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PIERROT'S THEOREM FOR SINGULAR RIEMANIAN FOLIATIONS

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Abstract

Let $\mathcal{F}$ be a singular Riemannian foliation on a compact connected Riemannian manifold M. We demonstrate that global foliated vector fields generate a distribution tangent to the strata defined by the closures of leaves of $\mathcal{F}$ and which, in each stratum, is transverse to these closures of leaves.

The aim of this short note is to prove M. Pierrot's theorem for singular Riemanian foliations, cf. [**5**], namely.

Theorem 1. *Let $\mathcal{F}$ be an SRF on a compact manifold M. Then the vector space of global foliated vector fields is transitive to the closures of leaves in each closure stratum.*

1. Preliminaries

First we recall some and prove other results about SRF-s (singular Riemannian foliations), cf. [**3**] and [**4**].

Assume that the manifold M is compact and connected (or the metric is complete). Then the closure of any leaf is a submanifold.

Let k be any number between 0 and n. Define

$$\Sigma_k = \{x \in M : x \in L_\alpha, \dim L_\alpha = k\}.$$

The leaves of $\mathcal{F}$ is Σ_k are of the same dimension, however they can have holonomy. P. Molino demonstrated that the sets Σ_k or rather their connected components are submanifolds of M and $\overline{\Sigma_k} \subset \bigcup_{i\leq k} \Sigma_i$. Note that for some i the sets Σ_i can be empty. Moreover, let k_0 be the maximum dimension of leaves of $\mathcal{F}$. Then the set Σ_{k_0} is open and dense in M. It is the principal stratum. In fact, the partition $\{\Sigma_k\}_0^n$ is an abstract stratification.

Let W be a compact submanifold of M. The geodesics define the exponential mapping $\exp : N(W) \to M$. Denote by $S_r(W) = \{v \in N(W) : \|v\| = r\}$ (resp. $D_r(W) = \{v \in N(W) : \|v\| \leq r\}$) and by $S(W,r)$ (resp. $D(W,r)$) its image by exp. If W is a closed leaf or the closure of a stratum then it is not difficult to notice that leaves of the foliation $\mathcal{F}$ live on $S(W,r)$, cf. [**3**], [**4**]. Moreover, the homotethies (along the geodesics) $h_\lambda : D(W,r) \to D(W,|\lambda|r)$, $h_\lambda(\exp(v)) = \exp(\lambda v)$ preserve the foliation. The leaf passing through $\exp(v)$ has the same dimension and holonomy as the leaf passing through $\exp(\lambda v)$.

Connected components of Σ_i are submanifolds of M. They can be of different codimension and it can happen that some connected component of Σ_i is a compact submanifold. Since the foliation is Riemannian the closure of a leaf from a stratum Σ_i remains in it. In fact, let $\partial\Sigma_i = V_1 \cup \cdots \cup V_k$ where $\bigcup_{s=1}^k V_s = \overline{\Sigma}_i - \Sigma_i$, each V_s being a connected submanifold of M. In a tubular neighbourhood of V_s leaves of $\mathcal{F}$ live on the sphere bundles $S(V_s, r)$. Thus if $L \subset S(V_s, r)$, so does its closure $\overline{L}$. Therefore for all our purposes the foliation $\mathcal{F}|\Sigma_i$ behaves like a RF on a compact manifold. Therefore we can define the subspaces

$$\Sigma_{ij} = \{x \in \Sigma_i : x \in L \in \mathcal{F},\, \dim \overline{L} = j\}.$$

Each Σ_{ij} is a submanifold of Σ_i and $\partial\Sigma_{ij_0} \subset \bigcup_{s<i} \Sigma_s \bigcup_{j<j_0} \Sigma_{ij}$. The closures of leaves of $\mathcal{F}$ induce a regular RF $\mathcal{F}_{ij}$ of compact leaves on Σ_{ij}. The leaves of $\mathcal{F}_{ij}$ have finite holonomy. Using the exponential mapping restricted to the normal bundle of a leaf one easily learns that the holonomy of a leaf is conjugated to the linear holonomy of this leaf. The linear holonomy is a finite subgroup of the linear orthogonal group. The linear holonomy groups $h(L,x)$ at different points x of a given leaf L are conjugated; let us denote this conjugacy class by $h(L)$. If α denotes a conjugacy class of a subgroup of the linear orthogonal group then let $\Sigma_{ij\alpha} = \{x \in \Sigma_{ij} : x \in L \in \mathcal{F}_{ij}, h(L) = \alpha\}$.

