackson (1994) characterizing *all* strategy-proof social choice functions on the domain of single-peaked preferences. Their characterization includes social choice functions whose range is not convex; however, the characterization is open because it relays on a family of *strategy-proof* tie-breaking rules (used to select between the two extremes of the discontinuity jumps). Our characterization is closed because it explicitly describes the exact family of admissible tie-breaking rules needed to preserve strategy-proofness. Yet, we are able to provide this closed description because our domain contains only symmetric preferences. The proof of our result relays at some point on Berga and Serizawa (2000)'s characterization of all strategy-proof and onto social choice functions on a minimally rich domain as the class of generalized median voter schemes;[7] we use their result in the easier case when the given strategy-proof social choice function is continuous. In addition, our proof is substantially simpler than it would have been if we were not able to use Barberà, Berga, and Moreno (2009) result identifying conditions of preference domains under which (individual) strategy-proofness is equivalent to group strategy-proofness. Their result allows us to avoid many steps of individual changes of preferences by instead moving simultaneously the preferences of all members of a given coalition.

The paper is organized as follows. In Section 2 we present the preliminary notation and the most basic definitions. In Section 3 we state some preliminary results and give the main definitions and intuitions in order to understand why and how the class of generalized median voter schemes has to be enlarged in order to identify the full class of strategy-proof social choice functions on the domain of symmetric single-peaked preferences. In Section 4 we state and prove our main result characterizing the complete class of strategy-proof social choice functions on the domain of symmetric single-peaked preferences (Theorem 1). After presenting some preliminaries of the proof in Subsection 4.2, we prove Theorem 1 in Subsection 4.3. In Section 5 we first state as corollaries of Theorem 1 the corresponding characterizations under strategy-

[7] A domain is *minimally rich* if (i) it is a subset of the single-peaked domain, (ii) for each alternative x there is a preference relation in the domain with top at x, and (iii) for any pair of alternatives x and y ($x \neq y$) there is a preference in the domain that strictly orders x and y and whose top lies between x and y. Obviously, the set of symmetric single-peaked preferences is a minimally rich domain.

proofness and anonymity (Corollary 1) and under strategy-proofness, anonymity and efficiency (Corollary 2). We then argue about the importance for applications of allowing for non-onto social choice functions which were ruled out by the combination of strategy-proofness and tops-onlyness in Moulin (1980)'s characterization under single-peaked preferences and state Corollary 3 characterizing all strategy-proof social choice functions that are efficient relative to a given closed set of feasible alternatives. We finish with the remark that, as the consequence of the main result in Barberà, Berga, and Moreno (2009), the four statements hold if we replace in them strategy-proofness by group strategy-proofness.

2 Preliminary notation and definitions

Let $N = \{1, ..., n\}$ be the set of *agents* of a society that has to choose an *alternative* x from the interval $[0, 1]$.[8] The *preference* of each agent $i \in N$ on the set of alternatives $[0, 1]$ is a complete, reflexive, and transitive binary relation (a complete preorder) R_i on $[0, 1]$. Let $\mathcal{R}$ be the set of complete preorders on $[0, 1]$. A *preference profile* $R = (R_1, ..., R_n) \in \mathcal{R}^N$ is a n-tuple of preferences. To emphasize the role of agent i or subset of agents T, a preference profile R will be represented by (R_i, R_{-i}) or (R_T, R_{-T}), respectively. As usual, let P_i and I_i denote the strict and indifference preference relations induced by R_i, respectively. Given $R_i \in \mathcal{R}$, the *top of* R_i (if any) is the unique alternative $t(R_i)$ that is strictly preferred to any other alternative; *i.e.*, $t(R_i)P_ix$ for all $x \in [0, 1]\backslash\{t(R_i)\}$.

Given a subset of preferences $\mathcal{S} \subseteq \mathcal{R}$, a *social choice function* f on $\mathcal{S}$ is a function $f : \mathcal{S}^N \to [0, 1]$ selecting an alternative for each preference profile in $\mathcal{S}^N$. We will refer to this Cartesian product set $\mathcal{S}^N$ (or to the set $\mathcal{S}$ itself) as a domain of preferences. Given a social choice function $f : \mathcal{S}^N \to [0, 1]$, denote its range by r_f; *i.e.*, $r_f = \{x \in [0, 1] \mid$there exists $R \in \mathcal{S}^N$ such that $f(R) = x\}$.

We will be interested in social choice functions that induce truth-telling as a (weakly) dominant strategy in their associated (ordinal) direct revelation game.

Definition 1 A social choice function $f : \mathcal{S}^N \to [0, 1]$ is *strategy-proof* if for all $R \in \mathcal{S}^N$, all $i \in N$, and all $R'_i \in \mathcal{S}$,

$$f(R_i, R_{-i})R_if(R'_i, R_{-i}).$$

[8]Our results also hold for any linearly ordered metric space of alternatives. In particular, for any set of alternatives which is a closed interval of real numbers (as well as for the set $\mathbb{R} \cup \{-\infty, +\infty\}$).

If $f(R_i', R_{-i})P_i f(R)$ we say that i *manipulates* f at R via R_i'. A social choice function $f : \mathcal{S}^N \to [0,1]$ is *group strategy-proof* if for all $R \in \mathcal{S}^N$, all $T \subseteq N$, all $R_T' \in \mathcal{S}^T$, and all $i \in T$,

$$f(R_T, R_{-T})R_i f(R_T', R_{-T}).$$

If $f(R_T', R_{-T})P_i f(R)$ for all $i \in T$ we say that T *manipulates* f at R via R_T'.

We will also consider other properties of social choice functions. A social choice function $f : \mathcal{S}^N \to [0,1]$ is *anonymous* if it is invariant with respect to the agents' names; namely, for all one-to-one mappings $\sigma : N \to N$ and all $R \in \mathcal{S}^N$, $f(R_1, ..., R_n) = f(R_{\sigma(1)}, ..., R_{\sigma(n)})$. A social choice function $f : \mathcal{S}^N \to [0,1]$ is *dictatorial* if there exists $i \in N$ such that for all $R \in \mathcal{S}^N$, $f(R)R_i x$ for all $x \in r_f$. A social choice function $f : \mathcal{S}^N \to [0,1]$ is *efficient* if for all $R \in \mathcal{S}^N$, there is no $z \in [0,1]$ such that, for all $i \in N$, $zR_i f(R)$ and $zP_j f(R)$ for some $j \in N$. A social choice function $f : \mathcal{S}^N \to [0,1]$ is *unanimous* if for all $R \in \mathcal{S}^N$ such that $t(R_i) = x$ for all $i \in N$, $f(R) = x$. A social choice function $f : \mathcal{S}^N \to [0,1]$ is *onto* if for all $x \in [0,1]$ there is $R \in \mathcal{S}^N$ such that $f(R) = x$ (*i.e.*, $r_f = [0,1]$). A social choice function $f : \mathcal{S}^N \to [0,1]$ is *tops-only* if for all $R, R' \in \mathcal{S}^N$ such that $t(R_i) = t(R_i')$ for all $i \in N$, $f(R) = f(R')$. Let $\widehat{\mathcal{S}} \subseteq \mathcal{R}$ be any subset of preferences with the property that for each $x \in [0,1]$ there exists at least a preference $R_i \in \widehat{\mathcal{S}}$ such that $t(R_i) = x$. Then, $\widehat{\mathcal{S}}^N$ is called a *rich domain* (note that all minimally rich domains are rich) and with some abuse of notation, given a tops-only social choice function $f : \widehat{\mathcal{S}}^N \to [0,1]$ we will refer to it by its corresponding voting scheme $f : [0,1]^N \to [0,1]$.

The Gibbard-Satterthwaite Theorem states that a social choice function $f : \mathcal{R}^N \to [0,1]$, with $\#r_f \neq 2$, is strategy-proof if and only if it is dictatorial.[9] An implicit assumption of the Gibbard-Satterthwaite Theorem is that the domain of the social choice function is universal: the social choice function operates on all preference profiles, because all of them are reasonable. However, for many applications, a linear order structure on the set of alternatives naturally induces a domain restriction in which there always exists a top, and at each of the sides of the top the preference is strictly monotonic.

Definition 2 A preference $R_i \in \mathcal{R}$ is *single-peaked* if:

[9] See Gibbard (1973) and Satterthwaite (1975). Of course, the social choice function f that consists of preselecting two different alternatives $x, y \in [0,1]$ and deciding between them by majority voting (*i.e.*, for all $R \in \mathcal{R}^N$, $f(R) = x$ if and only if $\#\{i \in N \mid xR_i y\} \geq \#\{i \in N \mid yP_i x\}$) is strategy-proof but not dictatorial. Observe that the range of f is equal to two. Constant social choice functions (with only one alternative in the range) are also covered by the Gibbard-Satterthwaite Theorem because they are trivially strategy-proof and dictatorial (note that our notion of dictator is relative to the range of the social choice function).

(1) there exists the top $t(R_i)$ of R_i, and

(2) for all $x, y \in [0,1]$ such that $y < x \leq t(R_i)$ or $t(R_i) \leq x < y$, xP_iy.

Let $\mathcal{SP}$ be the set of single-peaked preferences on $[0,1]$. Observe that, given a single-peaked preference $R_i \in \mathcal{SP}$, yP_ix may hold even if $|t(R_i) - x| < |t(R_i) - y|$; but then, x and y are necessarily located in different sides of the top $t(R_i)$. Often, the linear order structure of the set of alternatives and a distance conveys to the preference a symmetric property around the top (coming for instance, from a location interpretation of the set of alternatives) that naturally induces the restriction that preferences respond to the distance as follows.

Definition 3 A preference $R_i \in \mathcal{R}$ is *symmetric single-peaked* if:

(1) there exists the top $t(R_i)$ of R_i, and

(2) for all $x, y \in [0,1]$, xP_iy if and only if $|t(R_i) - x| < |t(R_i) - y|$.

Obviously, a symmetric single-peaked preference is single-peaked. Let $\mathcal{SSP}$ be the set of symmetric single-peaked preferences on $[0,1]$. Given any alternative $x \in [0,1]$, there is a *unique* symmetric single-peaked preference R_i with its top $t(R_i) = x$ ($\mathcal{SSP}$ is a rich domain). Hence, there is a one-to-one mapping between the set of symmetric single-peaked preferences $\mathcal{SSP}$ and the set of alternatives $[0,1]$. Thus, we will use $t_i \in [0,1]$ to identify the (unique) $R_i \in \mathcal{SSP}$ such that $t(R_i) = t_i$ and $t = (t_1, ..., t_n)$ to denote the corresponding symmetric single-peaked preference profile $R = (R_1, ..., R_n)$ such that $t(R_i) = t_i$ for all $i \in N$. Note that, by this one-to-one identification, any social choice function $f : \mathcal{SSP}^N \to [0,1]$ is tops-only. Observe that $\mathcal{SSP}^N$ is also a minimally rich domain. Thus, we will also denote a social choice function $f : \mathcal{SSP}^N \to [0,1]$ by its corresponding voting scheme $f : [0,1]^N \to [0,1]$.

