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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS

Jerzy Popenda and Ewa Schmeidel

Abstract

In the paper sufficient conditions for the difference equation

$$\Delta x_n = \sum_{i=0}^{r} a_n^{(i)} x_{n+i}$$

to have a solution which tends to a constant, are given. Applying these conditions, an asymptotic formula for a solution of an m-th order equation is presented.

In this paper we give a sufficient condition for any linear difference equation which can be treated as a first order equation with advanced arguments to possess solutions which tend to arbitrary real constants. Using this theorem we shall study a linear m-th order difference equation and obtain a solution with a particular asymptotic behaviour. The main idea consists in seeing that any m-th order difference equation can be studied as a first order one with perturbed arguments.

To start with, we consider the difference equation

$$(E_1) \qquad \Delta x_n = \sum_{i=0}^{r} a_n^{(i)} x_{n+i}, \quad n \in N.$$

Here by N, R we denote the set of positive integers and reals respectively. For any function $y : N \to R$ the difference operator Δ is defined as follows

$$\begin{aligned} \Delta y_n &= y_{n+1} - y_n, && n \in N, \\ \Delta^i y_n &= \Delta(\Delta^{i-1} y_n), && \text{for } i > 1. \end{aligned}$$

Difference equations of the type (E_1) arise in numerical methods for solving differential equations with advanced arguments. In economics and biology there are simple discrete models describing processes in which actual growth of considered parameter could be characterized in terms of its future states. Such models can be used in forecasting, and also to control actual growth to get the desired quantities at some fixed periods. Moreover as it is shown below the results for (E_1) can be used with success in studying typical equations.

To simplify formulae we use the conventional assumption that the void sum is equal to zero, while the void product is equal to one, that is

$$\sum_{j=n}^{k} y_j := 0, \quad \prod_{j=n}^{k} y_j := 1$$

for any k, $n \in N$, $k < n$, and any sequence y.

Instead of $\lim_{n\to\infty} x_n = C$ we shall write $x_n = C + o(1)$ and if $\lim_{n\to\infty}(x_n/y_n) = C$ we write $x_n = y_n(C + o(1))$.

Theorem 1. *Let* $a^{(i)} : N \to R$, $a_n^{(0)} \neq -1$ *for every* $n \in N$, *and*

$$\sum_{j=1}^{\infty} |a_j^{(i)}| < \infty, \tag{1}$$

for $i = 0, 1, \ldots, r$. *Then for any arbitrary constant* $C \in R$, $C \neq 0$ *there exists a solution* x *of* (E_1) *such that*

$$x_n = C + o(1). \tag{2}$$

Proof: Let us see that if u is a solution of (E_1) such that $u_n = C + o(1)$ with $C > 0$, then the sequence $v_n = -u_n$, for all $n \in N$ is also a solution of (E_1) possessing the same type of asymptotic behavior with the limit $-C < 0$ instead of $C > 0$.

We prove our theorem for the case $C > 0$.

Since $C > 0$ there exists a positive constant ε such that $C - \varepsilon > 0$. Let us take

$$C_1 = C + \varepsilon, \quad I = [C - \varepsilon, C + \varepsilon],$$

$$\alpha_n = C_1 \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^{(i)}|, \quad n \in N. \tag{3}$$

From (1) it follows that there exists $n_1 \in N$ such that for all $n \geq n_1$ we have $\alpha_n \leq \varepsilon$.

Let l_∞ denote the Banach space of bounded sequences $x = (h_i)_{i=1}^\infty$ with the norm $\|x\| = \sup\limits_{i\geq 1}|h_i|$.

Moreover, let $T \subset l_\infty$ be any set such that

$$x = \{h_i\}_{i=1}^\infty \in T \text{ if } \begin{cases} h_t = C & \text{for } t = 1, 2, \ldots, n_1 - 1 \\ h_t \in I_t & \text{for } t \geq n_1 \end{cases},$$

where $I_t = [C - \alpha_t, C + \alpha_t]$.