In [**5**] M. Pierrot uses a slightly rougher stratification for regular RFs, namely

$$\Sigma_{pjk} = \{x \in L \in \mathcal{F} : \dim \overline{L} = j, \sharp h(\overline{L}, x) = k\}$$

where $p = \dim \mathcal{F}$, and the holonomy is considered in the stratum Σ_j. However, in a tubular neighbourhood of a compact leaf $\overline{L}$, the foliation $\overline{\mathcal{F}}$ by the closures of leaves, is conjugated to the natural foliation of the flat bundle $\overline{L} \times_G R^s$ where G is the linear holonomy group of the leaf $\overline{L}$ and $s = \operatorname{codim}_{\Sigma_j} \overline{L}$. It is not difficult to notice that in these tubular neighbourhoods leaves of $\overline{\mathcal{F}}$ have their linear holonomy groups conjugated to a subgroup of G. It means that for any α, $G \in \alpha$, $\sharp G = k\,\Sigma_{pj\alpha} \subset \Sigma_{pjk}$ and the submanifolds Σ_{pjk} are separated. If $\Sigma_{pj\alpha}$ and $\Sigma_{pj\beta}$ are two such sets then the lemma concerning the homotetheies, cf. [**3**], [**4**], ensures that $\overline{\Sigma_{pj\alpha}} \cap \overline{\Sigma_{pj\beta}} = \emptyset$. Therefore connected components of $\Sigma_{pj\alpha}$ are also connected components of Σ_{pjk}. Thus connected components of these sets define the same stratification $\{\Sigma_\gamma\}$. The stratification $\{\Sigma_\gamma\}$ possesses a natural partial order

$$\Sigma_\gamma \leq \Sigma_{\gamma'} \text{ iff } \Sigma_\gamma \subset \overline{\Sigma}_{\gamma'}.$$

The strata defined above we call the closure strata of the foliation $\mathcal{F}$ to distinguish them from the strata defined by the dimension of leaves.

In [**3**], [**4**] P. Molino describes a way of desingularization of SRFs. Let Σ be a minimal stratum. Σ is a closed submanifold. Let $N(\Sigma)$ be the normal bundle of Σ. Leaves of $\mathcal{F}$ also live on sphere bundles $S(\Sigma, r)$ over Σ. Take $M^0 = (M - \Sigma) \times \{0\}$, $M^1 = (M - \Sigma) \times \{1\}$ and $S = S(\Sigma, r) \times (-1, 1)$ for some $r > 0$. Then M^0, M^1 and S glue together to become a compact manifold M_1, i.e. $S(\Sigma, r) \times \{t\}$ is identified with $S(\Sigma, |t|r) \times \{0\} \subset M^0$ if $t < 0$ and with $S(\Sigma, |t|r) \times \{1\} \subset M^1$ if $t > 0$. M_1 projects onto M, $p : M_1 \to M$. Over $M - \Sigma\, p$ is a double covering and $p^{-1}(\Sigma) = S(\Sigma, r)$.

P. Molino proves that on M_1 there exists an SRF $\mathcal{F}_1$, which does not have leaves of the type encountered in Σ, and including the old foliation $\mathcal{F}$ on M^0 and M^1. After a finite number of steps we get a regular Riemannian foliation on a compact manifold M_s.

Using the exponential mapping it is quite easy to prove a following lemma.

Lemma 1. *For any $0 < \delta_1 < \delta_2 \leq \epsilon$ there exists a basic smooth function*

$$\lambda(\delta_1, \delta_2) : D(\Sigma, \epsilon) \to [0, 1]$$

such that $\operatorname{supp} \lambda(\delta_1, \delta_2) \subset D(\Sigma, \delta_2)$ *and* $\lambda(\delta_1, \delta_2)|D(\Sigma, \delta_1) \equiv 1$.