3 Preliminary results and main intuition

3.1 Preliminary results

Moulin (1980) characterizes the family of strategy-proof and tops-only social choice functions on the domain of single-peaked preferences as well as its anonymous subfamily.[10] The two characterizations are useful to develop helpful intuitions to understand our

[10] Moulin (1980) also characterizes the subfamily of strategy-proof, tops-only, anonymous and efficient social choice functions on the domain of single-peaked preferences. See Corollary 2 in Section 5 for the characterization of the same class of social choice functions on the domain of symmetric single-peaked preferences.

characterization of strategy-proof social choice functions (and its anonymous subfamily) on the domain of symmetric single-peaked preferences. To state them, we need to define the median of an odd set of numbers. Given a set of odd real numbers $\{x_1, ..., x_K\}$, define its median as $med\{x_1, ..., x_K\} = y$, where y is such that $\#\{1 \leq k \leq K \mid x_k \leq y\} \geq \frac{K}{2}$ and $\#\{1 \leq k \leq K \mid x_k \geq y\} \geq \frac{K}{2}$. Observe that since K is odd the median belongs to the set $\{x_1, ..., x_K\}$ and it is unique.

Proposition 1 (Moulin, 1980) *A social choice function $f : \mathcal{SP}^N \to [0, 1]$ is strategy-proof, tops-only and anonymous if and only if there exist $n + 1$ fixed ballots $0 \leq p_n \leq ... \leq p_0 \leq 0$ such that for all $R \in \mathcal{SP}^N$,*

$$f(R) = med\{t(R_1), ..., t(R_n), p_n, ..., p_0\}.$$

Proposition 2 (Moulin, 1980) *A social choice function $f : \mathcal{SP}^N \to [0, 1]$ is strategy-proof and tops-only if and only if there exists a monotonic family $\{p_S\}_{S \in 2^N}$ of fixed ballots, with $p_S \in [0, 1]$ for all $S \in 2^N$ and $p_Q \leq p_T$ if $T \subset Q$, such that for all $R \in \mathcal{SP}^N$,*

$$f(R) = \min_{S \in 2^N} \max_{i \in S} \{t(R_i), p_S\}.$$

The social choice functions identified in Propositions 1 and 2 are called *median voter schemes* and *generalized median voter schemes*, respectively. A simple way of interpreting them is as follows. Each generalized median voting scheme (and its associated family of monotonic fixed ballots) can be understood as a particular way of distributing the power among coalitions to influence the social choice. To see that, take an arbitrary coalition S and its fixed ballot p_S. Then, coalition S can make sure that, by all of its members reporting a top alternative below p_S, the social choice will be at most p_S, independently of the reported top alternatives of the members of the complementary coalition.[11] An alternative way of describing this distribution of power among coalitions is as follows. Fix a family of monotonic fixed ballots $\{p_S\}_{S \in 2^N}$ (*i.e.*, a generalized median voter scheme) and take a vector of tops $(t(R_1), ..., t(R_n))$. Start at the left extreme of the interval and push the outcome to the right until it reaches an alternative x for which the following two things happen simultaneously: (i) there exists a coalition of agents S such that all its members have reported a top alternative below or equal to x (*i.e.*, $t(R_i) \leq x$ for all $i \in S$) and (ii) the fixed ballot p_S associated to S is located also below x (*i.e.*, $p_S \leq x$). Median voter schemes are the anonymous subclass of generalized median

[11] See Barberà, Massó, and Neme (1997) for a similar interpretation for the case of a finite number of ordered alternatives.

voter schemes. Hence, the fixed ballots of any two coalitions with the same cardinality of any anonymous generalized median voter scheme are equal. From a monotonic family of fixed ballots $\{p_S\}_{S \in 2^N}$ associated to an anonymous generalized median voter scheme f we can identify the $n+1$ ballots $p_n \leq p_{n-1} \leq ... \leq p_0$ needed to describe f as a median voter scheme as follows: for each $0 \leq s \leq n$, $p_s = p_S$ for all $S \in 2^N$ such that $\#S = s$.

Moulin (1980) also shows that the class of group strategy-proof and tops-only social choice functions on the domain of single-peaked preferences coincides with the set of generalized median voter schemes. From the main result in Barberà, Berga, and Moreno (2009) we can conclude that any strategy-proof social choice function on the domain of symmetric single-peaked preferences is group strategy-proof as well. Since we will later use this fact we state it here as a remark.[12]

Remark 1 (Barberà, Berga, and Moreno, 2009) *Let $f : \mathcal{SSP}^N \to [0,1]$ be a strategy-proof social choice function. Then, $f : \mathcal{SSP}^N \to [0,1]$ is group strategy-proof.*

To see that in the statements of Propositions 1 and 2 tops-onlyness does not follow from strategy-proofness consider the social choice function $f : \mathcal{SP}^N \to [0,1]$ where for all $R \in \mathcal{SP}^N$,

$$f(R) = \begin{cases} 0 & \text{if } \#\{i \in N \mid 0R_i1\} \geq \#\{i \in N \mid 1P_i0\} \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Notice that f is strategy-proof and anonymous but it is not tops-only. It also violates efficiency, unanimity, and ontoness. In the last section of the paper we will describe how our characterization includes this class of rules on the domain of symmetric single-peaked preferences.

3.2 Main intuition and definitions

Consider Propositions 1 and 2 for the simplest case where $n = 1$.[13] Figure 1 depicts the voting scheme $f : [0,1] \to [0,1]$ of a strategy-proof and tops-only social choice function

[12] Barberà, Berga and Moreno (2009) gives sufficient conditions defining domains of preferences under which strategy-proofness is equivalent to group strategy-proofness. The domain of symmetric single-peaked preferences satisfies these sufficient conditions.

[13] When $n = 1$ anonymity does not have any bite. Indeed, we can uniquely identify the two fixed ballots of the propositions as $p_1 = p_{\{1\}}$ and $p_0 = p_\emptyset$.

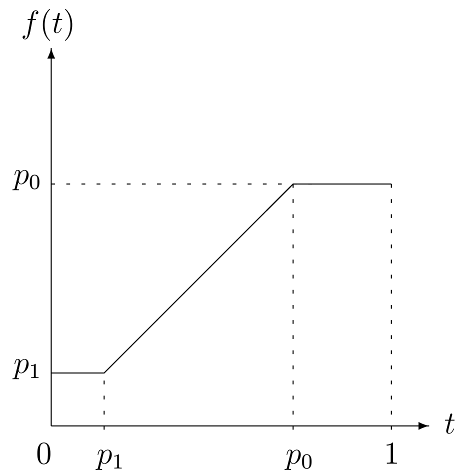

Figure 1

$f : \mathcal{SP} \to [0,1]$ with the two associated fixed ballots $0 < p_1 < p_0 < 1$. Observe that for any pair of fixed ballots $0 \leq p_1 \leq p_0 \leq 1$ the corresponding voting scheme $f : [0,1] \longrightarrow [0,1]$ is always increasing and continuous, and $r_f = [p_1, p_0]$. By Proposition 2 the following remark holds.

Remark 2 *Let $f : \mathcal{SP}^N \longrightarrow [0,1]$ be a strategy-proof and tops-only social choice function. Then, its corresponding voting scheme $f : [0,1]^N \longrightarrow [0,1]$ is increasing and continuous.*

Let $\mathcal{S}$ be any generic subset of $\mathcal{SP}$. A social choice function $f : \mathcal{S}^N \to [0,1]$ is increasing if $f(R) \leq f(R')$ for all $R, R' \in \mathcal{S}^N$ such that $t(R_i) \leq t(R_i')$ for all $i \in N$.

Lemma 1 below states that, for any $n \geq 1$, any strategy-proof social choice function is increasing on the domain of symmetric single-peaked preferences (observe that tops-only is not required explicitly since for each $x \in [0,1]$ there exists a unique $R_i \in \mathcal{SSP}$ such that $t(R_i) = x$).

Lemma 1 *Let $f : \mathcal{SSP}^N \to [0,1]$ be a strategy-proof social choice function. Then, f is increasing.*

Proof The statement follows from the iterated application of Claim A.

Claim A *Let $f : \mathcal{SSP}^N \to [0,1]$ be a strategy-proof social choice function. Let $t, t' \in \mathcal{SSP}^N$ be such that for some $i \in N$, $t_i < t_i'$ and $t_{-i} = t_{-i}'$. Then, $f(t) \leq f(t')$.*

Proof of Claim A Assume otherwise; that is, there exist $t, t' \in \mathcal{SSP}^N$ and $i \in N$ such that

$$t_i < t_i', \tag{2}$$

$t_{-i} = t'_{-i}$ and $f(t') < f(t)$. We distinguish among six possible cases. The first three cases (i) $f(t') < f(t) \leq t_i < t'_i$, (ii) $t_i \leq f(t') < f(t) \leq t'_i$, and (iii) $f(t') < t_i \leq f(t) \leq t'_i$ contradict strategy-proofness of f since in all three i manipulates f at t' via t_i. The two cases (iv) $t_i < t'_i \leq f(t') < f(t)$ and (v) $t_i \leq f(t') \leq t'_i \leq f(t)$ contradict strategy-proofness of f since in all two i manipulates f at t via t'_i. The remaining case is (vi) $f(t') \leq t_i < t'_i \leq f(t)$. Since $t_i, t'_i \in \mathcal{SSP}$ and f is strategy-proof,

$$\begin{aligned} f(t) - t_i &\leq t_i - f(t') \\ t'_i - f(t') &\leq f(t) - t'_i. \end{aligned}$$

Adding up,

$$\begin{aligned} f(t) - t_i + t'_i - f(t') &\leq t_i - f(t') + f(t) - t'_i \\ t'_i - t_i &\leq t_i - t'_i \\ t'_i &\leq t_i, \end{aligned}$$

a contradiction with (2). ■

We have shown that the monotonicity of strategy-proof social choice functions is preserved when we restrict the domain of single-peaked preferences to be symmetric. However, continuity (of its corresponding voting scheme) does not follow from strategy-proofness and tops-onlyness in this smaller domain. Indeed, a special class of discontinuities may arise. It is very easy to understand why when $n = 1$. First, take any $\tau, \delta \in (0, 1)$ such that $\delta \leq \min\{\tau, 1 - \tau\}$ and define the social choice functions $f^- : \mathcal{SSP} \longrightarrow [0, 1]$ and $f^+ : \mathcal{SSP} \longrightarrow [0, 1]$ where for each $t_i \in \mathcal{SSP}$,

$$f^-(t_i) = \begin{cases} \tau - \delta & \text{if } t_i \leq \tau \\ \tau + \delta & \text{if } \tau < t_i \end{cases}$$

and

$$f^+(t_i) = \begin{cases} \tau - \delta & \text{if } t_i < \tau \\ \tau + \delta & \text{if } \tau \leq t_i. \end{cases}$$

In Figure 2 we depict f^-. Both f^- and f^+ are strategy-proof on the domain of symmetric single-peaked preferences. At any $t_i \in \mathcal{SSP}$ such that either $t_i > \tau$ or $t_i < \tau$ agent i can not manipulate them. Let $t_i \in \mathcal{SSP}$ be such that $t_i = \tau$. Then, $(\tau - \delta) I_i (\tau + \delta)$ since $(\tau - \delta)$ and $(\tau + \delta)$ are at the same distance δ to τ.

