Publicacions Matemàtiques, Vol **38** (1994), 255–267.

ON APPROXIMATION AND INTERPOLATION OF ENTIRE FUNCTIONS WITH INDEX-PAIR (p, q)

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Abstract

In this paper we have studied the Chebyshev and interpolation errors for functions in $\mathcal{C}(E)$, the normed algebra of analytic functions on a compact set E of positive transfinite diameter. The (p, q)-order and generalized (p, q)-type have been characterized in terms of these approximation errors. Finally, we have obtained a saturation theorem for $f \in \mathcal{C}(E)$ which can be extended to an entire function of (p, q)-order 0 or 1 and for entire functions of minimal generalized (p, q)-type.

Introduction

Let E be a compact set in complex plane and $\xi^{(n)} = \{\xi_{n0}, \xi_{n1}, \dots, \xi_{nn}\}$ be a system of $(n+1)$ points of the set E such that

$$V(\xi^{(n)}) = \prod_{0 \le j \le k \le n} |\xi_{nj} - \xi_{nk}| \text{ and } \Delta^{(j)}(\xi^{(n)}) = \prod_{\substack{k=0 \\ k \ne j}}^{n} |\xi_{nj} - \xi_{nk}|, \quad j = 0, 1, \dots, n.$$

Again, let $\eta^{(n)} = \{\eta_{n1}, \eta_{n2}, \dots, \eta_{nn}\}$ be the system of $(n+1)$ points in E such that

$$\begin{aligned} V_n &\equiv V(\eta^{(n)}) \\ &= \sup_{\xi^{(n)} \subset E} V(\xi^{(n)}) \text{ and } \Delta^0(\eta^{(n)}) \le \Delta^{(j)}(\eta^{(n)}) \text{ for } j = 1, 2, \dots, n. \end{aligned}$$

*This work was supported by the Swedish Institute, Stockholm.

Such a system always exists and is called the nth *extremal system* of E. The polynomials

$$L^{(j)}(z,\eta^{(n)})=\prod_{\substack{k=0\\k\neq j}}^{n}\left(\frac{z-\eta_{nk}}{\eta_{nj}-\eta_{nk}}\right),\quad j=0,1,\dots,n,$$

are called Lagrange extremal polynomials and the limit $d\equiv d(E)=\lim_{n\to\infty}V_n^{2/n(n+1)}$ is called the *transfinite diameter* of E.

Let $\mathcal{C}(E)$ denote the algebra of analytic functions on the set E. Let us define the approximation errors as follows:

$$\mu_{n,1}(f;E)\equiv\mu_{n,1}(f)=\inf_{g\in\pi_n}\|f-g\|,$$

where $\|\cdot\|$ is the sup norm and $\pi_n(z)$ denotes the set of all polynomials of degree $\leq n$.

Further, we also define

$$\mu_{n,2}(f;E)\equiv\mu_{n,2}(f)=\|L_n-L_{n-1}\|,\quad n\geq 2$$
$$\mu_{n,3}(f;E)\equiv\mu_{n,3}(f)=\|L_n-f\|,$$

where $n\in N$ and

$$L_n(z)=\sum_{j=0}^{n}L^{(j)}(z,\eta^{(n)})f(\eta_{nj})$$

is the *Lagrange interpolation polynomial* of degree n.

Reddy [**8**], [**9**] connected classical order and type with polynomial approximation error of an entire function which is an extension of a continuous function defined on $[-1,1]$. Contemporarily, Rice [**10**] and Winiarski [**13**] studied these results for different approximation errors of a continuous function on the arbitrary domains. Later on, Massa [**6**] developed a simpler proof of Reddy's results. It has been noticed that these authors fail to compare the approximation errors of those entire functions which have the same positive finite order but their types are infinity. In order to include this important class of entire functions we shall utilise the concept of proximate order due to Levin [**5**]. Moreover, for the inclusion of entire functions of slow growth and fast growth their results will also be extended to the (p,q)-scale introduced by Juneja et al. ([**2**], [**3**]). It is significant to mention that Shah [**11**] and Kapoor and Nautiyal [**3**] have studied in this direction for continuous functions on the domain $[-1,1]$. They studied the results for (α,β)-orders. However they have to study separately the entire functions of slow and fast growth . That is why in our studies the (p,q)-growth has been preferred to the (α,β)-growth.