In our future considerations we shall need the following relations between basic functions on the foliated manifolds $(M, \mathcal{F})$ and $(M_1, \mathcal{F}_1)$.

Lemma 2. *Let f be a basic function on $(M_1, \mathcal{F}_1)$. Then for any point $x \in M^0$ there exists a foliated neighbourhood U of x in M^0 and a basic function f_U on $(M, \mathcal{F})$ such that $f_U p|U = f|U$.*

Proof: The set $D(\Sigma, \epsilon) - \Sigma = D^0(\Sigma, \epsilon)$ can be considered as (via p) an open subset of M^0. Therefore we have to consider two cases: (a) $x \notin D^0(\Sigma, \epsilon)$ and (b) $x \in D^0(\Sigma, \epsilon)$.

In the case (a) as U we can take $M - D(\Sigma, \delta_2)$, $0 < \delta_2 < \epsilon$ and as f_U the function

$$\begin{cases} f(z) & z \notin D^0(\Sigma, \epsilon) \\ (1 - \lambda(\delta_1, \delta_2))f(z) & z \in D^0(\Sigma, \epsilon),\ 0 < \delta_1 < \delta_2 \\ f(z) = 0 & z \in \Sigma. \end{cases}$$

In the case (b) let $x \in S(\Sigma, r)$, $0 < r \leq \epsilon$. Then we take $U = M - D(\Sigma, r/2)$ and define the function as in the case (a) taking $0 < \delta_1 < \delta_2 < r/2$. ■

Lemma 3. *Let f be a basic function on the foliated manifold $(M, \mathcal{F})$. Then for any point x of $M - \Sigma$ there exists an open foliated neighbourhood U of x in $M - \Sigma$ and a basic function f_U on $(M_1, \mathcal{F}_1)$ such that $f|Up = f_U|M^0 \cap p^{-1}(U)$.*

Proof: It is analogous to that of Lemma 2. Using this construction we obtain a basic function $\hat{f}_U$ with compact support on $(M^0, \mathcal{F}_1)$; we extend it to M_1 putting 0 on Σ and M^1. ■

Let us recall the definition of the 'musical' isomorphism, for example cf. [**1**].

$$\flat : TM \to T^*M$$

is given by: for $X \in TM_x X^\flat$ is the only 1-form such that

$$g(X, Y) = X^\flat(Y) \text{ for any } Y \in TM_x.$$

$$\sharp : T^*M \to TM$$

for any $\omega \in T^*M_x\, \omega^\sharp$ is the only vector for such that

$$g(\omega^\sharp, Y) = \omega(Y) \text{ for any } Y \in TM_x.$$

Therefore to any function f on M we associate a vector field X^f by the formula

$$g(X^f, Y) = df(Y) \text{ for any } Y \in TM \text{ or } X^f(x) = (df_x)^\sharp.$$

Now we shall study the properties of vector fields associated to basic functions. First let us notice that for any basic function f the vector field X^f is orthogonal to the leaves of the foliation. Moreover if the function f is global the vector field X^f is orthogonal to the closures of leaves.

Lemma 4. *If f is a basic function then the vector field X^f is an infinitesimal automorphism of the foliation.*

The proof is a straightforward calculation.

2. Regular case

Let $\mathcal{F}$ be an RF. We shall look at the existence of global basic functions. Denote $\mathcal{X}^\sharp(M,\mathcal{F})$ the vector space of global vector fields of the form X^f for some global basic function f on $(M,\mathcal{F})$.

The closures of leaves form an SRF and we can consider strata for this foliation, cf. [**5**]. These strata are just our closure strata for $\mathcal{F}$ as $\mathcal{F}$ being regular we have just the principal stratum for this foliation. It is obvious that global infinitesimal automorphisms must be tangent to the closure strata. Let Σ be one of these strata.