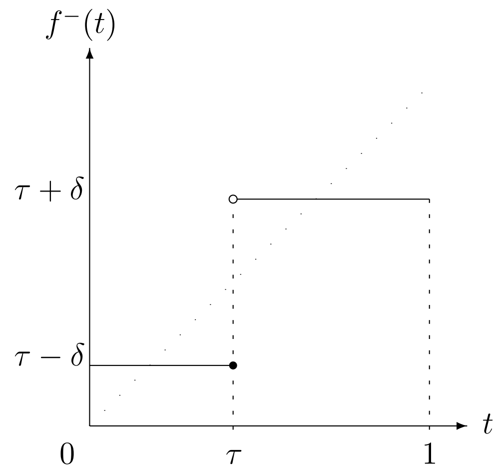

Figure 2

The function $f^-: \mathcal{SSP} \longrightarrow [0,1]$ is left-continuous while the function $f^+: \mathcal{SSP} \longrightarrow [0,1]$ is right continuous. Observe that neither f^- nor f^+ are strategy-proof on the domain of single-peaked preferences since, for instance, for $\tau = 1/2$, $\delta = 1/4$, and any $R_i \in \mathcal{SP}$ such that $t(R_i) = 3/8$ and $3/4 P_i 1/4$ agent i manipulates f^- and f^+ at R_i via any R_i' such that $t(R_i') = 7/8$ since $f^-(R_i') = f^+(R_i') = 3/4 P_i 1/4 = f^+(R_i) = f^-(R_i)$.

More generally, a strategy-proof social choice function $f : \mathcal{SSP} \longrightarrow [0,1]$ could have a countable number of discontinuities as long as the middle point of each discontinuity jump is the discontinuity point itself; namely, for the point $d \in [0,1]$ where f is discontinuous at d,

$$d = \frac{\lim_{x \to d^-} f(x) + \lim_{x \to d^+} f(x)}{2}$$

must hold, otherwise, f is not strategy-proof. Thus, discontinuity jumps have to be symmetric around the discontinuity point.

As we will show in Section 4, the class of strategy-proof social choice functions on the domain of symmetric single-peaked preferences is the class of generalized median voter schemes identified by Moulin (1980) plus the social choice functions obtained after perturbing each generalized median voter scheme by admitting these very particular kind of discontinuities. We will call them *disturbed minmax*. Formally,

Definition 4 Let $\{p_S\}_{S \in 2^N}$ be a family of monotonic fixed ballots. A collection of intervals $I = \{I_m\}_{m \in M}$ is a *family of discontinuity jumps compatible with* $\{p_S\}_{S \in 2^N}$ if:

(1) M is a countable set,

(2) for all $m \in M$, $I_m = (a_m, b_m) \subset [p_N, p_\emptyset]$,

(3) for all $m, m' \in M$ such that $m \neq m'$, $I_m \cap I_{m'} = \emptyset$,

(4) for all $S \in 2^N$, $p_S \notin \bigcup_{m \in M} I_m$.

Given a family of discontinuity jumps $I = \{I_m\}_{m \in M}$ we denote the *middle point* of each open interval $I_m = (a_m, b_m)$ by $d_m = \frac{a_m + b_m}{2}$ and we preliminary perturb the identity function as follows.

Definition 5 Given a family of discontinuity jumps $I = \{I_m\}_{m \in M}$, the corresponding *perturbation* function $\Pi^I : [0,1] \rightarrow [0,1]$ is defined as follows: for each $x \in [0,1]$,

$$\Pi^I(x) = \begin{cases} x & \text{if } x \notin \bigcup_{m \in M} I_m \\ a_m & \text{if } x \in (a_m, d_m] \\ b_m & \text{if } x \in (d_m, b_m). \end{cases} \quad (3)$$

Let I be a family of discontinuity jumps compatible with the family of monotonic fixed ballots $\{p_S\}_{S \in 2^N}$. A possible perturbation of the generalized median voter scheme associated to $\{p_S\}_{S \in 2^N}$ that preserves its strategy-proofness in the symmetric single-peaked domain is as follows: for each $t = (t_1, ..., t_n) \in \mathcal{SSP}^N$,

$$f(t_1, ..., t_n) = \Pi^I(\min_{S \in 2^N} \max_{i \in S} \{t_i, p_S\}).$$

We will show that these perturbed functions (of generalized median voter schemes) are the basis to characterize the class of all strategy-proof social choice functions on the domain of symmetric single-peaked preferences.

Figure 3 illustrates the perturbation for the case $n = 1$, $M = \{m\}$ and $I = \{I_m = (a_m, b_m)\}$; *i.e.*, $f(t) = \Pi^I(med\{t, p_1, p_0\})$.

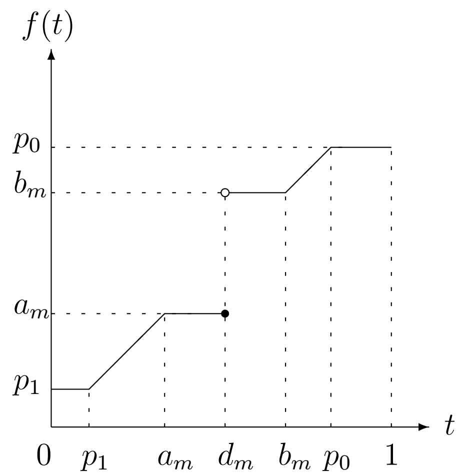

Figure 3

Notice that Π^I arbitrarily assigns the value a_m to the point d_m. If instead $\Pi^I(d_m) = b_m$, the perturbed median voter scheme would still be strategy-proof. When $n = 1$, there are just two ways of perturbing the generalized median voter scheme at each discontinuity jump while preserving its strategy-proofness. When $n > 1$ the process of assigning values to the discontinuity points in a way that maintains the strategy-proofness is more complex.

Figure 4 illustrates the perturbation of an anonymous social choice function for the case $n = 2$, $M = \{m\}$, $I = \{I_m\}$ and $0 < p_2 < a_m < d_m < b_m < p_1 < p_0 < 1$; *i.e.*, $f(t_1, t_2) = \Pi^I(med\{t_1, t_2, p_2, p_1, p_0\})$. The tops of the two agents are measured on the axes and in bold-italic is represented the value of the social choice function in each region. The bold line indicates the discontinuity points of the social choice function.

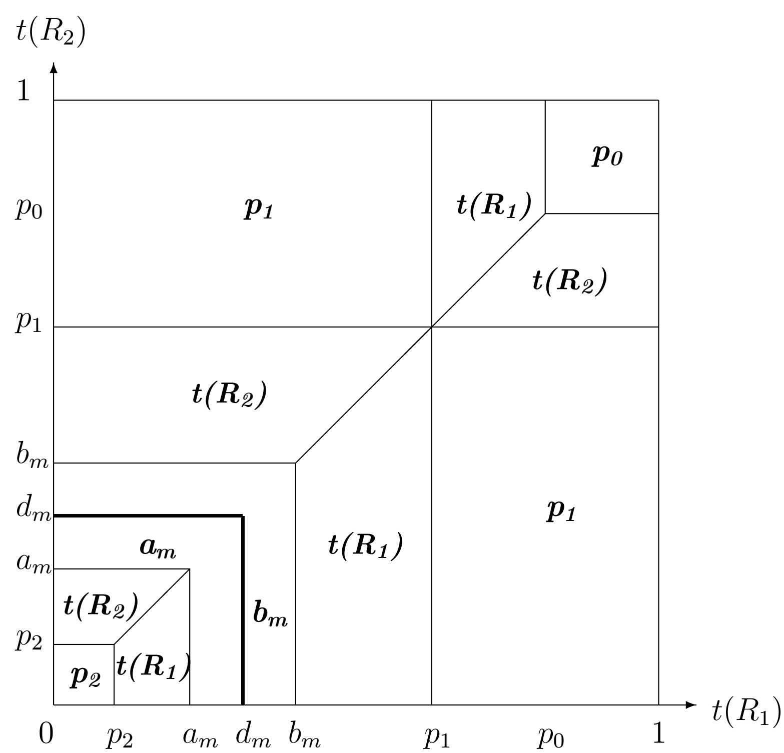

Figure 4

It is easy to see that if Π^I had assigned the value b_m, instead of a_m, to d_m the perturbation of the generalized median voter scheme would still have remained strategy-proof on the domain of symmetric single-peaked preferences. But now there are more ways of assigning values to the discontinuity points that preserve the strategy-proofness of f. For the particular case depicted in Figure 4, the social choice function would have remained strategy-proof and anonymous if it had assigned the value a_m to the points in the set $B_1 = \{(t_1, t_2) \in [0,1]^2 \mid 0 \leq t_1 < d_m \text{ and } t_2 = d_m\}$, as well as to the points in the set $B_2 = \{(t_1, t_2) \in [0,1]^2 \mid t_1 = d_m \text{ and } 0 \leq t_2 < d_m\}$, whereas it had assigned b_m to the point (d_m, d_m). Actually, if anonymity was not required then it could also have assigned the value a_m to the points in B_1, and b_m to the rest of points in $B_2 \cup (d_m, d_m)$. However assigning the value a_m to the point (d_m, d_m) and b_m to the rest of points in $B_1 \cup B_2$ would violate strategy-proofness because at any profile (t_1, d_m) with $0 < t_1 < d_m$ agent 1 could manipulate the social choice function via $t_1' = d_m$.

Intuitively, the perturbation of the generalized median voter scheme should preserve the increasing monotonicity of the social choice function; otherwise, some agent could manipulate it at some profile. We next formalize all these possibilities.

Consider a generalized median voter scheme with its associated family of monotone fixed ballots $\{p_S\}_{S \in 2^N}$. Let $I = \{I_m\}_{m \in M}$ be a family of discontinuity jumps compatible with $\{p_S\}_{S \in 2^N}$, and assume $M \neq \varnothing$. Fix $m \in M$ and define

$$D_m = \{t = (t_1, ..., t_n) \in \mathcal{SSP}^N \mid \min_{S \in 2^N} \max_{i \in S} \{t_i, p_S\} = d_m\};$$

namely, D_m is the set of symmetric single-peaked preference profiles at which the generalized median voter scheme will select d_m and thus the corresponding perturbation function Π^I will generate a discontinuity point. We refer to any set D_m as a *discontinuity set*. We want to determine the shape of the discontinuity sets because, in order to maintain strategy-proofness, we ought to preserve the increasing monotonicity of the function. To do that we need to track the agents with tops strictly bellow, equal, and strictly above d_m.

Note that, since no fixed ballot belongs to any discontinuity jump, if $t \in D_m$ then there is at least one agent $i \in N$ such that $t_i = d_m$.

