Publicacions Matemàtiques, Vol **38** (1994), 37–49.

# THE PROPERTY $(H_u)$ AND $(\tilde{\Omega})$ WITH THE EXPONENTIAL REPRESENTATION OF HOLOMORPHIC FUNCTIONS

NGUYEN MINH HA AND NGUYEN VAN KHUE

*Abstract*

The main aim of this paper is to prove that a nuclear Frechet space $E$ has the property $(H_u)$ (resp. $(\tilde{\Omega})$) if and only if every holomorphic function on $E$ (resp. on some dense subspace of $E$) can be written in the exponential form.

Let $E$ be a locally convex space. We say that $E$ has the property $(H_u)$ and write $E \in (H_u)$ if every holomorphic function $f$ on $E$ is of uniform type. This means that there exists a continuous semi-norm $\rho$ on $E$ such that $f$ can be factorized holomorphically through the canonical map $\omega_\rho : E \to E_\rho$, where $E_\rho$ denotes the Banach space associated to $\rho$. On the other hand, we recall that $E$ is called a space having the property $(\tilde{\Omega})$ if for every neighbourhood $U$ of $O \in E$ there exists a neighbourhood $V$ of $O \in E$ and $d > 0$ such that for every neighbourhood $W$ of $O \in E$ there exists $C > 0$ such that

$$\|u\|_V^{*1+d} \leq C\|u\|_W^*\|u\|_U^{*d}$$

for $u \in E^*$, the dual space of $E$, where

$$\|u\|_K^* = \sup\{|u(x)| : x \in K\}$$

for every subset $K$ of $E$.

The properties $(H_u)$ and $(\tilde{\Omega})$ were introduced and investigated by Meise and Vogt in [**5**]. In the present paper we investigate the property $(H_u)$ and $(\tilde{\Omega})$ by the relation with the exponential representation of entire functions.

## 1. The property $(H_u)$ and the exponential representation of entire functions

In this section we shall prove the following

**Theorem.** *Let $E$ be a Frechet space. Then $E$ is nuclear and has the property $(H_u)$ if and only if every entire function on $E$ with values in a Banach space $B$ can be written in the form*

$$(\mathrm{Exp})_B f(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$$

*where the series is absolutely convergent in the space $H(E, B)$ of holomorphic functions on $E$ with values in $B$ equipped with the compact-open topology.*

*Proof:* First prove sufficiency of the theorem. Given $f \in H(E, B)$ with $B$ is a Banach space. Since $E$ is a Frechet space we can find a continuous semi-norm $\rho$ on $E$ such that

$$\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_\rho^* < \infty,$$

with

$$\|u\|_\rho^* = \sup\{|u(x)| : \rho(x) < 1\}.$$

Indeed, in the converse case let $\{\|\cdot\|_p\}$ is a fundamental system of semi-norms on $E$. Then for every $p$ we have

$$\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_p^* = \infty.$$

Hence for every $p$ there exists $k_p$ such that

$$\sum_{k\leq k_p} \|\xi_k\| \exp \|u_k\|_p^* > p.$$

This inequality implies that for each $k \leq k_p$ there exists $x_k^p$ with $\|x_k^p\|_p \leq 1$ such that

$$\sum_{k\leq k_p} \|\xi_k\| \exp |u_k(x_k^p)| > p.$$

Put

$$K = \{x_1^1, \ldots, x_{k_1}^1, \ldots, x_1^p, \ldots, x_{k_p}^p, \ldots\} \cup \{0\}.$$

Then $K$ is compact in $E$ and

$$\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_K^* > p \text{ for every } p \geq 1.$$

This is impossible, because

$$\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_K^* < \infty.$$

Thus the form

$$\sum_{k\geq 1} \xi_k \exp u_k(x) \text{ for } x \in E_\rho, \qquad \|x\| < 1$$

defines a holomorphic function on $U_\rho$, the open unit ball in $E_\rho$ which is Gateaux holomorphic on $E/\operatorname{Ker}\rho$.