Lemma 5. *For any vector $X \in T\Sigma_x$ orthogonal to the closure S of the leaf L in Σ passing through x, there exists a global basic function f such that $df(X) \neq 0$.*

Proof: There exists $\epsilon > 0$ such that the mapping $\exp_S : B_\epsilon(X) \to M$ is an embedding. Then there is a leaf L', with the closure S', of the same stratum Σ on the geodesic with the initial condition X at the distance less than ϵ such that the mapping $\exp_{S'} : B_\epsilon(S') \to M$ is an embedding, cf. [**2**]. Then the function $f_{S'}(y) = d(y,S')^2$ is a smooth basic function on $\exp_{S'}(B_\epsilon(S'))$ for which $df_{S'}(X) \neq 0$. $f_{S'}$ can be easily extended to a global basic function. ■

Combining Lemmas 4 and 5 we get the following proposition which, in fact, is a variant of the theorem due to M. Pierrot, cf. [**5**].

Proposition 1. *Let $(M,\mathcal{F})$ be a compact foliated manifold with $\mathcal{F}$ being a regular RF. Then the vector space $\mathcal{X}^\sharp(M,\mathcal{F})$ is transitive to the closures of leaves in each closure stratum.*

3. Singular case

Now let $\mathcal{F}$ be an SRF on M. First we prove the singular version of Lemma 5.

Lemma 6. *Let $(M,\mathcal{F})$ be a compact foliated manifold with $\mathcal{F}$ being an SRF. Let Σ be a closure stratum of $\mathcal{F}$. For any vector $X \in T\Sigma_x$ orthogonal to the closure S of the leaf L passing through x there exists a basic function f such that $df(X) \neq 0$.*

Proof: Using the blowing up procedure and Lemma 2 we can reduce our considerations to the case where the point x belongs to the singular stratum Σ_0 of the foliation $\mathcal{F}$. Thus Σ is a submanifold of Σ_0 and a closure stratum of $(\Sigma_0, \mathcal{F})$ which is compact RM. Therefore according to Lemma 5 there exists a basic function f_0 on Σ_0 such that $df_0(X) \neq 0$. According to the next lemma this basic function can be easily extended to a global basic function on $(M, \mathcal{F})$. ■

Lemma 7. *Any basic function on a stratum Σ can be extended to a global basic function on M.*

Proof: Since the projection $p : B(\Sigma, \epsilon) \to \Sigma$ maps leaves onto leaves, for any basic function f on Σ, the function fp is basic on $B(\Sigma, \epsilon)$. Then using a function $\lambda(\delta_1, \delta_2)$ we can extend fp to a global basic function on $(M, \mathcal{F})$. ■

For vectors which are not tangent to strata we have the following lemma.

Lemma 8. *Let $\mathcal{F}$ be an SRF on a compact manifold M. If a vector field X is not tangent to the closure of a leaf L at a point x, then there exists a global basic function f such that the germ at x of the function $df(X)$ is not 0.*

Proof: Let S be the closure of the leaf L. It is a compact submanifold of M. Let $N(S)$ be its normal bundle. For some $\epsilon > 0$ the exponential mapping defined by the geodesics starting from vectors of $N(S)$ is a diffeomorphisms of $B_\epsilon(S) = \{v \in N(S) : \|v\| < \epsilon\}$ onto the image $B(S, \epsilon)$, cf. [**4**]. Using a similar method as in Lemma 2 we can extend any basic function on $B(S, \epsilon)$ to a global one. Therefore we have reduced our problem to a local one. Then the function

$$f_L(y) = d(L, y)^2$$

satisfies the conditions of the lemma. ■

4. Proof of Theorem 1

Let x be any point of a closure stratum Σ. Let V be the subspace of $T_x\Sigma$ orthogonal to T_xS, $S = \overline{L}_x$. We know that for any global basic function $fX_x^f \in V$. Lemma 6 ensures that there does not exist a vector in V which is orthogonal to all X_x^f. It means precisely that $SPAN\{X_x^f\} = V$. Therefore we have proved the following theorem:

Theorem 2. *Let M be a compact connected manifold and $\mathcal{F}$ be an SRF on M. Then the vector space $X^\sharp(M, \mathcal{F})$ is transitive to the closures of leaves in each closure stratum of $(M, \mathcal{F})$.*

Of course Theorem 2 is just a more detailed version of Theorem 1.

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