1. Definitions and auxiliary results

Before stating the auxiliary results it will be justified to introduce with the concept of (p,q)-scale, $p \geq q \geq 1$, and certain notations which will be frequently used in the text:

$$\begin{aligned}\exp^{[m]} x &= \log^{[-m]} x \\ &= \exp(\exp^{[m-1]} x) = \log(\log^{[-m-1]} x), \quad m = \pm 1, \pm 2, \ldots,\end{aligned}$$

$$\Lambda_{[r]}(x) = \prod_{i=0}^{r} \log^{[i]} x \text{ for } r = 0, 1, \ldots,$$

$$P(L(p,q)) = \begin{cases} L(p,q) & \text{if } q < p < \infty, \\ 1 + L(p,q) & \text{if } p = q = 2, \\ \max(1, L(p,q)) & \text{if } 3 \leq p = q, \\ \infty & \text{if } p = q = \infty. \end{cases}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. We set $M(r,f) = \max_{|z|=r} |f(z)|$; $M(r,f)$ is called the maximum modulus of $f(z)$ on the circle $|z| = r$.

Definition 1. An entire function $f(z)$ is said to be of (p,q)-order $\rho(p,q)$ if it is of *index-pair* (p,q) such that

$$\limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} = \rho(p,q),$$

and the function $f(z)$ having (p,q)-order $\rho(p,q)(b < \rho(p,q) < \infty)$ is said to be of (p,q)-type $T(p,q)$ if

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,f)}{(\log^{[q-1]} r)^{\rho(p,q)}} = T(p,q), \quad 0 \leq T(p,q) \leq \infty,$$

where $b = 1$ if $p = q$, $b = 0$ if $p > q$.

Nandan et al. [**7**] has extended the idea of *proximate order* to entire functions with (p,q)-growth as

Definition 2. A positive function $\rho_{p,q}(r)$ defined on $[r_o, \infty)$, $r_o > \exp^{[q-1]} 1$, is said to be of the proximate order of an entire function with index-pair (p,q) if

(i) $\rho_{p,q}(r) \to \rho(p,q)$ as $r \to \infty$, $b < \rho(p,q) < \infty$,
(ii) $\Lambda_{[q]}(r)\rho'_{p,q}(r) \to 0$ as $r \to \infty$; $\rho'_{p,q}(r)$ denotes the derivative of $\rho_{p,q}(r)$.

It is known [**7**, Thm. 4] that $(\log^{[q-1]} r)^{\rho_{p,q}(r)-A}$ is a monotonically increasing function of r for $r > r_0$, where $A = 1$ if $(p,q) = (2,2)$ and $A = 0$ otherwise.

Hence we can define the function $\phi(x)$ for $x > x_0$ to be the unique solution of the equation,

$$x = (\log^{[q-1]} r)^{\rho_{p,q}(r)-A} \Longleftrightarrow \phi(x) = \log^{[q-1]} r.$$

Definition 3. Let $f(z)$ be an entire function of (p,q)-order $\rho(p,q)(b < \rho(p,q) < \infty)$ such that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,f)}{(\log^{[q-1]} r)^{\rho_{p,q}(r)}} = T^*(p,q), \quad 0 \le T^*(p,q) \le \infty.$$

If the quantity $T^*(p,q)$ is different from zero and infinite then $\rho_{p,q}(r)$ is said to be the proximate order of a given entire function $f(z)$ and $T^*(p,q)$ as its *generalized* (p,q)*-type*. Clearly, proximate order and the corresponding generalized (p,q)-type of an entire function are not uniquely determined. For example, if we add $c/\log^{[q]} r$, $0 < c < \infty$ to the proximate order $\rho_{p,q}(r)$ then $\rho_{p,q}(r) + c/\log^{[q]} r$ is also a proximate order satisfaying (i) and (ii) and consequently, the generalized (p,q)-type turns out to be $e^c T^*(p,q)$.

Definition 4. An entire function with index-pair (p,q) is said to be of *minimal, normal* and *maximal* (p,q)-type with respect to a proximate order according as $T^*(p,q)$ as zero, positive finite and infinite respectively.

Let E_r be the curve defined by

$$E_r = \{z \in C : |\psi(z)|d = r\},$$

where $\omega = \psi(z)$ is holomorphic and maps the unbounded component of the complement of E on $|\omega| > 1$ such that $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. Also, we set $\overline{M}(r,f) = \sup_{z\in E_r} |f(z)|$ for $r > 1$.