It is easy to check that T is bounded, convex, and closed in l_∞. Furthermore by (1) and (3) it follows that $\operatorname{diam} I_t = 2\alpha_t \to 0$ with $t \to \infty$. So, for arbitrary $\varepsilon_1 > 0$ we can set up a finite ε_1-net for the set T. Hence by Hausdorff's theorem T is compact. Define now some operator A by $Ax = y = \{b_n\}_{n=1}^\infty$, where

$$b_n = \begin{cases} C, & \text{for } n = 1, 2, \ldots, n_1 - 1 \\ C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^{(i)} h_{j+i} & \text{for } n \geq n_1 \end{cases}$$

Let us see, using (1), that A is well defined on the space l_∞. Furthermore for $x \in T$ we obtain

$$|b_n - C| \leq \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^{(i)}|\, |h_{j+i}| \leq C_1 \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^{(i)}|,$$

because $h_{j+i} \in I_{j+i} \subset I$, for all $j \geq n_1$, $i \in \{0, \ldots, r\}$. Hence, by (3)

$$C - \alpha_n \leq b_n \leq C + \alpha_n,$$

which means that $b_n \in I_n$ for $n \geq n_1$. That is, A maps the set T into T. We now prove that A is continuous on T:

Take $\varepsilon_1 > 0$ and $\delta_1 = \frac{C_1 \varepsilon_1}{\alpha_{n_1}}$.

Let $x = \{h_i\}_{i=1}^\infty$ and $y = \{g_i\}_{i=1}^\infty$ be any two elements of the set T such that $\|x - y\| < \delta_1$. Then the absolute convergence of the series

$$\sum_{i=0}^{r} \sum_{j=n_1}^{\infty} a_j^{(i)} h_{j+i}, \quad \sum_{i=0}^{r} \sum_{j=n_1}^{\infty} a_j^{(i)} g_{j+i}$$

yields

$$\|Ax - Ay\| = \sup_{n \geq n_1} \left| \left[C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^{(i)} h_{j+i} \right] - \left[C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^{(i)} g_{j+i} \right] \right| \leq$$

$$\leq \sup_{n \geq n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^{(i)}| \, |h_{j+i} - g_{j+i}| \leq$$

$$\leq \sup_{n \geq n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^{(i)}| \, \|x - y\| \leq$$

$$\leq \delta_1 \sup_{n \geq n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^{(i)}| = \varepsilon_1.$$

Therefore the operator A is continuous on T, and by Schauder fixed point theorem we obtain that there exists in the set T a solution of the equation $x = Ax$.

Let $z = \{d_i\}_{i=1}^{\infty}$ denote such a solution.

Since $z \in T$, it can be written as follows

$$z = \{C, \ldots, C, d_{n_1}, d_{n_1+1}, \cdots\}$$

and

$$Az = \left\{ C, \ldots, C, \, C - \sum_{i=0}^{r} \sum_{j=n_1}^{\infty} a_j^{(i)} d_{j+i}, \ldots, C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^{(i)} d_{j+i}, \cdots \right\}.$$

Therefore

$$d_n = C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^{(i)} d_{j+i}, \quad \text{for } n \geq n_1. \tag{5}$$

Applying the operator Δ to (5) we obtain

$$\Delta d_n = \sum_{i=0}^{r} a_n^{(i)} d_{j+i}, \quad n \geq n_1.$$

This means that the sequence $\{d_n\}_{n=1}^{\infty}$ fulfills equation (E_1) but for $n \geq n_1$ only.

The equation (E_1) is a readily transformed to

$$x_n = -(1 + a_n^{(0)})^{-1} \left[(a_n^{(1)} - 1) x_{n+1} + \sum_{i=2}^{r} a_n^{(i)} x_{n+i} \right], \quad n \in N. \tag{6}$$

Substituting in (6) $n = n_1 - 1$, $x_n = d_n$ for $n \geq n_1$ we obtain x_{n_1-1}. Proceding in this way we find $x_{n_1-2}, \ldots, x_1$ one after the other. Consequently we get the sequence which fulfills (E_1) for all $n \in N$. Moreover this sequence coincides with z for $n \geq n_1$, hence it has the asymptotic behaviour (2) because $d_n \in I_n$ and $\operatorname{diam} I_n \to 0$ as $n \to \infty$. ■

A similar method and property for the difference equation $\Delta^2 x_n + a_n F(x_n) = 0$ can be found in [**1**].