Let $x \in E_\rho$. Put

$$W = \{(1-t)y + tx : t \in \mathbb{C}\backslash\{0\},\, y \in U_\rho\}.$$

Then $W$ is a non-empty open set in $E_\rho$. Hence there exists $z \in W \cap E/\operatorname{Ker}\rho$.

Let

$$z = (1-t_0)y_0 + t_0x_0$$

with $y_0 \in U_\rho$, $t_0 \in \mathbb{C}\backslash\{0\}$.

Then

$$x = z/t_0 + ((1-t_0)/t_0)y_0$$

and hence

$$\begin{aligned}
&\sum_{k\geq 1} \|\xi_k\| \exp |u_k(x)| \leq \\
\leq &\sum_{k\geq 1} \|\xi_k\| \exp[|(1/t_0)|\,|u_k(z)| + |(1-t_0)/t_0)|\,|u_k(y_0)|] \leq \\
\leq &\sum_{k\geq 1} \|\xi_k\| [\exp(2/|t_0|)|u_k(z)| + \exp 2|(1-t_0)/t_0|\,|u_k(y_0)|] < \\
< &\infty.
\end{aligned}$$

Thus

$$g = \sum_{k\geq 1} \xi_k \exp u_k$$

is a Gateaux holomorphic function on $E_\rho$. Since $g$ is holomorphic on $U_\rho$ by the Zorn Theorem [**6**], $g$ is holomorphic on $E_\rho$. Obviously $f = g\omega_\rho$ and hence $f$ is of uniform type.

To prove the nuclearity of $E$ for every continuous semi-norm $\rho$ on $E$ write the canonical map $\omega_\rho : E \to E_\rho$ in the form

$$\omega_\rho(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$$

in which

$$\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_K^* < \infty$$

for every compact set $K$ in $E$.

Then

$$\omega_\rho(x) = \sum_{k\geq 1} \xi_k u_k(x) \text{ for } x \in E$$

and

$$\sum_{k\geq 1} \|\xi_k\| \, \|u_k\|_K^* < \infty \text{ for every compact set } K \subset E.$$

As above there exists a continuous semi-norm $\beta > \rho$ on $E$ such that

$$\sum_{k\geq 1} \|\xi_k\| \, \|u_k\|_\beta^* < \infty.$$

This means that the canonical map $\omega_{\beta,\rho}$ from $E_\beta$ to $E_\rho$ is nuclear. Hence $E$ is nuclear.

Assume that $E$ is nuclear and has the property $(H_u)$. Given $f \in H(E,B)$, with $B$ is a Banach space. By hypothesis there exists a continuous semi-norm $\rho$ on $E$ and a holomorphic function $g$ on $E_\rho$ such that $f = g\omega_\rho$. Take a continuous semi-norm $\beta > \rho$ on $E$ such that $T = \omega_{\beta,\rho}$ is nuclear. Write

$$T(x) = \sum_{j\geq 1} t_j u_j(x) e_j$$

with

$$a = \sum_{j\geq 1} |t_j| < \infty \text{ and } \|u_j\| + \|e_j\| \leq 1 \text{ for } j \geq 1.$$

Consider the Taylor expansion of $g$ at $O \in E$,

$$g(x) = \sum_{n\geq 0} P_n g(x)$$

with

$$P_n g(x) = (1/2\pi i) \int_{|t|=r} (g(tx)/t^{n+1})\, dt.$$

Choose the two sequences $\{\xi_k\}$ and $\{\alpha_k\}$ in $\mathbb{C}$ such that

$$z = \sum_{k \geq 1} \xi_k \exp \alpha_k z \text{ for } z \in \mathbb{C}$$

and

$$C_r = \sum_{k \geq 1} |\xi_k| \exp r|\alpha_k| < \infty \text{ for all } r \geq 0.$$