The following auxiliary results will be utilised in the sequel:

Lemma 1. *If $f(z)$ is an entire function of (p,q)-order $\rho(p,q)$, then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \overline{M}(r,f)}{\log^{[q]} r} = \rho(p,q),$$

and for $\rho(p,q)(b<\rho(p,q)<\infty)$, $T^*(p,q)$ is given by

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}\overline{M}(r,f)}{(\log^{[q-1]}r)^{\rho_{p,q}(r)}}=T^*(p,q).$$

Proof: Let z_o be a fixed point of the set E, and $r>1$. Then from [**13**],

$$r-2|E|-|z_o|\le|z|\le r+|E|+|z_o|\quad z\in E_r.$$

For $p\ge q\ge 1$, $\alpha<1$ and $\beta>1$, using $\log^{[q]}kx\simeq\log^{[q]}x$ as $x\to\infty$, $0<k<\infty$,

$$\frac{\log^{[p]}M(\alpha r)}{\log^{[q]}r}\le\frac{\log^{[p]}\overline{M}(r)}{\log^{[q]}r}\le\frac{\log^{[p]}M(\beta r)}{\log^{[q]}r}\text{ for }r>r_o.$$

Also, for $\rho(p,q)(b<\rho(p,q)<\infty)$, using $\log^{[q]}(k+x)\simeq\log^{[q]}x$ as $x\to\infty$,

$$\frac{\log^{[p-1]}M(r-a)}{(\log^{[q-1]}r)^{\rho_{p,q}(r)}}\le\frac{\log^{[p-1]}\overline{M}(r)}{(\log^{[q-1]}r)^{\rho_{p,q}(r)}}\le\frac{\log^{[p-1]}M(r+c)}{(\log^{[q-1]}r)^{\rho_{p,q}(r)}},$$

where

$$a=2|E|+z_o,\quad c=|E|+|z_o|.$$

After passing to the limits and taking the Definition 1 and Definition 3 into account this lemma is established.

Lemma 2. *If a function f is defined and bounded on a compact set E, then*

$$\mu_{n,1}(f)\le\|f-L_n\|\le(n+2)_{\mu_{n,1}}(f)$$

and

$$\|L_n-L_{n-1}\|\le 2(n+2)_{\mu_{n-1,1}}(f),\ \text{for } n=2,3,\ldots,$$

whre L_n is the Lagrange interpolation polynomial with nodes at extremal points η_{nj}.

The proof of this lemma is available as Lemma 3.2 in [**13**].

Lemma 3. *Let $f\in\mathcal{C}(E)$. Then f can be extended to an entire function if and only if*

$$\mu_{n,j}^{1/n}(f)\to 0 \text{ as } n\to\infty,\quad j=1,2,3.$$

This lemma is a direct consequence of Lemma 1, equation (4.5) of Winiarski [**13**] and an inequality due to Walsh [**12**, p. 77].

Lemma 4. *For every $f \in \mathcal{C}(E)$ and $\mu_{n,j}(f)$, $j = 1, 2, 3$, there exists an entire function $g_j(z) = \sum_{n=0}^{\infty} \mu_{n,j}(f) z^{n+1}$ such that*

$$\overline{M}(r, f) \leq a_o + 2g_j(r/d).$$

Proof: Define the function

$$\tilde{f}(z) = \pi_o + \sum_{k=0}^{\infty} \{\pi_{n+1}(z) - \pi_n(z)\}. \tag{1.1}$$

Obviously, $\tilde{f}(z) = f(z)$ for all $z \in E$. We prove that $\tilde{f}(z) = f(z)$ for all z in the complex plane. For this it is enough to show that this series converges uniformly on every compact subset of the complex plane, since

$$\begin{aligned} |\pi_{n+1}(z) = \pi_n(z)| &\leq \|\pi_{n+1} - \pi_n\| \,\forall\, z \in E \\ &\leq \mu_{n+1,1}(f) + \mu_{n,1}(f) \leq 2\mu_{n-1}(f), \end{aligned}$$

using Walsh inequality [**12**, p. 77], we have

$$|\pi_{n+1}(z) - \pi_n(z)| \leq 2\mu_{n,1}(f) \left(\frac{r}{d}\right)^{n+1} \text{ for } z \in E_r.$$

Hence

$$\begin{aligned} |\tilde{f}(z)| &\leq |\pi_o(z)| + \sum_{n=1}^{\infty} |\pi_{n+1}(z) - \pi_n(z)| \\ &\leq a_o + 2\sum_{n=0}^{\infty} \mu_{n,1}(f) \left(\frac{r}{d}\right)^{n+1} \text{ for } z \in E_r. \end{aligned} \tag{1.2}$$