Now we use the previous theorem to study solutions of the m-th order difference equation

$$(E_2) \qquad \Delta^m x_n = a_n x_n, \quad n \in N, \quad m \geq 2.$$

Theorem 2. *Let $a : N \to R$ be such that $(-1)^m a_n \neq 1$ for all $n \in N$ and*

$$\sum_{n=1}^{\infty} \prod_{j=1}^{k-1} |1 + (-1)^{m+1} a_{n+j}| < \infty \text{ for } k = 2, \ldots, m,$$

then for arbitrary constant $C \neq 0$ there exists a solution x of (E_2) which possesses the asymptotic behaviour

$$x_n = \left\{ m^{-n} \prod_{j=1}^{n-1} [1 + (-1)^{m+1} a_j] \right\} (C + o(1)), \quad n \in N.$$

Proof: Similarly as in the proof of Theorem 1 we shall concentrate on the case $C > 0$.

By formula

$$\Delta^m y_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} y_{k+m-i},$$

we can transform equation (E_2) to the following form

$$(-1)^{m-1} m x_{n+1} + (-1)^m x_n - a_n x_n = - \sum_{i=0}^{m-2} (-1)^i \binom{m}{i} x_{n+m-i}, \quad n \in N.$$

Hence

(7)

$$(-1)^{m-1} m x_{n+1} - [(-1)^{m+1} + a_n] x_n = \sum_{i=0}^{m-2} (-1)^{i+1} \binom{m}{i} x_{n+m-i}, \quad n \in N.$$

Multiplying (7) by $(-1)^{m+1}m^n$ and setting $z_n = m^n x_n$ we obtain from (7)

$$(8)\quad z_{n+1} - [1 + (-1)^{m+1}a_n]z_n = \\ = \sum_{i=0}^{m-2} (-1)^{m+i} \binom{m}{i} m^{-m+i} z_{n+m-i}, \quad n \in N.$$

Now multiplication by

$$\prod_{j=1}^{n} [1 + (-1)^{m+1}a_j]^{-1}$$

and the substitution

$$v_n = z_n \prod_{j=1}^{n-1} [1 + (-1)^{m+1}a_j]^{-1}$$

yield

$$(9)\quad \Delta v_n = \\ = \sum_{i=0}^{m-2} (-1)^{m+i} \binom{m}{i} m^{-m+i} \left\{ \sum_{j=n+1}^{n+m-i-1} [1 + (-1)^{m+1}a_j] \right\} v_{n+m-i}, \; n \in N.$$

It is evident that (9) is of the form (E_1) and all assumptions of Theorem 1 hold. Therefore for arbitrary constant $C \neq 0$ there exists a solution v of (9) such that

$$v_n = C + o(1),$$

and as

$$v_n = x_n m^n \prod_{j=1}^{n-1} [1 + (-1)^{m+1}a_j]^{-1}$$

we have in conclusion

$$x_n m^n \prod_{j=1}^{n-1} [1 + (-1)^{m+1}a_j]^{-1} = C + o(1). \quad \blacksquare$$

As an example consider the equation

$$\Delta^2 x_n = \left(1 - \frac{1}{n^2}\right) x_n, \quad n \in N.$$

By Theorem 2 it follows that for arbitrary $C \neq 0$ there exists a solution of this equation such that

$$x_n = 2^{-n} \frac{1}{(n-1)!^2} (C + o(1)), \quad n \in N.$$

References

1. A. DROZDOWICZ AND J. POPENDA, Asymptotic behaviour of solutions of the second order difference equation, *Proc. Amer. Math. Soc.* **99(1)** (1987), 135–140.

Institute of Mathematics
Technical University
ul. Piotrowo 3a
60-965 Poznan
POLAND

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