For each $t \in D_m$ define the vector of *extreme votes* $ev^m(t) = (ev_1^m(t), ..., ev_n^m(t)) \in \{0, d_m, 1\}^N$, where for each $i \in N$,

$$ev_i^m(t) = \begin{cases} 0 & \text{if } 0 \leq t_i < d_m \\ d_m & \text{if } t_i = d_m \\ 1 & \text{if } d_m < t_i \leq 1. \end{cases}$$

The vector $ev^m(t)$ describes at the profile t the location of the top of each agent relative to d_m (0 if it is strictly below, 1 if it is strictly above, and d_m if it is exactly located at d_m). Let $EV(D_m)$ denote the set $\{ev^m \in \{0, d_m, 1\}^N \mid ev^m = ev^m(t) \text{ for some } t \in D_m\}$. To describe $EV(D_m)$ in a more useful way, define

$$\Omega_m = \{S \in 2^N \mid p_S < d_m\}$$

as the family of subsets of N whose associated fixed ballots are strictly below d_m. Since the family of fixed ballots $\{p_S\}_{S\in 2^N}$ is monotonic, $S \in \Omega_m$ and $S \subsetneq T$ imply $T \in \Omega_m$. To describe the set $EV(D_m)$ we may restrict our attention to the family of coalitions in Ω_m, those that may induce the value d_m at some preference profile.

Remark 3 *Let $m \in M$. Then,*

$$D_m = \{t = (t_1, ..., t_n) \in \mathcal{SSP}^N \mid \min_{S\in\Omega_m} \max_{i\in S}\{t_i\} = d_m\}$$

and

$$EV(D_m) = \{ev^m \in \{0, d_m, 1\}^N \mid \text{ there exists } S \in \Omega_m \text{ such that (i) } ev_i^m = d_m \text{ for some } i \in S, \text{ (ii) } ev_j^m \in \{0, d_m\} \text{ for all } j \in S \text{ and (iii) for all } T \in \Omega_m \backslash S, \text{ there exists } j \in T \text{ such that } ev_j^m \in \{d_m, 1\}\}.$$

Namely, the set $EV(D_m)$ describes all the extreme votes at which d_m is chosen by the generalized median voter scheme associated to the family of monotonic fixed ballots $\{p_S\}_{S\in 2^N}$. We call them extreme because given $t \in D_m$, $ev^m(t)$ reallocates agents' tops on the set $\{0, d_m, 1\}$ with the property that $\min_{S\in 2^N} \max_{i\in S}\{t_i, p_S\} = d_m = \min_{S\in 2^N} \max_{i\in S}\{ev_i^m(t), p_S\}$. To know them, we have to look at the set of coalitions who have the possibility of inducing the generalized median voter scheme to choose d_m by all of its members reporting a top at d_m or below, at least one of its members reporting a top exactly at d_m, and all other coalitions with a fixed ballot strictly below d_m must contain an agent reporting a top at d_m or above.

We now turn to describe how strategy-proof social choice functions on the symmetric single-peaked domain may choose between a_m and b_m at those profiles that induce a discontinuity at $d_m = \frac{a_m+b_m}{2}$. Define the preorder $\preceq$ on $\mathbb{R}^N$ as follows: for all $x, x' \in \mathbb{R}^N$,

$$x \preceq x' \Leftrightarrow x_i \leq x_i' \text{ for all } i \in \{1, ..., N\}$$

and, given $m \in M$, denote the restriction of $\preceq$ on the set $EV(D_m)$ by $\preceq_m$. Observe that the natural preorder $\preceq$ on $\mathbb{R}^N$ induces an incomplete, reflexive, and transitive binary relation $\preceq_m$ on $EV(D_m)$ with the property that $\widehat{ev}^m \preceq_m ev^m$ if and only if ev^m represents a shift to the right of some of the extreme votes of $\widehat{ev}^m$. Thus, $\succeq_m$ can be read as the relation "to be more rightist than".

Let Y_m be a non-empty subset of $EV(D_m)$. Denote by $X_m = U(Y_m)$ the upper contour set of Y_m (according to $\preceq_m$) as

$$X_m = U(Y_m) = \{ev^m \in EV(D_m) \mid \widehat{ev}^m \preceq_m ev^m \text{ for some } \widehat{ev}^m \in Y_m\}.$$

By convention, set $U(\varnothing) = \varnothing$. Now, given $X_m \subseteq EV(D_m)$ with the property that $X_m = U(X_m)$, define $g^{X_m} : D_m \longrightarrow \{a_m, b_m\}$ as follows: for every $t \in D_m$,

$$g^{X_m}(t) = \begin{cases} b_m & \text{if } ev^m(t) \in X_m \\ a_m & \text{otherwise.} \end{cases}$$

The functions g^{X_m} cover all different ways of assigning values a_m and b_m to the preference profiles that generate a discontinuity point at d_m preserving the monotonicity of the perturbation. For each particular $m \in M$ there are many such functions because there are many subsets $X_m \subseteq EV(D_m)$ with the property that $X_m = U(X_m)$. Given a family of discontinuity jumps $I = \{I_m\}_{m \in M}$ we say that $\{X_m\}_{m \in M}$ is a *family of tie-breaking sets of* M if for all $m \in M$, $X_m \subseteq EV(D_m)$ and $X_m = U(X_m)$.

4 Characterization

We are now ready to define disturbed minimax social choice functions and state and prove that they constitute the class of all strategy-proof social choice functions on the domain of symmetric single-peaked preferences.

4.1 Definition and statement

Definition 6 A social choice function $f : \mathcal{SSP}^N \longrightarrow [0,1]$ is a *disturbed minmax* if there exist:

(1) a monotonic family of fixed ballots $\{p_S\}_{S \in 2^N}$;

(2) a family of discontinuity jumps $I = \{I_m\}_{m \in M}$ compatible with $\{p_S\}_{S \in 2^N}$; and

(3) a family of tie-breaking sets $\{X_m\}_{m \in M}$ of M

such that, for all $t = (t_1, ..., t_n) \in \mathcal{SSP}^N$,

$$f(t) = \begin{cases} \Pi^I(\min\limits_{S \in 2^N} \max\limits_{i \in S} \{t_i, p_S\}) & \text{if } \min\limits_{S \in 2^N} \max\limits_{i \in S} \{t_i, p_S\} \neq d_m \text{ for all } m \in M \\ g^{X_m}(t_1, ..., t_n) & \text{if } \min\limits_{S \in 2^N} \max\limits_{i \in S} \{t_i, p_S\} = d_m \text{ for an } m \in M. \end{cases} \tag{4}$$

Theorem 1 *A social choice function* $f : \mathcal{SSP}^N \longrightarrow [0,1]$ *is strategy-proof if and only if it is a disturbed minmax.*

Before moving to the proof of Theorem 1 consider again the social choice function f defined in (1) but restricted to the domain of symmetric single-peaked preferences, where for all $R \in \mathcal{SSP}^N$,

$$f(R) = \begin{cases} 0 & \text{if } \#\{i \in N \mid 0R_i1\} \geq \#\{i \in N \mid 1P_i0\} \\ 1 & \text{otherwise.} \end{cases}$$

Observe that for any $R_i \in \mathcal{SSP}$, $0R_i1$ if and only if $t(R_i) \leq \frac{1}{2}$. It is easy to see that in the domain of single-peaked preferences f is strategy-proof and anonymous but it is not tops-only. Hence, while it is excluded in Moulin (1980)' characterization under the domain of single-peaked preferences stated above as Proposition 2, it has the following representation as a disturbed minmax under the domain of symmetric single-peaked preferences. Its family of monotonic fixed ballots is

$$p_S = \begin{cases} 0 & \text{if } \#S \geq \lceil \frac{n}{2} \rceil \\ 1 & \text{if } \#S < \lceil \frac{n}{2} \rceil, \end{cases}$$

where $\lceil \frac{n}{2} \rceil$ is the smallest integer larger or equal to $\frac{n}{2}$. The family I of discontinuity jumps compatible with the monotonic family of fixed ballots contains only one discontinuity interval $I_m = (a_m, b_m) = (0, 1)$ with $d_m = \frac{1}{2}$, and the tie-breaking set of $M = \{m\}$ is $X_m = \{ev \in \{0, \frac{1}{2}, 1\}^N \mid \#\{i \in N \mid ev_i \in \{0, \frac{1}{2}\}\} < \lceil \frac{n}{2} \rceil\}$.

4.2 Preliminaries of the proof of Theorem 1

We start with some additional notation. Given $x \in [0,1]$, $S \subseteq N$, and $t \in \mathcal{SSP}^N$, define $x^S \equiv (\underbrace{x, ..., x}_{\#S-\text{times}})$ and $t_S \equiv (t_j)_{j \in S}$. Thus, $(x^S, t_{-S}) \equiv (y_1, ..., y_n)$, where $y_j = x$ if $j \in S$ and $y_j = t_j$ if $j \notin S$. Let $f : \mathcal{SSP}^N \longrightarrow [0,1]$ be a social choice function and $S \subseteq N$. Define the social choice function $\Delta_f^S : [0,1] \times \mathcal{SSP}^{N \setminus S} \longrightarrow [0,1]$ as follows. For all $(x, t_{-S}) \in [0,1] \times \mathcal{SSP}^{N \setminus S}$,

$$\Delta_f^S(x, t_{-S}) = f(x^S, t_{-S}).$$

We will denote the diagonal function associated to f by $\Delta_f \equiv \Delta_f^N$.

Given $t \in [0,1]^N$ and $x \in [0,1]$, define the subset of profiles of tops $\mathcal{C}_{t,x}$ as:

$$\mathcal{C}_{t,x} = \{t' \in \mathcal{SSP}^N \mid x \leq t'_i \leq t_i \text{ for all } i \text{ such that } x \leq t_i \text{ and } t_i \leq t'_i \leq x \text{ for all } i \text{ such that } t_i \leq x\};$$

namely, $\mathcal{C}_{t,x}$ is the set of profiles t' with the property that the top t'_i of each agent i lies between t_i and x. Given a social choice function $f : \mathcal{SSP}^N \longrightarrow [0,1]$, a subset $\mathcal{T} \subseteq \mathcal{SSP}^N$, and $x \in [0,1]$ the notation $f \mid_{\mathcal{T}} \equiv x$ means that for all $t \in \mathcal{T}$, $f(t) = x$.

As a consequence of Remark 1 and Lemma 1 the following statements hold.

Remark 4 *Let $f : \mathcal{SSP}^N \longrightarrow [0,1]$ be a strategy-proof social choice function. Then,*
(R4.1) *f is unanimous on its range r_f; namely, $x \in r_f$ implies $f(x^N) = x$;*
(R4.2) *for all $S \subseteq N$, $\Delta_f^S : [0,1] \times \mathcal{SSP}^{N \setminus S} \longrightarrow [0,1]$ is strategy-proof; and*
(R4.3) *if $t \in \mathcal{SSP}^N$ is such that $f(t) = x$ then, $f|_{\mathcal{C}_{t,x}} \equiv x$.*

The two first statements follow from group strategy-proofness (Remark 1) and the last one from monotonicity (Lemma 1) and (R4.1).