Such sequence exist by [**2**]. Formally we have

$$\begin{aligned}
(gT)(x) = g(Tx) &= \sum_{n \geq 0} P_n g(Tx) = \sum_{n \geq 0} P_n g \left( \sum_{j \geq 1} t_j u_j(x) e_j \right) = \\
&= \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} t_{j_1} \dots t_{j_n} u_{j_1}(x) \dots u_{j_n}(x) P_n g(e_{j_1}, \dots, e_{j_n}) = \\
&= \sum_{n \geq 0} \sum_{j_1, \dots, t_{j_n} \geq 1} t_{j_1} \dots t_{j_n} P_n g(e_{j_1}, \dots, e_{j_n}) \\
&\quad \left( \sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_1}(x) \right) \dots \left( \sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_n}(x) \right) = \\
&= \sum_{n \geq 0} \sum_{\substack{j_1, \dots, j_n \geq 1 \\ k_1, \dots, k_n \geq 1}} t_{j_1} \dots t_{j_n} \dots \xi_{k_1} \dots \xi_{k_n} \cdot \\
&\quad \cdot P_n g(e_{j_1}, \dots, e_{j_n}) \exp[\alpha_{k_1} u_{j_1}(x) + \dots + \alpha_{k_n} u_{j_n}(x)].
\end{aligned}$$

It remains to check that the right hand side is absolutely convergent in $H(E, B)$. For each $r > 0$ take $s > C_r$ a.e. Since

$$\|P_n g(e_{j_1}, \dots, e_{j_n})\| \leq (n^n / n! s^n) \|g\|_s$$

where

$$\|g\|_s = \sup\{\|g(x)\| : \|x\| < s\},$$

and without loss of generality by the nuclearity of $E$, we may assume that $g$ is bounded on every bounded set in $E_\rho$, we have

$$\begin{aligned}
&\sum_{n \geq 0} \sum_{\substack{j_1, \dots, j_n \geq 1 \\ k_1, \dots, k_n \geq 1}} |t_{j_1}| \dots |t_{j_n}| \, |\xi_{k_1}| \dots |\xi_{k_n}| \cdot \\
&\qquad \cdot \|P_n g(e_{j_1}, \dots, e_{j_n})\| \exp r[|\alpha_{k_1}| + \dots + |\alpha_{k_n}|] \leq \\
&\qquad\qquad \leq \left( \sum_{n \geq 0} C_r^n a^n n^n / n! s^n \right) \|g\|_s < \infty \text{ for } \|x\| \leq r.
\end{aligned}$$

The theorem is completely proved. ■

## 2. The property $(\tilde{\Omega})$ and the exponential representation of entire functions

The relation between the property $(\tilde{\Omega})$ and the exponential representation of entire functions is given by

**Theorem 2.1.** *Let $E$ be a nuclear Frechet space having the approximation property. Then $E$ has the property $(\tilde{\Omega})$ if and only if there exists a balanced convex compact set $B$ in $E$ such that*

(i) *$E(B)$ is dense in $E$, where $E(B)$ denotes the Banach space spanned by $B$,*

(ii) *every holomorphic function on $(E(B), \tau_E)$, where $\tau_E$ is the topology of $E(B)$ induced by the topology of $E$, can be written in the form*

$$(\text{Exp}) : \sum_{k\geq 1} \xi_k \exp u_k$$

*in which the series is absolutely convergent in $H(E(B), \tau_E)$.*

*Proof:* Since every nuclear Frechet space having the property $(\tilde{\Omega})$ has also the property $(H_u)$ [**5**], and since every holomorphic function on $(E(B), \tau_E)$ can be extended holomorphically to $E$ [**5**], where $B$ is a balanced compact set in $E$ as in [**5**], the necessity of the theorem is as in Theorem 1.1.

Conversely, by [**5**] it suffices to show that every holomorphic function on $(E(B), \tau_E)$ is holomorphic on $E$. As in Theorem 1.1 there exists a continuous semi-norm $\rho$ on $E$ such that

$$\sum_{k\geq 1} |\xi_k| \exp \|u_k\|^*_{U_\rho \cap E(B)} < \infty.$$

Since $E(B)$ is dense in $E$, it follows that $U_\rho \cap E(B)$ is dense in $U_\rho$, and hence

$$\sum_{k\geq 1} |\xi_k| \exp \|u_k\|^*_{U_\rho} < \infty.$$

Given $x \in E$. As in Theorem 1.1 put

$$W = \{(1-t)y - tx : t \in \mathbb{C}\backslash\{0\},\, y \in U_\rho\}.$$

Then $W$ is an non-empty open set in $E$ and hence there exists $z \in W \cap E(B)$. Let

$$z = (1 - t_0)y_0 + t_0 x \text{ with } t_0 \in \mathbb{C}\backslash\{0\}, \qquad y_0 \in U_\rho.$$