The last series converges for every r and therefore, the series on the right of (1.1) converges uniformly on every compact subset of C and so $\tilde{f}(z)$ is entire and $\tilde{f}(z) = f(z)$. Consider the function

$$g_j(z) = \sum_{n=0}^{\infty} \mu_{n,j}(f) z^{n+1}.$$

Since $\lim_{n\to\infty} \mu_{n,j}^{1/n}(f) = 0$, $j = 1, 2, 3$, by Lemma 3, it follws that each $g_j(z)$ is entire and further, (1.2) implies the required inequality. ∎

2. Main results

Theorem 1. *If $f \in \mathcal{C}(E)$ can be extended to an entire function with index-pair (p,q), (p,q)-order $\rho(p,q)(b < \rho(p,q) < \infty)$ and generalized (p,q)-type $T^*(p,q)$, then for every $\mu_{n,j}(f)$, there exists an entire functions $g_j(z) = \sum_{n=0}^{\infty} \mu_{n,j}(f) z^{n+1}$ such that*

$$\rho(p,q,f) = \rho(p,q,g_j) \text{ and } T^*(p,q,f) = \beta T^*(p,q,g_j),$$

where $\beta = d^{-\rho(p,1)}$ for $q = 1$ and $\beta = 1$ for $q > 1$.

Proof: It has been shown in Lemma 3 and Lemma 4 that $\tilde{f}(z) = f(z)$ for all z in the complex plain and for each $\mu_{n,j}(f)$, the function $g_j(z) = \sum_{n=0}^{\infty} \mu_{n,j}(f) z^{n+1}$ is an entire function. Winiarski [**13**, p. 266] has proved that for any $\varepsilon > 0$,

$$\mu_{n,3}(f) \leq k\overline{M}(r,f)\left(\frac{de^{\varepsilon}}{r}\right)^n, \tag{2.1}$$

where k is a constant and $d > 0$ is the transfinite diameter of E.

Using (2.1) in the power series expansion of $g_j(z)$ for $j = 3$, it is inferred that

$$\begin{aligned} g_3\left(\frac{r}{de^{2\varepsilon}}\right) &= \sum_{n=0}^{\infty} \mu_{n,3}(f)\left(\frac{r}{de^{2\varepsilon}}\right)^{n+1} \\ &\leq \frac{kr\overline{M}(r,f)}{de^{2\varepsilon}} \sum_{n=0}^{\infty} \frac{1}{e^{n\varepsilon}} \\ &\leq \frac{kr\overline{M}(r,f)}{de^{\varepsilon}(e^{\varepsilon}-1)}, \end{aligned}$$

or

$$\log g_3\left(\frac{r}{de^{2\varepsilon}}\right) \leq O(1) + \log \overline{M}(r,f) + \log r.$$

Thus, in view of the above inequality and Lemma 1, for $p \geq 2$ and $q = 1$,

$$\rho(p,1,g_3) \leq \rho(p,1,f) \text{ and } T^*(p,1,g_3) \leq e^{2\varepsilon\rho(p,1)} d^{\rho(p,1)} T^*(p,1,f),$$

and for $p \geq 2$, and $q > 1$,

$$\rho(p,q,g_3) \leq \rho(p,q,f) \text{ and } T^*(p,q,g_3) \leq T^*(p,q,f).$$

Since $\varepsilon > 0$ is arbitrary, both inequalities imply that for all (p,q),

$$\rho(p,q,g_3) \leq \rho(p,q,f) \text{ and } \beta T^*(p,q,g_3) \leq T^*(p,q,f). \tag{2.2}$$

Further, using the inequality $\overline{M}(r,f) \leq a_o + 2g_j\left(\frac{r}{d}\right)$, we observe that for $q = 1$,

$$\rho(p,1,f) \leq \rho(p,1,g_j) \text{ and } T^*(p,1,f) \leq d^{-\rho(p,1)} T^*(p,1,g_j),$$

and for $q > 1$,

$$\rho(p,q,f) \leq \rho(p,q,g_j) \text{ and } T^*(p,q,f) \leq T^*(p,q,g_j).$$

Hence, for all index-pairs (p,q),

$$\rho(p,q,f) \leq \rho(p,q,g_j) \text{ and } T^*(p,q,f) \leq \beta T^*(p,q,g_j). \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\rho(p,q,f) = \rho(p,q,g_3) \text{ and } T^*(p,q,f) = \beta T^*(p,q,g_3)$$

and further, application of Lemma 2 makes this result valid for $j = 1$ and $j = 2$ also. ■