We now state and prove the following three lemmata that will be useful in the proof of Theorem 1. Lemma 2 says that the range of a strategy-proof social choice function and the range of its associated diagonal function coincide and it is a closed subset of $[0,1]$ (see also Zhou (1991)).

Lemma 2 *Let $f : \mathcal{SSP}^N \longrightarrow [0,1]$ be a strategy-proof social choice function. Then, $r_f = r_{\Delta_f}$. Moreover, r_f is closed.*

Proof By definition of Δ_f, $r_{\Delta_f} \subseteq r_f$. Take $x \in r_f$. Then, by (R4.1), $f(x^N) = x$. Thus, $x \in r_{\Delta_f}$. Let $\{x_k\} \to x$ be such that $x_k \in r_f$ for all $k \geq 1$ and assume $x \notin r_f$. Define $y = f(x^N) \neq x$ and let x_k be such that $|x_k - x| < |y - x|$. By (R4.1), $f(x_k^N) = x_k$. Thus, N manipulates f at x via x_k. ■

Lemmata 3 and 3′ roughly say that if a strategy-proof social choice function is constant and equal to x on one variable over some interval containing this constant x, but it is not constant over the whole interval $[0,1]$, then there is a discontinuity at some point z and the discontinuity leaves indifferent the agent with top at z (see Figures 2 and 3). In the proof of Theorem 1, z will correspond to the middle point d_m of a discontinuity jump $I_m = (a_m, b_m)$, where $a_m = x$ and $b_m = 2z - x$.

Lemma 3 *Let $f : \mathcal{SSP}^N \longrightarrow [0,1]$ be a strategy-proof social choice function with the property that there are $i \in N$, $x \in [a,b) \subset [0,1]$, and $t_{-i} \in \mathcal{SSP}^{N \setminus \{i\}}$ such that*
(3.1) *$f(t_i, t_{-i}) = x$ for all $t_i \in [a,b)$ and*
(3.2) *$f(1, t_{-i}) = y > x$.*
Then, there exists $z \in [b, \frac{x+y}{2}]$ such that $f(\cdot, t_{-i})$ is discontinuous at z and

$$f|_{[a,z)\times\{t_{-i}\}} \equiv x$$
$$f|_{(z,2z-x]\times\{t_{-i}\}} \equiv 2z - x.$$

Proof Let $i \in N$, $x \in [a,b)$, and $t_{-i} \in \mathcal{SSP}^{N \setminus \{i\}}$ be such that conditions (3.1) and (3.2) hold for f. First note that the interval $[b, \frac{x+y}{2}]$ is not empty since $b \leq \frac{x+y}{2}$. If

$b > \frac{x+y}{2}$ then b would be closer to y than to x and for a small enough $\epsilon > 0$, i would manipulate f at $(b-\epsilon, t_{-i})$ via 1.

Define $z = \sup\{t_i \in [0,1] \mid f(t_i, t_{-i}) = x\}$. The supremum is well defined because, condition (3.1) holds and, by Lemma 1, f is increasing. Obviously $z \geq b > x$ and, by the monotonicity of f, $\lim_{t_i \to z^-} f(t_i, t_{-i}) = x$ and $f|_{[a,z)\times\{t_{-i}\}} \equiv x$. We now prove that $\lim_{t_i \to z^+} f(t_i, t_{-i}) = 2z - x$. Suppose that $\lim_{t_i \to z^+} f(t_i, t_{-i}) < 2z - x$. Then, there exists $\epsilon > 0$ such that $f(z+\epsilon, t_{-i}) + 2\epsilon < 2z - x$ and $f(z-\epsilon, t_{-i}) = x$. But then, $f(z+\epsilon, t_{-i}) - (z-\epsilon) < (z-\epsilon) - x = (z-\epsilon) - f(z-\epsilon, t_{-i})$, and hence, i would manipulate f at $(z-\epsilon, t_{-i})$ via $z+\epsilon$. Similarly, if $\lim_{t_i \to z^+} f(t_i, t_{-i}) > 2z - x$, there exists $\epsilon > 0$ such that $f(z+\epsilon, t_{-i}) - 2\epsilon > 2z - x$ and $f(z-\epsilon, t_{-i}) = x$. But then $f(z+\epsilon, t_{-i}) - (z+\epsilon) > (z+\epsilon) - x = (z+\epsilon) - f(z-\epsilon, t_{-i})$ and hence, i would manipulate f at $(z+\epsilon, t_{-i})$ via $z-\epsilon$. Thus, $\lim_{t_i \to z^+} f(t_i, t_{-i}) = 2z - x$ and $f(\cdot, t_{-i})$ is discontinuous at z. Now by (R4.3), $f|_{(z,2z-x]\times\{t_{-i}\}} \equiv 2z - x$. Finally, by monotonicity of f, $2z - x \leq y$, and hence, $z \in [b, \frac{x+y}{2}]$. ■

Lemma 3′ *Let $f : \mathcal{SSP}^N \longrightarrow [0,1]$ be a strategy-proof social choice function with the property that there are $i \in N$, $x \in (a,b] \subset [0,1]$, and $t_{-i} \in \mathcal{SSP}^{N\setminus\{i\}}$ such that*

(3.1′) *$f(t_i, t_{-i}) = x$ for all $t_i \in (a,b]$ and*

(3.2′) *$f(0, t_{-i}) = y < x$.*

Then, there exists $z \in [\frac{x+y}{2}, a]$ such that $f(\cdot, t_{-i})$ is discontinuous at z and

$$f|_{(z,b]\times\{t_{-i}\}} \equiv x$$
$$f|_{[2z-x,z)\times\{t_{-i}\}} \equiv 2z - x.$$

Proof Omitted since it is symmetric to the proof of Lemma 3. ■

4.3 Proof of Theorem 1

It is easy to check that any disturbed minimax social choice function is strategy-proof on the symmetric single-peaked domain.

Let $f : \mathcal{SSP}^N \longrightarrow [0,1]$ be a strategy-proof social choice function. To show that f is a disturbed minmax we first have to identify its associated monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$, family $I = \{I_m\}_{m\in M}$ of discontinuity jumps compatible with $\{p_S\}_{S\in 2^N}$, and family of tie-breaking sets $\{X_m\}_{m\in M}$ of M. Then, we will show that f coincides with the disturbed minmax social choice function obtained by (4) in Definition 6, applied to all of them.

For each $S \in 2^N$, define its associated fixed ballot by setting

$$p_S \equiv f(0^S, 1^{N\setminus S}); \tag{5}$$

i.e., p_S is the image of f at the profile where all agents in S have their top at 0 and all agents not in S have their top at 1.

Consider the diagonal function $\Delta_f : \mathcal{SSP} \longrightarrow [0,1]$ associated to f. By (R4.2) and Lemma 1, Δ_f is strategy-proof and increasing. Hence, it has at most a countable number of discontinuities. Denote by $\{d_m\}_{m\in M}$ the discontinuity points of Δ_f, where M is a countable set. For each $m \in M$, define $a_m = \lim_{x\to d_m^-} \Delta_f(x)$, and $b_m = \lim_{x\to d_m^+} \Delta_f(x)$. Since Δ_f is discontinuous at d_m and increasing on $[0,1]$, a_m and b_m exist and $a_m < b_m$. Moreover, since Δ_f is strategy-proof, d_m must be the middle point of $I_m \equiv (a_m, b_m)$; *i.e.*, $d_m = \frac{a_m+b_m}{2}$. Notice that the family of discontinuity jumps $I = \{I_m\}_{m\in M}$ is compatible with $\{p_S\}_{S\in 2^N}$ since, by (5) and Lemma 2, for each $S \in 2^N$, $p_S \in r_f = r_{\Delta_f}$, $r_f \cap (a_m, b_m) = r_{\Delta_f} \cap (a_m, b_m) = \emptyset$ and by the monotonicity of Δ_f and the definition of a_m and b_m, $I_m \cap I_{m'} = \emptyset$ for any $m' \in M\backslash\{m\}$. In fact,

$$r_f = r_{\Delta_f} = [p_N, p_\varnothing]\backslash\{\bigcup_{m\in M} I_m\}. \tag{6}$$

If M is empty (*i.e.*, Δ_f is continuos and its range is equal to $[p_N, p_\varnothing]$), the statement of Theorem 1 follows because f is a generalized median voter scheme defined on the minimally rich domain $\mathcal{SSP}^N$ (see Theorem 1 in Berga and Serizawa (2000)).[14]

Assume M is non-empty and fix $m \in M$. To identify the element X_m in the family of tie-breaking sets of M, remember that $\Omega_m = \{S \in 2^N \mid p_S < d_m\}$ and consider first the previously defined discontinuity set

$$D_m = \{t = (t_1, ..., t_n) \in \mathcal{SSP}^N \mid \min_{S\in\Omega_m} \max_{i\in S}\{t_i\} = d_m\},$$

the set of profiles of extreme votes that induce d_m through the minmax

$$EV(D_m) = \{ev^m \in \{0, d_m, 1\}^N \mid \begin{array}{l}\text{there exists } S \in \Omega_m \text{ such that (i) } ev_i^m = d_m \text{ for some} \\ i \in S, \text{ (ii) } ev_j^m \in \{0, d_m\} \text{ for all } j \in S \text{ and (iii) for all} \\ T \in \Omega_m\backslash S, \text{ there exists } j \in T \text{ such that } ev_j^m \in \{d_m, 1\}\},\end{array}$$

and its associated preorder $\preceq_m$. Then, define

$$X_m = \{ev^m \in EV(D_m) \mid f(ev^m) > d_m\}. \tag{7}$$

[14] Observe that all results in Berga and Serizawa (2000) refer only to onto social choice functions. More precisely, the restriction of $\mathcal{SSP}$ on the interval $[p_N, p_\varnothing]$ is a symmetric single-peaked domain (on $[p_N, p_\varnothing]$) and it is a minimally rich domain (on $[p_N, p_\varnothing]$). Denote it by $\mathcal{SSP}\mid_{[p_N,p_\varnothing]}$. Thus, we can identify the notation of Berga and Serizawa (2000) for the image set $Z = [\alpha, \beta]$ with our identified interval $[p_N, p_\varnothing]$ and apply their Theorem 1 to the social choice function $f^* : (\mathcal{SSP}\mid_{[p_N,p_\varnothing]})^N \longrightarrow [p_N, p_\varnothing]$. Finally, observe that their generalized median voter schemes (defined through a left-coalition system) satisfy voter sovereignty and hence, $r_{f^*} = [p_N, p_\varnothing]$.

By Lemma 1, f is increasing and therefore X_m coincides with its upper contour set relative to $\preceq_m$; *i.e.*, $X_m = U(X_m)$.

So far we have identified from f the monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$, the family $I = \{I_m\}_{m\in M}$ of discontinuity jumps compatible with $\{p_S\}_{S\in 2^N}$ (we are now assuming that $M \neq \varnothing$), and the family $\{X_m\}_{m\in M}$ of tie-breaking sets of M (and hence, its corresponding family of tie-breaking functions $\{g^{X_m} : D_m \longrightarrow \{a_m, b_m\}\}_{m\in M}$). Given all of them, let F be the social choice function defined by condition (4) in Definition 6. We want to show that $f = F$.