Hence

$$\sum_{k\geq 1} |\xi_k| \exp |u_k(x)| \leq \sum_{k\geq 1} |\xi_k| \exp[|u_k(z)/t_0| + |(t_0 - 1)/t_0|\, |u_k(y_0)|] \leq$$
$$\leq \sum_{k\geq 1} |\xi_k| \exp 2|u_k(z)/t_0| + \exp 2|t_0 - 1/t_0|\, |u_k(y_0)|] < \infty.$$

By the Zorn Theorem [**6**], it follows that $f$ is holomorphic on $E$. Theorem 2.1 is proved. ■

## 3. The property $(H_u)$ and $(\tilde{\Omega})$

**Proposition 3.1.** *Let $E$ be a Frechet-Schwartz space with the property $(H_u)$. Then every holomorphic function on $E$ with values in a Banach space is of uniform type.*

*Proof:* Write $E = \operatorname{limproj} E_n$, where $E_n$ are Banach spaces such that $E$ is dense in $E_n$ for every $n \geq 1$ and the canonical maps $\omega_{n+1,n} : E_{n+1} \to E_n$ are compact. By hypothesis the canonical map

$$S : \operatorname{limind} H_b(E_n) \longrightarrow [H(E)]_{\text{bor}}$$

where $[H(E)]_{\text{bor}}$ denotes the bornological space associated to $H(E)$ and $H_b(E_n)$ for each $n \geq 1$ is the Frechet space of holomorphic functions on $E_n$ which are bounded on every bounded set in $E_n$, is a continuous bijection. Since $H(E)$ is complete, $[H(E)]_{\text{bor}}$ is untrabornological. By the open mapping theorem $S$ is an isomorphism. Given $f : E \to B$ a holomorphic function, where $B$ is a Banach space. Consider the continuous linear map $\hat{f} : B^* \to H(E)$ associated to $f$. Then $\hat{f} : B^* \to [H(E)]_{\text{bor}}$ is continuous. Since $S$ is isomorphic, we can find $n_0$ such that $\operatorname{Im} \hat{f} \subseteq H_b(E_{n_0})$ and $\hat{f} : B^* \to H_b(E_{n_0})$ is continuous. This yields

$$\sup\{|uf(x)| : \|u\| \leq 1,\ \|x\| \leq r\} =$$
$$= \sup\{|\hat{f}(u)(x) : \|u\| \leq 1,\ \|x\| \leq r\} < \infty$$

for all $r \geq 0$.

Thus $f$ induces a holomorphic function $g : E_{n_0} \to B$ such that $g\omega_{n_0} = f$. ■

**Remark.** Proposition 3.1 is a particular case of a recent result of Galindo, Garcia and Maestre [**3**].

**Theorem 3.2.** *Let $E$ be a nuclear Frechet space with the property $(\tilde{\Omega})$ and $F$ a Schwartz space with $F \in (H_u)$. Then $E \times F \in (H_u)$.*

We need the following

**Lemma 3.3.** *Let $E$ be a nuclear Frechet space with the property $(\tilde{\Omega})$ and $F$ a Banach space. Then every holomorphic function on $F \times E$ which is bounded on every bounded set in $F \times E$ is of uniform type.*

*Proof:* Lemma 3.3 will be proved as in [**5**] by use Lemmas 3.1 and 3.2 in [**5**]. Indeed, choose $p$ and $\delta > 0$ such that if $f$ is bounded on $B_\delta \times U_p$, where $f$ is a holomorphic function on $F \times E$ as in the lemma and $B_\delta = \{z \in F : \|z\| < \delta\}$. Since $E \in (\tilde{\Omega})$, by Vogt [**8**] there exists a balanced convex compact set $K$ in $E$ such that

$$\|\cdot\|_q^{*1+d} \leq \|\cdot\|_K^* \|\cdot\|_p^*$$

for some $q > p$ and $d > 0$.