Theorem 2. *Let $f(z) \in C(E)$. Then $f(z)$ can be extended to an entire function of (p,q)-order $\rho(p,q)(b < \rho(p,q) < \infty)$ if and only if*

$$\rho(p,q) = P(L(p,q)),$$

where

$$L(p,q) = \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \mu_{n,j}^{-1/n}(f)}.$$

Proof: In Lemma 3 and Lemma 4 we have concluded that $f \in C(E)$ can be extended to an entire function if and only if $g_j(z)$ is an entire function. Moreover, by Theorem 1, $f(z)$ and $g_j(z)$ have the same (p,q)-order. Applying Corollary 1 by Juneja et al. [**1**, p. 62] to the function $g_j(z) = \sum_{n=0}^{\infty} \mu_{n,j}(f) z^{n+1}$ Theorem 2 follows. ■

Remarks.

(a) For $(p,q)=(2,1)$, this theorem includes Theorem 2 by Winiarski [**13**] and further for $(p,q)=(2,1)$, $j=1$ and $E=[-1,1]$ this also contains a result by Massa [**6**].

(b) On setting $(p,q)=(2,2)$, $j=1$ and $E=[-1,1]$ this theorem may also be considered as an alternative proof of Theorem 5 by Reddy [**7**].

Theorem 3. *Let $f(z)\in\mathcal{C}(E)$. Then $f(z)$ can be extended to an entire function of (p,q)-order $\rho(p,q)(b<\rho(p,q)<\infty)$ and generalized (p,q)-type $T^*(p,q)(0<T^*(p,q)<\infty)$ if and only if*

$$\frac{T^*(p,q)}{\beta M(p,q)}=\limsup_{n\to\infty}\left[\frac{\phi(\log^{[p-2]}n)}{\log^{[q-1]}\mu_{n,j}^{-1/n}(f)}\right]^{\rho(p,q)-A},$$

where β is defined in Theorem 1 and

$$M(p,q)=\begin{cases}(\rho(2,2)-1)^{\rho(2,2)-1}/(\rho(2,2))^{\rho(2,2)} & \text{if } (p,q)=(2,2),\\ 1/e\rho(2,1) & \text{if } (p,q)=(2,1),\\ 1 & \text{otherwise.}\end{cases}$$

Proof: To prove this theorem we apply Theorem 1 by Kasana [**4**] to the function $g_j(z)$ defined in Lemma 4 and the resulting characterisation of $T^*(p,q,g_j)$ in terms of $\mu_{n,j}(f)$ and the relation $T^*(p,q;f)=\beta T^*(p,q,g_j)$ taking together prove the theorem. ■

Taking $\rho_{p,q}(r)=\rho(p,q)\,\forall\, r>r_o$ and $\phi(x)=x^{1/(\rho(p,q)-A)}$, we have the following corollary which gives a formula for (p,q)-type $T(p,q)$ in terms of the approximation errors of an entire function $f(z)$.

Corollary. *Let $f(z)\in\mathcal{C}(E)$. Then $f(z)$ is the restriction of an entire function having (p,q)-order $\rho(p,q)(b<\rho(p,q)<\infty)$ and (p,q)-type $T(p,q)(0<T(p,q)<\infty)$ if and only if*

$$\frac{T(p,q)}{\beta M(p,q)}=\limsup_{n\to\infty}\frac{\log^{[p-2]}n}{(\log^{[q-1]}\mu_{n,j}^{-1/n}(f))^{\rho(p,q)-A}}.$$

(a) for $(p,q)=(2,1)$ this corollary gives the Theorem 1 of Winiarski [**13**] *as a particular case.*

(b) On the domain $E=[-1,1]$ and for the approximation error $\mu_{n,3}$ this corollary also includes some of the theorems of Massa [**6**] *and Reddy* [**7**], *respectively for $(p,q)=(2,1)$ and $(p,q)=(2,2)$.*

Theorem 4. *If $f(z) \in \mathcal{C}(E)$ can be extended to an entire function of (p,q)-order $\rho(p,q)$ such that $\rho(p,q) = b$, then for every $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{(\log^{[p-2]} n)^{\delta}}{\log^{[q-1]} \mu_{n,j}^{-1/n}(f)} = 0.$$

Further, if $\rho(p,q) > b$ and $f(z)$ is of minimal generalized (p,q)-type, then

$$\limsup_{n\to\infty} \frac{\phi(\log^{[p-2]} n)}{\log^{[q-1]} \mu_{n,j}^{-1/n}(f)} = 0,$$

where b has the same meaning as in Definition 2.