Let $t = (t_1, ..., t_n) \in \mathcal{SSP}^N$ be arbitrary. To show that $f(t) = F(t)$ define $q = \min_{T\in 2^N} \max_{i\in T}\{t_i, p_T\}$. We distinguish between the case where $q \notin \{t_1, ..., t_n\}$ and the remaining case where $q \in \{t_1, ..., t_n\}$.

First assume that $q = \min_{T\in 2^N} \max_{i\in T}\{t_i, p_T\} \notin \{t_1, ..., t_n\}$. Then, $S = \{i \in N \mid t_i < q\}$ satisfies that $p_S = q$. To see that observe that if $p_S < q$ then $\max_{i\in S}\{t_i, p_S\} < q$ contradicting the definition of q. On the other hand, since $q = \min_{T\in 2^N} \max_{i\in T}\{t_i, p_T\} \notin \{t_1, ..., t_n\}$, there exists $\bar{T} \in 2^N$, such that $p_{\bar{T}} = q$ and $t_j < p_{\bar{T}}$ for all $j \in \bar{T}$. But then, $\bar{T} \subseteq S$ and, by the monotonicity of $p = \{p_T\}_{T\in 2^N}$, $p_S \leq p_{\bar{T}}$. Therefore, by the definition of q, $p_S = p_{\bar{T}} = q$.

By definition of S and the assumption that $q \notin \{t_1, ..., t_n\}$, $t_j > p_S$ for all $j \notin S$. Then, $t \in \mathcal{C}_{(0^S, 1^{N\setminus S}), p_S}$ and, by (R4.3) and the definition of p_S, $f|_{\mathcal{C}_{(0^S,1^{N\setminus S}),p_S}} \equiv p_S$. Therefore,

$$f(t) = p_S. \tag{8}$$

Moreover, by (6), $p_S \notin \cup_{m\in M} I_m$. Hence, by (4) in Definition 6 and the definition of Π^I in (3),

$$F(t) = \Pi^I(\min_{T\in 2^N} \max_{i\in T}\{t_i, p_T\}) = \min_{T\in 2^N} \max_{i\in T}\{t_i, p_T\} = p_S.$$

Thus, $f(t) = F(t)$.

Assume now that $q = \min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\} = t_i$ for some $i \in N$. We distinguish between two cases.

Case 1: $f(t) = t_i$. Then, $t_i \in r_f$ and therefore, by (6), $t_i \notin \bigcup_{m\in M} I_m$. By (4) in Definition 6 and the definition of Π^I in (3),

$$F(t) = \Pi^I(\min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\}) = \Pi^I(t_i) = t_i.$$

Thus, $f(t) = F(t)$.

Case 2: $f(t) \equiv x \neq t_i$. Define $S_i^< = \{j \in N \mid t_j < t_i\}$, $S_i^= = \{j \in N \mid t_j = t_i\}$ and $S_i^> = \{j \in N \mid t_j > t_i\}$. We will denote $S_i^\leq = S_i^< \cup S_i^=$ and $S_i^\geq = S_i^> \cup S_i^=$.

Because $t_i = \min_{T \in 2^N} \max_{j \in T}\{t_j, p_T\}$, the following condition holds: $t_i \in [p_{S_i^{\leq}}, p_{S_i^{<}}]$. If $t_i < p_{S_i^{\leq}}$ then, since for any coalition $S \subsetneq S_i^{\leq}$ it holds that $t_i < p_{S_i^{\leq}} \leq p_S$, we have a contradiction with $t_i = \min_{T \in 2^N} \max_{j \in T}\{t_j, p_T\}$. On the other hand, if $p_{S_i^{<}} < t_i$, then $\max_{j \in S_i^{<}}\{t_j, p_{S_i^{<}}\} < t_i$ again contradicting $t_i = \min_{T \in 2^N} \max_{j \in T}\{t_j, p_T\}$. We now show that

$$f(t_i^N) \in [p_{S_i^{\leq}}, p_{S_i^{<}}]. \tag{9}$$

If $f(t_i^N) < p_{S_i^{\leq}} \leq t_i$ then, N manipulates f at t_i^N via $(0^{S_i^{\leq}}, 1^{S_i^{>}})$ since $f(0^{S_i^{\leq}}, 1^{S_i^{>}}) = p_{S_i^{\leq}}$, and if $t_i \leq p_{S_i^{<}} < f(t_i^N)$ then, N manipulates f at t_i^N via $(0^{S_i^{<}}, 1^{S_i^{\geq}})$ since $f(0^{S_i^{<}}, 1^{S_i^{\geq}}) = p_{S_i^{<}}$.

Suppose first that $t_i \notin \cup_{m \in M}\{d_m\}$. We prove in Claim 1 below that in this case $f(t) = f(t_i^N)$ must hold.

Claim 1 *Assume* $t_i \notin \cup_{m \in M}\{d_m\}$. *Then,* $f(t_i^N) = x$.[15]

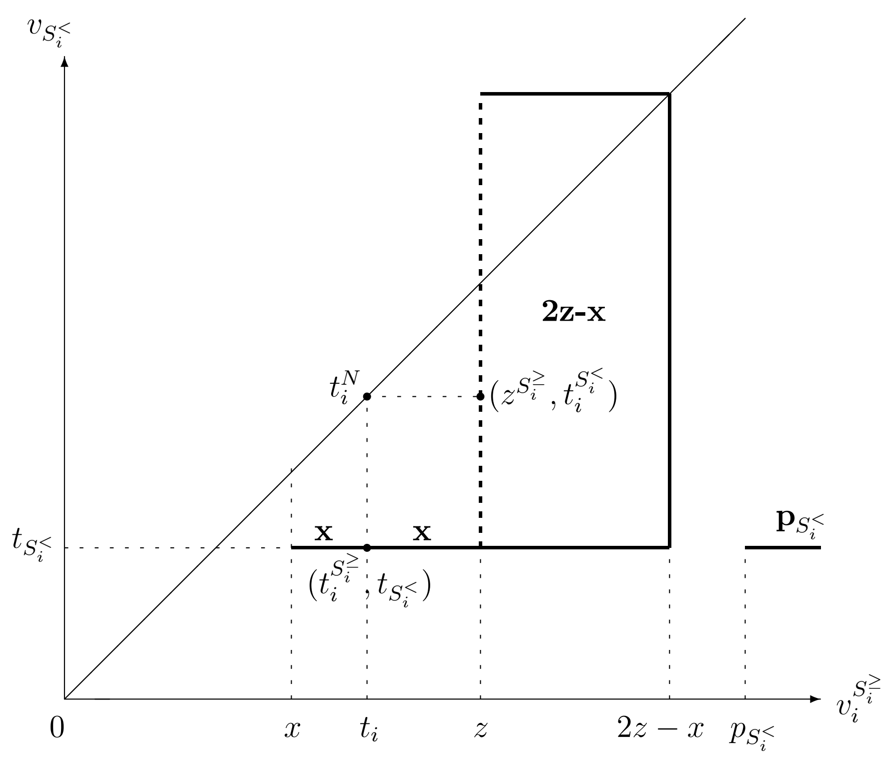

Figure 5

[15] Along the proof of Claim 1 it will be useful to look at Figure 5 where the argument is shown in two dimensions. On the axes, v represents a generic profile of tops whereas t is the profile of tops we are looking at. Bold letters represent the value of the social choice function at the corresponding preference profile whereas italic letters represent preferences, preference profiles and alternatives.

PROOF OF CLAIM 1 To obtain a contradiction, suppose $f(t_i^N) \neq x$. By (9), either $x < f(t_i^N) \leq p_{S_i^<}$ or $p_{S_i^\leq} \leq f(t_i^N) < x$.

Case 2.1: $x < f(t_i^N) \leq p_{S_i^<}$. Condition $x < f(t_i^N)$ implies $x < t_i$ since we are assuming that $x \neq t_i$ holds and if $x > t_i$ then, N would manipulate f at t_i^N via t. By (R4.3), the definition of $S_i^\geq$ and $f(t) = x$,

$$\Delta_f^{S_i^\geq}(\tau, t_{S_i^<}) = x \text{ for all } \tau \in [x, t_i].^{16}$$

On the other hand, since $t_j < t_i \leq p_{S_i^<}$ for all $j \in S_i^<$, $(\tau^{S_i^\geq}, t_{S_i^<}) \in \mathcal{C}_{(0^{S_i^<}, 1^{S_i^\geq}), p_{S_i^<}}$ for all $\tau \in [p_{S_i^<}, 1]$, and therefore by (5) and (R4.3),

$$\Delta_f^{S_i^\geq}(\tau, t_{-S_i^\geq}) = p_{S_i^<} \text{ for all } \tau \in [p_{S_i^<}, 1].^{17}$$

By Lemma 3, applied to the strategy-proof social choice function $\Delta_f^{S_i^\geq} : [0,1] \times [0,1]^{S_i^<} \longrightarrow [0,1]$, where $[a,b) = [x, t_i)$ and $y = p_{S_i^<}$, there exists $z \in [t_i, \frac{x + p_{S_i^<}}{2}]$ such that $\Delta_f^{S_i^\geq}(\cdot, t_{S_i^<})$ is discontinuous at z and

$$\Delta_f^{S_i^\geq}|_{[x,z) \times \{t_{S_i^<}\}} \equiv x \text{ and } \Delta_f^{S_i^\geq}|_{(z, 2z-x] \times \{t_{S_i^<}\}} \equiv 2z - x.$$

Applying (R4.3) again, for all $\tau \in (z, 2z-x]$ and for all $t'_j \in [t_j, 2z-x]$, for all $j \in S_i^<$,

$$\Delta_f^{S_i^\geq}(\tau, t'_{S_i^<}) = 2z - x.^{18} \tag{10}$$

Note that z is a discontinuity point of Δ_f as well. To see that observe that by (10), $f(w^N) = 2z - x$ for all $w \in (z, 2z-x]$. On the other hand, $f(t) = x$, and hence, $x \in r_f$ and by (R4.1), $f(x^N) = x$. Assume that there exists $\hat{w} \in (x, z)$ such that $f(\hat{w}^N) \neq x$. By monotonicity of f, $x < f(\hat{w}^N) \leq 2z - x$. Then, either $f(\hat{w}^N) = 2z - x$ and N manipulates f at $\hat{w}^N$ via x^N, or $f(\hat{w}^N) < 2z - x$ and for any $0 < \epsilon < z - \hat{w}$, N manipulates f at $(z + \epsilon)^N$ via $\hat{w}^N$. Thus, $f(\hat{w}^N) = x$. Therefore, Δ_f has the property that

$$\Delta_f(w) = \begin{cases} x & \text{if } w \in [x, z) \\ 2z - x & \text{if } w \in (z, 2z - x]. \end{cases}$$

This means that Δ_f is discontinuous at z and hence there exists $m \in M$ such that $d_m = z$. By the hypothesis of CLAIM 1, $t_i \neq z$ and therefore, by the definition of z, $t_i < z$.