We can assume that $E(K)$, $E_q$ and $E_p$ are Hilbert spaces. Write the canonical map $A$ from $E(K)$ to $E_p$ in the form

$$A(x) = \sum_{j\geq 1} \lambda_j (x|e_j)_{E(K)} y_j$$

where $\{e_j\}$ is a complete orthonormal system in $E$ and $\{y_j\}$ a orthonormal system in $E_p$ and $\lambda = (\lambda_j) \in s$. Let $\varphi_j$ denote the continuous linear functional on $E_q$ induced by $y_j$. Then

$$\|\varphi_j\|^{*1+d} \leq |\lambda_j| \text{ for } j \geq 1.$$

Take $0 < \varepsilon < \delta$ such that for $\mu = (\varepsilon/j)$ we have

$$\left\{x \in E : x = \sum_{j\geq 1} \xi_j y_j : |\xi_j| \leq \mu_j \text{ for } j \geq 1\right\} \subset \{x \in E_p : \|x\| < 1\}.$$

Put

$$M = \{m = (m_1, \dots, m_n, 0, \dots)\}$$

for each $k \geq 0$ and $m \in M$ put

$$\begin{aligned} a_{k,m}(z) =& (1/2\pi i)^{n+1} \int_{|\tau|=1} \int_{|\rho_1|=\mu_1} \cdots \int_{|\rho_n|=\mu_n} \\ & \frac{g(\tau z, \rho_1 y_1 + \cdots + \rho_n y_n)}{\tau^{k+1} \rho_1^{m_1+1} \dots \rho_n^{m_n+1}} d\tau d\rho_1 \dots d\rho_n \\ =& (1/\lambda^m)(1/2\pi i)^{n+1} \int_{|\tau|=1} \int_{|w_1|=r_1} \cdots \int_{|w_n|=r_n} \\ & \frac{f(z, w_1 e_1 + \cdots + w_n e_n)}{\tau^{k+1} w_1^{m_1+1} \dots w_n^{m_n+1}} d\tau dw_1 \dots dw_n \end{aligned}$$

where $g$ is the holomorphic function on $B_\delta \times \{y \in E_p : \|y\| < 1\}$ is induced by $f$ and

$$\lambda^m = \lambda_1^{m_1} \dots \lambda_n^{m_n}.$$

For $s$, $t > 0$ put

$$B(s,t) = B_s \times \left\{ x \in E : x = \sum_{j \geq 1} \xi_j e_j,\ |\xi_j| \leq t\mu_j \text{ for } j \geq 1 \right\}.$$

By hypothesis

$$N(s,t) = \sup\{|f(w)| : w \in B(s,t)\} < \infty$$

and hence

$$\sup\{|a_{k,m}(z)| : \|z\| < s\} \leq N(s,t)/\lambda^m \mu^m t^{|m|}$$

with

$$|m| = m_1 + \dots + m_n.$$

Let $\eta = 1/1+d$, $\nu = \gamma = \eta/2$, $\beta = 1-\gamma$. Given $s > 0$. Take $\sigma > 0$, such that $\sigma^\gamma \varepsilon^\beta > s$.

Since $\lambda \in s$, the sequence $(\lambda_j^\nu/\mu_j) = (j\lambda_j^\nu/\varepsilon) \in l^1$ and hence

$$R = \sup\{|\lambda_k^\nu| \mu_k^{-1} : k \geq 1\} < \infty.$$

Put $t = (2Rr)^{1/\gamma}$. Then as in [**5**] we have

$$\begin{aligned} &\sum_{m \in M} \sum_{k \geq 0} r^{|m|} \sup_{z \in B_s} |a_{k,m}(z)| \prod_{j \geq 1} \|\varphi_j\|^* m_j \leq \\ &\leq \sum_{m \in M} \sum_{k \geq 0} r^{|m|} ((s/\sigma)^k N(\sigma,t)/\mu^{m_t |m|})^\gamma (|\lambda|^m)^\nu (M(s/\varepsilon)^k/\mu^m)^\beta = \\ &\qquad = N(\sigma,t)^\gamma M^\beta \left[ \sum_{k \geq 0} (s/\sigma^\gamma \varepsilon^\beta)^k \right] \prod_{k \geq 1} (1 - |\lambda_k^\nu/2R\mu_k)^{-1} < \infty \end{aligned}$$