Proof: Since $\rho(p,q) = b$, it follows from Lemma 1 that for given $\varepsilon > 0$ and $r > r_o$,

$$\log \overline{M}(r,f) < \exp^{[p-2]}(\log^{[q-1]} r)^{b+\varepsilon}.$$

Using (2.1) in above, we get

$$\log \mu_{n,3}^{1/n}(f) < \log k + \log de^{\varepsilon_1} - \log r + \frac{\exp^{[p-2]}(\log^{[q-1]} r)^{b+\varepsilon}}{n}. \tag{2.4}$$

Choose the value of r satisfying

$$r = \exp^{[q-1]}\left(\log^{[p-2]} \frac{n}{b+\varepsilon}\right)^{1/(b+\varepsilon)}. \tag{2.5}$$

For $(p,q) = (2,1)$, (2.5) implies $r = (n/\varepsilon)^{1/\varepsilon}$ and using this value in (2.4), we get

$$\mu_{n,3}(f) < k(de^{\varepsilon_1})^n \left(\frac{e\varepsilon}{n}\right)^{n/\varepsilon},$$

or

$$\mu_{n,3}^{1/n}(f) < \frac{k^{1/n} de^{\varepsilon_1}(e\varepsilon)^{1/\varepsilon}}{n^{1/\varepsilon}},$$

which on taking limits gives

$$\limsup_{n\to\infty} n^{1/\varepsilon} \mu_{n,3}^{1/n}(f) < \infty. \tag{2.6}$$

In case of $(p,q)=(2,2)$ we observe that $\log r = \left(\frac{n}{1+\varepsilon}\right)^{1/(1+\varepsilon)}$ satisfies (2.5) and (2.4) is reduced to

$$\log \mu_{n,3} < \log k + \frac{n}{1+\varepsilon} + n\left(\log de^{\varepsilon_1} - \left(\frac{n}{1+\varepsilon}\right)^{1/(1+\varepsilon)}\right),$$

or

$$\log_{n,3}^{-1/n} > (1+o(1))\left(\frac{n}{1+\varepsilon}\right)^{1/(1+\varepsilon)}.$$

Thus,

$$\limsup_{n\to\infty} \frac{n^{1/(1+\varepsilon)}}{\log \mu_{n,3}^{-1/n}(f)} \leq 1. \tag{2.7}$$

Finally, for $(p,q)\neq(2,1)$ and $(p,q)\neq(2,2)$, (2.4) and (2.5) give

$$\log^{[q-1]} \mu_{n,3}^{-1/n}(f) > (1+o(1))\left(\log^{[p-2]} \frac{n}{\varepsilon}\right)^{1/\varepsilon}, \quad p>q,$$

or

$$\log^{[q-1]} \mu_{n,3}^{-1/n}(f) > (1+o(1))\left(\log^{[p-2]} \frac{n}{1+\varepsilon}\right)^{1/(1+\varepsilon)}, \quad p=q.$$

It means that for all $p \geq q \geq 3$,

$$\limsup_{n\to\infty} \frac{(\log^{[p-2]} n)^{1/(b+\varepsilon)}}{\log^{[q-1]} \mu_{n,3}^{-1/n}(f)} \leq 1. \tag{2.8}$$

Clearly, (2.6), (2.7) and (2.8) combine to give

$$\limsup_{n\to\infty} \frac{(\log^{[p-2]} n)^{\delta}}{\log^{[q-1]} \mu_{n,j}^{-1/n}(f)} < \infty. \tag{2.9}$$

for every $\delta>0$ and $j=1,2,3$.

If limit superior in (2.9) is finite and positive for some $\delta>0$ then for every $\alpha>0$, we have

$$\limsup_{n\to\infty} \frac{(\log^{[p-2]} n)^{\delta+\alpha}}{\log^{[q-1]} \mu_{n,j}^{-1/n}(f)} = \infty.$$

This is a contradiction to what we obtained in (2.9) and hence the first part is proved. ■

For the second part, we put $T^*(p,q) = 0$ in Theorem 2 to get the required result.

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Primera versió rebuda el 3 de Maig de 1993,
darrera versió rebuda el 24 d'Octubre de 1994