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[16] In Figure 5 this corresponds to the horizontal line $[x, t_i]^{S_i^\geq} \times \{t_{S_i^<}\}$.

[17] In Figure 5 this corresponds to the horizontal line $[p_{S_i^<}, 1]^{S_i^\geq} \times \{t_{S_i^<}\}$.

[18] In Figure 5 this corresponds to the rectangle $[z, 2z-x]^{S_i^\geq} \times (\prod_{j \in S_i^<} [t_j, 2z-x])$.

By monotonicity of f and (10), $f(t_i^N) \leq \Delta_f^{S_i^{\geq}}(z+\epsilon, t_i^{S_i^{<}}) = 2z - x$ for all sufficiently small $\epsilon > 0$ (later on we will find an upper bound for such ϵ's). We want to show that the inequality is strict; *i.e.*, $f(t_i^N) < 2z - x$ holds. Suppose $f(t_i^N) = 2z - x$; then, since $t_i < z$, $t_i - x < 2z - x - t_i$ holds and N would manipulate f at t_i^N via t which contradicts strategy-proofness of f.

To sum up, we have shown that $x < f(t_i^N) < 2z - x$ and $\lim_{\tau \to z^+} \Delta_f^{S_i^{\geq}}(\tau, t_i^{S_i^{<}}) = 2z - x$. But then it is easy to see that for a small $\epsilon > 0$, $S_i^{\geq}$ manipulates f at $((z+\epsilon)^{S_i^{\geq}}, t_i^{S_i^{<}})$ via $t_i^{S_i^{\geq}}$. Namely, if $0 < \epsilon < \frac{f(t_i^N)-x}{2}$, then $-(2z-x-(z+\epsilon)) < f(t_i^N)-(z+\epsilon) < 2z-x-(z+\epsilon)$ where the first inequality follows from the fact that $\epsilon < \frac{f(t_i^N)-x}{2}$ implies $2\epsilon + x - (z+\epsilon) < f(t_i^N) - (z+\epsilon)$ and this, in turn, implies $-(2z - x - (z+\epsilon)) < f(t_i^N) - (z+\epsilon)$, and the second inequality follows from $f(t_i^N) < 2z - x$. Therefore,

$$\left| f(t_i^N) - (z+\epsilon) \right| < 2z - x - (z+\epsilon),$$

which means that $S_i^{\geq}$ manipulates f at $((z+\epsilon)^{S_i^{\geq}}, t_i^{S_i^{<}})$ via $t_i^{S_i^{\geq}}$; a contradiction. Thus, $f(t_i^N) = x$.

Case 2.2: $p_{S_i^{\leq}} \leq f(t_i^N) < x$.

We omit its proof because it is symmetric to the previous one using Lemma 3′ instead of Lemma 3. This concludes the proof of Claim 1. □

Continuing with the main proof, we have shown that if $f(t) = x \neq t_i$ and $t_i \notin \cup_{m \in M}\{d_m\}$ then $f(t_i^N) = \Delta_f(t_i) = x \neq t_i$. By strategy-proofness of f, Δ_f is strategy-proof and hence, by (R4.1), $t_i \notin r_{\Delta_f}$. By (6), there exists $m \in M$ such that $t_i \in (a_m, b_m)$. By (R4.1), $\Delta_f(a_m) = a_m$ and $\Delta_f(b_m) = b_m$. Since, by (R4.2), Δ_f is strategy-proof,

$$x = \Delta_f(t_i) = \begin{cases} a_m & \text{if } a_m < t_i < d_m \\ b_m & \text{if } d_m < t_i < b_m, \end{cases} \tag{11}$$

which coincides with the value of $F(t) = \Pi^I(\min_{T \in 2^N} \max_{j \in T}\{t_j, p_T\} = \Pi^I(t_i) = x$. Thus, $F(t) = f(t)$.

The last case to be considered is when $f(t) = x \neq t_i$ and $t_i = d_m$ for some $m \in M$; that is, when t_i is a discontinuity point of Δ_f. Denote by $I_m = (a_m, b_m)$ the discontinuity jump corresponding to d_m. Denote $S_m^{=} = \{j \in N \mid t_j = d_m\}$, $S_m^{<} = \{j \in N \mid t_j < d_m\}$ and $S_m^{>} = \{j \in N \mid t_j > d_m\}$, and let ϵ be such that $0 < \epsilon < \min_{j \in S_m^{<},\, k \in S_m^{>}}\{d_m - a_m, d_m - t_j, t_k - d_m\}$. Given this $\epsilon > 0$, consider the two profiles of tops $t^{\epsilon-} = (t_{S_m^{<}}, (d_m - \epsilon)^{S_m^{=}}, t_{S_m^{>}})$ and $t^{\epsilon+} = (t_{S_m^{<}}, (d_m + \epsilon)^{S_m^{=}}, t_{S_m^{>}})$. By construction of $t^{\epsilon-}$, $t^{\epsilon+}$, the fact that $t_i = \min_{T \in 2^N} \max_{j \in T}\{t_j, p_T\}$, and since $p_T \notin I_m$ for all $T \in 2^N$,

$\min_{T\in 2^N} \max_{j\in T}\{t_j^{\epsilon-}, p_T\} = d_m - \epsilon$ and $\min_{T\in 2^N} \max_{j\in T}\{t_j^{\epsilon+}, p_T\} = d_m + \epsilon$. Both $d_m - \epsilon$ and $d_m + \epsilon$ belong to I_m and therefore they do not belong to r_f. Moreover, since $I_m \cap I_{m'} = \emptyset$, neither $d_m - \epsilon$ nor $d_m + \epsilon$ are discontinuity points of Δ_f. We are therefore under the assumptions of Claim 1, which says that

$$f(t^{\epsilon-}) = \Delta_f(d_m - \epsilon) = a_m$$
$$f(t^{\epsilon+}) = \Delta_f(d_m + \epsilon) = b_m,$$

where the second equality in both statements follow from the strategy-proofness of Δ_f. By monotonicity, $f(t^{\epsilon-}) \leq f(t) \leq f(t^{\epsilon+})$, which together with (6) implies that $f(t) \in \{a_m, b_m\}$. Thus, we have shown that if t is such that $\min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\} = t_i = d_m$ for some $m \in M$ then,

$$f(t) \in \{a_m, b_m\}. \tag{12}$$

To show that $f(t) = F(t)$, assume first that t is such that $ev^m(t) \notin X_m$. By definition of F, $F(t) = a_m$. Since $ev^m(t) \notin X_m$, by (7), $f(0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>}) \leq d_m$ which means, by (6), that $f(0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>}) \leq a_m$. Moreover, $t' = (0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>})$ is such that $\min_{T\in 2^N} \max_{j\in T}\{t'_j, p_T\} = d_m$ and, by (12), $f(0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>}) = a_m$. By (R4.3),

$$f(0^{S_m^<}, d_m^{S_m^=}, t_{S_m^>}) = a_m. \tag{13}$$

If $S_m^< = \varnothing$, then $(0^{S_m^<}, d_m^{S_m^=}, t_{S_m^>}) = t$, and $f(t) = a_m$. If $S_m^< \neq \varnothing$ then $f(t) = a_m$ or otherwise $S_m^<$ manipulates f at t via $0^{S_m^<}$. Thus, we have shown that $f(t) = a_m = F(t)$. Symmetrically, we can show that if t is such that $ev^m(t) \in X_m$ then $f(t) = F(t) = b_m$.

This finishes the proof of Theorem 1.

5 Final remarks

As direct consequences of Theorem 1, Corollaries 1, 2 and 3 below characterize three relevant subclasses of strategy-proof social choice functions on the domain of symmetric single-peaked preferences.

5.1 Anonymity and efficiency

Corollaries 1 and 2 characterize two nested subclasses: the class of strategy-proof and anonymous social choice functions (Corollary 1) and the class of strategy-proof, anonymous and efficient social choice functions (Corollary 2).

To state Corollary 1 we first need to translate the definitions of extreme votes and tie-breaking sets of M to the anonymous case. Consider the family of $n+1$ fixed ballots

$0 \leq p_n \leq ... \leq p_1 \leq p_0 \leq 1$ associated to a median voter scheme and let $m \in M$. The set of profiles at which the median voter scheme will select d_m is

$$\widetilde{D}_m = \{t = (t_1, ..., t_n) \in \mathcal{SSP}^N \mid \text{med}\{t_1, ..., t_n, p_n, ..., p_0\} = d_m\}.$$

By anonymity, we only need to track the number of agents with tops strictly bellow, equal, and strictly above d_m. Hence, for each $t = (t_1, ..., t_n) \in \mathcal{SSP}^N$, define the triple $l^m(t) = (l^m_<(t), l^m_=(t), l^m_>(t))$ where:
(1) $l^m_<(t) = \#\{i \in N \mid t_i < d_m\}$,
(2) $l^m_=(t) = \#\{i \in N \mid t_i = d_m\}$, and
(3) $l^m_>(t) = \#\{i \in N \mid t_i > d_m\}$.
Observe that $l^m_<(t) + l^m_=(t) + l^m_>(t) = n$ and since fixed ballots do not belong to any discontinuity jump, if $t \in \widetilde{D}_m$ then, there is $i \in N$ such that $t_i = d_m$ (*i.e.*, $l^m_=(t) \geq 1$). Let $\nabla^n = \{(x, y, z) \in \{0, 1, ..., n\}^3 \mid x + y + z = n \text{ and } y \geq 1\}$ be the set of triples with positive integer components adding up to n and whose middle component is equal or larger than 1 and define $L(\widetilde{D}_m) = \{l^m(t) \in \nabla^n \mid t = (t_1, ..., t_n) \in \widetilde{D}_m\}$; namely, $L(\widetilde{D}_m)$ describes all anonymous distributions of tops (number of tops strictly below d_m, number of tops at d_m, number of tops strictly above d_m) at which the median voter selects d_m. Define the preorder $\widetilde{\preceq}$ on $\{0, 1, ..., n\}^3$ as follows: for all $(x, y, z), (x', y', z') \in \{0, 1, ..., n\}^3$,

$$(x', y', z') \widetilde{\preceq} (x, y, z) \Leftrightarrow z' \leq z \text{ and } x' \geq x.$$

Denote the restriction of the preorder $\widetilde{\preceq}$ on the set $L(\widetilde{D}_m)$ by $\widetilde{\preceq}_m$ and let $\widetilde{Y}_m$ be a non-empty subset of $L(\widetilde{D}_m)$. Denote by $\widetilde{X}_m = U(\widetilde{Y}_m)$ the upper contour set of $\widetilde{Y}_m$ (according to $\widetilde{\preceq}_m$) as the set of triples in $L(\widetilde{D}_m)$ such that they are more rightist than some triple in $\widetilde{Y}_m$; namely,

$$\widetilde{X}_m = U(\widetilde{Y}_m) = \{(l_<, l_=, l_>) \in L(\widetilde{D}_m) \mid (x, y, z) \widetilde{\preceq}_m (l_<, l_=, l_>) \text{ for some } (x, y, z) \in \widetilde{Y}_m\}.$$

By convention, set $U(\varnothing) = \varnothing$. Given $\widetilde{X}_m \subseteq L(\widetilde{D}_m)$ with the property that $\widetilde{X}_m = U(\widetilde{X}_m)$, define $g^{\widetilde{X}_m} : \widetilde{D}_m \longrightarrow \{a_m, b_m\}$ as follows: for every $t \in \widetilde{D}_m$,

$$g^{\widetilde{X}_m}(t) = \begin{cases} b_m & \text{if } l^m(t) \in \widetilde{X}_m \\ a_m & \text{otherwise.} \end{cases}$$

Given a family of discontinuity jumps $I = \{I_m\}_{m \in M}$ we say that $\{\widetilde{X}_m\}_{m \in M}$ is an *anonymous family of tie-breaking sets of* M if for all $m \in M$, $\widetilde{X}_m \subseteq L(\widetilde{D}_m)$ and $\widetilde{X}_m = U(\widetilde{X}_m)$.