where

$$N = \sup\{|f(w)| : w \in B_\delta \times U_p\}.$$

As in [**5**] this implies the series

$$\sum_{m \in M} \sum_{k \geq 0} a_{k,m}(z) \prod_{j \geq 1} \varphi_j(x)^{m_j}$$

converges normanly on all sets

$$B_s \times \{x \in E_q : \|x\| < r\}, \qquad s, r > 0.$$

Hence it defines a holomorphic function $h$ on $F \times E_q$ such that

$$f(z,x) = h(z, \omega_q(x)) \text{ for } (z,x) \in F \times E.$$

The lemma is proved. ■

Now we can prove Theorem 3.2 as follows.

Given $f \in H(F \times E)$. (i) First show that there exists a neighbourhood $U$ for $O \in E$ such that $f$ is bounded on $B \times U$ for every bounded set $B$ in $F$. In the converse case for each $p$ there exists a bounded set $K_p$ in $F$ such that $f$ is not bounded on $K_p \times U_p$. Choose $\varepsilon_j \downarrow 0$ such that

$$K = \overline{\text{conv}} \bigcup_{j \geq 1} \varepsilon_j K_j$$

is bounded. Consider the holomorphic function $g = f | F(K) \times E$. Since every bounded set in $F(K)$ is bounded in $F$, it follows that $g$ is bounded on every bounded set in $F(K) \times E$. Lemma 3.3 implies there exists a neighbourhood $U$ of $O \in E$ such that $g$ is bounded on $B \times U$ for every bounded set $B$ in $F$. This is impossible.

(ii) Consider the function $\bar{f} : E \to H(F)$ associated to $f$. Then $\bar{f}$ is holomorphic and by (i) it is bounded at $O \in E$. Then as in [**5**] or as in Lemma 3.3 we can find $p$ such that $f$ can be factorized holomorphically through the canonical map $\omega_p$ from $E$ to $E_p$. Take $q > p$ such that $\omega_{q,p} : E_q \to E_p$ is nuclear. Write

$$\omega_{q,p}(z) = \sum_{j \geq 1} u_j(z) e_j$$

with

$$a = \sum_{j \geq 1} \|u_j\| \, \|e_j\| < \infty.$$

Consider the Taylor expansion of $f$ at $O \in E$ in the variable $z \in E$

$$f(z,x) = \sum_{n \geq 0} P_n f(z;x)$$

where

$$P_n f(z;x) = (1/2\pi i) \int_{|t|=r} (f(tz,x)/t^{n+1}) \, dt$$

for $(z,x) \in E \times F$.

We have

$$f(\omega_{q,p}(z),x) = \sum_{n\geq 0} P_n f\left(\sum_{j\geq 1} u_j(z)e_j; x\right) =$$
$$= \sum_{n\geq 0} \sum_{j_1,\ldots,j_n\geq 1} Pn_f(e_{j_1},\ldots,e_{j_n};x)u_{j_1}(z)\ldots u_{j_n}(z).$$