Definition 7 A social choice function $f : \mathcal{SSP}^N \longrightarrow [0, 1]$ is a *disturbed median* if there exist:

(1) a family of $n+1$ fixed ballots $0 \leq p_n \leq ... \leq p_1 \leq p_0 \leq 1$;

(2) a family of discontinuity jumps $I = \{I_m\}_{m \in M}$ compatible with $p_n, ..., p_1, p_0$; and

(3) an anonymous family of tie-breaking sets $\{\widetilde{X}_m\}_{m \in M}$ of M

such that, for all $t = (t_1, ..., t_n) \in \mathcal{SSP}^N$,

$$f(t) = \begin{cases} \Pi^I(med\{t_1, ..., t_n, p_n, ..., p_0\}) & \text{if } med\{t_1, ..., t_n, p_n, ..., p_0\} \neq d_m \text{ for all } m \in M \\ g^{\widetilde{X}_m}(t_1, ..., t_n) & \text{if } med\{t_1, ..., t_n, p_n, ..., p_0\} = d_m \text{ for an } m \in M. \end{cases}$$

Corollary 1 *A social choice function $f : \mathcal{SSP}^N \longrightarrow [0,1]$ is strategy-proof and anonymous if and only if it is a disturbed median.*

Corollary 2 *A social choice function $f : \mathcal{SSP}^N \longrightarrow [0,1]$ is strategy-proof, anonymous, and efficient if and only if it is a median voter scheme with the property that $p_n = 0$ and $p_0 = 1$.*

Efficiency requires that f respects unanimity and hence, $r_f = [0,1]$. Thus, (i) its associated family of $n+1$ fixed ballots has the property that $0 = p_n \leq p_{n-1} \leq ... \leq p_0 = 1$ and (ii) the family of discontinuity sets M is empty. Observe that since $p_n = 0$ and $p_0 = 1$ they cancel each other out in the computation of the median at any profile t and therefore, the generalized median voter scheme can also be described as the median of the n tops and the $n-1$ fixed ballots $p_{n-1} \leq ... \leq p_1$. This corresponds to Moulin (1980)'s characterization of the class of strategy-proof, anonymous and efficient social choice functions on the domain of single-peaked preferences. Thus, the reduction of the domain does not generate in this case new strategy-proof, anonymous and efficient social choice functions.

5.2 Feasibility constraints

Our result has important implications for the design of strategy-proof social choice functions on the domain of symmetric single-peaked preferences under feasibility constraints. Often, some subsets of alternatives (although conceivable) can not be chosen due to feasibility constraints. Then, discontinuities are compulsory rather than pathological because discontinuity jumps on the range of strategy-proof social choice functions are necessary. Our result precisely describes their nature and how the strategy-proof social choice function may select its value at these discontinuity points. However, if f is a strategy-proof and discontinuous social choice function then, $r_f \subsetneq [0,1]$ and hence, f will not be efficient; in particular, f will not respect unanimity. Social choice functions that are not efficient but they are efficient relative to the feasible set of alternatives are specially interesting. Thus, let $A \subsetneq [0,1]$ be a closed set of feasible alternatives.[19] A social choice function $f : \mathcal{SSP}^N \longrightarrow [0,1]$ is *efficient relative to* A if $r_f \subseteq A$ and for all $R \in \mathcal{SSP}^N$ there is no $z \in A$ such that, for all $i \in N$, $zR_i f(R)$ and $zP_j f(R)$ for some $j \in N$. The following result follows from Theorem 1.

Corollary 3 *Let A be a closed subset of $[0,1]$. A social choice function $f : \mathcal{SSP}^N \longrightarrow [0,1]$ is strategy-proof and efficient relative to A if and only if it is a disturbed minmax with $r_f = A$.*

Note that the requirement $r_f = A$ imposes certain conditions on the family of fixed ballots $\{p_S\}_{S \in 2^N}$ and on the discontinuity jumps. For instance $p_N = \min\{x \in A\}$, $p_\varnothing = \max\{x \in A\}$ and $p_S \in A$ for all $S \in 2^N$. Moreover since A is closed the set $[p_N, p_\varnothing] \backslash A$ is open and therefore it can be written as a countable and disjoint union of open intervals: $[p_N, p_\varnothing] \backslash A = \cup_{m \in M} I_m$ where I_m is an open interval for all $m \in M$ and $I_m \cap I_{m'} = \varnothing$ for all $m, m' \in M$. This representation is unique up to permutations in M, and in fact the requirement $r_f = A$ implies that the family of discontinuity jumps compatible with $\{p_S\}_{S \in 2^N}$ is exactly $I = \{I_m\}_{m \in M}$.

As an illustration of Corollary 3, suppose that the set of feasible alternatives is $A = \{0\} \cup \{0.1\} \cup [0.2, 0.8] \cup \{0.9\}$. In that case the only general requirements on the fixed ballots are that $p_N = 0$, $p_\varnothing = 0.9$ and p_S has to belong to A for all $S \in 2^N$. The family of discontinuity jumps is given by $I_1 = (0, 0.1)$, $I_2 = (0.1, 0.2)$, and $I_3 = (0.8, 0.9)$, and therefore the discontinuity points are $d_1 = 0.05$, $d_2 = 0.15$ and $d_3 = 0.85$. To proceed with the illustration and in order to design a particular strategy-proof and anonymous social choice function f whose range r_f be equal to A let $N = \{1, 2, 3\}$ be the set of agents and let $p_3 = p_2 = 0$ and $p_1 = p_0 = 0.9$ be the family of four fixed ballots. In this particular case the ballots cancel each other and hence, for all $(t_1, t_2, t_3) \in \mathcal{SSP}^{\{1,2,3\}}$, $med\{t_1, t_2, t_3, 0, 0, 0.9, 0.9\} = med\{t_1, t_2, t_3\}$. For each discontinuity point d_m the set $L(\widetilde{D}_m)$ consists of four triplets: $L(\widetilde{D}_m) = \{(1, 2, 0), (0, 3, 0), (1, 1, 1), (0, 2, 1)\}$ where for example, the triplet $(1, 2, 0)$ means that one top is strictly below d_m and the remaining two tops are exactly equal to d_m. Note, that in all the four cases the median of the tops coincides with d_m, and hence all the profiles of tops that are represented by $L(\widetilde{D}_m)$ result in discontinuity points. Moreover, and since $L(\widetilde{D}_1) = L(\widetilde{D}_2) = L(\widetilde{D}_3)$, $\widetilde{\preceq}_1 = \widetilde{\preceq}_2 = \widetilde{\preceq}_3$ as well. Denote it by $\widetilde{\preceq}'$ and observe that $(1, 2, 0) \widetilde{\preceq}' (1, 1, 1) \widetilde{\preceq}' (0, 2, 1)$, $(1, 2, 0) \widetilde{\preceq}' (0, 3, 0) \widetilde{\preceq}' (0, 2, 1)$ and that $(1, 1, 1)$ and $(0, 3, 0)$ are not comparable by $\widetilde{\preceq}'$. To assign a value to the social choice function on these discontinuity points preserving the

[19] Remember that, by Lemma 2, strategy-proof social choice functions have a closed range.

monotonicity of the social choice function f we need to select for each d_m a tie-breaking set $\widetilde{X}_m$ such that $\widetilde{X}_m = U(\widetilde{X}_m)$. Given $L(\widetilde{D}_m)$, there are six different ways of doing so: $\widetilde{X}_m \in \{\emptyset, \{(0,2,1)\}, \{(1,1,1),(0,2,1)\}, \{(0,3,0),(0,2,1)\}, \{(1,1,1,),(0,3,0),(0,2,1)\}, L(\widetilde{D}_m)\}$. For instance, choose $\widetilde{X}_1 = \{(1,1,1),(0,2,1)\}$, $\widetilde{X}_2 = \{(0,2,1)\}$, and $\widetilde{X}_3 = L(\widetilde{D}_m)$. Thus, the disturbed median f that we may define applying Definition 7 to the family of four fixed ballots $0 = p_3 = p_2 < p_1 = p_0 = 0.9$, the family of discontinuity jumps $I_1 = (0, 0.1)$, $I_2 = (0.1, 0.2)$, and $I_3 = (0.8, 0.9)$, and the anonymous family of tie-breaking sets $\widetilde{X}_1 = \{(1,1,1),(0,2,1)\}$, $\widetilde{X}_2 = \{(0,2,1)\}$, and $\widetilde{X}_3 = L(\widetilde{D}_3)$ has range equal to A and it is efficient relative to A. The disturbed median f could also be defined as follows. For all $t = (t_1, t_2, t_3) \in \mathcal{SSP}^{\{1,2,3\}}$, and after setting $y \equiv med\{t_1, t_2, t_3\}$,

$$f(t) = \begin{cases} 0 & \text{if } y < 0.05 \text{ or } y = 0.05 \text{ and } \#\{i \mid t_i \leq 0.05\} = 3 \\ 0.1 & \text{if } y = 0.05 \text{ and } \#\{i \mid t_i \leq 0.05\} < 3 \text{ or } 0.05 < y < 0.15 \\ & \quad \text{or } y = 0.15 \text{ and either } \exists j \text{ s.t. } t_j < 0.15 \text{ or } t_1 = t_2 = t_3 = 0.15 \\ 0.2 & \text{if } y = 0.15 \text{ and } \#\{i \mid t_i \geq 0.15\} = 3 \text{ and } \exists j \text{ s.t. } t_j > 0.15 \\ & \quad \text{or } 0.15 < y < 0.2 \\ y & \text{if } 0.2 \leq y \leq 0.8 \\ 0.8 & \text{if } 0.8 < y < 0.85 \\ 0.9 & \text{if } y \geq 0.85. \end{cases}$$

The complexity of this description indicates the usefulness of Theorem 1's characterization.

Finally, by Remark 1, the four statements above (Theorem 1 and Corollaries 1, 2 and 3) also hold after replacing strategy-proofness by group strategy-proofness.

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