Moreover

$$\sum_{n\geq 0} s^n \sum_{j_1,\ldots,j_n\geq 1} \|u_{j_1}\|\ldots\|u_{j_n}\|\|P_n f(e_{j_1},\ldots,e_{j_n},\ldots)\|_K =$$
$$= \sum_{n\geq 0} s^n \sum_{j_1,\ldots,j_n\geq 1} \|u_{j_1}\|\,\|e_{j_1}\|\ldots\|u_{j_n}\|\,\|e_{j_n}\|$$
$$\|P_n f(e_{j_1/\|e_{j_1}\|},\ldots,e_{j_n/\|e_{j_n}\|},\ldots)\|_K \leq$$
$$\leq \left(\sum_{n\geq 0} s^n a^n n^n/\rho^n n!\right) \|f\|_{B_\rho\times K} < \infty$$

for all $\rho >$ aes and all compact set $K$ in $F$, where

$$\|f\|_{B_\rho\times K} = \sup\{|f(z,x)|;\ \|z\| < \rho,\ x\in K\}$$

and

$$\|P_n f(e_{j_1},\ldots,e_{j_n},\ldots)\|_K = \sup\{|P_n f(e_{j_1},\ldots,e_{j_n},x)|;\ x\in K\}.$$

Let $B = \{z \in E_q : \|z\| = 1\}$. Consider the function

$$f : \mathbb{C}\times\tilde{F} \to 1^\infty(B) \text{ with } F \cong \mathbb{C}\times\tilde{F},$$

given by

$$\tilde{f}(t,x) = \left\{\sum_{n\geq 0} t^n \sum_{j_1,\ldots,j_n\geq 1} P_n f(e_{j_1},\ldots,e_{j_n},x)u_{j_1}(z)\ldots u_{j_n}(z)\right\}_{z\in B}.$$

For each $N \in \mathbb{N}$ put

$$S_N(t,x) = \left\{\sum_{n\leq N} t^n \sum_{j_1,\ldots,j_n\geq 1} P_n f(e_{j_1},\ldots,e_{j_n},x)u_{j_1}(z)\ldots u_{j_n}(z)\right\}_{z\in B}.$$

Since for every $k \geq 1$, the functions

$$S_{n,k}(t,x) = \sum_{n \leq N} t^n \sum_{j_1+\cdots+j_n \leq k} P_n f(e_{j_1}, \ldots, e_{j_n}, x) u_{j_1}(z) \ldots u_{j_n}(z)$$

are holomorphic on $F$ with values in $1^\infty(B)$ and

$$S_{n,k} \to S_N \text{ as } k \to \infty$$

uniformly on every compact set in $F$, we infer that $S_N$ is holomorphic for $N \geq 1$. On the other hand, since $S_N \to \tilde{f}$ uniformly on compact set in $F$, it follows that $\tilde{f}$ is holomorphic. By Proposition 3.1 there exists a continuous semi-norm $\rho$ on $F$ and a holomorphic function $\tilde{g}$ on $F$ with values in $1^\infty(B)$ such that

$$\tilde{f}(t,x) = \tilde{g}(t, \omega_\rho(x)) \text{ for } (t,x) \in \mathbb{C} \times \tilde{F}.$$

We may assume that $\tilde{g}$ is bounded on every bounded set in $\mathbb{C} \times \tilde{F}$, because $F$ is Schwartz. Then

$$\begin{aligned}
&\sup\{|f(z,x)| : \|z\| \leq s,\ \rho(x) \leq s\} = \\
&= \sup\{|f(tz,x)| : |t| \leq s,\ z \in B,\ \rho(x) \leq s\} = \\
&= \sup\left\{\left|\sum_{n\geq 0} t^n \sum_{j_1,\ldots,j_n \geq 1} P_n f(e_{j_1}, \ldots, e_{j_n}, x) \cdot u_{j_1}(z) \ldots u_{j_n}(z)\right| :\right. \\
&\qquad\qquad\qquad\qquad \left. : |t| \leq s,\ z \in B,\ \rho(x) \leq s \right\} = \\
&= \sup\{\|\tilde{f}(t,x)\| : |t| \leq s,\ \rho(x) \leq s\} = \\
&= \sup\{\|\tilde{g}(t,x)\| : |t| \leq s,\ \rho(x) \leq s\} < \infty
\end{aligned}$$

for all $s \geq 0$.

Consequently $f$ is of uniform type.

The theorem is proved. ■

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Department of Mathematics
Pedagogical University Hanoi I
Tu liem, Hanoi
VIETNAM

Primera versió rebuda el 30 d'Abril de 1993,
darrera versió rebuda el 30 d'Agost de 1993